Topology of Complex Polynomials and Jacobian Conjecture

A.D.R. Choudary

Mathematics Department, Central Washington University, Ellensburg, WA 98926, USA

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Abstract


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To understand the problems, let us first look at some examples.

Example 1. Let \( f(x, y) = xy; x, y \in \mathbb{C} \). Then \( a = (0, 0) \) is the unique critical point with critical value \( f(a) = 0 \). Now consider the topology of the fiber \( F_t = f^{-1}(t) \).

If \( t = 0 \), then
\[
F_0 = \{(x, y) \in \mathbb{C}^2: xy = 0\} = \mathbb{C} \cup \mathbb{C}.
\]

If \( t \neq 0 \), then
\[
F_t = \{(x, y) \in \mathbb{C}^2: xy = t\}.
\]

Using the parameterization \( y = t/x \), we see that the map \( \phi: \mathbb{C}^* \rightarrow F_t \), defined as \( \phi(x) = (x, t/x) \), is a complex analytic isomorphism. In particular, \( \phi \) is a homeomorphism. Here \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \), therefore \( H^1(F_t; \mathbb{C}) = H^1(\mathbb{C}^*; \mathbb{C}) = H^1(S^1; \mathbb{C}) = \mathbb{C} \) and \( H^1(F_0; \mathbb{C}) = 0 \).
This example is not surprising, because we know in general, for a smooth mapping \(g : M \rightarrow N\), \(M, N\) smooth manifolds, that \(g^{-1}(t)\) is a smooth submanifold if \(t\) is not a critical value of \(g\).

**Example 2.** \(f(x, y) = x^2y + x; \ x, y \in \mathbb{C}\). There is no critical point in this case, so all the fibers are smooth. Now we have a closer look at the topology of the fibers.

\[
F_0 = \{(x, y) \in \mathbb{C}^2 : x^2y + x = 0\} = \mathbb{C} \sqcup \mathbb{C}^* \quad \text{(disjoint union)}.
\]

For \(t \neq 0\), using the parameterization technique as in example 1, we again find that \(F_t\) is homeomorphic to \(\mathbb{C}^*\), we denote it by \(F_t \sim \mathbb{C}^*\).

Moreover, \(H^1(F_0; \mathbb{C}) = H^1(F_t : \mathbb{C}) = \mathbb{C}\). In this case, the first cohomology group could not distinguish the special fiber \(F_0\) from the general fiber \(F_t\).

To understand what happens in the second example, we have to look for singular points at infinity.

For this, in our polynomial \(f : \mathbb{C}^2 \rightarrow \mathbb{C} : f(x, y) = x^2 + x\), we replace \(\mathbb{C}^2\) by \(P^2\), the projective complex plane and use the homogeneous coordinates \([x : y : z]\). \(F_t\) has equation \(x^2y + x - t = 0\) in \(\mathbb{C}^2\). If \(\overline{F_t}\) is the projective closure of \(F_t\) in \(P^2\), then we obtain the equation of \(\overline{F_t}\) in \(P^2\) by homogenization as \(x^2y + xz^2 - tz^3 = 0\), which is homogeneous in \(x, y, z\).

Let \(g_t(x, y, z) = x^2y + xz^2 - tz^3\). The singular points of the projective plane curve \(\overline{F_t}\) are given by the solutions of the system \(g_t,x = 0, g_t,y = 0, g_t,z = 0\), where \(g_t,x = \partial g_t / \partial x, \ etc\). There is only one solution namely \(p = [0 : 1 : 0]\).

We have in fact a family of plane curves \((\overline{F_t})_t \in \mathbb{C}\), all of them singular at the point \(p\).

For understanding why the value \(t = 0\) is special, we take a deeper look at the singular point \(p\). To do this, we work in local coordinates, \(u = x/y, v = z/y (y \neq 0\) in a neighborhood of \(p\). In these coordinates we set

\[
\frac{g_t(x, y, z)}{y^3} = u^2 + uv^2 - tv^3 = h(u, v).
\]

Let \(\mu(\overline{F_0}, p)\) be the Milnor number for the isolated hypersurface singularity. Then we have the well-known formula

\[
\mu(\overline{F_0}, p) = \dim \mathbb{C}[u, v] \overline{h_u, h_v} \quad \text{(by dim here, we refer to complex dimension)}.
\]

Where \(\mathbb{C}[u, v]\) represents the convergent power series in \(u, v\) and \((h_u, h_v)\) is the ideal spanned by \(\partial h / \partial u = h_u\) and \(\partial h / \partial v = h_v\).

To compute the Milnor number is quite difficult in general. However, if we have a weighted homogeneous polynomial, then there is a simple formula for this Milnor number.

We say that \(h(u, v)\) is weighted homogeneous of type \((\alpha, \beta, D)\) if \(\alpha, \beta,\) and \(D\) are positive integers,

\[
h(u, v) = \sum c_{ij} u^i v^j \quad \text{then} \quad c_{ij} \neq 0 \Rightarrow i \alpha + j \beta = D.
\]

If \(h\) is weighted homogeneous as above, then

\[
\mu = \frac{(D - \alpha)(D - \beta)}{\alpha \beta}.
\]
If $t = 0$, then $h = u^2 + uv^2$. Take $\alpha = 2$, $\beta = 1$, $D = 4$, we get $\mu = 3$.

Now for $t \neq 0$, things are a bit more complicated. The points $(i,j)$ are on a line if and only if $h$ is weighted homogeneous. But for $t \neq 0$, this is not the case. However, if there is a system of weights such that the polynomial $h$ has monomials only on a line and above it and such that the sum $S$ of the monomials on the line still defines an isolated singularity, then the same formula as above works.

In our case $S(u,v) = u^2 - tv^3$ and we have $\alpha = 3$, $\beta = 2$, and $D = 6$ for the sum $S$. This yields $\mu = 2$. So, for $t = 0$, $\mu = 3$ and $t \neq 0$, $\mu = 2$.

Hence for $t = 0$, the singularity at $p$ is worse than for $t \neq 0$, i.e., there is an extra singularity at the infinity for $t = 0$.

Assume that we are in the general case $f : \mathbb{C}^2 \to \mathbb{C}$. Then Thom remarked that there are only finitely many special fibers for $f$, i.e., there exist a finite subset $B$ in $\mathbb{C}$ such that:

$$S = \mathbb{C} \setminus B, \quad X = f^{-1}(S).$$

Then $f : X \to S$ is a $\mathbb{C}^\infty$ locally trivial fibration. In particular, all the fibers $F_t, t \in S$ are diffeomorphic.

The following result is a well-known theorem (Ha–Lê, 1984):

Assume that $f$ has only isolated singularities. Then for $t \in S$ we have, $F_t$ is a connected smooth affine curve such that:

$$\dim H^1(F_t; \mathbb{C}) = \sum \mu_a(f) + \sum \lambda_{\infty, p}(f),$$

where $a \in \{ f_x = f_y = 0 \} \subseteq \mathbb{C}^2$, is a finite set and $p \in L_{\infty}$ = the line at infinity, $\lambda_{\infty, p}$ is the jump in Milnor number at $p$.

In our example, this jump is 1.

We are working on the following open problem:

Find a similar description for the Betti numbers of general fiber in the case $f : \mathbb{C}^3 \to \mathbb{C}$.

Of course, the parameterization in this case is not useful. The new difficulty here is that the projective closure of the general fiber has in general Non-Isolated Singularities, and hence the use of Milnor numbers is no longer enough.

**The relation to the Jacobian Conjecture.** Statement of Jacobian Conjecture is as follows:

Let $f, g \in \mathbb{C}[x, y]$ such that

$$j(f, g) = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = \lambda \in \mathbb{C}^*,$$

then $(f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ is a bijection.

The condition $j(f, g) = \lambda \in \mathbb{C}^*$ implies that $f$ has no critical points.

If we manage to prove that there are no jumps of Milnor numbers at infinity, then we are done!

Because in this case, using the Ha–Lê theorem we get that the Betti numbers $b_1(F_t) = \dim H^1(F_t; \mathbb{C}) = \sum \mu_a(f) + \sum \gamma_{\infty, p}(f) = 0$. Now the Euler number $\chi(F_t) = b_0(F_t) - b_1(F_t) = 1$ and a result of Choudary–Dimca, 1994, states that if $\chi(F_t) = 1$ then $F_t \cong \mathbb{C}$. 
Also in this situation, theorem of Abhyankar–Moh allows that we can choose coordinates on $\mathbb{C}^2$ such that $x = f$. For this choice

$$j(f, g) = \begin{vmatrix} 1 & 0 \\ gs & gy \end{vmatrix} = g_y = \lambda \Rightarrow g(x, y) = \lambda y + k(x).$$

Therefore, $(f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ defined as $(x, y) \to (x, \lambda + k(x))$ is a bijection.

References