Journal of Combinatorial Theory, Series B **72**, 104–109 (1998)
Article No. TB971794

Independent Dominating Sets and a Second Hamiltonian Cycle in Regular Graphs

Carsten Thomassen

Department of Mathematics, Technical University of Denmark, Building 303, DK-2800, Lyngby, Denmark

Received January 22, 1997

In 1975, John Sheehan conjectured that every Hamiltonian 4-regular graph has a second Hamiltonian cycle. Combined with earlier results this would imply that

1. INTRODUCTION

A classical result of Cedric Smith (see [12]) says that every edge in a 3-regular graph is contained in an even number of Hamiltonian cycles. So, if a 3-regular graph has a Hamiltonian cycle, then it has a second (in fact even a third) Hamiltonian cycle. In 1978 Andrew Thomason [9] extended Smith's theorem to all r-regular graphs where r is odd (in fact, to all graphs in which all vertices have odd degree). Sheehan [8] made the conjecture that every Hamiltonian 4-regular graph has a second Hamiltonian cycle. As every r-regular graph (r even) is the union of pairwise edge-disjoint spanning 2-regular graphs, Sheehan's conjecture combined with the results of Smith and Thomason implies that every Hamiltonian regular graph other than a cycle has a second Hamiltonian cycle. (Smith's theorem and Sheehan's conjecture were rediscovered by Chen [2].)

Besides being of interest in its own right, a second Hamiltonian cycle may have an application to the chromatic polynomial. If C is a Hamiltonian cycle in the graph G and e is an edge not in C but in a second Hamiltonian cycle, then, clearly, both $G-e$ and G/e are Hamiltonian. In [10] it is conjectured that every Hamiltonian graph of minimum degree at least 3 contains an edge e such that both $G-e$ and G/e are Hamiltonian. If true, the chromatic polynomial of a Hamiltonian graph cannot have a root between 1 and 2.

Reference [11] contains a general sufficient condition for the existence of a second Hamiltonian cycle. In [11] that condition is combined with a

0095-8956/98 $25.00

result of Fleischner and Stiebitz [6] to show that every longest cycle in every 3-connected, 3-regular graph has a chord. An intermediate step verifies Sheehan's conjecture for those 4-regular graphs which are the union of a Hamiltonian cycle and pairwise disjoint triangles. In this note we prove the last statement of the abstract by combining the general condition in [11] with Lovász' Local Lemma [4].

2. RED-INDEPENDENT, GREEN-DOMINATING SETS AND A SECOND HAMILTONIAN CYCLE

Let G be a graph in which each edge is colored red or green. A vertex set S is called *red-independent* if no two vertices of S are joined by a red edge. We say that S is *green-dominating* if every vertex not in S is joined to a vertex in S by a green edge. With this notation we have:

THEOREM 2.1 [11]. *Let C be a Hamiltonian cycle in the graph H. Color all edges in C red and all edges not in C green. If H has a vertex set S which in both red-independent and green-dominating, then H has a second Hamiltonian cycle.*

We shall establish the existence of the set S in a large class of regular graphs using the version of Lovász' Local Lemma [4] presented in [7, page 79].

THEOREM 2.2 (Lovász' Local Lemma). *Let $A_1, A_2, ..., A_n$ be events in a probability space. Let G be a graph with vertex set $A_1, A_2, ..., A_n$ such that, for each $i = 1, 2, ..., n$, A_i is independent of any combination of events that are not neighbors of A_i in G. Suppose there exist positive real numbers $x_1, x_2, ..., x_n$ less than 1 such that, for each $i = 1, 2, ..., n$, the probability $P(A_i)$ of A_i satisfies*

$$P(A_i) < x_i \Pi (1 - x_j) \tag{1}$$

(*where the product is taken over all j for which A_j is a neighbor of A_i*). *Then*

$$P(\bar{A}_1 \wedge \bar{A}_2 \wedge \cdots \wedge \bar{A}_n) > 0.$$

THEOREM 2.3. *Let H be a graph whose edges are colored red or green. Assume that the spanning red subgraph is r-regular and the spanning green subgraph is k-regular. If $r \geqslant 3$ and*

$$k > 200 r \log r,$$

(*where* log *is the natural logarithm*), *then* H *has a vertex set* S *which is red-independent and green-dominating.*

Proof. Let p be a fixed real number, $0<p<1$. We define a probability space on all colorings of $V(H)=\{v_1, v_2, ..., v_m\}$ in colors 0 and 1. We define the probability of a coloring where a particular set of q vertices have color 1 and the remaining vertices have color 0 as

$$p^q(1-p)^{m-q}$$

For each red edge e of H we let A_e denote the event that the ends of e both have color 1.

Clearly

$$P(A_e)=p^2.$$

For each vertex v of H, let A_v denote the event that v and the vertices of H joined to v by green edges all have color 0. Clearly

$$P(A_v)=(1-p)^{k+1}.$$

The existence of S will follow from a proof that there is a positive probability that neither of the events A_e or A_v defined above occur. For this we use Theorem 2.2.

Let G be the graph whose vertices are the events A_e, A_v defined above such that two events are neighbors in G if and only if the corresponding vertex sets intersect. Clearly, G satisfies the assumption of Theorem 2.2.

We shall now associate real numbers to the events so that (1) in Theorem 2.2 is satisfied.

To each event A_e we associate a real number $x(0<x<1)$, and to each event A_v we associate a real number $y(0<y<1)$. For each event of the form A_e, condition (1) is implied by

$$p^2<x(1-x)^{2r-2}(1-y)^{2k+2} \tag{2}$$

For each event of the form A_v condition (1) is implied by

$$(1-p)^{k+1}<y(1-x)^{(k+1)r}(1-y)^{k^2}. \tag{3}$$

Now, if $p=1/5r$, $x=1/10r^2$ and $y=1/k^2$, then (2) is satisfied for all $k\geqslant 10$, and (3) is satisfied for $k>200r\log r$. ∎

Theorem 2.3 is formulated for $r\geqslant 3$ only. We now consider the cases $r=1, 2$.

3. THE CASE $r = 2$

If we combine the proof of Theorem 2.3 with Theorem 2.1 we get

THEOREM 3.1. *If G has a Hamiltonian cycle and is m-regular with $m \geqslant 300$, then G has a second Hamiltonian cycle.*

Proof. If $r = 2$, $k = 298$, and p, x, y are as in the proof of Theorem 2.3, then (2) and (3) are satisfied. ▮

The constant 300 in Theorem 3.1 can be lowered to a number closer to 200 than 300. With a better choice of p, x, y it can be further improved. A referee says that Theorem 2.3 works for $r = 2$ and $k = 80$ by taking $y = 0, 0001$, $x = (0, 179)^2$ and $p = 0, 171$. Adrian Bondy (private communication) says that it works for $k = 71$ by taking $y = (0, 89)^{72}$, $x = (0, 25)^2$ and $p = 0, 2305$.

If C is a (red) Hamiltonian cycle in a 4-regular graph G and $G - E(C)$ is the union of (green) cycles of length 4, then a green-dominating set must contain at least half of the vertices of G. So, if it is also red-independent it must contain every second vertex of C. But, it is easy to construct G such none of the two sets consisting of every second vertex of C is green-dominating. Therefore, Sheehan's conjecture can not be obtained by the method of this paper. But, perhaps the 4-regular case is the only case that needs a special method.

Problem 3.2. Does Theorem 2.3 hold for $r = 2$ and $k = 4$?

A Hamiltonian graph of minimum degree 3 need not contain any second Hamiltonian cycle, see [3, 5]. Bondy [1, Problem 7.14] asked if Sheehan's conjecture extends to graphs of minimum degree 4. It would also be of interest to verify this for 10^{10}, say, instead of 4. A result of this type does not seem to follow from the method of this note since Theorem 2.3 does not extend to graphs of large minimum degree k even for $r = 1$. To see this, let $e_1, e_2, ..., e_k$ denote k red edges with no end in common. For each choice $x_1, x_2, ..., x_k$, where x_i is an end of e_i for $i = 1, 2, ..., k$, we add two vertices v_1, v_2, joined by a red edge such that both v_1, v_2 are joined to precisely $x_1, x_2, ..., x_k$ by green edges. The resulting graph has green minimum degree k and it has no red-independent, green-dominating set.

Adrian Bondy and Bill Jackson (private communication) have shown that there exists a constant c such that the following holds: If H is a graph with n vertices and with minimum degree $c \log n$, and the edges of H are colored red and green such that the red edges form a Hamiltonian cycle, then H has a red-independent, green-dominating set. By the above construction this is best possible except for the value of c.

4. THE CASE $r = 1$

THEOREM 4.1. *If the edges of a 3-regular multigraph with no loops are colored red and green such that the red subgraph is a perfect matching, then G has a red-independent green-dominating set of vertices S.*

Moreover,

(i) *if v is any vertex of G, then S can be chosen such that $v \in S$, and*

(ii) *if e is any green edge of G and we insert a new vertex u of degree 2 on e, then the resulting graph G' has a red-independent green-dominating set S such that $u \notin S$.*

Proof (by Induction on $|V(G)|$). We call the vertex v in (i) the *special* vertex and the vertex u in (ii) the *dummy* vertex. If $|V(G)| \leqslant 4$, the statement is easily verified so assume that $|V(G)| \geqslant 5$. We may also assume that G is connected.

Consider first the case where G has two vertices x_1, x_2 joined by two edges. Let y_1 (respectively y_2) be the neighbor of x_1 (respectively y_2) distinct from x_2 (respectively x_1). If $y_1 \neq y_2$, then we delete x_1, x_2 and add the edge $y_1 y_2$ and complete the proof be induction. (The color of $y_1 y_2$ is that of $y_1 x_1$) If $y_1 = y_2$, then we let x_3 denote the neighbor of y_1 distinct from x_1, x_2. We may assume that x_3 is not incident with a double edge as this case has been disposed of. The edge $y_1 x_3$ is red.

We now delete x_1, x_2, y_1. If we thereby delete the special vertex or the edge with the dummy vertex, we think of x_3 as a new dummy vertex. Otherwise, we also delete x_3 and add a green edge between the two neighbors. Now the proof is easily completed by induction. So we may assume that G has no multiple edges.

If G has a special vertex v, then we let u denote the vertex joined to v by a red edge. We now delete v and add a green edge between the two neighbors of v distinct from u. We apply induction to the resulting graph. We may therefore assume that G has no special vertex. In other words, it remains only to prove (ii). Let C be the green cycle of G containing u, and let C' be the corresponding cycle in G'. (Note that C may have red chords.)

Consider first the case where C has even length. Recolor every second edge of C by the color blue such that e remains green. We delete C and replace every maximal path which is alternately colored red and blue by a new red edge. We apply the induction hypothesis to the resulting graph.

Each new red edge e' corresponds to a red-blue path P. We may assume that S contains precisely one end of e'. That vertex will also be in the final S. Moreover, the final S will contain every second vertex of P. By an appropriate choice of a special vertex we may assume that at least one end of e is in the final S. This completes the proof when C has even length.

Consider finally the case where C has odd length. We now recolor each second edge of C' blue. Note that there is precisely one maximal blue-red path P_0 starting at u. We delete C' and replace each maximal blue-red path other that P_0 by a new red edge. We think of the end of P_0 other than u as a new dummy vertex. We now complete the proof as in the previous case. ∎

REFERENCES

1. J. A. Bondy, Basic graph theory, *in* "Handbook of Combinatorics" (M. Grötschel, L. Lovász, R. L. Graham, Eds.), pp. 3–110, Amsterdam, North-Holland, 1995.
2. C. C. Chen, On edge-hamiltonian property of Cayley graphs, *Discrete Math.* **72** (1988), 29–33.
3. R. C. Entringer and H. Swart, Spanning cycles of nearly cubic graphs, *J. Combin. Theory Ser. B* **29** (1980), 303–309.
4. P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, *Colloq. Math. Soc. János Bolyai* **10** (1973), 609–627.
5. H. Fleischner, Uniqueness of maximal dominating cycles in 3-regular graphs and of Hamiltonian cycles in 4-regular graphs, *J. Graph Theory* **18** (1994), 449–459.
6. H. Fleischner and M. Stiebitz, A solution to a colouring problem, *Discrete Math.* **101** (1992), 39–48.
7. R. L. Graham, B. L. Rotschild, and J. H. Spencer, "Ramsey Theory," Wiley, New York, 1980.
8. J. Sheehan, The multiplicity of Hamiltonian circuits in a graph , *in* "Recent Advances in Graph Theory" (M. Fiedler, Eds.), pp. 447–480, Academia, Prague, 1975.
9. A. G. Thomason, Hamiltonian cycles and uniquely edge colourable graphs, *Ann. Discrete Math.* **3** (1978), 259–268.
10. C. Thomassen, On the number of Hamiltonian cycles in bipartite graphs, *Combin. Probab. and Comput.* **5** (1996), 437–442.
11. C. Thomassen, Chords of longest cycles in cubic graphs, *J. Combin. Theory Ser. B* **71** (1997), 211–214.
12. W. T. Tutte, On Hamiltonian circuits, *J. London Math. Soc.* **21** (1946), 98–101.