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Quadratic Dynamical Systems and Algebras

MICHAEL K. KINYON*

Department of Mathematics, University of Utah, Salt Lake City, Utah 84112

AND

ARTHUR A. SAGLE†

Department of Mathematics, University of Hawaii at Hilo, Hilo, Hawaii 96720-4091

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Quadratic dynamical systems come from differential or discrete systems of the form $\dot{X} = Q(X)$ or $X(k+1) = Q(X(k))$, where $Q : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree 2; i.e., $Q(\alpha X) = \alpha^2 Q(X)$ for all $\alpha \in \mathbb{R}$, $X \in \mathbb{R}^n$. Defining a bilinear mapping $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $\beta(X, Y) := \frac{1}{2}[Q(X+Y) - Q(X) - Q(Y)]$, we view $XY \equiv \beta(X, Y)$ as a multiplication, and thus consider $A = (\mathbb{R}^n, \beta)$ to be a commutative, non-associative algebra. The quadratic systems are then studied with the general theme that the structure of the algebras helps determine the behavior of the solutions. For example, semisimple algebras give a decoupling of the original system into systems occurring in simple algebras, and solvable algebras give solutions to differential systems via linear differential equations; the general three-dimensional example of the latter phenomenon is described. There are many classical examples and the scope of quadratic systems is large; every polynomial system can be embedded into a higher dimensional quadratic system such that solutions of the original system are obtained from the quadratic system. For differential systems, nilpotents of index 2 ($N^2 = 0$) are equilibria and idempotents ($E^2 = E$) give ray solutions. The origin is never asymptotically stable, and the existence of nonzero idempotents implies that the origin is actually unstable. Nonzero equilibria are not hyperbolic, but can be studied by standard algebra techniques using nondegenerate bilinear forms as Lyapunov functions. Periodic orbits lie on "cones." They cannot occur in dimension 2 or in power-associative algebras. No periodic orbit can be an attractor but "limit cycles" (invariant cones) can exist. Automorphisms of the algebra A leave equilibria, periodic orbits, and domains of attraction invariant. Also, explicit solutions can be given by the action of automorphisms on an initial point; the general three-dimensional example of this is described. Thus if there are sufficient automorphisms, Hilbert's sixteenth problem in $\mathbb{R}^3$ has the following answer: if the

* Current address: Dept. of Mathematics and Computer Science, Indiana University South Bend, South Bend, IN 46634; email: mkinyon@peabody.iusb.indiana.edu.

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periodic orbits of fixed period are isolated, then there is only one cone of periodic solutions; this cone is almost an attractor. For discrete systems there are many similarities to the differential systems. For example, orbits can be given by automorphisms, and again, the general three-dimensional example of this is described. However, distinctions become more obvious using algebras; for example, if the algebra A is nilpotent, then for the differential system, the solutions are unbounded, but for the discrete system, the trajectories iterate to zero in A; also idempotents $E^2 = E$ are the fixed points for the discrete system, and E is unstable if there exist suitable nilpotents $N^2 = 0$. The interplay between algebras and dynamical systems can solve old problems, but more importantly, it can create new opportunities in both areas. © 1995 Academic Press, Inc.

1. INTRODUCTION

Quadratic dynamical systems are defined by quadratic mappings $E : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$E(X) = C + TX + Q(X), \tag{1}$$

where C is a constant vector, $T: \mathbb{R}^n \to \mathbb{R}^n$ is linear, and $Q : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree 2 ($Q(\alpha X) = \alpha^2 Q(X)$ for all $\alpha \in \mathbb{R}$ and $X \in \mathbb{R}^n$). One can think of such a mapping E as a vector field and thus study the associated quadratic differential system

$$\dot{X} = E(X). \tag{2}$$

(Here $\dot{X} = (d/dt)X$.) One can also think of E as being an iteratable mapping and thus study the associated quadratic discrete system

$$X(k+1) = E(X(k)). \tag{3}$$

From Q, one obtains a unique, symmetric, bilinear mapping $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\begin{aligned} \beta(X, Y) &= \frac{1}{2!} Q^{(2)}(0) \cdot (X, Y) \\ &= \tfrac{1}{2}[Q(X+Y) - Q(X) - Q(Y)]. \end{aligned}$$

For example, the Lorenz equations give the following quadratic system in $\mathbb{R}^3$:

$$\begin{aligned} \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{bmatrix} \\ &\equiv C + TX + Q(X). \end{aligned}$$

Thus the associated symmetric bilinear map is given by

$$\beta(X, Y) = \begin{bmatrix} 0 \\ -\frac{1}{2}(x_1 y_3 + y_1 x_3) \\ \frac{1}{2}(x_1 y_2 + y_1 x_2) \end{bmatrix} \quad \text{for} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \; Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

In general, we think of the bilinear mapping β as being a multiplication for $\mathbb{R}^n$, and thus the structure $A = (\mathbb{R}^n, \beta)$ is a commutative, *nonassociative* algebra [36]. Well-studied examples of nonassociative algebras include associative algebras ("non-" means "not necessarily"), alternative algebras (such as the octonians), Jordan algebras (such as the space of real symmetric matrices with anticommutator as product), and Lie algebras.

If we make the further abbreviation $XY := \beta(X, Y)$, we can think of (2) and (3) as being quadratic systems *occurring in* A, and write

$$\dot{X} = C + TX + X^2, \qquad X(k+1) = C + TX(k) + X(k+1)^2.$$

Our theme is to study the interrelationship between properties of the dynamical systems (2) and (3) and the properties of the corresponding algebra. For quadratic differential systems, this theme began with the seminal paper of L. Markus [23], who gave the first classification (up to affine equivalence) of system (2) in the *homogeneous* case ($C = 0$, $T = 0$). Other important contributions in this area include the papers of Kaplan and Yorke [14], Koecher [18], Röhrl [27], [28], and the recent monograph of Walcher [38]. This last contains an extensive bibliography of related papers.

We will now give examples to show that many well-known differential systems can be considered to be quadratic differential systems occurring in an algebra. In Section 6, we will give examples of quadratic discrete systems occurring in algebras.

EXAMPLES. (1) By Taylor's theorem, quadratic differential systems occur as quadratic approximations to more general systems $\dot{X} = F(X) \sim F(0) + F^{(1)}(0)X + F^{(2)}(0)\, X^{(2)}/2!$, where $\beta(X, X) = F^{(2)}(0)\, X^{(2)}/2!$ as before.

(2) Let $A = (\mathbb{R}^n, \beta)$ be an algebra with $\{X_1, ..., X_n\}$ as a basis, then for $X = \sum x_i X_i$, $Y = \sum y_j X_j$ bilinearity gives $\beta(X, Y) = \sum x_i y_j \beta(X_i, X_j)$. Thus it suffices to know the product of the basis vectors to know $\beta(X, Y)$. This may be given in terms of a multiplication table. For the Lorenz equation with $\{e_1, e_2, e_3\}$ the natural basis we have

β	e_1	e_2	e_3
e_1	0	$\frac{1}{2}e_3$	$-\frac{1}{2}e_2$
e_2	$\frac{1}{2}e_3$	0	0
e_3	$-\frac{1}{2}e_2$	0	0

(3) Quadratic equations occur in linear control systems with a quadratic cost function. For $X \in \mathbb{R}^n$ (the states) and $U \in \mathbb{R}^q$ (the inputs), let the linear system be given by $dX/dt = FX + GU$ for suitable constant matrices F and G, and $t \in [a, b]$. Let the quadratic cost function be given by $J(U) = \int_a^b L(X(t), U(t))\, dt$, where $L(X, U) = \frac{1}{2}(X^tQX + U^tRU)$ for suitable symmetric matrices Q and R. The system that is optimal over $[a, b]$ relative to the cost $J(U)$ is given by the feedback law $U(t) = -R^{-1}G^tP(t)\,X(t)$, where the $n \times n$ symmetric matrix $P(t)$ satisfies the quadratic Riccati matrix equation $dP/dt = -Q - PF - F^tP + P(GR^{-1}G^t)P$. The multiplication $\beta(P, P) = P(GR^{-1}G^t)P$ makes the vector space of $n \times n$ symmetric matrices into a Jordan algebra.

(4) The Euler equation for the motion of a rotating rigid body with no external forces is given by the quadratic system in $\mathbb{R}^3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} ((I_3 - I_2)/I_1)\, x_2 x_3 \\ ((I_1 - I_3)/I_2)\, x_1 x_3 \\ ((I_2 - I_1)/I_3)\, x_1 x_2 \end{bmatrix},$$

where the nonzero moments of inertia I_j satisfy $I_1 \neq I_2 \neq I_3 \neq I_1$.

(5) The differential geometry of invariant Lagrangian systems is given in terms of quadratic systems and extends the preceding example. Let G be a connected Lie group and let H be a closed (Lie) subgroup with Lie algebras g and h, respectively. The homogeneous space G/H is *reductive* if there is a subspace m of g such that $g = m + h$ (direct sum) and $(\mathrm{Ad}\, H)(m) \subseteq m$; i.e., $[h, m] \subseteq m$. For example, let g and h be semisimple and $m = h^\perp$ relative to the Killing form of g. For a reductive space, there is a bijective correspondence between the set of G-invariant connections ∇ on G/H and the set of algebras (m, α) with $\mathrm{Ad}\, H \subseteq \mathrm{Aut}(m, \alpha)$ which is the automorphism group of the algebra. In particular, a curve $\sigma(t)$ in G/H is a geodesic if its tangent field $X(t) = \dot{\sigma}(t)$ satisfies the quadratic equation

$$\dot{X} + \alpha(X, X) = 0.$$

Next let G/H be a configuration space for an invariant system with nondegenerate Lagrangian. Then a solution $\sigma(t)$ to the corresponding Euler–Lagrange equation satisfies an extended Euler field equation which reduces to the above geodesic equation when the Lagrangian is given by kinetic energy. More general quadratic equations occur when the Lagrangian is not given by kinetic energy. The algebras occurring in this example are usually noncommutative; see [3, 25, 26, 31, 32].

(6) Relative growth rate problems and related Lotka–Volterra predator–prey models are given by quadratic equations which lead

naturally to noncommutative algebras. Let the relative growth rate of n species be given by

$$\dot{x}_i/x_i = g_i(x_1, ..., x_n) \sim c_i + \sum_j b_{ij} x_j$$

for $i = 1, ..., n$ with the growth rate functions g_i having the indicated linear approximation. For $X \in \mathbb{R}^n$, this gives the quadratic equation $\dot{X} = TX + \beta(X, X)$:

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} c_1 & & 0 \\ & \vdots & \\ 0 & & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} x_1 \sum b_{1j} x_j \\ \vdots \\ x_n \sum b_{nj} x_j \end{bmatrix}.$$

The (noncommutative) growth rate algebra multiplication β on $\mathbb{R}^n$ is given by $\beta(X, Y) = (\text{diag } X)\, BY$, where diag X is the diagonal matrix formed from the vector X and $B = (b_{ij})$. Some properties of the algebra $A = (\mathbb{R}^n, \beta)$ are given as follows: For $Z \in A$, define the right multiplication function $R(Z) : A \to A : X \mapsto \beta(X, Z)$ and note that

$$\begin{aligned} R(Z)X &= \beta(X, Z) = (\text{diag } X) BZ \\ &= \begin{bmatrix} x_1 \sum b_{1j} z_j \\ \vdots \\ x_n \sum b_{nj} z_j \end{bmatrix} \\ &= (\text{diag}(BZ))\, X; \end{aligned}$$

i.e., $R(Z) = \text{diag}(BZ)$. The bilinearity of β implies that $R(Z)$ is a linear transformation satisfying $R(c_1 Z_1 + c_2 Z_2) = c_1 R(Z_1) + c_2 R(Z_2)$. For $X, Y \in A$ we see $R(X)$ and $R(Y)$ are given by diagonal matrices and, consequently, $R(X)\, R(Y) = R(Y)\, R(X)$. This implies that A satisfies the identity $\beta(\beta(Z, X), Y) = R(Y)\, R(X) Z = R(X)\, R(Y) Z = \beta(\beta(Z, Y), X)$; frequently algebras are defined by identities. If B^{-1} exists, then A has the right identity element $e = B^{-1} I$ where $I = (1, ..., 1)' \in A$; just note that $\beta(X, e) = (\text{diag } X)\, B(B^{-1} I) = X$. The vector field $E(X) = TX + \beta(X, X)$ has equilibrium point N given by the solution to $BN + c = 0$, where $c = (c_1, ..., c_n)' \in A$. Furthermore, analogous to the logistics equation, we have

$$E(X) = \beta(X, X - N)$$

expressing $E(X)$ in terms of A, for $\beta(X, X - N) = \beta(X, X) - \beta(X, N) = \beta(X, X) - (\text{diag } X) BN = \beta(X, X) - (\text{diag } X)(-c) = TX + \beta(X, X) = E(X)$.

Next we give the definitions of the various algebraic concepts that we will need in this paper. Let A be an algebra and let B, C be subspaces of A; the product $BC = \{\sum b_i c_i : b_i \in B, c_i \in C\}$ is the subspace spanned by all products from B and C. A *subalgebra* B of an algebra A is a subspace of

A such that $B^2 \subseteq B$. For $X \in A$, $\mathbb{R}[X]$ denotes the subalgebra generated by X and we shall see that the solution $X(t)$ to $\dot{X} = X^2$ with $X(0) = X$ is in $\mathbb{R}[X]$. An algebra A is *power-associative* if $\mathbb{R}[X]$ is associative for every $X \in A$; for example, Jordan algebras are power-associative. An *ideal* I of an algebra A is a subspace of A such that $IA \subseteq I$ and $AI \subseteq I$. As in associative algebras, the *quotient algebra* A/I can be formed and the map $A \to A/I$: $X \mapsto X + I$ is an algebra homomorphism. A is a *simple* algebra if $A^2 \neq 0$ and A has no proper ideals; i.e., no proper homomorphisms.

(7) For example, the vector field in $\mathbb{R}^3$ given by

$$E(X) = \begin{bmatrix} a^1_{23} x_2 x_3 \\ a^2_{13} x_1 x_3 \\ a^3_{12} x_1 x_2 \end{bmatrix}$$

for $a^i_{jk} \in \mathbb{R}$ gives the algebra $A = (\mathbb{R}^3, \beta)$, where

$$2\beta(X, Y) = E(X + Y) - E(X) - E(Y) = \begin{bmatrix} a^1_{23}(x_2 y_3 + y_2 x_3) \\ a^2_{13}(x_1 y_3 + y_1 x_3) \\ a^3_{12}(x_1 y_2 + y_1 x_2) \end{bmatrix}.$$

This generalizes the algebra obtained from the Lorenz and Euler equations, and it is simple, provided that $a^1_{23} a^2_{13} a^3_{12} \neq 0$ as we now show. From the above formula the following table is obtained for the natural basis $\{e_1, e_2, e_3\}$ of A:

β	e_1	e_2	e_3
e_1	0	$\frac{1}{2}a^3_{12}e_3$	$\frac{1}{2}a^2_{13}e_2$
e_2	$\frac{1}{2}a^3_{12}e_3$	0	$\frac{1}{2}a^1_{23}e_1$
e_3	$\frac{1}{2}a^2_{13}e_2$	$\frac{1}{2}a^1_{23}e_1$	0

Now suppose that $0 \neq X \in B$ which is an ideal of A. Let $X = \sum x_i e_i$, then from the table $e_1 X = \frac{1}{2} x_2 a^3_{12} e_3 + \frac{1}{2} x_3 a^2_{13} e_2 \in B$ and $e_1(e_1 X) = \frac{1}{4} a^3_{12} a^2_{13}(x_2 e_2 + x_3 e_3) \in B$. Thus $a^3_{12} a^2_{13} x_1 e_1 = a^3_{12} a^2_{13} X - 4e_1(e_1 X) \in B$ and if $x_1 \neq 0$, then $e_1 \in B$. From the table this gives $B = A$. Similar calculations hold if $x_1 = 0$ but $x_2 \neq 0$ or $x_3 \neq 0$ [34].

An algebra is *semisimple* if it is the direct sum of ideals which are simple algebras. The *radical*, Rad A, of an algebra is the smallest ideal of A such that $A/\text{Rad}\, A$ is semi-simple or the zero algebra. The radical is given by Rad $A = (\text{Rad}\, M)A$, where M is the associative algebra generated by the right and left multiplication functions $R(Z): X \to \beta(X, Z)$ and $L(W)$: $X \to \beta(W, X)$. An algebra A is *nilpotent* if there exists an integer N such that all products with N factors are zero. A is *solvable* if for $A^{(1)} = A$,

$A^{(2)} = AA, ..., A^{(k+1)} = A^{(k)}A^{(k)}$, there is an integer N such that $A^{(N)} = 0$; i.e., $A \supseteq A^{(2)} \supseteq A^{(3)} \supseteq \cdots \supseteq A^{(N)} = 0$. The radical is usually associated with a nilpotent or solvable ideal.

(8) The algebra in example (7) is solvable if $a_{23}^1 = 0$. Thus from the table we see $A = \{e_1, e_2, e_3\} \supset A^{(2)} = \{e_2, e_3\} \supset A^{(3)} = 0$. If $a_{13}^2 a_{12}^3 \neq 0$, then A is not nilpotent since $L(e_1)$ is not a nilpotent linear transformation.

We conclude this Introduction with an outline of the sequel. In Section 2, we consider the general quadratic differential system (2) and show how it is possible to embed this in a homogeneous quadratic system. This will justify us in restricting our attention to the homogeneous case. Next we discuss the series solution to $\dot{X} = X^2$ and show how one can use this to determine explicit solutions in the cases where the algebra A is power-associative or when the initial point is an idempotent. We conclude with a theorem that shows that every polynomial differential equation can be embedded in a quadratic differential system so that solutions to the polynomial equation can be obtained from solutions to the quadratic system.

In Section 3, we study equilibria and periodic trajectories for $\dot{X} = X^2$ occurring in an algebra A. If N is an equilibrium, then $N^2 = 0$, so N is a nilpotent of index 2, and conversely. Also we show that N is not hyperbolic. We discuss stability/instability of the origin in the presence of idempotents and identity elements and in the power-associative case. In particular, if A has a nonzero idempotent, then the origin is an unstable equilibrium. We also show how algebras A with positive definite symmetric bilinear forms have natural Lyapunov functions for the associated differential system. Next we consider periodic trajectories and show how the homogeneity of the differential system implies restrictions on the behavior of such solutions. Thus two-dimensional systems have no periodic solutions (a well-known result) and in higher dimensional systems, periodic trajectories live on "cones." No periodic orbit can be an attractor. Power-associative algebras cannot have periodic orbits. We consider a parameterized three-dimensional example in some detail, and we find that it has periodic trajectories for certain parameter values. The cone obtained from these periodic trajectories is almost an attractor; this is illustrated with a figure.

In Section 4, we discuss the connection between algebra structure and solution behavior. Thus in semisimple algebras A, $\dot{X} = X^2$ decouples into a system of equations occurring in simple ideals. If $A = S + R$, a Wedderburn–Levi decomposition, with S a semisimple subalgebra and R the radical, the equation decouples into an equation in S and a nonautonomous equation in R. In nilpotent algebras, the solution is given by a polynomial. In solvable algebras (when the associated descending chain of subalgebras are actually ideals), the solution can be obtained by solving finitely many linear equations. We prove this last result in the presence of a nontrivial linear term and give the most general three-dimensional example.

In Section 5, we study automorphisms and derivations of vector fields and algebras. Automorphisms preserve equilibria, periodic orbits (preserving the period), and domains of attraction. We also show that explicit solutions of $\dot{X} = X^2$ may be obtained by the action of a one-parameter subgroup of automorphisms on an initial vector. Sufficient conditions that guarantee that this must occur are given. We give all two-dimensional bounded solutions which occur in this manner. We give the general three-dimensional algebra admitting a periodic solution of this form; it turns out to be the example of Section 3. Furthermore, if there are sufficient automorphisms and the periodic trajectories of period τ are isolated, then they are given by this automorphism solution, they are limit cycles (i.e., ω-limit sets), and they are located on an explicit cone in $\mathbb{R}^3$. This cone is a symmetric space [10] which is described in terms of a bilinear form, the group that preserves this form, and the automorphism group. Thus extending Hilbert's sixteenth problem to $\mathbb{R}^3$, the isolated trajectories of fixed period are located on this single cone of limit cycles.

In Section 6, we turn our attention to discrete systems $X(k+1) = X(k)^2$ occurring in A. While many of the basic results are analogous to the differentiable case, the use of algebras shows many striking differences. For example, the stability of equilibria (nilpotents of index 2) is of interest in the differentiable case, the stability of fixed points (idempotents) are of interest in the discrete case. We discuss many such results, and then turn our attention once again to automorphisms and derivations. In particular, we examine conditions for explicit orbits to be given by the action of an iterated automorphism on an initial vector. We find all the two-dimensional examples and find the general three-dimensional example that gives a periodic orbit. We also discuss the stability/instability of this orbit. Another example is given in which each iterate lives in a *different* algebra and has a cone as an attractor. Finally, we close by discussing the *non*-chaotic dynamics of the squaring map on S^3 (the unit quaternions) and S^7 (the unit octonians).

2. Solutions and Polynomial Differential Equations

In this section we discuss the series solution to the equation $\dot{X} = X^2$ and how a solution to an nth-order autonomous polynomial differential equation may be obtained from a quadratic system. First we consider the general quadratic equation

$$\dot{X} = E(X) \equiv C + TX + X^2$$

in A and show how to homogenize the equation so that it is of the form $\dot{\tilde{X}} = \tilde{X}^2$ in an algebra $\tilde{A}$. Our discussion follows and expands upon that of [38, p. 22].

Set $E_u(X) = u^2C + uTX + X^2$, so that $\dot{X} = E_1(X)$ is the original system. Let $F_t^u(X)$ denote the flow for E_u. Let $\tilde{A} = A \times \mathbb{R}$ and define a multiplication $\tilde{\beta}$ on $\tilde{A}$ by

$$\tilde{\beta}(\tilde{X}, \tilde{X}) = \begin{pmatrix} E_u(X) \\ 0 \end{pmatrix} \equiv \tilde{E}(\tilde{X}), \qquad \text{where} \quad \tilde{X} = \begin{pmatrix} X \\ u \end{pmatrix}.$$

Note that

$$\tilde{\beta}(s\tilde{X}, s\tilde{X}) = \begin{pmatrix} E_{su}(sX) \\ 0 \end{pmatrix} = s^2\tilde{\beta}(\tilde{X}, \tilde{X})$$

so that $\tilde{\beta}$ is quadratic and gives a commutative algebra multiplication on $\tilde{A}$. Let $F_t(\tilde{X})$ be the flow for $\dot{\tilde{X}} = \tilde{E}(\tilde{X}) = \tilde{X}^2$. Since the last component of $\tilde{E}(\tilde{X})$ is zero, the last component of $F_t(\tilde{X})$ is a constant and, hence, must be u. Thus $F_t(\tilde{X}) = \binom{G_t(\tilde{X})}{u}$ for some function $G_t(\tilde{X})$. Now $(d/dt)\, G_t(\tilde{X}) = E_u(G_t(\tilde{X}))$ and $G_0(\tilde{X}) = X$, so by uniqueness, $G_t(\tilde{X}) = F_t^u(X)$, the flow for $\dot{X} = E_u(X)$. Therefore,

$$F_t(\tilde{X}) = \begin{pmatrix} F_t^u(X) \\ u \end{pmatrix}.$$

To recover the solution to the original system, restrict the flow to the hyperplane given by $u \equiv 1$.

Remarks. (1) We remind the reader of the notation $L(U)\,V := UV$ for the multiplication operator in an algebra.

(2) The linear transformation T does not get lost in the homogenization process. For $\tilde{X} = \binom{X}{u}$, $\tilde{Y} = \binom{Y}{v}$ in $\tilde{A}$, the bilinearization gives

$$\tilde{\beta}(\tilde{X}, \tilde{Y}) = \begin{pmatrix} uvC + (1/2)(uTY + vTX) + \beta(X, Y) \\ 0 \end{pmatrix}.$$

Let $\tilde{M} = \binom{0}{1}$ in $\tilde{A}$, then

$$2L(\tilde{M})\,\tilde{Y} = \begin{pmatrix} 2vC + TY \\ 0 \end{pmatrix}.$$

In particular, the action of T in A is given by the action of $2L(\tilde{M})$ on the subalgebra $A \subset \tilde{A}$ (that is, when $v=0$).

The preceding procedure justifies us in restricting our attention to the equation $\dot{X}=X^2$ occurring in some commutative algebra A. The solution $X(t)$ with $X(0)=X$ has a Taylor's series expansion $X(t)=X(0)+X^{(1)}(0)t+X^{(2)}(0)(t^2/2!)+\cdots$ for t in suitable interval about zero in $\mathbb{R}$. Thus computing the derivatives from $\dot{X}=X^2$ in the algebra A, we obtain

$$X(0)=X \qquad (\text{the initial condition}),$$

$$X^{(k)}(0)=\sum_{i=0}^{k-1}\binom{k-1}{i}\beta(X^{(i)}(0), X^{(k-i-1)}(0)) \qquad \text{for} \quad k=1,2,\ldots.$$

Thus

$$X(t)=X+X^2t+XX^2t^2+\cdots$$

which is in the subalgebra $\mathbb{R}[X]$ generated by the initial condition X. Setting $X^k=XX^{k-1}$, $k=2,3,\ldots$, we can write

$$X(t)=X+X^2t+X^3t^2+(X^2X^2+2X^4)\frac{t^3}{3}+\cdots.$$

If A is power-associative, then we have

$$\begin{aligned} X(t) &= X+X^2t+X^3t^2+X^4t^3+\cdots \\ &= (I+tL(X)+t^2L(X)^2+t^3L(X)^3+\cdots)X \\ &= (I-tL(X))^{-1}X, \end{aligned}$$

which is valid for t in some interval containing 0; see [18; 38, Proposition 2.7, p. 29].

In Section 5, we will give other solution forms using automorphisms of A. The following result indicates the scope of quadratic systems.

PROPOSITION 2.1. *Let $z^{(n)}=p(z, z^{(1)}, \ldots, z^{(n-1)})$ be a polynomial differential equation in $\mathbb{R}$; i.e., $p(z_1, \ldots, z_n)$ is a polynomial in the z_i's. Then the solution to this equation may be obtained from the solution of a quadratic system $\dot{Y}=Y^2$ occurring in a suitable algebra A.*

Proof. Write $z^{(n)}=p(z, \ldots, z^{(n-1)})$ as a system

$$\dot{X}=\begin{pmatrix}\dot{x}_1\\ \vdots\\ \dot{x}_n\end{pmatrix} \qquad \text{with} \quad X=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\equiv\begin{pmatrix}z\\ \vdots\\ z^{(n-1)}\end{pmatrix}$$

$$= \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ p(x_1, ..., x_n) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ p(0) \end{pmatrix} + \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ p^{(1)}(0) X \end{pmatrix} + \sum_{k=2}^{N} \frac{1}{k!} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ p^{(k)}(0) X^{(k)} \end{pmatrix};$$

expanding p in its Taylor's series gives $\sum_{k=0}^{N} p_k(X)$, where each p_k is homogeneous of degree k.

Next, homogenize this equation by letting $\tilde{V} = \mathbb{R}^n \times \mathbb{R}$, and $\tilde{X} = \binom{X}{u}$ for all $X \in \mathbb{R}^n$ and $u \in \mathbb{R}$. Let

$$\tilde{p}(\tilde{X}) = \begin{pmatrix} \sum_{k=0}^{N} u^{N-k} p_k(X) \\ 0 \end{pmatrix}$$

and, as in the earlier discussion, $\tilde{p}$ is homogeneous of degree N. The solution to the homogeneous equation $\dot{\tilde{X}} = \tilde{p}(\tilde{X})$ gives the solution to the original system $\dot{X} = \sum p_k(X)$ when $u = 1$. We simplify notation by dropping the tildes and writing the homogeneous system as $\dot{Z} = p(Z) = (p_1(Z), ..., p_m(Z))^t$ in $\mathbb{R}^m$, where $m = n + 1$.

Now solutions to the homogeneous system $\dot{Z} = p(Z)$ of degree N can be obtained in terms of a quadratic system. Before proving this in general, we first consider the example $\dot{Z} = Z^3$ in $\mathbb{R}$. Let $Y_1 = Z$ and $Y_2 = Z^2$; then

$$\dot{Y}_1 = \dot{Z} = Y_1 Y_2$$

and, by the product rule,

$$\dot{Y}_2 = 2Z\dot{Z} = 2ZZ^3 = 2Y_2 Y_2.$$

Thus the original cubic equation is now given by the quadratic system in $\mathbb{R}^2$,

$$\dot{Y}_1 = Y_1 Y_2$$
$$\dot{Y}_2 = 2Y_2 Y_2.$$

This is the idea behind the following general procedure. Let $Y_i = Z_i$ for $i = 1, ..., m$. There are $\binom{N+m+1}{N}$ monomials (in the variables $Z_1, ..., Z_m$) that are homogeneous of degree $N - 1$. Order these in some fashion, labeling them Y_{m+1} through Y_l, where $l = m + \binom{N+m+1}{N}$. Any monomial (in

$Z_1, \ldots, Z_m)$ which is homogeneous of degree N can be written (non-uniquely) as some Z_i times a monomial of degree $N-1$, hence as a product $Y_i Y_j$, where $1 \leqslant i \leqslant m$ and $m+1 \leqslant j \leqslant l$. Thus for $i = 1, \ldots, m$, we obtain the quadratic equations

$$\dot{Y}_i = \dot{Z}_i = \sum a_{ijk} Y_j Y_k.$$

Suppose $Y_{m+1} = Z_{i_1} \cdots Z_{i_{N-1}}$. Then by the product rule,

$$\begin{aligned}\dot{Y}_{m+1} &= \dot{Z}_{i_1} \cdots Z_{i_{N-1}} + \cdots + Z_{i_1} \cdots \dot{Z}_{i_{N-1}}\\ &= p_{i_1}(Z) Z_{i_2} \cdots Z_{i_{N-1}} + \cdots + Z_{i_1} \cdots Z_{i_{N-2}} p_{i_{N-1}}(Z),\end{aligned}$$

using the differential equations. The right side consists of monomials, each containing $N + (N-2) = 2(N-1)$ Z_i's. The Z_i's in these monomials can be reassociated and written as a product of the form $Y_i Y_j$, since all possible products of $N-1$ Z_i's occur as some Y_k. Thus,

$$\dot{Y}_{n+1} = \sum a_{n+1,ij} Y_i Y_j$$

is quadratic in the Y_i's. Extend this to the other derivatives $\dot{Y}_j$ for $m+2 \leqslant j \leqslant l$. This gives the quadratic system

$$\dot{Y} = \beta(Y, Y) = Y^2 \qquad \text{in } \mathbb{R}^l.$$

The solution to this is given by the series solution $Y(t) = Y + tY^2 + \cdots$ with $Y = Y(0)$. Solving the equations $Y_i = Z_i$ for $i = 1, \ldots, m$ and $Y_{m+1} = Z_{i_1} \cdots Z_{i_{N-1}}$, etc. gives a solution to $\dot{Z} = p(Z)$.

Remark. The preceding theorem extends to the system $X^{(n)} = E(X, X^{(1)}, \ldots, X^{(n-1)})$, where $E(Z_1, \ldots, Z_n)$ is a $\mathbb{R}^n$-valued function of $Z_i \in \mathbb{R}^n$ which has a truncated Taylor's series.

As is seen from the proof, the dimension of the resulting quadratic system can be quite large. The next example shows that a clever choice of variables can keep the size of the quadratic system down.

EXAMPLE. For the Van der Pol equation $\ddot{x} = (3cx^2 + d)\dot{x} + abx = p(x, \dot{x})$, let

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \\ x^2 \end{pmatrix},$$

then

$$\dot{X} = \begin{pmatrix} \dot{x} \\ \ddot{x} \\ 2x\dot{x} \end{pmatrix} = \begin{pmatrix} x_2 \\ (3cx_3 + d)x_2 + abx_1 \\ 2x_1x_2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 0 \\ ab & d & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 3cx_2x_3 \\ 2x_1x_2 \end{pmatrix}$$
$$= TX + X^2$$

which is a quadratic differential equation but which is not homogeneous. Finally we homogenize this equation to obtain a system $\dot{\tilde{X}} = \tilde{X}^2$ occurring in the algebra $\tilde{A} = \mathbb{R}^2 \times \mathbb{R}$. The solution to the original equation is obtained as discussed earlier.

3. Equilibria and Periodic Orbits

These solutions are briefly discussed here, and later more results will be obtained using automorphisms. We begin with equilibria. From Section 1, recall that N is in the set $\mathscr{E}$ of equilibrium points of $\dot{X} = E(X) \equiv X^2$, provided that $N^2 = 0$. From Section 2, the linearization $E'(N) = 2L(N)$ so that $E'(N)N = 2N^2 = 0$; thus N is not hyperbolic. Thus standard linearization methods cannot be effectively used, and we now see how the structure of A might help determine the stability properties of an equilibrium solution.

We begin with a well-known folk result that follows from the uniqueness of solutions to initial value problems. One can find this, e.g., in [17, Lemma 2.1, p. 484 or 9, Chap. 5, p. 90]. It can also be seen by considering the series solution.

LEMMA 3.1. *Let $F_t(X)$ denote the flow of $\dot{X} = X^2$ through $X \in A$. Then for $a \in \mathbb{R}$ we have $F_t(aX) = aF_{at}(X)$ whenever the right side is defined.*

The analysis of many algebras depends on idempotents $E = E^2$ and the Peirce decomposition [36] of $A = \sum A(\lambda)$ into the generalized eigenspaces relative to $L(E)$. For example, if A is power-associative and commutative, then A decomposes into eigenspaces $A = A(0) + A(1) + A(\frac{1}{2})$ and the multiplication table for A is obtained for these eigenspaces.

LEMMA 3.2. *Let $E \in A$ be an idempotent: $E^2 = E \neq 0$. Then $F_t(E) = (1/(1-t))E$; this solution is unbounded and in fact blows up in finite time* [14].

Proof. Let $G(t) = g(t)E$, where $g(t)$ satisfies $\dot{g} = g^2$, $g(0) = 1$; i.e., $g(t) = 1/(1-t)$. Then $\dot{G} = \dot{g}E = g^2E^2 = (gE)^2 = G^2$, so that $G(t)$ is the solution to $\dot{X} = X^2$, $X(0) = E$. The other assertions follow from the form of the solution. This result can also be seen by considering the power series solution

$$\begin{aligned} F_t(E) &= E + tE^2 + \cdots \\ &= (1 + t + t^2 + \cdots)E \\ &= (1/(1-t))E \qquad \text{for} \quad |t| < 1. \end{aligned}$$

COROLLARY 3.3. *Let $0 \neq P \in A$ satisfy $P^2 = \alpha P$ for some $\alpha \neq 0$. Then the solution $F_t(P)$ of $\dot{X} = X^2$ is unbounded and blows up in finite positive time for $\alpha > 0$ and in finite negative time for $\alpha < 0$.*

Proof. $E = \alpha^{-1}P$ satisfies $E^2 = \alpha^{-2}P^2 = \alpha^{-2}\alpha P = E$. Thus $F_t(P) = F_t(\alpha E) = \alpha F_{\alpha t}(E) = \alpha(1/(1-\alpha t))\, E = (1/(1-\alpha t))P$, using Lemmas 3.1 and 3.2. This establishes the result.

PROPOSITION 3.4. *If A has a nonzero idempotent, then the origin $0 \in A$ is an unstable equilibrium point for the system $\dot{X} = X^2$.*

Proof. Let $\delta > 0$ and let $E^2 = E \neq 0$. For $P = \delta E$ we have from Corollary 3.2 that $F_t(P)$ is unbounded and blows up in finite time. Since $\|P\| = \delta\ \|E\|$, every neighborhood of the origin has a solution that starts in that neighborhood and that blows up; thus the origin is unstable.

Remark. It follows that if the origin is stable for the system $\dot{X} = X^2$ in A, then A does not have an identity element.

The existence of nonzero nilpotent elements $N^2 = 0$ also affects the stability of the origin. First we note that if $N \neq 0$ is a nilpotent of A of index 2 (and, hence, an equilibrium for $\dot{X} = X^2$), then so is every element of the line $\mathscr{L}(N) = \{sN : s \in \mathbb{R}\}$ (thus $\mathscr{L}(N) \subseteq \mathscr{E}$). To see this note that $(sN)^2 = s^2N^2 = 0$. An immediate consequence of this is the following observation.

PROPOSITION 3.5. *If A has a nonzero nilpotent element of index 2, then the origin $0 \in A$ is not asymptotically stable for the system $\dot{X} = X^2$.*

The following theorem is due to Markus in the odd-dimensional case [23, p. 188]. In the general case, a degree theoretic treatment can be found in [14], and an algebraic treatment can be found in [30].

THEOREM 3.6. *A real, commutative algebra A has a nonzero idempotent or a nonzero nilpotent of index 2.*

COROLLARY 3.7. *The origin* $0 \in A$ *is not asymptotically stable for* $\dot{X} = X^2$.

Proof. If A has a nonzero idempotent, then this follows from Proposition 3.4. If not, then by Theorem 3.6, A must have a nonzero nilpotent of index 2, in which case the result follows from Proposition 3.5.

Remarks. (1) Corollary 3.7 has also been observed by Koditschek and Narenda [17, Corollary 2.1, p. 784].

(2) It also follows from Theorem 3.6 that the system $\dot{X} = X^2$ is not dissipative; the existence of a nonzero idempotent implies the existence of an unbounded solution, while the existence of a nonzero nilpotent of index 2 implies that for any ball containing the origin, there exists an equilibrium outside the ball.

For the stability of nonzero equilibria, we have the following.

COROLLARY 3.8. *Let* $E \neq 0$ *be an idempotent in* A, *and let* N *be an equilibrium for* $\dot{X} = X^2$. *If* $N \in A(\frac{1}{2})$, *which is an eigenspace; then* N *is unstable.*

Proof. The argument is similar to that of Proposition 3.4. Let $\delta > 0$ and let $P = \delta E + N$; then

$$\begin{aligned} P^2 = (\delta E + N)^2 &= \delta^2 E^2 + 2\delta EN + N^2 \\ &= \delta(\delta E + 2EN) = \delta P, \end{aligned}$$

using $L(E)N = \frac{1}{2}N$. By Corollary 3.3, the solution $F_t(P)$ blows up in finite time and, since $\|P - N\| = \delta\, \|E\|$, we again obtain that N is unstable.

Examples. Semisimple Jordan algebras (especially those occurring in the matrix Riccati equation) and commutative division algebras have nonzero idempotents since they have identity elements. In the Jordan algebra A of 2×2 matrices with multiplication $\beta(X, Y) = 1/2(XY + YX)$, $E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = E^2 = \beta(E, E)$ and $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in A(\frac{1}{2})$ satisfies $N^2 = \beta(N, N) = 0$. Thus N is an unstable equilibrium for $\dot{X} = X^2$ in A.

The analysis of many semisimple algebras involves nondegenerate bilinear forms satisfying certain relations [13, 35]. Also the analysis of nonhyperbolic equilibrium points frequently uses Lyapunov functions obtained from positive definite bilinear forms. The following result combines these two ideas.

THEOREM 3.9. *Let* $\dot{X} = X^2$ *occur in an algebra* A *that has a symmetric, positive definite, bilinear form* $C\colon A \times A \to \mathbb{R}$. *If* C *satisfies*

$$C(X, X^2) = 0 \qquad \textit{for all} \quad X \in A, \tag{*}$$

the origin $0 \in A$ *is stable.*

Proof. Assume (*) and let $V(X) = C(X, X)$. For a trajectory $X(t)$, we compute the derivative

$$\dot{V}(X) := \frac{d}{dt} V(X(t)) = (DV)(X) \cdot \dot{X} = 2C(X, \dot{X}) = 2C(X, X^2) = 0$$

on A. Thus V is conserved and, since $V(0) = 0$ and V is positive definite, V is a Lyapunov function and the origin is stable.

Remarks. (1) It would seem that the condition (*) could be replaced by the condition "$C(X, X^2) \leqslant 0$ for all $X \in A$". However, this condition is equivalent to the one given, for if $C(Z, Z^2) < 0$ for some $Z \in A$, then $C(-Z, (-Z)^2) = -C(Z, Z^2) > 0$.

(2) Condition (*) and the positive definiteness of C imply that the only idempotent in A is 0: for if $P = P^2$, then $0 = C(P, P^2) = C(P, P)$ so that $P = 0$.

(3) If C is symmetric and *associative*, i.e., $C(XZ, Y) = C(X, ZY)$ for all $X, Y, Z \in A$, then one can actually show that $\dot{X} = X^2$ is a gradient system; see [38, Proposition 5.9, pp. 79–80].

EXAMPLES. For the Jordan algebra of symmetric matrices, $C(X, Y) = \operatorname{trace} L(XY)$ is positive definite and $C(X, X^2) \neq 0$ for $X = I$. In fact, since I is an idempotent, the origin is unstable (Proposition 3.4).

Variants of the Euler equation from invariant Lagrangian mechanics and corresponding differential geometry satisfy condition (*) of the theorem. First a basic three-dimensional example occurs when $C(X, Y) = b_1 x_1 y_1 + b_2 x_2 y_2 + b_3 x_3 y_3$, where $b_j > 0$ and the quadratic system is given by

$$\dot{x}_1 = a^1_{23} x_2 x_3, \qquad \dot{x}_2 = a^2_{13} x_1 x_3, \qquad \dot{x}_3 = a^3_{12} x_1 x_2,$$

where $\sum a^i_{jk} b_i = 0$. In this case $C(X, X^2) = b_1 x_1 (a^1_{23} x_2 x_3) + b_2 x_2 (a^2_{13} x_1 x_2) + b_3 x_3 (a^3_{12} x_1 x_2) = (\sum a^i_{jk} b_i)\, x_1 x_2 x_3 = 0$ and the origin is stable.

More generally [3, 25, 31, 32], let G/H be a reductive homogeneous space with Lie algebra decomposition $g = m + h$ for some Ad H-invariant subspace m as in Section 1. Using the bijection between the set of G-invariant connections ∇ on G/H and the set of nonassociative algebras (m, α) with Ad $H \subseteq \operatorname{Aut}(m, \alpha)$, a curve $\sigma(t)$ in G/H is a geodesic relative to ∇ if its tangent field $X(t) = \dot{\sigma}(t)$ satisfies $\dot{X} + \alpha(X, X) = 0$, where $\alpha: m \times m \to m$ is the algebra multiplication. In case ∇ is a Riemannian connection, there exists a positive definite form $C(X, Y)$ on m such that $C(X, \alpha(X, X)) = 0$ for all $X \in m$. This implies that the origin is stable. From Section 1, the results also hold when G/H is the configuration space for an invariant Lagrangian system given by kinetic energy which is a positive definite bilinear form.

We now consider periodic solutions and give a general example with figures.

THEOREM 3.10. *Let $\dot{X} = X^2$ occur in an algebra A of dimension n. Then*

(1) *The trajectory through $P \in A$ does not pass through aP for any $a \leqslant 0$. If P lies on a periodic trajectory, the trajectory through P does not pass through aP for any $a \neq 1$.*

(2) *For $n = 2$, the system $\dot{X} = X^2$ does not have a periodic solution.*

(3) *If $\gamma \subset A$ is a periodic orbit with least period τ, then $a\gamma = \{aP : P \in \gamma\}$ is a periodic trajectory with least period $\tau/|a|$ for $a \neq 0$. Thus scalar multiples of periodic orbits are periodic, and solutions of any period exist, provided that one periodic orbit exists.*

(4) *If $n = 3$, then periodic trajectories lie on cones; for arbitrary n, they are on $(n-1)$-cones.*

Proof. (1) Suppose that for some $r \in \mathbb{R}$, $F_r(P) = aP$, where $a \leqslant 0$. Then since $0 < a/(a-1) < 1$, the calculation

$$\begin{aligned} F_{(a/(a-1))r}(P) &= F_{r/(a-1)+r}(P) = F_{r/(a-1)}(F_r(P)) \\ &= F_{r/(a-1)}(aP) = aF_{ar/(a-1)}(P), \end{aligned}$$

is valid, using Lemma 3.1. This implies that $F_{ar/(a-1)}(P) = 0$ which is impossible unless $P = 0$. If P lies on a periodic trajectory, then this calculation is valid for all $a \neq 0, 1$, because the solution extends as far as necessary.

(2) This follows from (1) since a periodic trajectory in the plane must intersect some line through the origin at least twice.

(3) Note that for $a \neq 0$, $F_{\tau/a}(aP) = aF_\tau(P) = aP$, so that aP is a periodic point of period $\tau/|a|$.

(4) This follows from (3). If P is a periodic point, then the line $\mathscr{L}(P)$ through the origin $0 \in A$ and P generates a cone with the periodic solution $F_t(P)$ as its "base."

Remarks. (1) Let $\mathscr{P}_\tau$ denote the set of periodic trajectories of $\dot{X} = X^2$ which are of period τ. If $\mathscr{P}_\tau$ is finite, then for any other period σ, $\mathscr{P}_\sigma$ is finite, and they both contain the same number of trajectories. This follows from part (3) of the theorem.

(2) We note that a *bounded* solution $X(t)$ to $\dot{X} = X^2$ in a two-dimensional algebra A must be a heteroclinic orbit. We briefly sketch the proof of this. By the Poincaré–Bendixson theorem, we have $X(t) \to N_1 \in \mathscr{E}$ as

$t \to \infty$, or $X(t)$ is a periodic orbit, or $X(t)$ approaches a limit cycle as $t \to \infty$. Since periodic solutions cannot exist in A, only the first option obtains. Similarly, $X(t) \to N_2 \in \mathscr{E}$ as $t \to -\infty$. To see that $X(t)$ is not homoclinic, suppose that $N_1 = N_2$. If $X(t)$ looped around the origin at least once, then $X(t)$ would have to cross the line of equilibria collinear with N_1 and the origin, which is impossible. Otherwise, $X(t)$ had to return to N_1 without looping around the origin, but this would mean that it would cross a ray twice, which is also impossible, by part (1) of Theorem 3.10.

EXAMPLE. The referee has provided the following example to show that the conclusion of part (1) of the theorem is false in the nonperiodic case if $a > 0$. Consider the homogeneous quadratic system

$$\dot{x} = x(x + y - 2z) - yz$$
$$\dot{y} = y(x + y - 2z) + xz$$
$$\dot{z} = z(x + y - 2z)$$

in $\mathbb{R}^3$. The cone $x^2 + y^2 - z^2 = 0$ is seen to be invariant:

$$x\dot{x} + y\dot{y} - z\dot{z} = (x^2 + y^2 - z^2)(x + y - 2z).$$

If ρ denotes the radial distance from the origin, then

$$\rho\dot{\rho} = \rho(x + y - 2z),$$
$$x\dot{y} - y\dot{x} = z(x^2 + y^2),$$

so that the radial and tangential components of the flow are nonzero on the cone. Thus orbits spiral around the cone, approaching the origin for $z > 0$ and moving away from the origin for $z < 0$. Each orbit crosses any generator of the cone infinitely often.

THEOREM 3.11. *If A is power-associative, then $\dot{X} = X^2$ has no periodic solutions.*

Proof. From the discussion in Section 2, the solution is given by $F_t(P) = (I - tL(P))^{-1} P$. Let P be periodic of period τ. Then $(I - \tau L(P))^{-1} P = P$ implies that $P = (I - \tau L(P)) P = P - \tau P^2$. Thus $P^2 = 0$, which is impossible (otherwise P would be an equilibrium).

This says that some of the most interesting dynamics cannot be found in the classical power-associative algebras such as associative, alternative, or Jordan algebras.

THEOREM 3.12. *Let $\dot{X} = X^2$ occur in an algebra A. Then no periodic orbit is an attractor.*

Proof. Let $\gamma(t) = F_t(P)$ be a periodic solution through P and let $\mathscr{U}$ be a neighborhood of γ. Then the line $\mathscr{L}(P)$ from the origin $0 \in A$ through P intersects $\mathscr{U}$. Since $\mathscr{U}$ is open, there is a point $aP \in \mathscr{L}(P) \cap \mathscr{U}$ and this point determines a periodic solution $F_t(aP)$. Thus $\lim_{t \to \infty} \|F_t(aP) - \gamma\| \neq 0$, so that γ is not an attractor.

We shall give an example of a quadratic system whose periodic solutions are limit sets and the cone formed from these periodic solutions is almost an attractor.

EXAMPLE. Let $A = A(\lambda, \mu, c)$ $(c \neq 0)$ be the three-dimensional commutative algebra with basis $\{X_0, X_1, X_2\}$ and multiplication table

	X_0	X_1	X_2
X_0	λX_0	cX_2	$-cX_1$
X_1	cX_2	μX_0	0
X_2	$-cX_1$	0	μX_0

where $\lambda, \mu \in \mathbb{R}$. First we explore this as an algebra, and then we discuss the associated quadratic differential system.

LEMMA 3.13. (1) *If $\mu \neq 0$, then A is simple.*

(2) *If $\mu = 0$, then $J \equiv \mathbb{R}X_1 + \mathbb{R}X_2$ is a nilpotent ideal of A.*

(3) *If $\mu = 0$ and $\lambda = 0$, then A is solvable, but not nilpotent.*

(4) *If $\mu = 0$ and $\lambda \neq 0$, then $A = \mathbb{R}X_0 \oplus J$ (direct sum of subalgebras). In this case J is the radical of A.*

Proof. Let B be an ideal of A with nonzero element $X = x_0 X_0 + x_1 X_1 + x_2 X_2$ in B. Then

$$X_0(X_0 X) = X_0(x_0 \lambda X_0 + c x_1 X_2 - c x_2 X_1)$$
$$= x_0 \lambda^2 X_0 - x_1 c^2 X_1 - x_2 c^2 X_2 \in B.$$

Thus

$$c^2 X + X_0(X_0 X) = x_0(c^2 + \lambda^2)\, X_0 \in B.$$

If $x_0 \neq 0$, then $X_0 \in B$. But then since B is an ideal, $X_0 X_1$, $X_0 X_2 \in B$, and using the table $X_1, X_2 \in B$, so $B = A$. If $x_0 = 0$, then we compute

$$X_1 X = \mu x_1 X_0, \qquad X_2 X = \mu x_2 X_0.$$

Now at least one of x_1 and x_2 is nonzero. If $\mu \neq 0$, then $X_0 \in B$, and so $B = A$ before. This proves (1).

If $\mu = 0$, then $B = \mathbb{R}X_1 + \mathbb{R}X_2 = J$. The calculations directly show that J is nilpotent. This proves (2).

If $\mu = \lambda = 0$, then from the table $A \supset A^{(2)} = AA = J \supset A^{(3)} = A^{(2)}A^{(2)} = JJ = \{0\}$, so that A is solvable. A is not nilpotent since the matrix of $L(X_0)$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -c \\ 0 & c & 0 \end{pmatrix}$$

which is not nilpotent. This proves (3).

If $\mu = 0$, but $\lambda \neq 0$, then from the table, $\mathbb{R}X_0$ is a (simple) subalgebra, and the direct sum decomposition follows. That J is the radical of A follows because a straightforward calculation shows that J is the only proper ideal of A and $A/J \cong \mathbb{R}X_0$ which is simple. This proves (4) and completes the proof of the lemma.

We introduce some geometry into the algebra A by considering the trace form [36]

$$C(X, Y) = \text{trace } L(XY).$$

For $X = \sum x_i X_i$, the matrix for $L(X)$ in the $\{X_0, X_1, X_2\}$ basis is

$$\begin{pmatrix} \lambda x_0 & \mu x_1 & \mu x_2 \\ -cx_2 & 0 & -cx_0 \\ cx_1 & cx_0 & 0 \end{pmatrix}.$$

Thus for $X = \sum x_i X_i$ and $Y = \sum y_i X_i$,

$$XY = (\lambda x_0 y_0 + \mu(x_1 y_1 + x_2 y_2))\, X_0 - c(x_0 y_2 + x_2 y_0)\, X_1 + c(x_0 y_1 + y_0 x_1)\, X_2,$$

so the matrix for $L(XY)$ is

$$\begin{pmatrix} \lambda(\lambda x_0 y_0 + \mu(x_1 y_1 + x_2 y_2)) & -\mu(x_0 y_2 + x_2 y_0)c & \mu(x_0 y_1 + x_1 y_0)c \\ -c^2(x_0 y_1 + x_1 y_0) & 0 & -c(\lambda x_0 y_0 + \mu(x_1 y_1 + x_2 y_2)) \\ -c^2(x_0 y_2 + x_2 y_0) & c(\lambda x_0 y_0 + \mu(x_1 y_1 + x_2 y_2)) & 0 \end{pmatrix}.$$

Thus

$$C(X, Y) = \lambda^2 x_0 y_0 + \lambda\mu(x_1 y_1 + x_2 y_2).$$

This form vanishes identically when $\lambda = 0$. For $\lambda \neq 0$, C is clearly nondegenerate if and only if $\mu \neq 0$, i.e., if and only if A is simple. If $\mu = 0$, then

C has a radical (i.e., A has a nontrivial subspace that is C-orthogonal to all of A). This radical coincides with J, the radical of A. The restriction of C to the simple summand $\mathbb{R}X_0$ is nondegenerate. Finally, note that C is positive definite if and only if $\lambda\mu > 0$. If $\lambda\mu < 0$, then C has a null cone given by

$$C(X, X) = \lambda x_0^2 + \mu(x_1^2 + x_2^2) = 0.$$

The differential system associated with the algebra $A(\lambda, \mu, c)$ is given in coordinates by

$$\dot{x}_0 = \lambda x_0^2 + \mu(x_1^2 + x_2^2)$$
$$\dot{x}_1 = -2c x_0 x_2$$
$$\dot{x}_2 = 2c x_0 x_1.$$

We next give the explicit solution to this system in the various cases. Comparing the theorem below with Lemma 3.11, there is an apparent correspondence between solution forms and algebra structures. We will make more comments on this in the next section.

THEOREM 3.14. *Let $\dot{X} = X^2$ occur in the above algebra A and let $X = x_0 X_0 + x_1 X_1 + x_2 X_2$. Then the solution through X is given by*

$$F_t(X) = x_0(t)\, X_0 + r(\cos(2c\theta(t) + \psi)\, X_1 + \sin(2c\theta(t) + \psi)\, X_2),$$

where $r = (x_1^2 + x_2^2)^{1/2}$, $\theta(t) = \int_0^t x_0(s)\, ds$, ψ is chosen so that $\cos\psi = x_1/r$ and $\sin\psi = x_2/r$, and

(1) *if $\lambda = \mu = 0$, then $x_0(t) = x_0$,*

(2) *if $\lambda = 0$, $\mu \neq 0$, then $x_0(t) = x_0 + \mu r^2 t$,*

(3) *if $\lambda \neq 0$, $\mu = 0$, then $x_0(t) = x_0/(1 - \lambda x_0 t)$,*

(4) *if $\lambda\mu > 0$, then $x_0(t) = (\beta_0/\lambda)(\beta_0 \sin\beta_0 t + \lambda x_0 \cos\beta_0 t)/(\beta_0 \cos\beta_0 t - \lambda x_0 \sin\beta_0 t)$,*

(5) *if $\lambda\mu < 0$, then $x_0(t) = (\beta_1/\lambda)(-\beta_1 \sinh\beta_1 t + \lambda x_0 \cosh\beta_1 t)/(\beta_1 \cosh\beta_1 t - \lambda x_0 \sinh\beta_1 t)$, where $\beta_i = ((-1)^i \lambda\mu)^{1/2} r$, $i = 0, 1$.*

Proof. Tedious but straightforward calculations yield these results.

We now describe without detailed proofs the dynamics that occur in each of the various cases listed in the previous theorem. Recall that $\mathscr{E}$ denotes the set of equilibria for a given vector field. First, as is easily seen from the general solution formula for all cases, if $X \notin \mathscr{E}$, then $F_t(X)$ lies on a circular cylinder with axis X_0 and radius r.

(1) $\lambda=0$, $\mu=0$: $\mathscr{E}=\{x_1x_2\text{-plane}\}\cup\{x_0\text{-axis}\}$. For all nonequilibrium points X, the trajectory through X is a periodic orbit of period $\pi/|cx_0|$ (because the system reduces to a two-dimensional linear system).

(2) $\lambda=0$, $\mu\neq 0$: $\mathscr{E}=\{x_0\text{-axis}\}$. For all nonequilibrium points X, the trajectory through X is a helix, which reverses rotation about the x_0-axis on passing through the plane $x_0=0$.

(3) $\lambda\neq 0$, $\mu=0$: $\mathscr{E}=\{x_1x_2\text{-plane}\}$. If $\lambda x_0<0$, then for each $X\notin\mathscr{E}$, the maximal interval of existence for $F_t(X)$ is $(1/\lambda x_0, \infty)$; the ω-limit set of X is the curve $x_1^2+x_2^2=r^2$, $x_0=0$, which is not a trajectory; $F_t(X)$ blows up in negative time. If $\lambda x_0>0$, then for each $X\notin\mathscr{E}$, the maximal interval of existence for $F_t(X)$ is $(-\infty, 1/\lambda x_0)$; the α-limit set of X is the curve $x_1^2+x_2^2=r^2$, $x_0=0$ which is not a trajectory; $F_t(X)$ blows up in positive time.

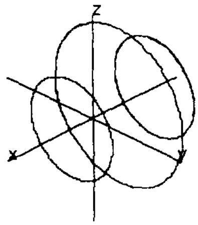
(4) $\lambda\mu>0$: $\mathscr{E}=\{0\}$. When $\lambda x_0<0$, the maximal interval of existence for $F_t(X)$ is $((1/\beta_0)\tan^{-1}(\beta_0/\lambda x_0), (1/\beta_0)\cot^{-1}(\beta_0/\lambda x_0)))$; when $\lambda x_0>0$, it is $((1/\beta_0)\cot^{-1}(\beta_0/\lambda x_0), (1/\beta_0)\tan^{-1}(\beta_0/\lambda x_0))$. In either case, $F_t(X)$ blows up in both positive and negative times.

(5) $\lambda\mu<0$: $\mathscr{E}=\{0\}$. If X lies on the cone $\mathscr{C}=\{\lambda x_0^2+\mu(x_1^2+x_2^2)=0\}$, then $|x_0|=(-\mu/\lambda)^{1/2}r=\beta_1/|\lambda|$. Thus $F_t(X)$ is periodic with period $\pi|\lambda|/|c|\beta_1$. The maximal intervals of existence for all other points X is given in the table,

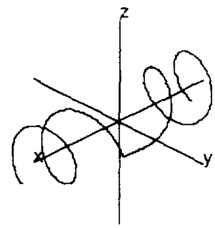
Parameter values	Maximal interval
$\lambda>0$, $\lambda x_0^2+\mu r^2<0$	$(-\infty, \infty)$
$\lambda<0$, $\lambda x_0^2+\mu r^2>0$	$(-\infty, \infty)$
$\lambda>0$, $\lambda x_0^2+\mu r^2>0$, $x_0>0$	$(-\infty, \omega)$
$\lambda<0$, $\lambda x_0^2+\mu r^2<0$, $x_0<0$	$(-\infty, \omega)$
$\lambda>0$, $\lambda x_0^2+\mu r^2>0$, $x_0<0$	(ω, ∞)
$\lambda<0$, $\lambda x_0^2+\mu r^2<0$, $x_0>0$	(ω, ∞)

where $\omega=(1/\beta_1)\tanh^{-1}(\beta_1/\lambda x_0)$. In all cases where t approaches a finite end-point, the solution $F_t(X)$ blows up. In all cases where t approaches $\pm\infty$, an ω (resp. α)-limit set exists, namely the periodic orbit(s) on $\mathscr{C}$ that lie on the same cylinder as X.

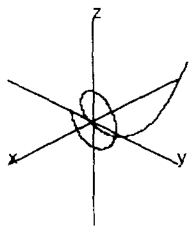
Remarks. In the subcase of case (5), where, say $\lambda<0$, we see that the nappe $\mathscr{C}^+=\mathscr{C}\cap\{x_0>0\}$ is an attracting set: $\mathscr{C}^+$ is invariant and attracts nearby solutions as they spiral along a cylinder toward the nappe. However, the closure $\mathscr{C}^+\cup\{0\}$ is not an attractor. The origin is unstable since A has a nonzero idempotent $E=(1/\lambda)X_0$; see Proposition 3.4. Similarly $\mathscr{C}^-=\mathscr{C}\cap\{x_0<0\}$ is a repelling set, but $\mathscr{C}^-\cup\{0\}$ is not a repellor. Note that orbits starting in $\{\lambda x_0^2+\mu r^2>0\}$ spiral away from one periodic orbit on $\mathscr{C}^-$ to the corresponding periodic orbit on $\mathscr{C}^+$ having the



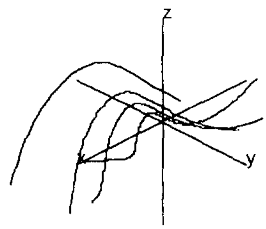
Case 1: $\lambda = 0, \mu = 0, c = 1$



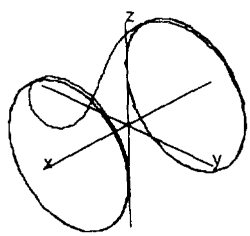
Case 2: $\lambda = 0, \mu = 1, c = 1$



Case 3: $\lambda = 1, \mu = 0, c = 1$



Case 4: $\lambda = 1, \mu = 1, c = 1$



Case 5: $\lambda = 1, \mu = -1, c = 1$

FIGURE 1

same period. We discuss case (5) and the cone $\mathscr{C}$ further in Section 5 using automorphisms.

4. STRUCTURE

In the theory of algebras [1, 36], one is often concerned with semisimple algebras, radicals, and how to put them together as a direct sum to obtain the "structure" of a class of algebras. A semi-simple algebra, S, is a direct sum of ideals, each of which is a simple algebra. The simple commutative algebras often contain idempotent elements which help determine their nature by various "Peirce decompositions." On the other hand, the radical, R, of an algebra is often given by a solvable or nilpotent ideal [2]. Thus nilpotent elements of index >2 may occur. Then one can often show a Wedderburn or Levi type decomposition $A = S + R$ with R the radical and S a semisimple subalgebra of A. We now consider the effect of these structures on the quadratic equation $\dot{X} = X^2$.

PROPOSITION 4.1. *Let $\dot{X} = X^2$ occur in the algebra A.*

(1) *If $A = A_1 \oplus \cdots \oplus A_k$ is semisimple, then the equation $\dot{X} = X^2$ decouples into a system of equations $\dot{X}_i = X_i^2$ in the simple algebras A_i* [23, 38].

(2) *If $A = S + R$ (subspace direct sum), where S is a semisimple subalgebra and R is the radical ideal (i.e., A has a Wedderburn–Levi decomposition), then the equation $\dot{X} = X^2$ can be solved by solving an autonomous equation $\dot{X}_S = X_S^2$ in S and a nonautonomous quadratic equation $\dot{X}_R = X_R^2 + 2X_S X_R$ in R.*

Proof. (1) Let $X = X_1 + \cdots + X_k \in A$. Then since $X_i X_j \in A_i \cap A_j = 0$ if $i \neq j$, we have

$$\begin{aligned} X_1^2 + \cdots + X_k^2 &= (X_1 + \cdots + X_k)^2 \\ &= X^2 \\ &= \dot{X} = \dot{X}_1 + \cdots + \dot{X}_k. \end{aligned}$$

Consequently $\dot{X}_i = X_i^2$ for $i = 1, ..., k$.

(2) Next suppose that $A = S + R$, where S is a semisimple subalgebra and the ideal R is the radical. Write the solution to $\dot{X} = X^2$ in A as $X(t) = X_S(t) + X_R(t)$, $X_S \in S$, $X_R \in R$. We compute

$$\begin{aligned} \dot{X}_S + \dot{X}_R &= \dot{X} \\ &= X^2 \\ &= (X_S + X_R)^2 \\ &= X_S^2 + 2X_S X_R + X_R^2 \\ &= X_S^2 + 2L(X_S)\, X_R + X_R^2, \end{aligned}$$

where $X_S^2 \in S$ and $2L(X_S)\, X_R + X_R^2 \in R$. Thus to solve $\dot{X} = X^2$ in A, one must solve $\dot{X}_S = X_S^2$ in S and the nonautonomous equation $\dot{X}_R = 2L(X_S(t))\, X_R + X_R^2$ in R.

Remark. In general, using (2) of the previous proposition is almost as formidable a task as solving the original equation, but in the special case where the radical R is itself a zero algebra, the second equation reduces to a nonautonomous linear system $\dot{X}_R = 2L(X_S(t))\, X_R$. As an example, we note that this was precisely the situation of the example in Section 3 in the case where $\mu = 0$ and $\lambda \neq 0$.

Next we consider the connections between boundedness of solutions and nilpotence.

PROPOSITION 4.2. *Let $P \in A$ be nilpotent (of index > 2); i.e., the subalgebra $\mathbb{R}[P]$ generated by P is nilpotent. Then the solution $F_t(P)$ of $\dot{X} = X^2$ exists for all time and is unbounded.*

Proof. Since P is nilpotent, there exists an integer $N > 1$ so that all products of length N involving P are zero. From the series solution $F_t(P) = P + tP^2 + \cdots + t^{N-2}P^{(N-1)}$, where $P^{(N-1)} \in \mathbb{R}[P]$ is homogeneous of degree $N-1$. Thus $F_t(P)$ is a polynomial in t and consequently unbounded.

Next we consider solvability.

THEOREM 4.3. *Let A be a solvable commutative algebra with $A \supset A^{(2)} \supset A^{(3)} \supset \cdots \supset A^{(N)} = 0$. Assume that $A^{(k+1)} = A^{(k)}A^{(k)}$ are ideals of A and $TA^{(k)} \subset A^{(k)}$, where $T: A \to A$ is a linear map. Then the solution to the quadratic equation $\dot{X} = TX + X^2$ can be obtained by solving finitely many linear equations.*

Proof. The proof is given by constructing the sequence of linear equations; see [38] for related results. We shall make use of the multiplication operator $L(X)\,Y := XY$ in the proof.

(1) Let W_1 be a subspace of A with $A = W_1 + A^{(2)}$, a direct sum, and let $u_1(t) = w_1(t) + u_2(t)$ be a solution to the above quadratic equation with $w_1(t) \in W_1$, $u_2(t) \in A^{(2)}$. Then

$$\begin{aligned} \dot{w}_1 + \dot{u}_2 = \dot{u}_1 &= T(w_1 + u_2) + (w_1 + u_2)^2 \\ &= Tw_1 + Tu_2 + (w_1 + u_2)^2 \end{aligned}$$

and $\dot{u}_2$, $(w_1 + u_2)^2$, and Tu_2 are in $A^{(2)}$, using $TA^{(2)} \subseteq A^{(2)}$. Thus in the quotient space $A/A^{(2)} \cong W_1$, we obtain the linear equation

$$\dot{\overline{w_1}} = \overline{Tw_1} = \bar{T}\overline{w_1},$$

where $\bar{w} = w + A^{(2)}$ is a coset and $\bar{T}$ is the induced linear map in $A/A^{(2)}$. Thus we solve this linear equation to obtain

$$\overline{w_1}(t) = (\exp \bar{T})\, \overline{w_1}(0) = w_1(t) + A^{(2)}$$

and note that $\dot{w}_1 - Tw_1$ is in $A^{(2)}$. Having obtained w_1, we see that u_2 satisfies in $A^{(2)}$ the equation

$$\begin{aligned}\dot{u}_2 &= -\dot{w}_1 + Tw_1 + Tu_2 + (w_1 + u_2)^2 \\ &= -\dot{w}_1 + Tw_1 + w_1^2 + Tu_2 + 2w_1 u_2 + u_2^2 \\ &= -\dot{w}_1 + Tw_1 + w_1^2 + (T + 2L(w_1))\, u_2 + u_2^2 \\ &\equiv a_2 + T_2 u_2 + u_2^2,\end{aligned}$$

where $a_2 = -\dot{w}_1 + Tw_1 + w_1^2$ is in $A^{(2)}$ and $T_2 = T + 2L(w_1)$ maps $A^{(2)}$ into $A^{(2)}$ using the fact that $A^{(2)}$ is a T-invariant ideal of A.

(2) To solve

$$\dot{u}_2 = a_2 + T_2 u_2 + u_2^2, \qquad (*)$$

let W_2 be a subspace of $A^{(2)}$ with $A^{(2)} = W_2 + A^{(3)}$ a direct sum, and let $u_2 = w_2 + u_3$ be a solution to $(*)$. Then

$$\begin{aligned}\dot{w}_2 + \dot{u}_3 = \dot{u}_2 &= a_2 + T_2(w_2 + u_3) + (w_2 + u_3)^2 \\ &= a_2 + T_2 w_2 + T_2 u_3 + (w_2 + u_3)^2,\end{aligned}$$

where $\dot{u}_3$, $(w_2 + u_3)^2$, and $T_2 u_3$ are in $A^{(3)}$ which is a T-invariant ideal of A.

As in the preceding case, we obtain in the quotient space $A^{(2)}/A^{(3)}$ the linear equation

$$\dot{\overline{w_2}} = \overline{a_2} + \overline{T}\,\overline{w_2}.$$

We solve this linear equation to obtain $\overline{w_2}(t) = w_2(t) + A^{(3)}$, where $\dot{w}_2 - a_2 - Tw_2$ is in $A^{(3)}$. Having obtained w_2, we see that u_3 satisfies the equation in $A^{(3)}$,

$$\begin{aligned}\dot{u}_3 &= -\dot{w}_2 + a_2 + T_2 w_2 + w_2^2 + (T_2 + 2L(w_2))\, u_3 + u_3^2 \\ &\equiv a_3 + T_3 u_3 + u_3^2,\end{aligned}$$

where $a_3 = -\dot{w}_2 + a_2 + T_2 w_2 + w_2^2$ is in $A^{(3)}$ and $T_3 = T_2 + 2L(w_2)$ maps $A^{(3)}$ into $A^{(3)}$ using the fact that $A^{(3)}$ is a T-invariant ideal of A.

To solve

$$\dot{u}_3 = a_3 + T_3 u_3 + u_3^2$$

and the resulting sequence of equations, we continue this process of passing to the quotient $A^{(k)}/A^{(k+1)}$ to reduce the quadratic equation to a linear equation. But when $A^{(N)} = 0$, we already have a linear equation to solve!

Substituting backward, we obtain the original solution as a sum of solutions to linear equations.

EXAMPLE. The example of Section 3 in the case $\lambda = 0$, $\mu = 0$ is easily solved by this method. In lieu of the details, we instead pass on to the following.

EXAMPLE. The following is the general three-dimensional commutative solvable algebra $A = (\mathbb{R}^3, \beta)$ such that the $A^{(k+1)} = A^{(k)}A^{(k)}$ are ideals of A and $\dim A^{(k)}/A^{(k+1)} = 1$. Let A have basis $\{f_1, f_2, f_3\}$ with multiplication table

	f_1	f_2	f_3
f_1	$a_{11}^2 f_2 + a_{11}^3 f_3$	$a_{12}^2 f_2 + a_{12}^3 f_3$	$a_{13}^3 f_3$
f_2	$a_{12}^2 f_2 + a_{12}^3 f_3$	$a_{22}^3 f_3$	$a_{23}^3 f_3$
f_3	$a_{13}^3 f_3$	$a_{23}^3 f_3$	0

Thus $A = \{f_1, f_2, f_3\} \supset A^{(2)} = \{f_2, f_3\} \supset A^{(3)} = \{f_3\} \supset A^{(4)} = 0$.

Now we wish to solve $\dot{X} = X^2$ in A using the procedure of the theorem and extend to $\dot{X} = TX + X^2$.

(1) Write $A = W_1 + A^{(2)}$, where $W_1 = \mathbb{R}f_1$, and let $u_1 = w_1 + u_2$ be a solution, where $w_1(t) = x_1(t) f_1$,. Then

$$\dot{w}_1 + \dot{u}_2 = \dot{u}_1 = (w_1 + u_2)^2$$

is in $A^{(2)}$; therefore we obtain the linear equation $\dot{w}_1 = 0$. That is, $\dot{x}_1 = 0$ and $x_1(t) = c_1$ is contant which gives $w_1(t) = c_1 f_1$. Thus in $A^{(2)}$, solve the equation

$$\begin{aligned} \dot{u}_2 = (w_1 + u_2)^2 &= w_1^2 + 2w_1 u_2 + u_2^2 \\ &= (c_1 f_1)^2 + 2c_1 f_1 u_2 + u_2^2. \end{aligned}$$

(2) To solve this latter equation in $A^{(2)}$, write $A^{(2)} = W_2 + A^{(3)}$, where $W_2 = \mathbb{R}f_2$, and let $u_2 = w_2 + u_3$ be a solution, where $w_2(t) = x_2(t) f_2$. Then

$$\begin{aligned} \dot{w}_2 + \dot{u}_3 = \dot{u}_2 &= w_1^2 + 2w_1(w_2 + u_3) + (w_2 + u_3)^2 \\ &= w_1^2 + 2w_1 w_2 + 2w_1 u_3 + (w_2 + u_3)^2, \end{aligned} \tag{1}$$

where $\dot{u}_3$, $2w_1 u_3$, $(w_2 + u_3)^2$ are in $A^{(3)}$. Thus in $A^{(2)}/A^{(3)}$, we have

$$\dot{\overline{w_2}} = \overline{w_1}^2 + 2\overline{w_1 w_2},$$

using the fact that $A^{(3)}$ is an ideal in A. Since $w_2 = x_2 f_2$, $\bar{w}_2 = x_2 f_2 + A^{(3)}$, and using $\overline{w_1} = c_1 f_1 + A^{(3)}$, the preceding equation becomes

$$\begin{aligned}\dot{x}_2 f_2 + A^{(3)} &= (c_1 f_1)^2 + 2(c_1 f_1)(x_2 f_2) + A^{(3)}\\ &= c_1^2(a_{11}^2 f_2 + a_{11}^3 f_3) + 2c_1 x_2(a_{12}^2 f_2 + a_{12}^3 f_3) + A^{(3)}\\ &= (a_{11}^2 c_1^2 + 2a_{12}^2 c_1 x_2) f_2 + A^{(3)},\end{aligned}$$

using the table and the fact that $f_3 \in A^{(3)}$. Thus, we solve the linear equation

$$\dot{x}_2 = a_{11}^2 c_1^2 + 2a_{12}^2 c_1 x_2$$

to obtain $w_2(t) = x_2(t) f_2$. With this value of w_2, we now solve for $u_3(t) = x_3(t) f_3$:

$$\begin{aligned}\dot{x}_3 f_3 &= \dot{u}_3\\ &= w_1^2 + 2w_1 w_2 - \dot{w}_2 + 2w_1 u_3 + w_2^2 + 2w_2 u_3 + u_3^2,\\ &\qquad \text{using (1)}\\ &= w_1^2 + 2w_1 w_2 - \dot{w}_2 + w_2^2 + 2(w_1 + w_2)\, u_3,\\ &\qquad \text{using } u_3^2 \in A^{(4)} = 0\\ &= (c_1 f_1)^2 + 2(c_1 f_1)(x_2 f_2) - \dot{x}_2 f_2 + (x_2 f_2)^2\\ &\quad + 2(c_1 f_1 + x_2 f_2)\, x_3 f_3\\ &= c_1^2(a_{11}^2 f_2 + a_{11}^3 f_3) + 2x_1 x_2(a_{12}^2 f_2 + a_{12}^3 f_3)\\ &\quad - (a_{11}^2 c_1^2 + 2a_{12}^2 c_1 x_2) f_2 + x_2^2 a_{22}^3 f_3\\ &\quad + 2c_1 x_3 a_{13}^3 f_3 + 2x_2 x_3 a_{23}^3 f_3\\ &\qquad \text{using } c_1 = x_1 \text{ and the table,}\\ &= (a_{11}^3 c_1^2 + a_{22}^3 x_2^2 + 2c_1 a_{12}^3 x_2 + 2(a_{13}^3 c_1 + a_{23}^3 x_2)\, x_3) f_3,\\ &\qquad \text{noting that } x_1 = c_1.\end{aligned}$$

Thus we solve this linear equation for x_3 to obtain $u_3 = x_3 f_3$ and, consequently, the solution

$$u_1 = w_1 + u_2 = c_1 f_1 + (w_2 + u_3) = c_1 f_1 + x_2 f_2 + x_3 f_3.$$

Thus we can solve in terms of coordinates or directly in terms of the algebra as in the proof of the theorem. The above coordinate equations are exactly what is obtained from writing $X = \sum x_i f_i$ and expanding $\dot{X} = X^2$, using the multiplication table.

For this algebra, a system $\dot{X} = TX + X^2$ which can be solved by solving a sequence of linear equations is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ a_{11}^2 x_1^2 + 2a_{12}^2 x_1 x_2 \\ a_{11}^3 x_1^2 + a_{22}^3 x_2^2 + 2a_{12}^3 x_1 x_2 + 2(a_{13}^3 x_1 + a_{23}^3 x_2) x_3 \end{bmatrix}$$

relative to the $\{f_1, f_2, f_3\}$ basis. The triangular form of the matrix for T is due to $TA^{(k)} \subseteq A^{(k)}$

Remark. The Lorenz system and the quadratic Duffing system cannot be solved by this method; the algebra A is solvable with $A^{(k)}$ ideals in A, but the condition $TA^{(k)} \subseteq A^{(k)}$ is not satisfied.

5. Automorphisms

The automorphisms of an equation or algebra measure its structure and symmetries. For the system $\dot{X} = X^2$, the automorphism group of the vector field X^2 is the same as the automorphism group of the algebra. This can help locate equilibrium points, periodic orbits, and domains of attraction in an algebraic manner.

Definition. An *automorphism* of an algebra $A = (\mathbb{R}^n, \beta)$ is an invertible linear transformation $\phi \in \mathrm{GL}(\mathbb{R}^n)$, the general linear group, such that $\phi\beta(X, Y) = \beta(\phi X, \phi Y)$ for all $X, Y \in A$. A *derivation* of an algebra A is a linear transformation $D: A \to A$ satisfying the product rule $D\beta(X, Y) = \beta(DX, Y) + \beta(X, DY)$ for all $X, Y \in A$.

Remark. The set Aut A of all automorphisms of A is a closed (Lie) subgroup of $\mathrm{GL}(\mathbb{R}^n)$, and the set Der A of all derivations of A is a Lie subalgebra of $\mathrm{gl}(\mathbb{R}^n)$ which is the Lie algebra of $\mathrm{GL}(\mathbb{R}^n)$. For any $D \in \mathrm{Der}\, A$, $\exp D = I + D + D^2/2! + \cdots$ is in Aut A; that is, the Lie algebra of Aut A is Der A [35]. If A is a semisimple algebra with identity, then Der A consists of inner derivations [36] which frequently have easy formulas; in this case, inner automorphisms are obtained.

A similar concept is the following.

Definition. An *automorphism* of a vector field $E(X)$ in $\mathbb{R}^n$ is an invertible linear transformation $\phi \in \mathrm{GL}(\mathbb{R}^n)$ such that $E(\phi X) = \phi E(X)$ for all $X \in \mathbb{R}^n$. A *derivation* of a vector field is a linear transformation $D: \mathbb{R}^n \to \mathbb{R}^n$ satisfying $DE(X) = E'(X)\, DX$ for all $X \in \mathbb{R}^n$.

Remark. As above, the set Aut E of all automorphisms of E is a closed (Lie) subgroup of $\mathrm{GL}(\mathbb{R}^n)$, and the set Der E of all derivations of E is a Lie subalgebra of $gl(\mathbb{R}^n)$. For any $D \in \operatorname{Der} E$, $\exp D = I + D + D^2/2! + \cdots$ is in Aut E; that is, the Lie algebra of Aut E is Der E [35].

These two concepts are related by the following.

THEOREM 5.1. *Let* $\dot{X} = E(X) \equiv TX + X^2$ *occur in a commutative algebra* $A = (\mathbb{R}^n, \beta)$. *Then* $\operatorname{Aut} E = \{\phi \in \operatorname{Aut} A : T\phi = \phi T\}$ *and* $\operatorname{Der} E = \{D \in \operatorname{Der} A : TD = DT\}$. *In particular, when* $E(X) = X^2$, $\operatorname{Aut} E = \operatorname{Aut} A$ *and* $\operatorname{Der} E = \operatorname{Der} A$.

Proof. Suppose that $\phi \in \operatorname{Aut} E$. Then from $E(X) = TX + X^2$, we obtain $T\phi X + (\phi X)^2 = E(\phi X) = \phi E(X) = \phi TX + \phi(X^2)$ which gives $T\phi = \phi T$ and $\phi(X^2) = (\phi X)^2$. Since A is commutative, the latter implies that $\phi\beta(X, Y) = \beta(\phi X, \phi Y)$. Thus $\phi \in \operatorname{Aut} A$. The reverse inclusion is obvious. A similar argument gives the other equality.

Except as noted, most of the rest of the results in this section hold for any vector field E.

LEMMA 5.2. *Let* $E(X)$ *be a vector field with flow* $F_t(X)$, *and let* $\phi \in GL(\mathbb{R}^n)$. *Then* $\phi \in \operatorname{Aut} E$ *if and only if* $\phi \circ F_t = F_t \circ \phi$.

Proof. Suppose that $\phi \in \operatorname{Aut} E$. For $X \in \mathbb{R}^n$, we note that $(d/dt)[(\phi \circ F_t)(X) - (F_t \circ \phi)(X)] = \phi(d/dt) F_t(X) - (d/dt) F_t(\phi X) = (\phi \circ F)(X) - (F \circ \phi)(X) = 0$, so $(\phi \circ F_t)(X) - (F_t \circ \phi)(X) = C$, a constant vector. Set $t = 0$ to get $C = \phi(X) - \phi(X) = 0$.

Conversely, suppose that $\phi \circ F_t = F_t \circ \phi$. Then

$$\begin{aligned} \phi E(X) &= \phi E(F_t(X))|_{t=0} \\ &= \phi \left.\frac{d}{dt}\right|_{t=0} F_t(X) \\ &= \left.\frac{d}{dt}\right|_{t=0} \phi F_t(X), \qquad \text{using } \phi \text{ linear} \\ &= \left.\frac{d}{dt}\right|_{t=0} F_t(\phi X) \\ &= E(F_t(\phi X))|_{t=0} \\ &= E(\phi X). \end{aligned}$$

PROPOSITION 5.3. *Let* $G \in \operatorname{Der} E$, *then* $(\exp tG)P$ *is a solution to* $\dot{X} = E(X)$ *if and only if* $GP = E(P)$.

Proof. We have

$$GP = E(P) \qquad \text{if and only if}$$

$$(\exp tG)\, GP = (\exp tG)\, E(P) \qquad \text{if and only if}$$

$$(d/dt)(\exp tG)\, P = E((\exp tG)\, P), \qquad \text{using } \exp tG \in \operatorname{Aut} E.$$

COROLLARY 5.4. *Suppose that* $(\exp tG)P$ *is a solution to* $\dot{X} = E(X)$.

(1) *P is an equilibrium point if and only if* $GP = 0$.

(2) *P is periodic point if and only if there exists* $\tau \in \mathbb{R}$ *such that* $(\exp \tau G)P = P$; *i.e., P is an eigenvector of* $\exp \tau G$ *with eigenvalue* 1.

COROLLARY 5.5. *Let* $G \in \operatorname{Der} E$ *and let* $F_t(P) = (\exp tG)P$ *be a solution. If there exists* $t_0 \in \mathbb{R}$ *such that* $GF_{t_0}(P) = 0$, *then* $GF_t(P) = 0$, *i.e.,* $F_t(P) \in \operatorname{Ker}(G)$ *for all t. If there exists* $t_0 \in \mathbb{R}$ *such that* $GF_{t_0}(P) \neq 0$, *then* $GF_t(P) \neq 0$ *for all t.*

Proof. $GX = 0$ implies that $e^{sG}X = X$ for all $s \in \mathbb{R}$. This gives $F_t(X) = F_t(e^{sG}X) = e^{sG}F_t(X)$ for all $s, t \in \mathbb{R}$. Differentiating at $s = 0$ gives $0 = GF_t(X)$ for all t in $\mathbb{R}$. The same argument with $=$ replaced by $\neq$ gives the other assertion.

The derivations giving a solution γ are unique up to their action on γ.

COROLLARY 5.6. *Let* γ *be a trajectory for* $\dot{X} = E(X)$ *and let* $P_i \in \gamma$ $(i = 1, 2)$ *be such that* $\gamma = \{(\exp tG_i)\, P_i\}$, *where* $G_i \in \operatorname{Der} E$ *and* $G_i P_i = E(P_i)$. *Then for any* $Q \in \gamma$, $G_1 Q = G_2 Q = E(Q)$. *In particular, if* γ *contains a basis of* $\mathbb{R}^n$, *then* $G_1 = G_2$.

Proof. Let $Q \in \gamma$, then writing $Q = (\exp s_i G_i)\, P_i (i = 1, 2)$, we have

$$\begin{aligned} G_i Q &= (\exp s_i G_i)\, G_i P_i \\ &= (\exp s_i G_i)\, E(P_i) \\ &= E((\exp s_i G_i)\, P_i), \exp s_i G_i \in \operatorname{Aut} E \\ &= E(Q). \end{aligned}$$

PROPOSITION 5.7. *Let* $\mathscr{P}_\tau$ *be the set of periodic solutions of* $\dot{X} = E(X)$ *which are of period* τ *and let* $\mathscr{E}$ *be the set of equilibrium points. Then* $(\operatorname{Aut} E)\, \mathscr{P}_\tau = \mathscr{P}_\tau$ *and* $(\operatorname{Aut} E)\mathscr{E} = \mathscr{E}$.

Proof. Let $F_t(X)$ be in $\mathscr{P}_\tau$ and let $\phi \in \operatorname{Aut} E$, then $\phi X = \phi F_\tau(X) = F_\tau(\phi X)$ and $F_t(\phi X)$ is a solution to the differential equation with initial condition ϕX. Thus $F_t(\phi X)$ is a periodic solution with period $\sigma \leqslant \tau$. But since $\phi X = F_\sigma(\phi X) = \phi F_\sigma(X)$ and ϕ is bijective, we have $X = F_\sigma(X)$ so that $\sigma = \tau$; i.e., $\phi \mathscr{P}_\tau \subseteq \mathscr{P}_\tau$. Since $F_t(X) = \phi(\phi^{-1} F_t(X))$ and $\phi^{-1} F_t(X)$ is in $\mathscr{P}_\tau$, we have $\mathscr{P}_\tau \subseteq \phi \mathscr{P}_\tau$.

Similarly, $N \in \mathscr{E}$, provided that $E(N) = 0$ and in this case $0 = \phi E(N) = E(\phi N)$, so that $\phi \mathscr{E} \subseteq \mathscr{E}$. Also $\mathscr{E} \subseteq \phi \mathscr{E}$, as above.

DEFINITION. $\mathscr{P}_\tau$ is said to consist of *isolated orbits* if for each $\gamma \in \mathscr{P}_\tau$, there is a tubular neighborhood $\mathscr{U}$ of γ so that no $\delta \in \mathscr{P}_\tau$ intersects $\mathscr{U}$.

We now obtain periodic solutions of $\dot{X} = E(X)$ in terms of automorphisms.

THEOREM 5.8. *Suppose that* $\mathscr{P}_\tau$ *consists of isolated orbits and* Der $E \neq \{0\}$.

(1) *If there is a* $\gamma \in \mathscr{P}_\tau$, $P \in \gamma$, *and* $D \in \operatorname{Der} E$ *such that* $DP \neq 0$, *then there is a nonzero* $a \in \mathbb{R}$ *such that* $\gamma(t) = (\exp tG)P$, *where* $G = a^{-1}D$.

(2) *Let* $(\operatorname{Aut} E)_0$ *denote the connected component of the identity in* Aut E. *For each* $\gamma \in \mathscr{P}_\tau$, $(\operatorname{Aut} E)_0 \gamma = \gamma$ *as sets.*

Proof. From the previous proposition, we know that automorphisms map periodic points of period τ onto periodic points of period τ. Now fix $\gamma \in \mathscr{P}_\tau$ and $P \in \gamma$, and let $\gamma(t) = F_t(P)$. Let $D \in \operatorname{Der} E$ be such that $DP \neq 0$ and define the map

$$k: \mathbb{R} \to A : s \to (\exp sD)P.$$

For each s, $k(s)$ is a periodic point of period τ. Since k is continuous, the image of k is connected. But since the orbits in $\mathscr{P}_\tau$ are isolated, there is $\tilde{\gamma} \in \mathscr{P}_\tau$ such that $k(s) \in \tilde{\gamma}$ for all $s \in \mathbb{R}$. Since $k(0) = P \in \gamma$, we have $\tilde{\gamma} = \gamma$. Consequently, there is $u(s) \in \mathbb{R}$ so that

$$(\exp sD)P = \gamma(u(s)) = F_{u(s)}(P). \tag{*}$$

The function $u: \mathbb{R} \to \mathbb{R} : s \to u(s)$ is differentiable and we may assume that $u(0) = 0$ as follows. Let $u(0) = b$ and write $u(s) = \tilde{u}(s) + b$, where $\tilde{u}(0) = 0$, then $(\exp sD)P = F_{\tilde{u}(s)+b}(P) = F_{\tilde{u}(s)}(F_b(P))$. Set $s = 0$ to obtain $P = F_b(P)$ and, consequently, $(\exp sD)P = F_{\tilde{u}(s)}(P)$.

Differentiating (*) we obtain

$$\begin{aligned}(\exp sD)\, DP &= (d/ds)((\exp sD)P) \\ &= (d/ds)\, F_{u(s)}(P) \\ &= (d/du)\, F_u(P)(du/ds), \qquad \text{chain rule} \\ &= E(F_u(P))\, u'(s),\end{aligned}$$

and setting $s=0$, $DP=E(F_{u(0)}(P))\,u'(0)=u'(0)\,E(P)$. Since $DP\neq 0$, we have $u'(0)=a\neq 0$. We set $G=a^{-1}D$ to obtain $GP=E(P)$. By Proposition 5.3, $F_t(P)=(\exp tG)P$, which yields (1).

To prove (2), fix a trajectory $\gamma\in\mathscr{P}_\tau$ and write $\gamma(t)=F_t(P)$. If $DP=0$ for all $D\in\operatorname{Der} E$, then $e^D F_t(P)=F_t(e^D P)=F_t(P)$. Consequently, for any $\phi=e^{D_1}\cdots e^{D_n}$ in $(\operatorname{Aut} E)_0$, we have $\phi F_t(P)=F_t(P)$. Next let $D\in\operatorname{Der} E$ be such that $DP\neq 0$. Using (1), we can write $F_t(P)=e^{tG}P$ with $G=bD$ for some nonzero $b\in\mathbb{R}$. Consequently, $e^{sD}F_t(P)=e^{sD}e^{tbD}P=e^{((s/b)+t)bD}P=F_{(s/b)+t}(P)$; thus, $e^{sD}\gamma\subseteq\gamma$. Conversely, $\gamma(t)=e^{tG}P=e^{sD}e^{(t-(s/b))G}P=e^{sD}\gamma(t-(s/b))$ so that $\gamma\subseteq e^{sD}\gamma$, and thus, $e^{sD}\gamma=\gamma$ as sets. For any $\phi\in(\operatorname{Aut} E)_0$, we have $\phi=\exp(s_1D_1)\cdots\exp(s_nD_n)$, and so the general case follows from the two cases just considered.

COROLLARY 5.9. *Let $\gamma\in\mathscr{P}_\tau$ be such that $(\operatorname{Der} E)\gamma\neq 0$ and let $\mathscr{P}_\tau$ have finitely many elements. Then $\gamma(t)=e^{tG}P$ for some $P\in\gamma$ and some $G\in\operatorname{Der} E$ with $GP=E(P)\neq 0$.*

EXAMPLE. The three-dimensional algebra A in Section 3 with $E(X)=X^2$ satisfies these conditions when $(\mu,\lambda)\neq 0$ and is discussed in more detail later in this section. Regarding polynomial systems in $\mathbb{R}^2$ having finitely many limit cycles and results of related interest, see [37, 7, 12, 8].

Remark. The referee raises the following interesting question: is there a counterexample to Theorem 5.8 when elements of $\mathscr{P}_\tau$ are not isolated from one another?

THEOREM 5.10. *If N be an isolated equilibrium of E, then $(\operatorname{Aut} E)_0 N = N$ and $(\operatorname{Der} E)N=\{0\}$.*

Proof. The map $k:\mathbb{R}\to A: s\to e^{sD}N$ is continuous and $k(s)$ is an equilibrium for all $s\in\mathbb{R}$. But $\mathbb{R}$ is connected; therefore, the image $k(\mathbb{R})$ is connected in $\mathscr{E}$. Since $k(0)=N$ and N is isolated, $k(s)=e^{sD}N=N$ for all $s\in\mathbb{R}$, and consequently $(\operatorname{Aut} E)_0 N=N$. Differentiating $k(s)$ and setting s equal to 0 gives $DN=0$, and consequently $(\operatorname{Der} E)N=\{0\}$.

COROLLARY 5.11. *Let $\mathscr{E}$ denote the set of all equilibria of E. If $\mathscr{E}$ contains a basis of $\mathbb{R}^n$ consisting of isolated points, then* $\operatorname{Der} E=\{0\}$. *(Thus no nontrivial solutions can be of the form $e^{tG}P$.)*

Remark. If $E(X)=X^2$ and $\mathscr{E}\neq 0$, then $\mathscr{E}$ has no isolated points, since for a given $N\in\mathscr{E}$, the line $\mathscr{L}(N)$ through the origin and N is contained in $\mathscr{E}$: a point Q is on the line if $Q=uN$ for some $u\in\mathbb{R}$ and $E(Q)=E(uN)=(uN)^2=u^2N^2=0$; see [23, Theorem 2].

COROLLARY 5.12. *Let $E(X) = X^2$ and let $\mathscr{E}$ consist of only finitely many lines, and let $0 \neq N \in \mathscr{E}$.*

(1) *If $\phi \in (\text{Aut } E)_0$, then $\phi N = u(\phi)N$ for some $u = u(\phi) \in \mathbb{R}$; i.e., $\phi\mathscr{L}(N) = \mathscr{L}(N)$.*

(2) *The mapping u_N: $(\text{Aut } E)_0 \to \mathbb{R} : \phi \mapsto u(\phi)$ is a homomorphism into the multiplicative group $\mathbb{R}^*$ of $\mathbb{R}$; i.e., u_N is a character of $(\text{Aut } E)_0$.*

(3) *If $(\text{Aut } E)_0$ is semi-simple, then $\phi N = N$ for all $\phi \in (\text{Aut } E)_0$.*

(4) *If $\mathscr{E}$ contains a basis, then all $\phi \in (\text{Aut } E)_0$ can be simultaneously diagonalized; in particular, $(\text{Aut } E)_0$ is commutative.*

Proof. (1) As in the proof of Theorem 5.10, the image $k(\mathbb{R})$ is in $\mathscr{E}$ and is connected. Since there are only finitely many lines in $\mathscr{E}$ and $k(0) = N \in \mathscr{L}(N)$, we have $k(\mathbb{R}) \subseteq \mathscr{L}(N)$. Thus $e^{sD}N = u(s)N$ fir $u(s) \in \mathbb{R}$ and, consequently, for $\phi \in (\text{Aut } E)_0$, $\phi N = uN$ for some $u = u(\phi) \in \mathbb{R}$.

(2) For $\phi_1, \phi_2 \in (\text{Aut } E)_0$ we have $u(\phi_1\phi_2)N = \phi_1\phi_2(N) = \phi_1(\phi_2 N) = \phi_1(u(\phi_2)N) = u(\phi_2)(\phi_1 N) = u(\phi_2)\, u(\phi_1)N$ so that $u(\phi_1\phi_2) = u(\phi_1)\, u(\phi_2)$ in $\mathbb{R}$. Next if $u(\phi) = 0$ for some ϕ, then $\phi N = u(\phi)N = 0$ implies that $N = 0$, contrary to the assumption that $N \neq 0$. Consequently, $u = u_N$ is a homomorphism into $\mathbb{R}^*$.

(3) Since u_N is a homomorphism, the derivative $u'_N = u'(I)$: $\text{Der } E \to \mathbb{R}$ is a homomorphism of Lie algebras, and for $G_1, G_2 \in \text{Der } E$ we have $u'_N([G_1, G_2]) = [u'_N G_1, u'_N G_2] = 0$, since $\mathbb{R}$ is an abelian Lie algebra. If $\text{Der } E$ is semi-simple, then $[\text{Der } E, \text{Der } E] = \text{Der } E$ so that $u'_N(\text{Der } E) = 0$. For any $G \in \text{Der } E$, $u_N(\exp G) = \exp(u'_N G) = 1$ and $(\exp G)N = u_N(\exp G)N = N$. Since $(\text{Aut } E)_0$ is generated by $\exp(\text{Der } E)$, $\phi N = N$ for all $\phi \in (\text{Aut } E)_0$.

(4) Let $\{X_1, ..., X_n\} \subseteq \mathscr{E}$ be a basis, then for $X_i = N_i$ in part (1), note that $\phi X_i = u_i X_i$, i.e., ϕ is diagonalizable.

Again, let E be an arbitrary vector field.

THEOREM 5.13. *Let γ be a trajectory for $\dot{X} = E(X)$ and let $\text{Att}(\gamma)$ denote its domain of attraction. Let $G_\gamma = \{\phi \in \text{Aut } E : \phi\gamma = \gamma \text{ (as sets)}\}$. Then $G_\gamma \text{Att}(\gamma) = \text{Att}(\gamma)$.*

Proof. For $\phi \in G_\gamma$ and $Z \in \text{Att}(\gamma)$, $\|F_t(\phi Z) - \gamma\| = \|F_t(\phi Z) - \phi\gamma\| = \|\phi F_t(Z) - \phi\gamma\| = \|\phi(F_t(Z) - \gamma)\| \leqslant \|\phi\| \, \|F_t(Z) - \gamma\| \to 0$ as $t \to \infty$. Thus $G_\gamma \text{Att}(\gamma) \subseteq \text{Att}(\gamma)$. Conversely, note that $Z = \phi(\phi^{-1}Z)$ and $\phi^{-1}Z \in \text{Att}(\gamma)$, so $\text{Att}(\gamma) \subseteq G_\gamma \text{Att}(\gamma)$.

COROLLARY 5.14. *Let $\mathscr{P}_\tau$ consist of isolated orbits, and let $\gamma \in \mathscr{P}_\tau$ be given by $\gamma(t) = e^{tG}P$. Then $(\text{Aut } E)_0 \text{Att}(\gamma) = \text{Att}(\gamma)$.*

Proof. From Theorem 5.8(2), we note that $(\text{Aut } E)_0 \subseteq G_\gamma$ and that the proof of the theorem holds for subgroups of G_γ.

Similarly we have the following results for equilibria.

THEOREM 5.15. *Let N be an equilibrium of $\dot{X} = E(X)$ and let $G_N = \{\phi \in \text{Aut } E : \phi N = N\}$, and let* $\text{Att}(N)$ *be the domain of attraction of N. Then G_N* $\text{Att}(N) = \text{Att}(N)$.

COROLLARY 5.16. *If $N \in \mathscr{E}$, where $\mathscr{E}$ consists only of isolated points, then* $(\text{Aut } E)_0 \text{ Att}(N) = \text{Att}(N)$.

COROLLARY 5.17. *Suppose that $E(X) = X^2$ and suppose that $\mathscr{E}$ consists of finitely many lines $\mathscr{L}(N)$, then* $(\text{Aut } E)_0 \text{ Att}(\mathscr{L}(N)) = \text{Att}(\mathscr{L}(N))$.

EXAMPLE. Let A be the commutative Euler algebra given in Section 1. A has a basis $\{X_1, X_2, X_3\}$ with multiplicative relations $X_1 X_2 = a^3_{12} X_3$, $X_1 X_3 = a^2_{13} X_2$, $X_2 X_3 = a^1_{23} X_1$ for $a^k_{ij} \neq 0$ and $X_j^2 = 0$ for $j = 1, 2, 3$. Solutions to $\dot{X} = X^2$ occurring in A are not of the form $e^{tG}P$ for any $G \in \text{Der } A$ since we now show that $\text{Der } A = 0$. Let $GX_k = \sum g_{ik} X_i$ for $k = 1, 2, 3$. Then

$$
\begin{aligned}
0 = G(X_1^2) &= 2X_1(GX_1) \\
&= 2X_1\left(\sum g_{i1} X_i\right) \\
&= \sum 2g_{i1} X_1 X_i \\
&= 2g_{21} X_1 X_2 + 2g_{31} X_1 X_3, \qquad \text{using } X_1^2 = 0 \\
&= 2g_{21} a^3_{12} X_3 + 2g_{31} a^2_{13} X_2
\end{aligned}
$$

which implies $g_{21} = g_{31} = 0$, since $a^k_{ij} \neq 0$. Similarly, $0 = G(X_2^2) = G(X_3^2)$ implies that $g_{13} = g_{23} = 0 = g_{12} = g_{32}$. Next we compute

$$
\begin{aligned}
G(X_1 X_2) &= G(a^3_{12} X_3) \\
&= a^3_{12} g_{33} X_3 \\
&\equiv (GX_1) X_2 + X_1(GX_2) \\
&= g_{11} X_1 X_2 + g_{22} X_1 X_2 \\
&= (g_{11} a^3_{12} + g_{22} a^3_{12}) X_3
\end{aligned}
$$

which implies, since $a^3_{12} \neq 0$, that $g_{33} = g_{11} + g_{22}$. Similarly, computing $G(X_1 X_2)$ and $G(X_2 X_3)$ gives $g_{22} = g_{11} + g_{33}$ and $g_{11} = g_{22} + g_{33}$. These equations imply that $g_{11} = g_{22} = g_{33} = 0$. Thus we conclude that $G = 0$.

We offer another proof: if $X=\sum x_i X_i$ is an equilibrium point, $0=X^2=2a^1_{32}x_2x_3X_1+2a^2_{13}x_1x_3X_2+2a^3_{12}x_1x_2X_3$, so that $x_1x_2=x_1x_3=x_2x_3=0$. Thus there are finitely many equilibrium lines in $\mathscr{E}$ and $\mathscr{E}$ contains the basis $\{X_1 X_2 X_3\}$ of A. By Corollary 5.12, Der A can be simultaneously diagonalized relative to this basis: $GX_i=\lambda_i(G)X_i$. Consequently,

$$\begin{aligned} a^3_{12}\lambda_3 X_3 &= a^3_{12} G X_3 \\ &= G(X_1 X_2) \\ &= (GX_1)X_2 + X_1(GX_2) \\ &= (\lambda_1+\lambda_2)a^3_{12}X_3 \end{aligned}$$

and $\lambda_3=\lambda_1+\lambda_2$. Similarly, $\lambda_1=\lambda_2+\lambda_3$, $\lambda_2=\lambda_1+\lambda_3$ which gives $\lambda_1=\lambda_2=\lambda_3=0$.

For the more general Euler/geodesic equation $\dot{X}+\alpha(X,X)=0$, there often exist nontrivial derivations. This equation lives in the algebra (m,α) that arises from the Nomizu correspondence [25] between invariant connections on reductive homogeneous spaces G/H (where at the Lie algebra level, $g=m+h$ with $(\mathrm{Ad}\,H)m\subseteq m$) and algebras (m,α) satisfying $\mathrm{Ad}\,H\subseteq \mathrm{Aut}(m,\alpha)$, or equivalently, $\mathrm{ad}\,h\subseteq \mathrm{Der}(m,\alpha)$. Usually $\mathrm{ad}\,h\neq\{0\}$; see [31–33].

EXAMPLE. We now determine the two-dimensional commutative algebras A which admit a bounded (but, of course, not periodic) solution to $\dot{X}=X^2$ of the form $F_t(P)=e^{tG}P$ for $G\in \mathrm{Der}\,A$. By Theorem 5.1, Proposition 5.3 applies, which motivates the following requirement: $GP=P^2$. The algebra we shall obtain is of type (2) in [23, p. 194; also see 24].

LEMMA 5.18. *If $P\neq 0$ and $P^2\neq 0$, then $\{P, P^2\}$ is a basis for A.*

Proof. Suppose that $P^2=aP$. Then

$$\begin{aligned} 0 &= G(P^2-aP)=GP^2-aGP=2P(GP)-aP^2, \\ &\qquad \text{using } G\in \mathrm{Der}\,A,\ GP=P^2 \\ &= 2PP^2-a^2P=2aP^2-a^2P=2a^2P-a^2P, \\ &\qquad \text{using } P^2=aP \\ &= a^2P. \end{aligned}$$

This implies $a=0$ so that P and P^2 are linearly independent and, consequently, form a basis of A.

This basis gives the multiplication table

	P	P^2
P	P^2	$aP+bP^2$
P^2	$aP+bP^2$	$cP+dP^2$

for a, b, c, d to be determined in $\mathbb{R}$. With this basis, we have the matrix representation

$$L(P)=\begin{pmatrix}0 & a\\ 1 & b\end{pmatrix}, \qquad L(P^2)=\begin{pmatrix}a & c\\ b & d\end{pmatrix}, \qquad G=\begin{pmatrix}0 & 2a\\ 1 & 2b\end{pmatrix},$$

using $GP=P^2$ and $GP^2=2P(GP)=2PP^2=2aP+2bP^2$. Now since $G\in \operatorname{Der} A$, we use the above matrices and $[G, L(X)]=L(GX)$ [36, p. 20] for any derivation G to compute

$$\begin{aligned}[G, L(P)] &= GL(P)-L(P)G\\ &=\begin{pmatrix}a & 0\\ b & -a\end{pmatrix}\\ &\equiv L(GP)=L(P^2)\\ &=\begin{pmatrix}a & c\\ b & d\end{pmatrix},\end{aligned}$$

so that $c=0$, $d=-a$. Similarly, using $[G, L(P^2)]=L(GP^2)=2aL(P)+2bL(P^2)$, we obtain $a=0$; that is, $a=c=d=0$.

As in the case of Lie algebras, $\operatorname{Ker} G\neq 0$. Otherwise A is a nilpotent algebra and the solution $F_t(P)$ is unbounded. So next let $Q'=uP+vP^2\in \operatorname{Ker} G$. Then

$$\begin{aligned}0=GQ' &= uGP+vGP^2\\ &=uP^2+v(2bP^2)\\ &=(u+2bv)\,P^2.\end{aligned}$$

Thus $u+2bv=0$, so $Q'=v(-2bP+P^2)$; let $Q=-2bP+P^2$ be a basis element for $\operatorname{Ker} G$ and now consider the multiplication table for A relative to the $\{Q, P^2\}$ basis.

Using the previous table with $a=c=d=0$, we compute to obtain $Q^2=(-2bP+P^2)^2=0$ and $QP^2=-2b^2P^2$, yielding the new table

	Q	P^2
Q	0	$-2b^2P^2$
P^2	$-2b^2P^2$	0

Relative to this $\{Q, P^2\}$ basis, we have the matrix representations

$$L(Q)=\begin{pmatrix}0 & 0\\ 0 & -2b^2\end{pmatrix},\qquad L(P^2)=\begin{pmatrix}0 & 0\\ -2b^2 & 0\end{pmatrix},\qquad G=\begin{pmatrix}0 & 0\\ 0 & 2b\end{pmatrix}.$$

LEMMA 5.19. *A is solvable, but not nilpotent.*

Proof. From the above table, $A=\operatorname{span}\{Q, P^2\}\supset A^{(2)}=\operatorname{span}\{P^2\}\supset A^{(3)}=0$, but $L(Q)$ is not nilpotent.

Next from $P=-1/2b(Q-P^2)$, the solution is given by

$$\begin{aligned}F_t(P)=e^{tG}P&=-1/2b(e^{tG}Q-e^{tG}P^2)\\ &=1/2b(Q-e^{2bt}P^2)\\ &=P+tP^2+O(b).\end{aligned}$$

THEOREM 5.20. *Let $\dot{X}=X^2$ occur in a two-dimensional algebra A which supports a solution of the form $F_t(P)=e^{tG}P$ for some $G\in \operatorname{Der} A$ with $GP=P^2$, and $P\neq 0$, $P^2\neq 0$. Then*

1. *A is solvable but not nilpotent and has a basis $\{Q, P^2\}$ of equilibria where Q, P^2 are eigenvectors of G with $GQ=0$ and $GP^2=2bP^2$.*

2. *The solution through the point P is given by $F_t(P)=e^{tG}P=-1/2b(Q-e^{2bt}P^2)$, where $Q=-2bP+P^2$. For $b<0$, this solution asymptotically approaches the equilibrium point $-1/2bQ$, for $b>0$, the solution is repelled by this equilibrium point. Near the bifurcation value $b=0$, the solution is approximately $P+tP^2$ and at $b=0$, the solution is P (since $P_2=0$ in this case).*

3. *Let $X=x_1Q+x_2P^2$ in A, then the general solution to $\dot{X}=X^2$ through X is given by $x_1(t)\equiv x_1$ (constant), $x_2(t)=x_2\exp(-4b^2x_1t)$. For $x_1>0$, $X(t)\to x_1Q$ as $t\to\infty$, where x_1Q is an equilibrium.*

EXAMPLE. Suppose that A is a three-dimensional commutative algebra with a derivation $G\neq 0$ and a nonzero point $P\in A$ such that $GP=P^2$ and such that the corresponding solution trajectory $F_t(P)=e^{tG}P$ is periodic of least period τ. We now determine A up to isomorphism. Since $F_t(P)$ also satisfies the linear equation $\dot{X}=GX$, there must exist a basis $\{X_0, X_1, X_2\}$ such that relative to this basis, G has the matrix representation

$$\begin{pmatrix}a & 0 & 0\\ 0 & 0 & b\\ 0 & -b & 0\end{pmatrix},$$

where $b = 2\pi/\tau$. If $a \neq 0$, then by decomposing the complexification $A_{\mathbb{C}}$ of A, relative to G,

$$A_{\mathbb{C}} = A_{\mathbb{C}}(a) + A_{\mathbb{C}}(bi) + A_{\mathbb{C}}(-bi) \qquad \text{(direct sum)}$$

the results in [13, p. 54, Example 8] can be used to show that the eigenspaces satisfy $A_{\mathbb{C}}(u)\, A_{\mathbb{C}}(v) \subseteq A_{\mathbb{C}}(u+v)$ if $u+v$ is an eigenvalue; otherwise $A_{\mathbb{C}}(u)\, A_{\mathbb{C}}(v) = 0$. By considering the possible eigenvalues $u, v \in \{a, bi, -bi\}$, we see that $u+v$ is not an eigenvalue, i.e., $A_{\mathbb{C}}(u)\, A_{\mathbb{C}}(v) = 0$. This gives $A_{\mathbb{C}}^2 = 0$ and $A^2 = 0$. Thus assuming that the algebra is nontrivial, we have $a = 0$. We now determine the general three-dimensional commutative algebra A having G as a derivation by giving a multiplication table in terms of $\{X_0, X_1, X_2\}$. From the above decomposition,

$$A_{\mathbb{C}}(0) = \{Z \in A_{\mathbb{C}} : GZ = 0\} = (\operatorname{Ker} G)_{\mathbb{C}},$$

$$A_{\mathbb{C}}(\pm bi) = \{Z \in A_{\mathbb{C}} : GZ = \pm biZ\}$$

are G-invariant subspaces which satisfy the relations

$$A_{\mathbb{C}}(0)\, A_{\mathbb{C}}(0) \subseteq A_{\mathbb{C}}(0), \tag{1}$$

$$A_{\mathbb{C}}(0)\, A_{\mathbb{C}}(\pm bi) = A_{\mathbb{C}}(\pm bi)\, A_{\mathbb{C}}(0) \subseteq A_{\mathbb{C}}(\pm bi), \tag{2}$$

$$A_{\mathbb{C}}(bi)\, A_{\mathbb{C}}(bi) = A_{\mathbb{C}}(-bi)\, A_{\mathbb{C}}(-bi) = 0, \tag{3}$$

$$A_{\mathbb{C}}(bi)\, A_{\mathbb{C}}(-bi) \subseteq A_{\mathbb{C}}(0). \tag{4}$$

We have $A_{\mathbb{C}}(0) = \mathbb{C} \cdot X_0$, $A_{\mathbb{C}}(bi) = \mathbb{C} \cdot (X_1 + iX_2)$, and $A_{\mathbb{C}}(-bi) = \mathbb{C} \cdot (X_1 - iX_2)$.

From (1), $X_0^2 = \lambda X_0$ for some $\lambda \in \mathbb{C}$, but since X_0 and X_0^2 are in $\mathbb{R}^3$, λ must be real. From (3), we have $X_1^2 - X_2^2 \pm 2i(X_1 X_2) = 0$, so $X_1^2 = X_2^2$ and $X_1 X_2 = 0$. From (4), $X_1^2 + X_2^2 = \alpha X_0$ for some α, which, as before, must be real. Thus $X_1^2 = X_2^2 = \mu X_0$, where $\mu = \alpha/2$. From (2), we have

$$X_0(X_1 + iX_2) = \beta(X_1 + iX_2) \tag{5}$$

for some $\beta \in \mathbb{C}$. Adding (5) to its own complex conjugate yields $X_0 X_1 = c_1 X_1 + c_2 X_2$, where $c_1 = \frac{1}{2}(\beta + \bar{\beta}) \in \mathbb{R}$, and $c_2 = (i/2)(\beta - \bar{\beta}) \in \mathbb{R}$. Taking (5) minus its complex conjugate yields $X_0 X_2 = -c_2 X_1 + c_1 X_2$. We thus have the following multiplication table for $A = A(\lambda, \mu, c_1, c_2)$.

	X_0	X_1	X_2
X_0	λX_0	$c_1 X_1 + c_2 X_2$	$-c_2 X_1 + c_1 X_2$
X_1	$c_1 X_1 + c_2 X_2$	μX_0	0
X_2	$-c_2 X_1 + c_1 X_2$	0	μX_0

The vector field associated with A for $X = x_0 X_0 + x_1 X_1 + x_2 X_2$ is

$$\begin{aligned} E(X) = X^2 &= (x_0 X_0 + x_1 X_1 + x_2 X_2) \\ &= [\lambda x_0^2 + \mu(x_1^2 + x_2^2)]\, X_0 + [2x_0(c_1 x_1 - c_2 x_2)]\, X_1 \\ &\quad + [2x_0(c_2 x_1 + c_1 x_2)]\, X_2 \\ &= \begin{pmatrix} \lambda x_0^2 + \mu(x_1^2 + x_2^2) \\ 2x_0(c_1 x_1 - c_2 x_2) \\ 2x_0(c_2 x_1 + c_1 x_2) \end{pmatrix}. \end{aligned}$$

Now we solve the equation $GP = P^2$ in A under the assumptions that $P \neq 0$ and $P^2 \neq 0$. For $P = p_0 X_0 + p_1 X_1 + p_2 X_2$, we have

$$\lambda p_0^2 + \mu(p_1^2 + p_2^2) = 0 \tag{6}$$

$$2p_0(c_1 p_1 - c_2 p_2) = b p_2 \tag{7}$$

$$2p_0(c_2 p_1 + c_1 p_2) = -b p_1. \tag{8}$$

If $p_1 = p_2 = 0$, then (6) shows that $p_0 = 0$ (in which case $P = 0$, a contradiction), or $\lambda = 0$. In the latter case $P = p_0 X_0$ and $P^2 = p_0^2 \lambda X_0 = 0$, another contradiction. Hence under our assumptions, at least one of p_1 and p_2 is nonzero. Thus we may write (7) and (8) as a homogeneous matrix-vector equation with a nontrivial solution:

$$\begin{pmatrix} 2p_0 c_1 & -2p_0 c_2 - b \\ 2p_0 c_2 + b & 2p_0 c_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Computing the determinant, we find that

$$4p_0^2 c_1^2 + (2p_0 c_2 + b)^2 = 0.$$

If $p_0 = 0$, then $b = 0$, a contradiction. Thus $c_1 = 0$ and $p_0 c_2 = -b/2$. We set $c = c_2$ and note that $p_0 = -b/2c$. Our algebra A is now the algebra $A(\lambda, \mu, c)$ of Section 3.

Now (6) restricts the location of P, depending on the values of λ and μ. Assume that $\lambda = \mu = 0$, so that the first equation is a triviality. Then we can choose p_1 and p_2 arbitrarily, and we have $P = (-b/2c)\, X_0 + p_1 X_1 + p_2 X_2$. The solution is given by $F_t(P) = e^{tG} P = (-b/2c)\, X_0 + (p_1 \cos bt + p_2 \sin bt)\, X_1 + (-p_1 \sin bt + p_2 \cos bt)\, X_2$. In this case, $\mathscr{P}_\tau$ does not consist of isolated trajectories.

Now assume that λ and μ are not both zero. Then, in fact, neither one of them can be zero. Indeed, if $\lambda = 0$ and $\mu \neq 0$, (6) shows that $p_1 = p_2 = 0$ and, as before, we have that $P^2 = 0$. If $\lambda \neq 0$ and $\mu = 0$, then (6) shows that

$p_0 = 0$, and (7) and (8) give $p_1 = p_2 = 0$; thus $P = 0$. Thus, neither λ nor μ is zero, and (6) shows that $\lambda\mu < 0$.

Therefore, the solutions P to $GP = P^2$ in $A(\lambda, \mu, c)$ lie in the plane $x_0 \equiv -b/2c$ on the circle centered at $(-b/2c, 0, 0)$ with radius $(b/2\,|c|)(-\lambda/\mu)^{1/2}$; i.e., $p_1^2 + p_2^2 = -\lambda p_0^2/\mu = -\lambda b^2/4\mu c^2$. We write

$$P = P(a) = -\frac{b}{2c} X_0 + r[(\cos a)\, X_1 + (\sin a)\, X_2],$$

where $0 \leqslant a < 2\pi$ and $r = (b/2\,|c|)(-\lambda/\mu)^{1/2}$. The solution trajectory through P is given by

$$\begin{aligned} F_t(P) = e^{tG} P &= -\frac{b}{2c} e^{tG} X_0 + (r \cos a)\, e^{tG} X_1 + (r \sin a)\, e^{tG} X_2 \\ &= -\frac{b}{2c} X_0 + r[\cos a(\cos btX_1 + \sin btX_2) \\ &\quad + \sin a(-\sin btX_1 + \cos btX_2)] \\ &= -\frac{b}{2c} X_0 + r[\cos(bt + a)\, X_1 + \sin(bt + a)\, X_2]. \end{aligned}$$

Summarizing this and previous results, including Theorem 5.8, we have the following.

THEOREM 5.21. *Let $\dot{X} = X^2$ occur in a three-dimensional algebra A supporting a periodic solution $F_t(P) = \exp(tG)P$ of least period $\tau \neq 0$. Then A is isomorphic to $A(\lambda, \mu, c)$. Moreover, $F_t(P)$ is isolated in $\mathscr{P}_\tau$ and $(\mathrm{Aut}\, A)_0\, P \neq P$ if and only if $\lambda\mu \neq 0$.*

We now show that curves in Aut A can determine *every* solution in the algebra $A(\lambda, \mu, c)$ with $\lambda\mu < 0$.

THEOREM 5.22. *Let $\dot{X} = X^2$ occur in $A(\lambda, \mu, c)$. Then the solution through $X = x_0 X_0 + x_1 X_1 + x_2 X_2$ is given by*

$$F_t(X) = x_0(t)\, X_0 + \left(\exp - 2\frac{c}{b}\, \theta(t) G\right) Y_0,$$

where $x_0(t)$ is the solution to $\dot{x}_0 = \lambda x_0^2 + \mu r^2$, $r^2 = x_1^2 + x_2^2$, $\theta(t) = \int_0^t x_0(s)\, ds$, and $Y_0 = x_1 X_1 + x_2 X_2$.

Proof. As in Section 3, $x_1(t)^2 + x_2(t)^2 = x_1^2 + x_2^2 = r^2$, a constant. Thus the differential system can be written in the form

$$\dot{x}_0 = \lambda x_0^2 + \mu r^2$$

$$\dot{Y} = -2\frac{c}{b} x_0 G Y,$$

where $Y(t) := x_1(t) X_1 + x_2(t) X_2 = X(t) - x_0(t) X_0$. The last equation in $Y(t)$ is a linear, nonautonomous system with the solution

$$Y(t) = \left(\exp - 2\frac{c}{b}\theta(t) G\right) Y_0.$$

This completes the proof.

The cone $\mathscr{C}$ obtained from the periodic solutions can be described using trace formulas and groups that leave them invariant. From our calculations in Section 3, we see that for $X = \sum x_i X_i$,

$$f(X) = \text{trace } L(X) = \lambda x_0$$

and $f: A \to \mathbb{R}$ is a linear functional which defines a plane

$$S_1(c) = \{X : f(X) = c = \text{const}\}.$$

Next recall that

$$C(X, Y) = \text{trace } L(XY) = \lambda^2 x_0 y_0 + \lambda\mu(x_1 y_1 + x_2 y_2)$$

defines a nondegenerate bilinear form on A which gives the quadric surfaces

$$S_2(k) = \{X : C(X, X) = k = \text{const}\}.$$

Now let $P \in A$ and $G \in \text{Der } A$; then the solution $F_t(P) = e^{tG}P$ is on the plane $S_1(c)$, where $c = \text{trace } L(P)$: for $\phi = e^{tG} \in \text{Aut } A$, note that $\phi(XY) = (\phi X)(\phi Y)$ implies that $\phi L(X) \phi^{-1} = L(\phi X)$, and, consequently, trace $L(e^{tG}P) = \text{trace } L(P)$; that is, $e^{tG}P \in S_1(c)$. Since $L(GX) = [G, L(X)]$, $F_t(P) \in S_2(0)$, which is the cone $\mathscr{C}$:

$$\begin{aligned} C(F_t(P), F_t(P)) &= \text{trace } L((e^{tG}P)^2) = \text{trace } L(e^{tG}P^2) \\ &= \text{trace } L(P^2) = \text{trace } L(GP) = 0, \end{aligned}$$

where the second and third equations use $e^{tG} \in \text{Aut } A$, the fourth uses $GP = P^2$, and the last uses trace $L(GX) = \text{trace}[G, L(X)] = 0$. These results give the following coordinate free location of the periodic trajectory $\gamma(t) = F_t(P)$.

PROPOSITION 5.23. *The periodic orbit γ lies on $S_1(k) \cap S_2(0)$.*

Remark. Since C is nondegenerate, there exists $Q \in A$ such that $f(X) = C(Q, X)$. Now $0 = \text{trace } L(GX) = f(GX) = C(Q, GX) = -C(GQ, X)$, where in the last equality we use the linearity of trace, L, and the identity $L(GY)=0$, so that G is G-skew-symmetric. From $0=C(GQ, X)$ for all $X\in A$, we see that $GQ=0$ and $Q\in \operatorname{Ker} G$. This gives the decomposition $A=\operatorname{Ker} G+\operatorname{Ker} f$ and $\gamma(t)=\gamma_1(t)+\gamma_2(t)$, where, as before, $\gamma_1(t)=(-b/2)\,X_0\in \operatorname{Ker} G$ and $\gamma_2(t)=(b/2)(\sin(bt+a)\,X_1+\cos(bt+a)\,X_2)\in \operatorname{Ker} f$.

Next let the Lie group $\mathscr{G}=\{\phi\in GL(A): C(\phi X, \phi X)=C(X, X),$ for all $X\in A\}$, and let $e^D\in\mathscr{G}$, where $D\in g=\{D\in \mathrm{gl}(A): C(DX, Y)+C(X, DY)=0\}$, the Lie algebra of $\mathscr{G}$. Then since $C(X, X)=\lambda^2x_0^2+\lambda\mu(x_1^2+x_2^2)$, $D\in g$ is of the form

$$D=\begin{pmatrix} 0 & \mu u & \mu v \\ -\lambda u & 0 & -w \\ -\lambda v & w & 0 \end{pmatrix},$$

where

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -w \\ 0 & w & 0 \end{pmatrix}\in \operatorname{Der} A.$$

The cone $\mathscr{C}$ of periodic trajectories has two components,

$$\mathscr{C}^+=\{P\in\mathscr{C}: p_0>0\},$$
$$\mathscr{C}^-=\{P\in\mathscr{C}: p_0<0\}.$$

The group $\mathscr{G}$ acts transitively on each component. Indeed, suppose that P and P' are points on, say, $\mathscr{C}^+$. Then

$$P=-\frac{b}{2c}X_0+r[(\cos a)\,X_1+(\sin a)\,X_2],$$

$$P'=-\frac{b'}{2c}X_0+r'[(\cos a')\,X_1+(\sin a')\,X_2],$$

where $a, a'\in[0, 2\pi)$, $b, b'\in\mathbb{R}$, b and b' both negative, $r=(b/2\,|c|)(-\lambda/\mu)^{1/2}$ and $r'=(b'/2\,|c|)(-\lambda/\mu)^{1/2}$. We shall exhibit group elements whose composition maps P to P'. First let

$$G'=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\in \operatorname{Der} A$$

and let $t_0 = a - \pi/2$. Then by the action of $\exp tG'$ on P,

$$\begin{aligned} e^{t_0 G'}P &= -\frac{b}{2c} X_0 + r[\cos(t_0 - a)\, X_1 + \sin(t_0 - a)\, X_2] \\ &= -\frac{b}{2c} X_0 + rX_2. \end{aligned}$$

Next let

$$D = \begin{pmatrix} 0 & 0 & \mu v \ln\left(\frac{b}{b'}\right) \\ 0 & 0 & 0 \\ -\lambda v \ln\left(\frac{b}{b'}\right) & 0 & 0 \end{pmatrix} \in g,$$

where $v = -b/2c\mu r = 2cr/\lambda b$. Then, as one can easily check, $De^{t_0 G'}P = \ln(b'/b)\, e^{t_0 G'}P$, so

$$e^D e^{t_0 G'} P = \frac{b}{b'} e^{t_0 G'} P = -\frac{b'}{2c} X_0 + r' X_2, \qquad \text{noting that } r'/r = b'/b.$$

Finally, let $t_1 = \pi/2 - a'$. Then

$$e^{t_1 G'} e^D e^{t_0 G'} P = -\frac{b'}{2c} X_0 + r'[(\cos a')\, X_1 + (\sin a')\, X_2] = P'.$$

PROPOSITION 5.24. *$\mathscr{C}^+$ is diffeomorphic to $\mathscr{G}/\mathrm{Aut}\, A$ which is a symmetric space.*

Proof. From the preceding, $\mathscr{G}$ acts transitively on $\mathscr{C}^+$. Aut A leaves any trajectory $\gamma \subset \mathscr{C}^+$ fixed so that, regarding $\mathscr{C}^+ = \bigcup\{F_t(P) : P \in \mathscr{P}_\sigma,\ \sigma > 0,\ t \in \mathbb{R}\}$, we obtain the diffeomorphism [35, 39]. Next note that $g = m + \mathrm{Der}\, A$, where m is the subspace of matrices of the form

$$\begin{pmatrix} 0 & \mu u & \mu v \\ -\lambda u & 0 & 0 \\ -\lambda v & 0 & 0 \end{pmatrix}.$$

We have $(\mathrm{Der}\, A)m \subseteq m$ and $[m, m] \subseteq \mathrm{Der}\, A$ so that $\mathscr{C}^+$ is a symmetric homogeneous space [10].

6. Discrete Systems

We now discuss discrete systems $X(k+1)=X(k)^2$ occurring in an algebra A. For $E(X)=X^2$ in A, the trajectory is now the orbit of iterates

$$O^+(X)=\{E^{(0)}(X), E^{(1)}(X), ..., E^{(k)}(X), ...\},$$

where $X=E^{(0)}(X)$, $E^{(1)}(X)=E(X)=X^2, ..., E^{(k)}(X)=E(E^{(k-1)}(X))$. Most authors stress the similarities between the continuous and discrete systems. However, the use of algebras shows many striking differences as reflected in the structure of the algebra. We consider the basic similarities involving semisimple algebras, homogenization, and automorphisms. Then we note the differences concerning periodic solutions, isolated trajectories, and attracting sets. In particular, we discuss the problem of periodic trajectories for quadratic systems and give examples where the same curve has attracting properties for a continuous and a discrete system. Finally, the (non)chaotic behavior of $E(X)=X^2$ on S^3 and S^7 is discussed using the quaternions and octonians (Cayley numbers).

We use the previous notation $(\mathbb{R}^n, \beta)=A$ to represent a commutative algebra and $X^2=\beta(X, X)$. The general form of the quadratic system in A is

$$X(k+1)=C+TX(k)+X(k)^2 \equiv E(X(k)),$$

where $C \in A$ and $T\colon A \to A$ is linear.

Examples. (1) The Henon map in $\mathbb{R}^2$ given by

$$\begin{aligned} E(X) &= \begin{pmatrix} a+bx_2+cx_1^2 \\ dx_1 \end{pmatrix} \\ &= \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ d & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} cx_1^2 \\ 0 \end{pmatrix} \\ &= C+TX+X^2 \end{aligned}$$

occurs in the algebra $A=(\mathbb{R}^2, \beta)$, where

$$\beta(X, Y)=\begin{pmatrix} cx_1y_1 \\ 0 \end{pmatrix}$$

gives the algebra multiplication.

(2) As an extension of the logistics equation, the discrete predator–prey model is given by

$$x_1(k+1) = c_1 x_1(k) + \sum b_{1j} x_1(k)\, x_j(k)$$

$$x_2(k+1) = c_2 x_2(k) + \sum b_{2j} x_2(k)\, x_j(k)$$

$$\vdots$$

$$x_n(k+1) = c_n x_n(k) + \sum b_{nj} x_n(k)\, x_j(k).$$

Thus $X(k+1) = TX(k) + X(k)^2$ in the algebra A, where

$$\beta(X, X) = \begin{pmatrix} \sum b_{1j} x_1 x_j \\ \vdots \\ \sum b_{nj} x_n x_j \end{pmatrix}.$$

For P satisfying $c + BP = I$ (using the notation of Section 1), the map $E(X) = TX + \beta(X, X)$ has P as a fixed point,

$$TP + \beta(P, P) = \begin{pmatrix} p_1\left(c_1 + \sum b_{1j} p_j\right) \\ \vdots \\ p_n\left(c_n + \sum b_{nj} p_j\right) \end{pmatrix} = P,$$

using $c_k + \sum b_{kj} p_j = 1$. Let $e = B^{-1} I$ be the right identity in Section 1, let $Q = P - e$, then in analogy with the continuous system we can write the map $E(X)$ in terms of the algebra and Q: $\beta(X, X - Q) = \beta(X, X) + \beta(X, e) - \beta(X, P) = \beta(X, X) + X - \text{diag}\, X)(BP) = \beta(X, X) + X - (\text{diag}\, X)(I - c) = \beta(X, X) + TX = E(X)$.

Remarks. Some basic similarities and differences between continuous and discrete quadratic systems are the following:

(1) If $A = A_1 \oplus \cdots \oplus A_m$ is semisimple, then the system $X(k+1) = X(k)^2$ in A decouples into $X_i(k+1) = X_i(k)^2$ occurring in the simple algebras A_i. Similarly, a Wedderburn–Levi decomposition $A = S + R$ as in Proposition 4.1 gives the systems $X_S(k+1) = X_S(k)^2$ in S and $X_R(k+1) = 2X_S(k)\, X_R(k) + X_R(k)^2$ in R.

(2) The scope of quadratic discrete systems is similar to that of continuous quadratic systems: a discrete polynomial system $X(n+1) = P(X(n), ..., X)$ can be embedded into a discrete quadratic system. However, the algebra may be infinite dimensional; for example, consider $X(k+1) = X(k)^3$ in $\mathbb{R}$. Let V be the vector space of all real sequences $(v_1, v_2, ..., v_j, ...)$. Let A be the following commutative infinite dimensional

algebra (V, β): for $Z = \sum z_i F_i$, let $\beta(Z, Z) = \sum z_i z_{i+1} F_i$, where $F_i = (0, ..., 0, 1, 0, ...)$ with 1 in the i-position and 0 elsewhere. Let $U(1) = \sum u_i(1) F_i$ be the initial condition vector in A, where $u_i(1) = X(1)$ and $u_{i+1}(1) = u_i(1)^2$. Let $U(k) = \sum u_i(k) F_i \in A$ with $u_1(k) = X(k)$ and $u_{i+1}(k) = u_i(k)^2$. Then we obtain the quadratic system $U(k+1) = \beta(U(k), U(k))$ in A and the system $X(k+1) = X(k)^3$ is obtained by considering the $u_1(k+1) = X(k+1)$ component in this product.

(3) A homogenization process similar to that for continuous systems exists for discrete systems. In this case the iterates of the original quadratic map in A are given by powers in the homogenized algebra $\tilde{A}$. Let $E(X) = C + TX + X^2$ in A, and let $\tilde{A} = A \times \mathbb{R}$ with the multiplication

$$\begin{aligned}\tilde{\beta}(\tilde{X}, \tilde{X}) &= \tilde{\beta}\left(\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} X \\ u \end{pmatrix}\right) \qquad \text{with } \tilde{X} = \begin{pmatrix} X \\ u \end{pmatrix} \\ &= \begin{pmatrix} u^2 C + uTX + X^2 \\ u^2 \end{pmatrix}\end{aligned}$$

for $X \in A$ and $u \in \mathbb{R}$. The operation $\tilde{\beta}$ is homogeneous quadratic and gives a commutative algebra $\tilde{A} = (A \times \mathbb{R}, \tilde{\beta})$.

The iterates of E in A are given by powers in $\tilde{A}$ as follows: The iterates in A are

$$\begin{aligned} X(1) &= X, \\ X(2) &= E(X(1)) = C + TX(1) + X(1)^2, \\ &\vdots \\ X(k+1) &= E(X(k)) = C + TX(k) + X(k)^2, \\ &\vdots \end{aligned}$$

Powers of $\tilde{X} = \binom{X}{1}$ in $\tilde{A}$ are

$$\begin{aligned} \tilde{X}^{(1)} &= \begin{pmatrix} X(1) \\ 1 \end{pmatrix} \qquad \text{for} \quad X(1) = X, \\ \tilde{X}^{(2)} &= \tilde{X}^2 = \begin{pmatrix} C + TX(1) + X(1)^2 \\ 1^2 \end{pmatrix} = \begin{pmatrix} E(X(1)) \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} X(2) \\ 1 \end{pmatrix} \\ &\vdots \\ \tilde{X}^{(k+1)} &= \tilde{\beta}(\tilde{X}^{(k)}, \tilde{X}^{(k)}) = \begin{pmatrix} E(X(k)) \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} X(k+1) \\ 1 \end{pmatrix}. \end{aligned}$$

Thus if $\pi: \tilde{A} \to A : \binom{X}{u} \mapsto X$, then we obtain $\pi \tilde{X}^{(n)} = X(n)$; that is, π(powers) = iterates. Because of this, we shall concentrate on discrete systems of the form $X(k+1) = E(X(k)) = X(k)^2$ in A.

(4) Note that a decreasing chain of derived subalgebras of A is obtained by setting $A^{(1)} = A$, $A^{(k+1)} = A^{(k)}A^{(k)}$, so that

$$A = A^{(1)} \supseteq A^{(2)} \supseteq \cdots \supseteq A^{(k+1)} \supseteq \cdots.$$

If A is finite dimensional, then there is an N such that $A^{(N)} = A^{(N+1)} = \cdots$. This is where the interesting dynamics occur.

(5) Let A be a nilpotent algebra, then we have the following interesting difference between continuous and discrete systems. From Proposition 4.2, the solution $X(t)$ of $\dot{X} = X^2$ is unbounded. In the discrete case, the orbit $X(n)$ of $X(k+1) = X(k)^2$ becomes 0 after finitely many iterations.

(6) Let $\{X(n)\}$ be the orbit of $X(k+1) = X(k)^2$ in A and assume that $P = \lim_{n \to \infty} X(n)$ exists. Then

$$P = \lim X(n+1) = \lim X(n)^2 = P^2;$$

that is, P is an idempotent of A. In particular, a nonzero equilibrium point N for the discrete system $X(k+1) = X(k)^2$ in A cannot be an equilibrium point for the differentiable system $\dot{X} = X^2$ in A: $N = N^2$ for a discrete system, but $N^2 = 0$ for a continuous system. Since the linearization of $E(Z) = Z^2$ at a point $Q \in A$ is given by $E'(Q) = 2L(Q)$, an equilibrium point for a continuous system cannot be hyperbolic. For a discrete system, we have the following result.

PROPOSITION 6.1. *Let the discrete system* $X(k+1) = E(X(k)) = X(k)^2$ *occur in an algebra* A. *Let* $P = P^2 \neq 0$ *be a fixed point. Then* P *is unstable.*

Proof. We have $E'(P)P = 2L(P)P = 2P^2 = P$; hence, $\lambda = 2$ is an eigenvalue of $E'(P)$. Thus the spectral radius of $E'(P)$ strictly exceeds 1, so P is unstable [19, Chap. 1, Theorem 9.14].

In contrast with the above result, the origin is an asymptotically stable fixed point in normed algebras.

PROPOSITION 6.2. *Let* $X(k+1) = X(k)^2$ *occur in an algebra* A *that has a (positive definite) norm* $\|\cdot\| : A \to \mathbb{R}$ *satisfying* $\|X^2\| \leqslant \|X\|^2$ *for all* $X \in A$. *Then the origin* $0 \in A$ *is an asymptotically stable fixed point.*

Proof. Let $V(X) = \|X\|$. For each $X \in A$, we have $\dot{V}(X) := V(X^2) - V(X) = \|X^2\| - \|X\| \leqslant 0$. Now $V(0) = 0$ and, since V is positive definite, $\|X\|^2 < \|X\|$ for all $X \neq 0$ such that $\|X\| < 1$, and so $\|X^2\| \leqslant \|X\|^2 < \|X\|$. Thus $\dot{V}(X) < 0$ for all nonzero X in the open neighborhood $\mathscr{U} = \{X \in A : \|X\| < 1\}$. Thus V is a Lyapunov function with V and $-\dot{V}$ both positive definite, and so the origin is asymptotically stable [21, Chap. 1, Corollary 2.7].

EXAMPLES (1) Let $M_n(\mathbb{R})$ be the associative algebra of real $n \times n$ matrices with norm $\|T\| = \sup\{\|TX\| : X \in \mathbb{R}^n \text{ and } \|X\| = 1\}$. We have $\|ST\| \leqslant \|S\| \|T\|$ for $S, T \in M_n$ so that $\|T^2\| \leqslant \|T\|^2$. Let M_n^+ be the commutative Jordan algebra with vector space M_n and multiplication $\beta(S, T) = \frac{1}{2}(ST + TS)$. Let A be a Jordan subalgebra of M_n^+, then for $T \in A$ we have $\|\beta(T, T)\| = \|T^2\| \leqslant \|T\|^2$ so that the origin is asymptotically stable for the system $X(k+1) = X(k)^2$ in A.

(2) Let A be the Cayley algebra as discussed in [5], then the function $V(X) = \sum x_i^2 = \|X\|^2$ satisfies $V(XY) = V(X)\, V(Y)$. Thus $\|X^2\| = \|X\|^2$ so that the origin is asymptotically stable for the system $X(k+1) = X(k)^2$ in A^+.

Next we consider automorphisms of discrete systems, particularly quadratic systems.

LEMMA 6.3. *Let $E: \mathbb{R}^n \to \mathbb{R}^n$, and let $\phi \in \mathrm{Aut}\, E$. Then ϕ is solution-preserving for $X(n+1) = E(X(n))$; i.e., $\{\phi E^{(n)}(X)\}$ is an orbit whenever $\{E^{(n)}(X)\}$ is an orbit.*

Proof. $\phi E^{(n)}(X) = E^{(n)}(\phi X)$.

PROPOSITION 6.4. *Let $E: \mathbb{R}^n \to \mathbb{R}^n$.*

1. *Let $P \in \mathbb{R}^n$ and $\phi \in \mathrm{Aut}\, E$ be such that $\phi P = E(P)$. Then the iterates $E^{(n)}(P) = \phi^{(n)} P$, where $\phi^{(n)} = \phi\phi \cdots \phi$ (composition n times).*

2. *Let $\mathscr{E}$ be the set of equilibria of E and let $\mathscr{P}_N$ be the set of points of period N of E. Then $\mathscr{E}$ and $\mathscr{P}_N$ are* Aut *E-invariant.*

Proof. For (1), note that $E^{(0)}(P) = \phi^{(0)} P = P$, $E^{(1)}(P) = E(P) = \phi^{(1)} P$, and, using induction,

$$\begin{aligned} E^{(k+1)}(P) &= E(E^{(k)}(P)) \\ &= E(\phi^{(k)} P) \\ &= \phi^{(k)}(E(P)), \qquad \phi^{(k)} \in \mathrm{Aut}\, E \\ &= \phi^{(k)}(\phi(P)) \\ &= \phi^{(k+1)} P. \end{aligned}$$

The proof of (2) is similar to the differentiable case.

COROLLARY 6.5. *Let $G \in \mathrm{Der}\, E$ and $P \in A$ be such that $e^G P = E(P)$; then the iterates $E^{(n)}(P) = e^{nG} P$.*

Remarks. (1) The formula $E^{(n)}(P)=e^{nG}P$ allows for easy numerical calculation with small error compared to computing the iterates.

(2) If $E^{(n)}(P)=e^{nG}P$, then we can iterate backward by $E^{(-n)}(P)=e^{-nG}P$. Thus for $Q=e^{-G}P$,

$$\begin{aligned} E(Q) &= E(e^{-G}P) \\ &= e^{-G}E(P), && \text{using } e^{-G}\in \operatorname{Aut} E \\ &= e^{-G}e^{G}P, && \text{using } E(P)=e^{G}P \\ &= P. \end{aligned}$$

If $E(X)=X^2$, then Q is a solution to $X^2=P$ in A; i.e., $Q=P^{1/2}$.

(3) If $E^{(n)}(P)=e^{nG}P$, then the orbit $\{P, E^{(1)}(P), ..., E^{(n)}(P), ...\}$ is always on the curve $k(s)=e^{sG}P$ for $s\in\mathbb{R}$.

LEMMA 6.6. *For $E(X)=X^2$ in A and $s\in\mathbb{R}$, $E^{(n)}(sX)=s^mE^{(n)}(X)$, where $m=2^n$.*

Proof. This is an easy calculation.

Next we consider some results on periodic orbits.

PROPOSITION 6.7. *For $E(X)=X^2$ in A, let $P\in\mathscr{P}_N$ and $N>1$. Then $\{P, E(P)\}$ is a linearly independent set of vectors.*

Proof. Suppose that $E(P)=aP$ for some $a\in\mathbb{R}$. Then

$$\begin{aligned} P &= E^{(N)}(P) \\ &= E^{(N-1)}(E(P)) \\ &= E^{(N-1)}(aP) \\ &= a^mE^{(N-1)}(P), \qquad \text{where} \quad m=2^{N-1}. \end{aligned}$$

Thus, we find that

$$\begin{aligned} aP &= E(P) \\ &= E(a^mE^{(N-1)}(P)) \\ &= a^{2m}E(E^{(N-1)}(P)) \\ &= a^{2m}E^{(N)}(P) \\ &= a^{2m}P. \end{aligned}$$

This implies that $a=1$ or 0, both of which contradict the fact that $P\in\mathscr{P}_N$.

The next result suggests that one should look for periodic orbits in simple algebras.

PROPOSITION 6.8. *If A is power-associative and has a periodic point, then A has an idempotent, i.e., if $X(k+1)=X(k)^2$ in A has a periodic orbit, then it has a fixed point.*

Proof. In a power-associative algebra, the iterate $E^{(k)}(X)=X^a$, where $a=2^k$. Thus if $P=E^{(N)}(P)=P^{2^N}$, let $Q=P^m$, where $m=2^N-1$. Next note that $QP=P^{2^N}=P$, $QP^2=P^2$, ..., QP^m; that is, $Q^2=Q$.

DEFINITION. The set $\mathscr{P}_N$ of points of period N is said to consist of *isolated* points if each P in $\mathscr{P}_N$ has a neighborhood $\mathscr{U}$ which contains no other points in $\mathscr{P}_N$; i.e., $\mathscr{U}\cap\mathscr{P}_N=\{P\}$.

The following result contrasts sharply with the corresponding continuous Theorem 5.8.

THEOREM 6.9. *Suppose P is an isolated periodic point of period N. Let $(\mathrm{Aut}\,E)_0$ denote the connected component of the identity in* Aut E. *Then $\phi E^{(n)}(P)=E^{(n)}(P)$ for all $\phi\in(\mathrm{Aut}\,E)_0$ and all nonnegative integers n.*

Proof. Let $D\in\mathrm{Der}\,E$, then from Proposition 6.4, $e^{sD}P\in\mathscr{P}_N$ for all $s\in\mathbb{R}$. Let $k\colon\mathbb{R}\to\mathbb{R}^n\colon s\to e^{sD}P$. As before, k is continuous, so $k(\mathbb{R})$ is connected. But $k(s)\in\mathscr{P}_N$ for all $s\in\mathbb{R}$, $k(0)=P$, and P is isolated in $\mathscr{P}_N$. Thus $e^{sD}P=P$ for all $s\in\mathbb{R}$. This proves the result since any $\phi\in(\mathrm{Aut}\,E)_0$ may be written as $\phi=e^{D_1}\cdots e^{D_k}$ for $D_i\in\mathrm{Der}\,E$.

Conjecture. If $\mathscr{P}_N$ is not isolated, $P\in\mathscr{P}_N$, and $(\mathrm{Der}\,E)P\neq\{0\}$, then there exists $\phi\in\mathrm{Aut}\,E$ such that $\phi P=E(P)$. Thus the iterates $E^{(n)}(P)=\phi^{(n)}P$.

EXAMPLE. The following are two-dimensional algebras A which support orbits of $X(k+1)=X(k)^2$ that are of the form $X(k)=(\exp kG)P$ for $G\in\mathrm{Der}\,A$ and $P\in A$ not a fixed point.

(1) Assume that 0 is an eigenvalue of G with $a\neq 0$ the other real eigenvalue. Thus A has a basis $\{X_0, X_a\}$ with $GX_0=0$ and $GX_a=aX_a$. The preliminary multiplication table is

	X_0	X_a
X_0	rX_0	sX_a
X_a	sX_a	0

for $r,s\in\mathbb{R}$ suitably related to a. This table uses $A(\alpha)A(\beta)\subseteq A(\alpha+\beta)$ if $\alpha+\beta$ is an eigenvalue of G; otherwise, $A(\alpha)A(\beta)=0$ [13, p. 54]. The element $P\in A$ with $P^2=(\exp G)P$ is $P=(1/r)X_0+p_aX_a$, where $p_a\neq 0$ in $\mathbb{R}$

and $2s/r=\exp(a)$. P is not periodic since the equation $P=(\exp nG)P$, when expressed in terms of the basis, implies $n=0$.

(2) Assume 0 is not an eigenvalue of G. For complex eigenvalues λ and $\bar{\lambda}$ of G, decompose the complexification

$$A_{\mathbb{C}}=A_{\mathbb{C}}(\lambda)+A_{\mathbb{C}}(\bar{\lambda}),$$

using the fact that G can be extended to a derivation of $A_{\mathbb{C}}$. Thus $A_{\mathbb{C}}(\lambda)^2=A_{\mathbb{C}}(\bar{\lambda})^2=0$, since 2λ and $2\bar{\lambda}$ are not eigenvalues of G. Also $A_{\mathbb{C}}(\lambda)\,A_{\mathbb{C}}(\bar{\lambda})=0$, since $\lambda-\bar{\lambda}$ is not an eigenvalue. Thus $A_{\mathbb{C}}^2=0$ and, consequently, $A^2=0$, a contradiction.

(3) For a repeated real eigenvalue $a\neq 0$, there exists a basis $\{X_1, X_2\}$ of A such that $GX_1=aX_1$ and $GX_2=aX_2+X_1$. Using that G is a derivation, a straightforward calculation gives $A^2=0$. The other case, where G is fully diagonalizable cannot occur, for then the eigenspaces would not multiply properly.

(4) Assume that 0 is a repeated eigenvalue of $G\in \operatorname{Der} A$. Then A has a basis $\{X_1, X_2\}$ with $GX_1=0$, $GX_2=X_1$ so that G has as its matrix $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and $G^2=0$. Since $\operatorname{Ker} G=\mathbb{R}X_1$ and $G(X_1^2)=2X_1(GX_1)=0$, we have $X_1^2=\lambda X_1$ for $\lambda\in\mathbb{R}$. Next $G(X_1X_2)=(GX_1)\,X_2+X_1(GX_2)=X_1X_1=\lambda X_1$ and $G(X_2^2)=2X_2(GX_2)=2X_2X_1$. From this $0=G^2(X_2^2)=2G(X_1X_2)$ so that $X_1X_2\in\operatorname{Ker} G$; thus $X_1X_2=\mu X_1$ for $\mu\in\mathbb{R}$. Using this and the above calculation $G(X_1X_2)=\lambda X_1$, note that $0=\mu G(X_1)=G(X_1X_2)=\lambda X_1$, which gives $\lambda=0$ and the table

	X_1	X_2
X_1	0	μX_1
X_2	μX_1	aX_1+bX_2

for μ, a, $b\in\mathbb{R}$. Next let $P=p_1X_1+p_2X_2$ satisfy $P^2=(\exp G)P$. Then, using the table,

$$\begin{aligned}P^2&=(ap_2^2+2\mu p_1p_2)\,X_1+bp_2^2X_2\\&\equiv(\exp G)P\\&=(I+G)P\\&=(p_1+p_2)\,X_1+p_2X_2\end{aligned}$$

which implies that $p_1=(b-a)/b(2\mu-b)$ and $p_2=1/b$ when $b(b-2\mu)\neq 0$, or $p_1\in\mathbb{R}$ and $p_2=1/b$ when $a=b=2\mu\neq 0$. In either case,

$$\begin{aligned}X(n)&=(\exp nG)P\\&=(I+nG)P, \qquad \text{using } G^2=0\\&=(p_1+np_2)\,X_1+p_2X_2.\end{aligned}$$

Then $E^{(n)}(P)\neq P$ for any $n=1, 2, \ldots$.

EXAMPLE. Suppose that A is a three-dimensional commutative algebra with a derivation $G \neq 0$ and a nonzero point $P \in A$ such that $e^G P = P^2$ and such that the corresponding orbit $X(n) = e^{nG}P$ is periodic of periodic $N > 1$. By the same argument as in the differentiable case (i.e., using the periodicity of the function $t \mapsto e^{tG}P$ and assuming that $A^2 \neq 0$), there must exist a basis $\{X_0, X_1, X_2\}$ such that the matrix representation of G is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & -b & 0 \end{pmatrix},$$

where $b = 2\pi k/N$ for any $k \in \mathbb{Z}$, $k \not\equiv 0 \pmod N$. Thus we must solve $e^G P = P^2$ in the algebra $A = A(\lambda, \mu, c_1, c_2)$ constructed in Section 5 following Eq. (5). Since we are assuming that $N > 1$, we impose the conditions $P \neq 0$ and $P^2 \neq P$.

Letting $P = p_0 X_0 + p_1 X_1 + p_2 X_2$, we have from $e^G P = P^2$,

$$\lambda p_0^2 + \mu(p_1^2 + p_2^2) = p_0, \tag{1}$$

$$2p_0(c_1 p_1 - c_2 p_2) = p_1 \cos b + p_2 \sin b, \tag{2}$$

$$2p_0(c_2 p_1 + c_1 p_2) = -p_1 \sin b + p_2 \cos b. \tag{3}$$

If $p_0 = 0$, then the right-hand sides of (2) and (3) can be solved uniquely for p_1 and p_2 to get $p_1 = p_2 = 0$; thus $P = 0$, a contradiction. On the other hand, if $p_1 = p_2 = 0$, then (1) implies that $\lambda p_0^2 = p_0$. Thus $P = p_0 X_0$ and $P^2 = p_0^2 \lambda X_0 = p_0 X_0 = P$, another contradiction. Thus we write (2) and (3) as a homogeneous matrix-vector equation with a nontrivial solution:

$$\begin{pmatrix} 2p_0 c_1 - \cos b & -2p_0 c_2 - \sin b \\ 2p_0 c_2 + \sin b & 2p_0 c_1 - \cos b \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Computing the determinant, we find that

$$(2p_0 c_1 - \cos b)^2 + (2p_0 c_2 + \sin b)^2 = 0.$$

Thus $2p_0 c_1 = \cos b$ and $2p_0 c_2 = -\sin b$. This gives us the compatibility condition

$$c_2 \cos b = -c_1 \sin b.$$

We thus set $c_1 = c \cos b$ and $c_2 = -c \sin b$ for some $c \in \mathbb{R}$ and we work in the algebra $A(\lambda, \mu, c \cos b, -c \sin b)$ whose table is given by

	X_0	X_1	X_2
X_0	λX_0	$c(\cos bX_1 - \sin bX_2)$	$c(\sin bX_1 + \cos bX_2)$
X_1	$c(\cos bX_1 - \sin bX_2)$	μX_0	0
X_2	$c(\sin bX_1 + \cos bX_2)$	0	μX_0

Note that $c \neq 0$; otherwise the determinant condition becomes $\cos^2 b + \sin^2 b = 0$. Thus $p_0 = 1/2c$. Equation (1) becomes

$$\mu(p_1^2 + p_2^2) = \frac{1}{2c}(1 - \lambda/2c). \tag{4}$$

If $\mu = 0$, then $\lambda = 2c$ (by (4)) and a short calculation using the table shows that $e^G P = P^2$ for any $P \neq p_0 X_0$. If $\mu \neq 0$, then for later reference, we note that (4) implies that $(1/2c\mu)(1 - \lambda/2c) \geqslant 0$, from which it follows that either $c\mu \geqslant 0$ and $\lambda/c \leqslant 2$, or $c\mu \leqslant 0$ and $\lambda/c \geqslant 2$. Thus the solutions P to $e^G P = P^2$ in $A(\lambda, \mu, \cos b, -\sin b)$ and the orbits $\{E^{(n)}(P)\}$ through each P are described in the following.

THEOREM 6.10. *Let A be a three-dimensional algebra with an orbit $\{X(n)\} = \{(\exp nG)P\}$ of $X(k+1) = X(k)^2$, where $G \in \operatorname{Der} A$ and $P \in \mathscr{P}_N$. Then $A \cong A(\lambda, \mu, c\cos b, -c\sin b)$, where $b = 2\pi k/N$ for some $k \in \mathbb{Z}$, $k \not\equiv 0 \pmod N$, and $c \neq 0$. When $\mu \neq 0$,*

$$P = \frac{1}{2c} X_0 + \rho[(\cos a) X_1 + (\sin a) X_2],$$

$$E^{(n)}(P) = \frac{1}{2c} X_0 + \rho[\cos(bn - a) X_1 - \sin(bn - a) X_2],$$

where $0 \leqslant a < 2\pi$, and $\rho = ((1/2\mu c)(1 - \lambda/2c))^{1/2} = (p_1 + p_2)^{1/2}$. When $\mu = 0$, the orbit through each $P \neq (1/2c) X_0$ is given by $\{e^{nG}P\}$.

We now assume that, for some $a \in [0, 2\pi)$, $P = P(a)$ is a critical point of E, i.e., $E'(P)$ is singular. We temporarily compute in Cartesian coordinates: $P(a) = p_0 X_0 + p_1 X_1 + p_2 X_2$. Recalling that $p_0 = 1/2c$, we have

$$E'(P) = \begin{pmatrix} \lambda/c & 2\mu p_1 & 2\mu p_2 \\ 2c(p_1 \cos b + p_2 \sin b) & \cos b & \sin b \\ 2c(-p_1 \sin b + p_2 \cos b) & -\sin b & \cos b \end{pmatrix},$$

so after some calculations we find

$$\det E'(P) = \frac{\lambda}{c} - 4c\mu[(p_1 \cos b + p_2 \sin b)^2 + (-p_1 \sin b + p_2 \cos b)^2]$$

$$= \frac{\lambda}{c} - 4c\mu(p_1^2 + p_2^2).$$

If $\mu = 0$, then $\lambda = 2c$ so $\det E'(P) = 2 \neq 0$. Thus in this case, $E'(P)$ cannot be singular, so we assume from now on that $\mu \neq 0$. But from Theorem 6.10, $p_1^2 + p_2^2 = \rho^2 = (1/2c\mu)(1 - (1/2c)\lambda)$, so $\lambda/c - 2(1 - (1/2c)\lambda) = 0$; i.e., $\lambda = c$.

From the sign condition of Eq. (4), $c\mu > 0$, so $\rho = 1/(2(c\mu)^{1/2})$ and in cylindrical coordinates

$$P(a) = \frac{1}{2c} X_0 + \frac{1}{2(c\mu)^{1/2}} [(\cos a) X_1 + (\sin a) X_2].$$

Conversely, if we assume that $\lambda = c$, then reversing the calculations shows that $E'(P)$ is singular for *any* $P \in \{P(a) : 0 \leqslant a < 2\pi\}$. This shows the following.

LEMMA 6.11. *The following are equivalent*:

(1) *For some* $a \in [0, 2\pi)$, $P(a)$ *is a critical point of* E,

(2) *For every* $a \in [0, 2\pi)$, $P(a)$ *is a critical point of* E.

We now consider the stability analysis of the discrete system $X(k+1) = X(k)^2$ in A. Let $S(1/2c)$ denote the circle with center $1/2c$ on the X_0-axis and with radius $\rho = 1/2(c\mu)^{1/2}$. We are interested in orbits starting near the circle $S(1/2c) = \{P(a) : 0 \leqslant a < 2\pi\}$ of periodic points as in Theorem 6.10. We shall assume that some (and, hence, every) point on $S(1/2c)$ is a critical point of E. As before, we use cylindrical coordinates and write

$$X = x_0 X_0 + x_1 X_1 + x_2 X_2 = x_0 X_0 + r[\cos \theta X_1 + \sin \theta X_2],$$

where $r^2 = x_1^2 + x_2^2$, $x_1 = r \cos \theta$, $x_2 = r \sin \theta$. We compute the iterates of $E(X) = X^2$ in A relative to these coordinates and we find, using $\lambda = c$ and the table,

$$\begin{aligned} E(X) &= (cx_0^2 + \mu(x_1^2 + x_2^2)) X_0 + 2cx_0(x_1 \cos b + x_2 \sin b) X_1 \\ &\quad + 2cx_0(-x_1 \sin b + x_2 \cos b) X_2, \\ &= (cx_0^2 + \mu r^2) X_0 + 2cx_0 r[\cos(\theta - b) X_1 + \sin(\theta - b) X_2], \\ &\equiv x_0(1) X_0 + r(1)[\cos \theta(1) X_1 + \sin \theta(1) X_2], \end{aligned} \tag{5}$$

where $x_0(1) = cx_0^2 + \mu r^2$, $r(1) = 2cx_0 r$, and $\theta(1) = \theta - b$. By induction, if the kth-iterate

$$E^{(k)}(X) = x_0(k) X_0 + r(k)[\cos \theta(k) X_1 + \sin \theta(k) X_2],$$

then

$$E^{(k+1)}(X) = x_0(k+1) X_0 + r(k+1)[\cos \theta(k+1) X_1 + \sin \theta(k+1) X_2],$$

where

(i) $x_0(k+1) = cx_0(k)^2 + \mu r(k)^2$,

(ii) $r(k+1) = 2cx_0(k) r(k)$,

and $\theta(k+1) = \theta - (k+1)b$.

The fixed points of the discrete system (i), (ii) are $(0, 0)$, $(1/c, 0)$, $(1/2c, 1/2(c\mu)^{1/2})$, and $(1/2c, -1/2(c\mu)^{1/2})$. Because we are considering r to be a radius, we disregard this last point. In the original system $X(k+1) = X(k)^2$, the first two fixed points correspond to the points $X = 0$ and $X = (1/c) X_0$, respectively. The fixed point $(1/2c, 1/2(c\mu)^{1/2})$ corresponds to the circle $S(1/2c)$ of periodic points. Thus to analyze the stability of the fixed points $X = 0$, $X = (1/c) X_0$, and the periodic points $S(1/2c)$, we are left with the task of analyzing the stability of the fixed points of system (i), (ii).

The derivative of the quadratic map

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} cu^2 + \mu v^2 \\ 2cuv \end{pmatrix}$$

is

$$F'\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2cu & 2\mu v \\ 2cv & 2cu \end{pmatrix}.$$

Evaluating this at each of the fixed points and computing eigenvalues we find the following.

Fixed point	Eigenvalues
$(0, 0)$	0, 0
$(1/c, 0)$	2, 2
$(1/2c, 1/2(c\mu)^{1/2})$	0, 2

Comparing these with 1, we see that $(0, 0)$ is asymptotically stable, $(1/c, 0)$ is unstable, and $(1/2c, 1/2(c\mu)^{1/2})$ is a saddle. We denote this latter fixed point by Q and compute the eigenvectors at Q in a convenient form. We find that the stable eigenvector at Q is $(-1/2c, 1/2(c\mu)^{1/2})$ and the unstable eigenvector at Q is $(-1/2c, -1/2(c\mu)^{1/2})$.

In particular, if an orbit of the system (i), (ii) starts at $(x_0, r) = Q + \alpha(-1/2c, 1/2(c\mu)^{1/2})$ for $\alpha \in (-1, 1)$, then any easy induction shows that

$$(x_0(n), r(n)) = Q + \alpha^{2^n}(1/2c, -1/2(c\mu)^{1/2}).$$

Thus, since $-1 < \alpha < 1$, $(x_0(n), r(n)) \to Q$ as $n \to \infty$. Note that all points in this orbit satisfy $(1/c - x_0)^2 = (\mu/c) r^2$.

Similarly, if an orbit starts at $(x_0, r) = Q + \alpha(-1/2c, -1/2(c\mu)^{1/2})$, then an easy induction shows that

$$(x_0(n), r(n)) = ((1-\alpha)^{2^n}/2c, (1-\alpha)^{2^n}/2(c\mu)^{1/2}).$$

Thus $(x_0(n), r(n)) \to (0, 0)$ as $n \to \infty$ if $0 < \alpha < 1$, while $(x_0(n), r(n)) \to (\infty, \infty)$ if $-1 < \alpha < 0$. Note that all points in this orbit satisfy $x_0^2 = (\mu/c) r^2$.

Finally, we interpret these results in terms of the original system $X(k+1)=X(k)^2$ in A. The stability analysis of the fixed points $(0,0)$ and $(1/c, 0)$ in the system (i), (ii) implies that the origin $X=0$ is asymptotically stable and the fixed point $X=(1/c)\,X_0$ is asymptotically unstable. On the other hand, the circle $S(1/2c)$ has both a stable cone and an unstable cone. Thus, if an orbit starts at a point X on the cone

$$\mathscr{C}_0=\left\{\sum x_i X_i : (1/c-x_0)^2=\frac{\mu}{c}(x_1^2+x_2^2)\right\}$$

in the region where $0<x_0<1/c$, then the orbit $\{E^{(n)}(X)\}$ spirals along $\mathscr{C}_0$ and approaches $S(1/2c)$ as n increases without bound. If an orbit starts at a point X on the cone

$$\mathscr{C}_1=\left\{\sum x_i X_i : x_0^2=\frac{\mu}{c}(x_1^2+x_2^2)\right\}$$

in the region where $0<x_0<1/c$ ($x_0\neq 1/2c$), then the orbit $\{E^{(n)}(X)\}$ spirals along $\mathscr{C}_1$ and is repelled from $S(1/2c)$.

This completes the stability analysis of the system $X(k+1)=X(k)^2$ in the algebra $A=A(\lambda, \mu, c\cos b, c\sin b)$.

Remark. In finding periodic trajectories for a discrete quadratic system, behavior similar to the differentiable case has occurred. For the latter, the solution $F_t(P)$ is the circle $S(-b/2c)$ which is an attracting limit set along a horizontal cylinder. For the discrete case, the orbits $\{E^{(n)}(P)\}$ are on the circle $S(1/2c)$ which is attracting for the cone $\mathscr{C}_0$.

EXAMPLE. We now modify the preceding example so that the quadratic system depends on a varying parameter. Let $\mathscr{C}=\bigcup S(1/2c)$, let $A(c)=A(c, \mu, c\cos b, c\sin b)$, and using Eq. (5), let $E_c(X)=X^2$ in $A(c)$. We now define a quadratic system on the open half-space $\{x_0>0\}\subset\mathbb{R}^3$ in terms of E_c so that $\mathscr{C}$ becomes an attracting set. Consider $X=x_0X_0+x_1X_1+x_2X_2$, $x_0\neq 0$ and write $x_0=1/2c$ uniquely. Writing $X=X(c)=(1/2c)\,X_0+x_1X_1+x_2X_2$, we define

$$Q\colon \{x_0>0\}\to\mathbb{R}^3 : X\to E_c(X(c)),$$

where $E_c(X(c))$ is computed in the algebra $A(c)$. Thus in cylindrical coordinates, write

$$X=X(c)=\frac{1}{2c}X_0+r[\cos\theta X_1+\sin\theta X_2] \qquad \text{in} \quad A(c)$$

and

$$
\begin{aligned}
Q(X) &= E_c(X(c)) \\
&= \left(c\left(\frac{1}{2c}\right)^2 + \mu r^2\right) X_0 + 2c\left(\frac{1}{2c}\right) r[\cos(\theta-b)\, X_1 + \sin(\theta-b)\, X_2] \\
&= \left(\frac{1}{4c} + \mu r^2\right) X_0 + r[\cos(\theta-b)\, X_1 + \sin(\theta-b)\, X_2] \\
&\equiv \frac{1}{2c(1)} X_0 + r[\cos\theta(1)\, X_1 + \sin\theta(1)\, X_2] \qquad \text{in} \quad A(c(1))
\end{aligned}
$$

where $1/2c(1) = 1/4c + \mu r^2$ and $\theta(1) = \theta - b$. Continuing by induction, if

$$
Q^{(k)}(X) = \frac{1}{2c(k)} X_0 + r[\cos\theta(k)\, X_1 + \sin\theta(k)\, X_2], \qquad \text{in} \quad A(c(k))
$$

then

$$
Q^{(k+1)}(X) = \frac{1}{2c(k+1)} X_0 + r[\cos\theta(k+1)\, X_1 + \sin\theta(k)\, X_2]
$$
$$
\text{in} \quad A(c(k+1)),
$$

where $1/2c(k+1) = 1/4c(k) + \mu r^2$ and $\theta(k+1) = \theta(k) - b = \theta - (k+1)b$.

The formula for $c(k+1)$ implies that $\lim c(n) = \beta$ exists and $\beta \neq 0$. Thus

$$
\frac{1}{2\beta} = \frac{1}{4\beta} + \mu r^2,
$$

so that $\beta = 1/4\mu r^2$. Thus $Q^{(n)}(X)$ spirals on the cylinder along the x_0-axis of radius r and approaches the circle $S(2\mu r^2)$ on the paraboloid $\mathscr{C}$. This paraboloid is also Q-invariant, and so we have the following.

PROPOSITION 6.12. *$\mathscr{C}$ is asymptotically stable for Q.*

EXAMPLE. The squaring map $z \mapsto z^2$ in the complex numbers $\mathbb{C}$ has attracted a lot of interest because of its connections with chaos [6, 4]. In particular, the restriction of this map to the unit circle S^1 has chaotic behavior. In [15], we discuss the squaring maps in the quaternions and the octonians (or Cayley numbers), and their restrictions to S^3 and S^7.

Devaney [6] has defined a map on a metric space to be a chaotic if (1) it is sensitive to initial conditions, (2) its periodic points are dense, and (3) it is topologically transitive. Banks *et al* [4] have shown that (2) and (3) automatically imply (1). However, the common wisdom is that sensitive dependence is the essential feature of chaos. This being the case, the result of [4] should be contrasted with the following [15].

PROPOSITION 6.13. *Let A denote either the quaternions or the octonians with S^n as a submanifold, $n = 3$ or 7. Then the squaring map $E(X) = X^2$ in A satisfies the following properties*:

(1) $E|_{S^n}$ *is sensitive to initial conditions*;

(2) *The periodic points of* $E|_{S^n}$ *are dense in* S^n;

(3) $E|_{S^n}$ *is not topologically transitive.*

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