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Invertible classes

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Abstract

This paper considers when one can invert general recursive operators which map a class of functions $\mathcal{F}$ to $\mathcal{F}$. In this regard, we study four different notions of inversion. We additionally consider enumeration of operators which *cover* all general recursive operators which map $\mathcal{F}$ to $\mathcal{F}$ in the sense that, for every general recursive operator Ψ mapping $\mathcal{F}$ to $\mathcal{F}$, there is a general recursive operator in the enumerated sequence which behaves the same way as Ψ on $\mathcal{F}$. Three different possible types of enumeration are studied.

1. Introduction

To invert a function or an operator is quite important in many applications, and it has been widely studied in mathematics. Typical examples from the more applied world are the following ones:

– *Cryptography*. Often the encryption algorithms are known, like the widely used "blowfish" algorithm [11], and we can intercept the encoded message, but can we get the message that resulted in the code?
– *Chemical analysis*. Many chemical processes are known. Assume we have the result of a chemical reaction. Can we find the ingredients that were used?
– *Customer modeling*. There are very good models of human motivation; cf. [8] for example. We can observe customer behaviour. But why did the customer actually buy or not buy the product? Where did he learn about the product and what advertisement measures were effective?

More precisely, in these scenarios, the process or operator which transforms the input to the output is known. Furthermore, the output can be accessed. But the input is unknown and should be reconstructed:

$$? \rightarrow \text{Process} \rightarrow \text{Output}.$$

So we know the process and can observe the output, but the question is whether we can recover the input. In the case that there are several inputs which the process translates to the same observed output, it is impossible to say which of them caused the output; therefore we just want to recover one of the possible inputs.

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In this paper we consider inversion of operators. Operators map functions to functions. Though operators are at a different level than functions, it would be interesting to consider when one can invert operators. Another way to look at operators could be for mapping real numbers to real numbers, where a real number (between 0 and 1) can be represented as a function from the set of natural numbers to $\{0, 1\}$. Operators have been used in various fields, for example in recursion theory [12], functions over real numbers, in computational learning theory [3], and so on.

It may not be reasonable to consider all inputs and outputs, but only those which fit into a special context. For example, often one considers the class of linear time computable, or polynomial time computable, or polynomial space computable functions. Thus, we fix a class $\mathcal{F}$ of such functions. It is required that $\mathcal{F}$ contains only total functions and that all input and output of the operator are from this class. In other words, we consider mainly $\mathcal{F}$-preserving recursive operators Φ, which map every $f \in \mathcal{F}$ to a total function in $\mathcal{F}$. In this paper Φ will mostly be general recursive, that is, it will map every total function to a total one, but in some special cases we investigate also $\mathcal{F}$-preserving operators which are not general recursive.

Following the above-mentioned scenario, we are interested in studying when an $\mathcal{F}$-preserving general recursive operator Φ can be inverted. That is, given $\Phi(f)$ as input, for $f \in \mathcal{F}$, when can we find a g such that $\Phi(g) = \Phi(f)$, via some computable mechanism? As the class $\mathcal{F}$ might often be computationally very difficult to hit, we do not require that g belongs to $\mathcal{F}$. Furthermore, given $\Phi(f)$, possible methods for finding such a g usually work via trial and error. Therefore we would mostly be using limit-recursive operators as methods for inverting Φ.

In Section 3 we study four different notions of inversion which form a hierarchy. In the following, let Φ be an $\mathcal{F}$-preserving general recursive operator and $\Psi = \lim_s \Psi_s$ be a limit-recursive operator (as defined in Section 2 below) to invert Φ:

- Ψ weakly inverts Φ iff, for all $f \in \mathcal{F}$, there exists a g such that $\Phi(f) = \Phi(g)$ and for all x, $\lim_s \Psi_s(\Phi(f))(x) = g(x)$;
- Ψ bounded weakly inverts Φ iff Ψ weakly inverts Φ and for all $f \in \mathcal{F}$, $\Psi(\Phi(f)) \leq_T \Phi(f)$;
- Ψ inverts Φ iff Ψ weakly inverts Φ and there are, for every $f \in \mathcal{F}$, only finitely many pairs (x, s) such that $\Psi_s(\Phi(f))(x) \neq \Psi(\Phi(f))(x)$;
- Ψ strongly inverts Φ iff Ψ inverts Φ and Ψ_0 is a general recursive operator.

In the formal Definition 4(b) below, "strongly inverts" is defined equivalently but in slightly different form. Note that in the case of weakly inverting a function, the requirement $\Psi(\Phi(f)) \leq_T \Phi(f)$ is not automatically guaranteed as Ψ is a limiting process — it is indeed a restriction. The motivation for the requirement is the following: it is a natural constraint to say that one can compute the original input function from the observed output function; however, one may not be able to perform these computations uniformly for all functions in the range of Φ and therefore may need a limit-recursive process to invert the data of the observed output. A class $\mathcal{F}$ is called invertible (weakly invertible, strongly invertible, bounded weakly invertible), if one can invert (weakly invert, strongly invert, bounded weakly invert) every $\mathcal{F}$-preserving general recursive operator.

In this paper we will show that the above notions of invertibility form a strict hierarchy. Theorem 5 shows that $\mathcal{R}$ is not weakly invertible. Proposition 6 shows that every subclass of $\{0, 1\}^\infty$ is weakly invertible. However, Example 7 shows that the class of binary functions is weakly invertible but not bounded weakly invertible. Theorem 8 extends the result to the class of recursive binary functions. Example 14 gives a class which is bounded weakly invertible but not invertible. Example 12 gives a class which is invertible but not strongly invertible. Examples 10 and 11 show that strong invertibility is not trivial by giving interesting infinite classes of recursive functions which are strongly invertible. In Proposition 16 we show that every recursively enumerable class is strongly invertible.

The question of whether an operator is invertible also depends on the variety of operators that are available. Therefore one might ask how difficult an enumeration has to be so that all possible restrictions of mappings from $\mathcal{F}$ to $\mathcal{F}$, which can be done by general recursive operators, also occur in this enumeration. We call this notion coverability and study it in Section 4.

- An enumeration $\Phi_0, \Phi_1, \ldots$ weakly covers $\mathcal{F}$, iff, for every $\mathcal{F}$-preserving general recursive operator Φ, there is an e such that Φ_e is general recursive and Φ_e, restricted to domain $\mathcal{F}$, is the same as Φ.
- An enumeration $\Phi_0, \Phi_1, \ldots$ covers $\mathcal{F}$, iff it weakly covers $\mathcal{F}$ and every Φ_e is total on $\mathcal{F}$.
- An enumeration $\Phi_0, \Phi_1, \ldots$ strongly covers $\mathcal{F}$, iff it weakly covers $\mathcal{F}$ and every Φ_e is general recursive.

$\mathcal{F}$ is (weakly, strongly) coverable, if some recursive enumeration of recursive operators (weakly, strongly) covers $\mathcal{F}$. Note that the recursive enumeration of all recursive operators trivially weakly covers every class $\mathcal{F}$.

Example 22 shows that there is a class which is coverable but not strongly coverable. Coverable classes of recursive functions are quite restrictive: every coverable class of recursive functions is contained in a recursively enumerable class of recursive functions. Example 19 gives a class of binary functions which is strongly coverable, but not bounded weakly invertible. Remark 20 extends this to general classes of functions which are strongly coverable but not weakly invertible.

Proposition 21 shows that there are even simple classes like $\{0^e 1^\infty : e \in \mathbb{N}\}$ which are not coverable. On the other hand, Example 23 shows that any class of functions which recursively approximates a 1-generic set below the halting problem is coverable. Even though not every recursively enumerable class is coverable, Proposition 24 shows that every recursively enumerable class is covered by some K'-recursive enumeration of recursive operators.

In Section 5 we pay special attention to the class of periodic functions, $\mathcal{F}_{per}$. Let $\Phi_0, \Phi_1, \ldots$ be an acceptable numbering of all recursive operators. Corollary 28 shows that the set

$$\{e : \Phi_e \text{ is } \mathcal{F}_{per}\text{-preserving}\}$$

is Π^0_3-complete.

In Section 6 we consider variants of the notion of inverting. We consider situations such as: What happens if Φ is not general recursive? Given an enumeration of operators, is it possible to invert all of the $\mathcal{F}$-preserving operators in this list on at least some of the functions in their range?

2. Basic notation

In this section, some basic notation and definitions are introduced. Notation not explained here is standard and follows the textbooks of Odifreddi [9] and Soare [12].

Let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$ of natural numbers. We often identify function f (with domain $\mathbb{N}$) with the infinite string $f(0)f(1)\ldots$; similarly, a finite string σ can be identified with the finite function η, which is defined on $x < |\sigma|$, such that $\sigma = \eta(0)\eta(1)\ldots\eta(x-1)$. Let $\varphi_0, \varphi_1, \ldots$ be an acceptable numbering of all partial-recursive unary functions and W_e be the domain of φ_e. $W_{e,s}$ denotes the set of all $x < s$ for which $\varphi_e(x)$ halts within s steps. K denotes the halting problem, $\{e : e \in W_e\}$. A' denotes the halting problem relative to A, that is $\{e : e \in W_e^A\}$ (where W_e^A is the set accepted by the e-th oracle Turing machine using the oracle A).

Definition 1. Given a function f or a string σ of length at least n, $f[n]$ and $\sigma[n]$ denote the first n elements of f and σ, respectively. Furthermore, λ denotes the empty string which coincides with $f[0]$ and $\sigma[0]$ for all functions f and strings σ.

Remark 2. For several examples, an effective version of Ramsey's Theorem is needed. In particular the following notion is used. An A-recursive 2-*colouring* is an A-recursive function R with the domain $\{(x, y) : x < y\}$ and range $\{\text{false}, \text{true}\}$. The members of the range are called the colours. A set E is 2-r-*cohesive relative to* A iff, for all A-recursive 2-colourings R, there are an $e \in E$ and a colour u such that, for all $x, y \in E$ with $e < x < y$, $R(x, y) = u$. One can generalize Ramsey's Theorem and show that, for every A and infinite B, B has an infinite subset which is 2-r-cohesive relative to A. Hummel and Jockusch [4] give an overview on 2-r-cohesive sets and generalize these notions with respect to the involved parameters.

As the main goal is to translate total functions into total functions, one can define recursive operators by the easiest approach and view them as oracle Turing machines following certain restrictions. Odifreddi [9, Section II.3] provides more information on recursive operators and introduces more variants of this model.

Definition 3 (*[12]*). (a) A *recursive operator* Φ is an oracle Turing machine which takes functions as an oracle. So $\Phi(f)(x)$ is the value of the function computed by Φ at x with oracle f.

Without loss of generality, $\Phi(f[s])(x)$ is defined and y iff $\Phi(f)(x) = y$, $x < s$, the computation converges in less than s steps and the computation queries f only below s. Otherwise $\Phi(f[s])(x)$ is undefined.

(b) Φ is a *general recursive operator* iff $\Phi(f)$, defined as $x \mapsto \Phi(f)(x)$, is total for every total function f.

(c) A *limit-recursive operator* Ψ is given by a recursive sequence $\Psi_0, \Psi_1, \ldots$ of recursive operators. For any total function f, one says that $\Psi(f)$ is defined and equal to g iff (1) for all s, $\Psi_s(f)$ is defined and (2) for all x there is an s such that for all $t > s$, $\Psi_t(f)(x) = g(x)$. If such a g does not exist, then $\Psi(f)$ is undefined.

Note that there are acceptable enumerations of all recursive operators and of all limit-recursive operators. The latter is given by an acceptable two-dimensional enumeration $\Psi_{e,s}$ of recursive operators such that $\Psi_e(f)(x)$ is defined to be y iff there is an s such that, for all $t > s$, the operator $\Psi_{e,t}(f)(x)$ converges to y.

3. Inverting operators

The mathematical model is taken from inductive inference which is the recursion-theoretic model of learning theory. In the following let $\mathcal{F}$ denote the class of functions under consideration. Given a general recursive operator Φ, there are several degrees of inversion.

Definition 4. (a) A general recursive operator Φ is called *$\mathcal{F}$-preserving* iff it maps every function from $\mathcal{F}$ to $\mathcal{F}$.

(b) Ψ *strongly inverts* Φ iff Ψ is a general recursive operator and, for every $f \in \mathcal{F}$, there exists a g such that g is a finite variant of $\Psi(\Phi(f))$ and $\Phi(g) = \Phi(f)$.

(c) Ψ *inverts* Φ iff Ψ is a limit-recursive operator such that, for every $f \in \mathcal{F}$ and for all $x \in \mathbb{N}$, the limit $g(x) = \lim_s \Psi_s(\Phi(f))(x)$ exists, $\Phi(g) = \Phi(f)$ and there are only finitely many pairs (x, s) with $\Psi_s(\Phi(f))(x) \neq g(x)$.

(d) Ψ *weakly inverts* Φ iff Ψ is a limit-recursive operator such that, for every $f \in \mathcal{F}$ and for all $x \in \mathbb{N}$, the limit $g(x) = \lim_s \Psi_s(\Phi(f))(x)$ exists and $\Phi(f) = \Phi(g)$.

(e) Ψ *bounded weakly inverts* Φ iff Ψ is a limit-recursive operator such that, for every $f \in \mathcal{F}$ and for all $x \in \mathbb{N}$, the limit $g(x) = \lim_s \Psi_s(\Phi(f))(x)$ exists, $\Phi(f) = \Phi(g)$ and $g \leq_T \Phi(f)$.

(f) The class $\mathcal{F}$ is called *invertible*, *strongly invertible*, *weakly invertible* or *bounded weakly invertible* iff, for every $\mathcal{F}$-preserving general recursive operator Φ, there is a Ψ such that Ψ inverts, strongly inverts, weakly inverts or bounded weakly inverts Φ, respectively.

Although the notion of weakly invertible has a certain interest in its own right, it is a limiting process where it is no longer possible to get g from $\Phi(f)$ by any effective means. Somehow, it might be natural also to consider the case where such a translation of $\Phi(f)$ into g at least exists, although it is not applied by Ψ. This additional requirement that $g \leq_T \Phi(f)$ is then considered in (e).

Note that strongly invertible implies invertible as follows. Suppose $\mathcal{F}$ is given and Ψ strongly inverts Φ. The new operator inverting Φ is given by taking as the s-th approximation the first finite variant g_s, in some standard enumeration of the finite variants of $\Psi(\Phi(f))$, found for which $\Phi(g_s)[s] = \Phi(f)[s]$. For $f \in \mathcal{F}$, $\Psi(\Phi(f))$ is a finite variant of some g with $\Phi(g) = \Phi(f)$ and thus the g_s converge to this g or some other finite variant with the same property.

The implication from invertible to bounded weakly invertible comes from the following argument. Whenever Ψ converges to g on input $\Phi(f)$ according to (c), then $g = \Psi_s(\Phi(f))$ for large enough s. Since Ψ_s is a recursive operator for every s, $g \leq_T \Phi(f)$.

The implication from bounded weakly invertible to weakly invertible is obvious since just one requirement on the process to get g from $\Phi(f)$ is dropped.

Theorem 5. *The class $\mathcal{R}$ of all recursive functions is not weakly invertible.*

Proof. Define an operator Φ by the equation

$$\Phi(f) = \begin{cases} (f(0))^\infty & \text{if } \forall s\ [|W_{f(0),f(s)}| \geq s] \text{ or } \forall s\ [|W_{f(0),s}| \leq f(1)]; \\ (f(0))^s(f(0)+1)^\infty & \text{if } s \text{ is the least positive number for which the first case fails.} \end{cases}$$

For every e there is a recursive f with $\Phi(f) = e^\infty$. In the case that W_e is finite, such an f is $e|W_e|0^\infty$; in the case that W_e is infinite, such an f can be obtained by letting $f(x) = \min(\{s : |W_{e,s}| \geq x\})$.

On the other hand, if W_e is finite, then only functions f with $f(0) = e \wedge f(1) \geq |W_e|$ are mapped to e^∞. Thus, if Ψ inverts Φ on e^∞ in the limit, then the function $F(e) = \lim_s \Psi_s(e^\infty)(1)$ is K-recursive and satisfies $F(e) \geq |W_e|$ whenever W_e is finite. It follows that $\{e : W_e \text{ is finite}\} = \{e : |W_e| \leq F(e)\}$ where the first set is Σ_2^0-complete and the second is K-recursive, a contradiction. Therefore $\mathcal{R}$ is not weakly invertible. $\square$

The above result used the fact that the function $e \mapsto |W_e|$ restricted to the domain of all e, where W_e is finite, is not dominated by any K-recursive function. So, although all involved functions are recursive, their initial growth from $f(0)$ to $f(1)$ cannot be captured even by a K-recursive function. One might ask what happens if growth conditions cannot be exploited because all functions involved are bounded. The next result implies that every such class is weakly invertible. The following proposition is based on Kreisel's work [6] and is a uniform version of [9, Proposition V.5.31] relativized to $\Phi(f)$.

Proposition 6 (*Based on Kreisel [6]*). *For every constant c, the class* $\{0, 1, \ldots, c\}^\infty$ *is weakly invertible.*

Proof. Let Φ be a general recursive operator and $f \in \{0, 1, \ldots, c\}^\infty$. The set

$$T(\Phi(f)) = \{\sigma \in \{0, 1, \ldots, c\}^* : \forall x \leq |\sigma| \text{ [if } \Phi(\sigma)(x) \text{ is defined then } \Phi(\sigma)(x) = f(x)]\}$$

forms an f-recursive tree. Now one can define a limit-recursive operator Ψ such that the function g, given by $g(x) = \lim_s \Psi_s(\Phi(f))(x)$, is in $\{0, 1, \ldots, c\}^\infty$, and is an infinite branch of $T(\Phi(f))$, that is, it satisfies $\Phi(g) = \Phi(f)$. This is done by choosing $\Psi_s(\Phi(f))(x)$ to be $\tau(x)$ for the lexicographic least string $\tau \in \{0, 1, \ldots, c\}^{x+s}$ which is in $T(\Phi(f))$. □

In the following, it is proven that the notions strongly invertible, invertible, bounded weakly invertible and weakly invertible form a strict hierarchy. The classes separating the levels of this hierarchy are subclasses of $\{0, 1\}^\infty$.

Example 7. The class $\{0, 1\}^\infty$ is weakly invertible but not bounded weakly invertible.

Proof. $\{0, 1\}^\infty$ is weakly invertible by Proposition 6. We now show that $\{0, 1\}^\infty$ is not bounded weakly invertible. Let the partial-recursive function ψ be defined as

$$\psi(e) = \begin{cases} 0 & \text{if } \varphi_e(e) \downarrow \geq 1; \\ 1 & \text{if } \varphi_e(e) \downarrow = 0; \\ \uparrow & \text{if } \varphi_e(e) \uparrow . \end{cases}$$

Note that ψ is partial recursive but has no total recursive extension, as the definition makes ψ inconsistent with all total recursive functions. Let ψ_s denote the finite part of ψ which is computed within s steps. Now define

$$\Phi(f)(s) = \begin{cases} 0 & \text{if } \psi_s \text{ and } f \text{ are consistent;} \\ 1 & \text{otherwise.} \end{cases}$$

This operator maps all total extensions of ψ to 0^∞ while it maps all recursive functions to $\{0^k 1^\infty : k \in \mathbb{N}\}$. So some functions are mapped to 0^∞ but none of them is recursive relative to 0^∞. Thus, the condition $g \leq_T \Phi(f)$ from Definition 4(e) cannot be satisfied. □

This result was of course induced by the fact that the class contains nonrecursive functions. So one could ask whether there is a class containing only recursive functions which is not bounded weakly invertible. The following example shows that this is indeed true.

Theorem 8. *The class* $\mathcal{R}_{0,1}$ *consisting of all* $\{0, 1\}$*-valued recursive functions is not bounded weakly invertible.*

Proof. As in Example 7, let ψ be a partial-recursive $\{0, 1\}$-valued function without total recursive extension and ψ_s be the finite part of it computed in time s. Similarly, let ξ^K a corresponding partial K-recursive $\{0, 1\}$-valued function without a total K-recursive extension. The function ξ^K has a $\{0, 1\}$-valued recursive approximation $\xi_0, \xi_1, \ldots$ so that, for all e in the domain of ξ^K, $\lim_s \xi_s(e) = \xi^K(e)$. For $e \in \mathbb{N}$ and $a \in \{0, 1\}$, define a function $\theta_{e,a}$ as follows:

$$\theta_{e,a}(x) = \begin{cases} 0 & \text{if } x < e; \\ 1 & \text{if } x = e \text{ or } x = e + 1; \\ a & \text{if } x = e + 2; \\ \psi_s(x) & \text{if } x > e + 2 \text{ and } s \text{ is the first } t > x \text{ found such that either } \psi_t(x) \text{ is defined or } \xi_t(e) = a. \end{cases}$$

Here, in the fourth case, $\theta_{e,a}(x)$ is undefined if either s is never found because the corresponding t does not exist or $\psi_s(x)$ is undefined. Now define Φ as follows:

$$\Phi(f) = \begin{cases} 0^\infty & \text{if } f = 0^\infty; \\ 0^e101^\infty & \text{if } f \text{ extends } 0^e10; \\ 0^e10^\infty & \text{if } f \text{ extends } \theta_{e,0} \text{ or } \theta_{e,1}; \\ 0^e10^{s+1}1^\infty & \text{if } f \text{ extends } 0^e11 \text{ but one finds in } s \text{ steps that the previous case fails.} \end{cases}$$

It is easy to verify that Φ is general recursive. For every e there is $a \in \{0, 1\}$ such that $\xi_s(e) = a$ for infinitely many s. For such a, let $h_{e,a}$ be defined such that $h_{e,a}[e+3] = 0^e11a$ and, for $x > e+2$, one does the following. One finds the first $s \geq x$ such that $\xi_s(e) = a$; if $\psi_s(x)$ is defined, then $h_{e,a}(x) = \psi_s(x)$, else $h_{e,a}(x) = 0$. Every total $h_{e,a}$ is a total and recursive extension of $\theta_{e,a}$, and for every e, either $h_{e,0}$ or $h_{e,1}$ is total.

Now assume by way of contradiction that $\Psi = \lim_s \Psi_s$ bounded weakly inverts Φ. Then $\Psi(0^e10^\infty)$ converges to a function g_e and g_e extends $\theta_{e,g_e(e+2)}$ (as Φ only maps extensions of $\theta_{e,0}$ or $\theta_{e,1}$ to 0^e10^∞). The function $e \mapsto g_e(e+2) = \lim_s \Psi_s(0^e10^\infty)(e+2)$ is K-recursive. As ξ^K has no K-recursive total extension, there is an e such that $\xi^K(e)$ is defined and different from $g_e(e+2)$. Also, there is an s such that $\xi_t(e) = \xi^K(e)$ for all $t \geq s$. As a consequence, by definition of $\theta_{e,g_e(e+2)}$, for all $x \geq s$ in the domain of ψ, $g_e(x) = \theta_{e,g_e(e+2)}(x) = \psi(x)$. Thus, g_e is a finite variant of an extension of ψ. This contradicts the fact that ψ has no recursive extension. Thus, Ψ cannot exist and $\mathcal{R}_{0,1}$ is not bounded weakly invertible. □

Note that, instead of $\mathcal{R}_{0,1}$, one could already use the class consisting of all functions 0^e10^∞, 0^e101^∞ and $h_{e,a}$ whenever the latter is total. The resulting class is finitely learnable, that is, there is a learner which outputs void hypotheses until it has seen enough data and outputs exactly one correct hypothesis from then on. Finite learning is one of the most restrictive learning criteria, but it still does not guarantee weak invertibility.

This contrasts with Proposition 16 below which says that all recursively enumerable classes are strongly invertible. The next section deals with recursively enumerable classes explicitly, but before that some further examples of invertible classes are presented.

Example 9. Every class $\{f\}$ consisting of only one function is strongly invertible. The reason is that every $\{f\}$-preserving Φ maps f to itself and so one can take Ψ as the identity.

One might ask whether this comes from the small cardinality of given class. It does not, as the following example of a similar class with cardinality $2^{\aleph_0}$ shows. The construction of a tree with the properties postulated in this example is well known [9, Exercise V.2.18 (b)] and thus omitted.

Example 10. There is a recursive tree $T \subseteq \{0, 1\}^*$, with uncountably many infinite branches, such that the class $\mathcal{F}$ of all its infinite branches satisfies the following statement: any two distinct members of $\mathcal{F}$ have incomparable Turing degrees. This class $\mathcal{F}$ is uncountable and strongly invertible.

Proof. As any two infinite branches of T are Turing incomparable, no infinite branch is recursive. Thus, if Φ is $\mathcal{F}$-preserving, then, for every infinite branch f of T, $\Phi(f) = f$. Thus, one can choose Ψ to be the operator which maps every function to itself. Note that all recursive infinite trees $T \subseteq \{0, 1\}^*$, without recursive infinite branches, have uncountably many infinite branches [9, Proposition V.5.27]. □

Note that the same Ψ works for all $\mathcal{F}$-preserving Φ. Furthermore, the given example consists of infinite branches of a recursive tree, so Φ and Ψ can even detect eventually whenever their input is not from $\mathcal{F}$. The next example consists only of recursive functions but has a similar flavour.

Example 11. Let $e_0, e_1, \ldots$ be an infinite sequence of minimal indices of total functions such that $\{e_0, e_1, \ldots\}$ is 2-r-cohesive relative to K'. Such a set exists by Remark 2. The class $\mathcal{F} = \{\varphi_{e_n} : n \in \mathbb{N}\}$ is strongly invertible.

Proof. Given $\mathcal{F}$ and Φ, one defines R as a {false, true}-valued colouring by

$$R(i, j) \Leftrightarrow (\varphi_i \text{ and } \varphi_j \text{ are total}) \wedge ((\Phi(\varphi_i) = \varphi_j) \vee (\Phi(\varphi_j) = \varphi_i)).$$

Note that $R \leq_T K'$ as one can use the oracle K' for deciding whether φ_i, φ_j are total, whether $\Phi(\varphi_i) = \varphi_j$ and whether $\Phi(\varphi_j) = \varphi_i$. By Remark 2 there is a number k and a colour u such that for all i, j with $k < i < j$,

$R(e_i, e_j) = u$. If $u =$ true, then $R(e_{k+1}, e_n) =$ true for all $n > k$. As $\Phi(\varphi_{e_{k+1}})$ takes only one value, one can conclude that $\Phi(\varphi_{e_n}) = \varphi_{e_{k+1}}$ for almost all n. If $u =$ false, then $R(e_m, e_n)$ does not hold whenever $k < m < n$.

Thus, in both cases, for all $n > k + 1$, either $\Phi(\varphi_{e_n}) = \varphi_{e_n}$ or $\Phi(\varphi_{e_n}) \in \{\varphi_{e_0}, \varphi_{e_1}, \ldots, \varphi_{e_{k+1}}\}$ or $\varphi_{e_n} = \Phi(\varphi_{e_{k+1}})$. So there is a finite set $\mathcal{G} \subseteq \mathcal{F}$ such that every function in $\Phi(\mathcal{F})$ is either the image of itself or in $\Phi(\mathcal{G})$. As $\mathcal{G}$ is finite, there is a finite set D giving the indices of the functions in $\mathcal{G}$. Thus, one can define a strongly inverting operator Ψ as follows:

$$\Psi(g)(x) = \begin{cases} g(x), & \text{if no member } d \text{ of } D \text{ satisfies } \Phi(\varphi_d)[x] = g[x]; \\ \varphi_d(x), & \text{if } d \text{ is the least member of } D \text{ such that } \Phi(\varphi_d)[x] = g[x]. \end{cases}$$

It is easy to verify that Ψ strongly inverts Φ. □

Example 12. Let $\Phi_0, \Phi_1, \ldots$ be an enumeration of all recursive operators and let G be the index set of the e where Φ_e is general recursive. Furthermore, let $F = \{2^n + \sum_{m<n} 2^m \cdot G(m)\}$ be a set of numbers coding initial parts of G by its binary digits. Let $\{e_0, e_1, \ldots\}$ be a subset of F which is 2-r-cohesive relative to K'. Let $\mathcal{F}$ contain, for every k, the functions $0^{e_k}101^\infty$, $0^{e_k}10^\infty$ and θ_{e_k}, where, for any e, x, $\theta_e(x)$ is defined as follows.

One chooses θ_e such that $\theta_e[e+2] = 0^e11$. For $x \geq e+2$, let $a < e$ be such that $x \equiv a$ modulo e. If $2^{a+1} \geq e$ then $\theta_e(x) = 0$. Otherwise determine the $a+1$-st least significant bit of e. If this bit is 0 then $\theta_e(x) = 0$ again. Otherwise

$$\theta_e(x) = \begin{cases} 0 & \text{if } \Phi_a(0^e10^\infty)(x)\downarrow > 0; \\ 1 & \text{if } \Phi_a(0^e10^\infty)(x)\downarrow = 0; \\ \uparrow & \text{otherwise.} \end{cases}$$

Note that the function θ_e is total for $e \in F$. The class $\mathcal{F}$ is invertible but not strongly invertible.

Proof. Given $\mathcal{F}$ and Φ, there is a k such that for all $n, m > k$ and all $a, b, a', b' \in \{0, 1\}$,

(a) if f extends $0^{e_n}1ab$, then $\Phi(f)$ extends $0^{e_m}1$ only if $m = n$;
(b) if f extends $0^{e_n}1ab$ and f' extends $0^{e_m}1ab$ and $\Phi(f)$ extends $0^{e_n}1a'b'$ and $\Phi(f')$ extends $0^{e_m}1a''b''$, then $a' = a''$ and $b' = b''$.

The above properties (a) and (b) are proven by considering the following K'-recursive colouring $R_{00}, R_{01}, R_{11}, S_{00}, S_{01}, S_{11}$. For these definitions, let $f^e_{01} = 0^e101^\infty$, $f^e_{00} = 0^e100^\infty$, $f^e_{11} = \theta_e$. Let

$$\begin{aligned} R_{ab}(i, j) &\Leftrightarrow (f^i_{ab} \text{ and } f^j_{ab} \text{ are total}) \wedge ((\Phi(f^i_{ab}) \text{ extends } 0^j1) \vee (\Phi(f^j_{ab}) \text{ extends } 0^i1)). \\ S_{ab}(i, j) &\Leftrightarrow (f^i_{ab} \text{ and } f^j_{ab} \text{ are total}) \wedge ((\Phi(f^i_{ab})(i+1) = \Phi(f^j_{ab})(j+1)) \text{ and} \\ &\qquad (\Phi(f^i_{ab})(i+2) = \Phi(f^j_{ab})(j+2)). \end{aligned}$$

It is easy to see that R_{ab} and S_{ab} can be computed relative to oracle K'. Thus, as $\{e_0, e_1, \ldots\}$ is 2-r-cohesive relative to K', there is a number k such that the following two conditions (c) and (d) hold; keep this number k fixed from now on.

(c) $R_{ab}(e_i, e_j) =$ false for all i, j with $j > i > k$ (and thus, for $j > k$, $\Phi(f^{e_j}_{ab})$ extends $0^{e_j}1$ or $\Phi(f^{e_j}_{ab})$ extends $0^{e_r}1$, for some $r \leq k$). This immediately gives (a) above.

Here note that $R_{ab}(e_i, e_j) =$ true for all i, j such that $j > i > k$ leads to a contradiction. This is so, because $R_{ab}(e_{k+1}, e_j) =$ true for all $j > k + 1$ implies

$$\Phi(f^j_{ab}) \text{ extends } 0^{k+1}1 \text{ for all but finitely many } j, \tag{$*$}$$

as $\Phi(f^{k+1}_{ab})$ can take at most one value. Similarly, $R_{ab}(e_{k+2}, e_j) =$ true for all $j > k + 2$ implies

$$\Phi(f^j_{ab}) \text{ extends } 0^{k+2}1 \text{ for all but finitely many } j. \tag{$**$}$$

However, ($*$) and ($**$) lead to a contradiction.

(d) $S_{ab}(e_i, e_j)$ is true for all i, j with $j > i > k$ (since $S_{ab}(e_i, e_j)$ cannot be false for all large enough i, j — otherwise for all large enough i, j, $\Phi(f^i_{ab})(i+1) \neq \Phi(f^j_{ab})(j+1)$ or $\Phi(f^i_{ab})(i+2) \neq \Phi(f^j_{ab})(j+2)$, which is impossible as there are only finitely many possibilities for $\Phi(f^i_{ab})(i+1)$ and $\Phi(f^i_{ab})(i+2)$). This immediately gives (b) above.

Now we continue with the proof. The indices of $0^{e_n}10^\infty$, $0^{e_n}101^\infty$ and θ_{e_n} can be computed from e_n.

An operator Ψ to invert Φ can work as follows. It first determines the e_n, a, b of the prefix $0^{e_n}1ab$ of the function to be inverted. Note that Ψ knows e_n without knowing the index n of it. If $e_n \le e_k$, the inversion can be handled as in Example 11. If $e_n > e_k$, then Ψ can conclude, from a finite table, which of the three functions $0^{e_n}10^\infty$, $0^{e_n}101^\infty$ or θ_{e_n} is mapped by Φ to the input function — Ψ can then invert the input.

Note that this algorithm is sensitive to the fact that the input function is $\Phi(f)$, for some $f \in \mathcal{F}$, and might be undefined on prefixes of other functions. Thus the resulting operator Ψ may not be general recursive. So it might only witness that $\mathcal{F}$ is invertible.

To see that $\mathcal{F}$ is not strongly invertible, consider the operator Φ as follows. Φ maps 0^∞ to itself. For all e, Φ maps all functions beginning with 0^e10 to 0^e101^∞. For all e, Φ maps all functions beginning with 0^e11 to the following: if the function is consistent with θ_e, then it is mapped to 0^e10^∞; otherwise it is mapped $0^e10^{s+1}1^\infty$, where s is the number of steps needed to detect the inconsistency.

So 0^e10^∞ is only the image of total extensions of θ_e, which of course is the function θ_e itself in the case that θ_e is total. Now, if Φ_e is a general recursive operator and $e_n > 2^{e+1}$ then one has that the functions $\Phi_e(0^{e_n}10^\infty)$ and θ_{e_n} differ at infinitely many places, although θ_{e_n} is the only function with $\Phi(\theta_{e_n}) = 0^{e_n}10^\infty$. Thus, no general recursive operator strongly inverts Φ and $\mathcal{F}$ is not strongly invertible. □

For the separation of bounded weakly invertible from invertible, the following result of Kaufmann [5, Theorem 5.2.2] is crucial, which is formulated such that it fits conveniently into the setting of the present work; the proof is nevertheless almost the same as by Kaufmann and thus omitted. The original result of Kaufmann applies to the constructed tree indices of the $e + 1$ partial-recursive functions $x \mapsto \Psi_{e',x}(0^e10^\infty)(x)$ with $e' \in \{0, 1, \dots, e\}$ instead of invoking them directly. However, a set of those indices could easily be generated from the parameter e and so Kaufmann's proof directly transfers to the current application.

Proposition 13 (*Kaufmann [5]*). *Let $\Psi_0, \Psi_1, \dots$ be an acceptable numbering of all limit-recursive operators such that $\Psi_e = \lim_s \Psi_{e,s}$. Then there is a uniformly recursive family $T_0, T_1, \dots$ of trees such that for every e the following holds:*

- *each T_e is a subset of $0^e11\{0,1\}^* \cup \{\lambda, 0, 00, \dots, 0^e, 0^e1\}$.*
- *for each e, n, $|T_e \cap \{0,1\}^n| \le e + 2$, that is, T_e has bounded width;*
- *for each e and each infinite branch A of T_e and each $e' \le e$, there are infinitely many x such that either $\Psi_{e',x}(0^e10^\infty)(x)$ is undefined or different from $A(x)$.*

Furthermore, each T_e has at least one infinite branch and all its infinite branches are recursive.

Using this result, one can now construct the separating class.

Example 14. Let T_e be as defined in Proposition 13. Let $\{e_0, e_1, \dots\}$ be a set which is uniformly 2-r-cohesive relative to K' and satisfies, for every n and every $m \ge n$, $\varphi_n^{K'}(e_m) < e_{m+1}$. Let $\Psi_0, \Psi_1, \dots$ be a recursive enumeration of all limit-recursive operators. Now let $\mathcal{F}$ contain the functions $0^{e_n}10^\infty$, $0^{e_n}101^\infty$ and the left-most infinite branch θ_{e_n} of T_{e_n} for all n. The class $\mathcal{F}$ is bounded weakly invertible but not invertible.

Proof. The argumentation that $\mathcal{F}$ is bounded weakly invertible is parallel to the argumentation of the corresponding class being invertible in Example 12. Here note that θ_e is computable from e using the oracle K. Thus, a limit-recursive operator can compute θ_e.

To see that $\mathcal{F}$ is not invertible, consider the general recursive operator Φ with $\Phi(f)$ being determined by the following case distinction:

$$\Phi(f) = \begin{cases} 0^\infty & \text{if } f = 0^\infty; \\ 0^e101^\infty & \text{if } f \text{ extends } 0^e10; \\ 0^e10^\infty & \text{if } f \text{ extends } 0^e11 \text{ and } f \text{ on } T_e; \\ 0^e10^s1^\infty & \text{if } f \text{ extends } 0^e11 \text{ and } s \text{ is the least number with } f[s] \notin T_e. \end{cases}$$

It is easy to see that Φ is $\mathcal{F}$-preserving. Furthermore, $\Phi(\theta_{e_n}) = 0^{e_n}10^\infty$ for all n. Assume now by way of contradiction that Ψ_e inverts Φ and $n > e$. Then, by the third condition listed in Proposition 13, the function g_{e_n} given by

$x \mapsto \Psi_{e,x}(0^{e_n}10^\infty)(x)$ differs from all infinite branches of T_{e_n} at infinitely many places. But for almost all s, the functions $\Psi_{e,s}(0^{e_n}10^\infty)$ are the same, thus their limit is a finite variant of g_{e_n} and not an infinite branch of T_{e_n}. As Φ maps only the infinite branches of T_{e_n} to $0^{e_n}10^\infty$, no finite variant of g_{e_n} is mapped to $0^{e_n}10^\infty$ and Ψ_e does not invert Φ in contradiction to the assumption. □

4. Enumerating operators and functions

It is quite natural to deal with classes where there is an indexing for all the functions involved. Such classes are known as "indexed families", "uniformly recursive classes" or "recursively enumerable classes", where the enumeration is now an enumeration of the involved functions and not of the elements of a set.

Definition 15. A class $\mathcal{F}$ is recursively enumerable iff there is a total recursive function $e, x \mapsto f_e(x)$ in two variables such that $\mathcal{F}$ equals the set of functions obtained by fixing the input e: $\mathcal{F} = \{f_0, f_1, \ldots\}$.

Such classes are quite easy to invert as one may define Ψ as follows. Given a general recursive operator Φ, $\Psi(\Phi(f))(x) = f_e(x)$, for the least e such that either $e = x$ or $\Phi(f_e)(y) = \Phi(f)(y)$ for all $y \le x$. It is well known that such an algorithm of "learning by enumeration" gives a general recursive operator which makes only finitely many errors.

Proposition 16. *Every recursively enumerable class of functions is strongly invertible.*

The question whether an operator can be inverted also depends on the variety of operators available. Therefore, one might ask how difficult an enumeration has to be so that all possible restrictions of mappings from $\mathcal{F}$ to $\mathcal{F}$ occur. This is formalized in the following definition.

Definition 17. (a) An enumeration $\Phi_0, \Phi_1, \ldots$ of recursive operators *weakly covers* $\mathcal{F}$ iff, for every $\mathcal{F}$-preserving general recursive operator Ψ, there is an e with Φ_e being general recursive and $\forall f \in \mathcal{F}\,[\Phi_e(f) = \Psi(f)]$.

(b) An enumeration $\Phi_0, \Phi_1, \ldots$ of recursive operators *covers* $\mathcal{F}$ iff it weakly covers $\mathcal{F}$ and every $\Phi_e(f)$ is total for every $f \in \mathcal{F}$. Furthermore, $\mathcal{F}$ is *coverable* iff some recursive enumeration of recursive operators covers $\mathcal{F}$.

(c) An enumeration $\Phi_0, \Phi_1, \ldots$ of recursive operators *strongly covers* $\mathcal{F}$ iff it weakly covers $\mathcal{F}$ and every Φ_e is a general recursive operator. Furthermore, $\mathcal{F}$ is *strongly coverable* iff some recursive enumeration of recursive operators strongly covers $\mathcal{F}$.

Note that every class is weakly covered by an acceptable numbering of all recursive operators. Clearly, $\{f\}$ is strongly coverable since an enumeration only needs to contain the identity operator in order to cover $\{f\}$. When considering classes of recursive functions, coverable classes are restricted to be contained in enumerable ones.

Theorem 18. *Every coverable class of recursive functions is a subclass of a recursively enumerable class of recursive functions.*

Proof. Let $\mathcal{F}$ contain only recursive functions, $f \in \mathcal{F}$ and $\Phi_0, \Phi_1, \ldots$ be an enumeration covering $\mathcal{F}$. Now, for every $g \in \mathcal{F}$, there is an operator Φ_e which maps every function to g and thus $\Phi_e(f) = g$. So $\mathcal{F} \subseteq \{\Phi_e(f) : e \in \mathbb{N}\}$ and the function $e, x \mapsto \Phi_e(f)(x)$ is total and recursive in both inputs. So $\mathcal{F}$ is a subclass of $\{\Phi_e(f) : e \in \mathbb{N}\}$, a recursively enumerable class of recursive functions. □

As a consequence, one has that every coverable class of recursive functions is also strongly invertible. One might therefore ask whether every strongly coverable class is also strongly invertible. This is unfortunately not the case.

Example 19. Let ψ be a partial-recursive $\{0, 1\}$-valued function without recursive total extension and f be a (nonrecursive) total extension of ψ. The class $\{0^\infty, 1^\infty, f\}$ is strongly coverable but not bounded weakly invertible.

Proof. One just has to find an enumeration of general recursive operators which covers each of the finitely many possibilities how an operator can map the three functions inside this class. So the class is strongly coverable. It is not bounded weakly invertible as one might consider any operator which satisfies that $\Phi(0^\infty) = \Phi(1^\infty) = 1^\infty$, $\Phi(f) = 0^\infty$ and $\Phi(g) \neq 0^\infty$ for any recursive g. The proof of Example 7 essentially describes how to construct such a Φ. Then, one can conclude, as in Example 7, that Φ is not bounded weakly invertible as $\Phi(f)$ is recursive, but the inverting algorithm cannot find any recursive g with $\Phi(f) = \Phi(g)$, as such a g does not exist. □

Remark 20. A similar result can also be obtained for unbounded functions. It is well known that there is a recursive tree $T \subseteq \{0, 1, \ldots\}^*$ which has infinite branches but no hyperarithmetic ones — this fact is for example mentioned by Marek, Nerode and Remmel [7]. This tree T has, in particular, no infinite branch which is limit recursive. One can select an infinite branch f of this tree T and then show, by an argument similar to the one used in the proof of Example 19, that $\{0^\infty, 1^\infty, f\}$ is strongly coverable but not weakly invertible.

Most recursively enumerable classes are not coverable. Examples for noncoverable classes are the class of all primitive recursive functions, the class of all characteristic functions of regular languages and $\{0^e 1^\infty : e \in \mathbb{N}\}$. These examples have in common that they have an infinite finitely learnable subclass: namely, the third example. Here a class $\mathcal{F}$ is said to be *finitely learnable* [1,2] iff there is a general recursive operator mapping every function $f \in \mathcal{F}$ to a function of the form $0^* e^\infty$ such that $e > 0 \wedge \varphi_e = f$. The next proposition provides a formal proof for the fact that such classes are not coverable.

Proposition 21. *If $\mathcal{F}$ is recursively enumerable and $\mathcal{F}$ has an infinite finitely learnable subclass then $\mathcal{F}$ is not coverable.*

Proof. Consider an enumeration $\Theta_0, \Theta_1, \ldots$ of recursive operators which are total on all functions in $\mathcal{F}$. It is shown that this enumeration does not cover $\mathcal{F}$. This is done following a distinction of two cases.

Case 1. There is a constant c such that the set $\{f[c] : f \in \mathcal{F}\}$ is infinite. Then there is a recursive sequence f_n of functions in $\mathcal{F}$ such that

- $f_n[c] \neq f_m[c]$ for all pairwise distinct n, m;
- $\{f_n[c] : n \in \mathbb{N}\}$ is recursive.

Now define

$$\Phi(g) = \begin{cases} f_0 & \text{if there is an } n \text{ such that } f_n[c] = g[c] \text{ and } \Theta_n(f_n)[c] \neq f_0[c]; \\ f_1 & \text{otherwise.} \end{cases}$$

It is easy to see that $\Theta_n(f_n) \neq \Phi(f_n)$ for all n. Furthermore, Φ is a general recursive operator. So Φ is not covered by the enumeration $\Theta_0, \Theta_1, \ldots$ from above.

Case 2. For every constant c the set $\{f[c] : f \in \mathcal{F}\}$ is finite. Furthermore, there is an infinite r.e. set of functions in $\mathcal{F}$ which is finitely learnable; the reason is that if the learner converges on some input σ, it can pick some member of $\mathcal{F}$ which extends σ. Now one can select from this r.e. set of functions a recursive sequence of pairs $f_n, g_n \in \mathcal{F}$ of pairwise distinct functions such that $f_n[n] = g_n[n]$. As $\{f_n, g_n : n \in \mathbb{N}\}$ is finitely learnable, there is a strictly increasing recursive function h such that f_n, g_n are learned after inspecting the first $h(n)$ function values. Hence $f_n[h(n)], g_n[h(n)]$ is not a prefix of any other function in the set $\{f_n, g_n : n \in \mathbb{N}\}$. Now one defines a general recursive operator Φ such that

$$\Phi(u) = \begin{cases} f_n & \text{if } u[h(n)] = f_n[h(n)] \text{ and } \Theta_n(f_n)[h(n)] \neq f_n[h(n)]; \\ g_n & \text{if } u[h(n)] = f_n[h(n)] \text{ and } \Theta_n(f_n)[h(n)] = f_n[h(n)]; \\ u & \text{otherwise.} \end{cases}$$

This operator Φ is general recursive: in the case that there is no $m \leq n$ with $u[h(m)] = f_m[h(m)]$ it knows that $\Phi(u)[n] = u[n]$. It can be seen, as in Case 1 above, that Φ is $\mathcal{F}$-preserving and not covered by the enumeration $\Theta_0, \Theta_1, \ldots$ from above. $\square$

Example 22. Let $\Phi_0, \Phi_1, \ldots$ be an acceptable enumeration of recursive operators and let h be a strictly increasing function which grows so fast that $\Phi_e(f[h(n)])(x)$ is defined whenever $e, x \leq n$, $f \in \{0, 1, 2\}^\infty$ and Φ_e is a general recursive operator. Let H be the range of h. Then the class

$$\mathcal{F} = \{f : \forall x\ [(x \notin H \Rightarrow f(x) = 0) \wedge (x \in H \Rightarrow f(x) \in \{1, 2\})]\}$$

is coverable but not strongly coverable.

Proof. For any function f define $h_f(n) = \max(\{x : |\{y < x : f(y) \neq 0\}| \leq n\})$, that is, $h_f(n)$ is the $n+1$-st position x where $f(x)$ is different from 0. The function h_f is partial recursive relative to the oracle f and total iff f is different from 0 infinitely often.

For showing that the class $\mathcal{F}$ is coverable, one defines an enumeration $\Psi_0, \Psi_1, \ldots$ covering $\mathcal{F}$ from the given enumeration $\Phi_0, \Phi_1, \ldots$ as follows.

To compute $\Psi_e(f)(x)$, one searches for the first s for which either $\Phi_e(f[s])(x)$ is defined or $s = h_f(x + e + 1)$ and then defines that

$$\Psi_e(f)(x) = \begin{cases} \Phi_e(f[s])(x) & \text{if } s \text{ is found and } \Phi_e(f[s])(x) \text{ is defined;} \\ 0 & \text{if } s \text{ is found and } \Phi_e(f[s])(x) \text{ is undefined;} \\ \uparrow & \text{if } s \text{ is not found.} \end{cases}$$

Thus, $\Psi_e(f)$ is total whenever either $\Phi_e(f)$ is total or $f(x) \neq 0$ for infinitely many x. In particular, $\Psi_e(f)$ is total for all $e \in \mathbb{N}$ and $f \in \mathcal{F}$.

If Φ_e is a general recursive operator then Ψ_e is also general recursive. This is because $\Phi_e(f)$ is total for every function f. For $f \in \mathcal{F}$, $h_f(e + x + 1) = h(e + x + 1)$ and by the choice of h, $\Phi_e(f[h(e + x + 1)])(x)$ is defined. It follows that $\Psi_e(f)(x) = \Phi_e(f)(x)$. So the operators Ψ_e, Φ_e have the same behaviour on $\mathcal{F}$. Thus, $\Psi_0, \Psi_1, \ldots$ covers $\mathcal{F}$.

For every r.e. set A of general recursive operators, there exists a recursive set E of indices such that (1) every operator in A is equal to some Φ_e with $e \in E$ and (2) all Φ_e with $e \in E$ are general recursive. Now one defines a function $h'(n)$ to be the least number t such that $\Phi_e(f[t])(x)$ is defined for all $x \leq n$, for all $e \leq n$ with $e \in E$ and for all $f \in \{0, 1, 2\}^\infty$. As all Φ_e with $e \in E$ are general recursive, h' is a recursive function. Furthermore $h'(n) \leq h(n)$ for all n. Now one defines

$$\Theta(f)(x) = \begin{cases} 0 & \text{if } x \neq h_f(n) \text{ for all } n \leq x; \\ 1 & \text{if } [x = h_f(e) \text{ for some } e \in E \text{ with } e \leq x] \text{ and } [\Phi_e(f[h'(x + e + 1)])(x)\downarrow \neq 1]; \\ 2 & \text{otherwise.} \end{cases}$$

First, the operator Θ is general recursive as h' is a total function and all other tests apply to bounded search. Second, for every $e \in E$ and $f \in \mathcal{F}$,

$$\Theta(f)(h(e)) = \begin{cases} 2 & \text{if } \Phi_e(f)(h(e)) = 1; \\ 1 & \text{otherwise.} \end{cases}$$

Thus, $\Theta(f) \neq \Phi_e(f)$ and Θ differs on $\mathcal{F}$ from every Φ_e with $e \in E$. Third, Θ is $\mathcal{F}$-preserving since, whenever $f \in \mathcal{F}$, $\Theta(f)(x) = 0$ for $x \notin H$ and $\Theta(f)(x) \in \{1, 2\}$ for $x \in H$. Thus, Θ is an $\mathcal{F}$-preserving general recursive operator different on $\mathcal{F}$ from all Φ_e with $e \in E$. So $\mathcal{F}$ is not strongly coverable. □

Recall that by the Schoenfield's limit lemma [9, Proposition IV.1.17] every set $F \leq_T K$ has an approximation. Here an *approximation of* F is a sequence of recursive functions $f_0, f_1, \ldots$ such that (a) $m, x \mapsto f_m(x)$ is a recursive two-place function and (b) for every x, there is an n such that, for all $m > n$, $f_m(x) = F(x)$.

Furthermore, F is 1-*generic* if, for every r.e. set A of strings, either some prefix $F[n]$ of F is in A or there is an n such that A does not contain any extension of $F[n]$ [10, Section XII.1].

Example 23. Let F be a 1-generic set below K. Let $f_0, f_1, \ldots$ be a sequence of recursive $\{0, 1\}$-valued functions approximating the characteristic function of F. Then $\{f_0, f_1, \ldots\}$ is strongly coverable.

Proof. Let $\Phi_0, \Phi_1, \ldots$ be an enumeration of those general recursive operators for which there are finitely many pairwise incomparable strings $\sigma_0, \sigma_1, \ldots, \sigma_k$ and functions $f_{a_0}, f_{a_1}, \ldots, f_{a_k}$ such that every function extending σ_ℓ is mapped to f_{a_ℓ} and every function not extending any σ_ℓ is mapped to itself.

It remains to show that this enumeration covers $\{f_0, f_1, \ldots\}$. Given a general recursive operator Φ, let A be the set of all binary σ such that $\Phi(\sigma)$ is inconsistent with σ. This set A is recursively enumerable. There are two cases.

Case 1. Some prefix $F[n]$ of F is in A. There are only finitely many functions which extend $\Phi(F[n])$. Furthermore, there are only finitely many m such that f_m does not extend $F[n]$. Hence $\{\Phi(f_0), \Phi(f_1), \ldots\}$ is a finite set. Since Φ

is general recursive and all relevant inputs are binary functions, there is a constant c such that, for every binary string $\sigma \in \{0,1\}^c$, $\Phi(\sigma)$ is consistent with at most one function in $\{\Phi(f_0), \Phi(f_1), \ldots\}$. It is easy to see that there is an operator Φ_e with $\Phi_e(f_\ell) = \Phi(f_\ell)$ for all ℓ, as this operator only depends on the values of the input function below c.

Case 2. There is an n such that no extension of $F[n]$ is in A. There is an index k such that all f_m with $m > k$ extend $F[n]$. Then $\Phi(f_m) = f_m$ for all $m > k$. (Otherwise there would be a prefix $f_m[y]$ of f_m such that $\Phi(f_m[y])$ is inconsistent with $f_m[y]$, contradicting the choice of n.) For the $\ell \le k$ there is a prefix σ_ℓ of f_ℓ which is so large that no other function in $\{f_0, f_1, \ldots\}$ extends σ_ℓ. Then one chooses $f_{a_\ell} = \Phi(f_\ell)$ for all $\ell \le k$. Now there is a general recursive operator Φ_e mapping all extensions of σ_ℓ to f_{a_ℓ} for $\ell = 0, 1, \ldots, k$ and mapping all other functions to themselves. It is easy to see that Φ and Φ_e coincide on $\{f_0, f_1, \ldots\}$. Thus, Φ is covered by the given list of operators. □

Although not every recursively enumerable class is coverable, the next result shows that it is at least coverable relative to K'. This relativized concept uses the notion of a K'-recursive enumeration of recursive operators. Here an enumeration $\Phi_0, \Phi_1, \ldots$ is K'-recursive iff there is a K'-recursive function h and an acceptable numbering $\Gamma_0, \Gamma_1, \ldots$ of recursive operators with $\Phi_e = \Gamma_{h(e)}$ for all e.

Proposition 24. *If $\mathcal{F}$ is recursively enumerable then some K'-recursive enumeration of recursive operators covers $\mathcal{F}$.*

Proof. Let $f_0, f_1, \ldots$ be an enumeration of $\mathcal{F}$ and $\Gamma_0, \Gamma_1, \ldots$ be an acceptable numbering of all recursive operators. Define a function h inductively as follows: $h(e)$ is the least number e', greater than all $h(e'')$ with $e'' < e$, such that

$$\forall x, y\ [\Gamma_{e'}(f_x)(y) \text{ is defined}].$$

This predicate is a Π_2^0 predicate as it is universally quantified over the Σ_1^0 condition whether a certain computation halts. Thus, the predicate can be evaluated with the oracle K'. Furthermore, it just selects the first index e' after all indices $h(e'')$ with $e'' < e$ such that $\Gamma_{e'}$ is total on $f_0, f_1, \ldots$ and therefore all general recursive operators plus some others are covered by the enumeration $\Gamma_{h(0)}, \Gamma_{h(1)}, \ldots$. □

Proposition 25. *Let $\mathcal{F} = \{f_0, f_1, \ldots\}$ be recursively enumerable and $\Phi_0, \Phi_1, \ldots$ be any recursive enumeration weakly covering $\mathcal{F}$. The set*

$$E = \{e : \forall f \in \mathcal{F}\ [\Phi_e(f) \text{ is total and } \Phi_e(f) \in \mathcal{F}]\}$$

of all recursive operators which preserve $\mathcal{F}$ is a Π_3^0-set.

Proof. Given the enumerations $f_0, f_1, \ldots$ of functions and $\Phi_0, \Phi_1, \ldots$ of recursive operators, the set E is defined by the following Π_3^0-formula:

$$\begin{aligned} e \in E \quad \text{iff} \quad & \forall n \forall x \exists k\ [\Phi_e(f_n[k])(x) \text{ is defined] and} \\ & \forall n \exists m \forall k, x\ [\text{if } \Phi_e(f_n[k])(x) \text{ is defined then } \Phi_e(f_n[k])(x) = f_m(x)]. \end{aligned}$$

So this formula says that e is in E iff for every n the function $\Phi_e(f_n)$ is total and there is an index m such that $\Phi_e(f_n)$ is consistent with f_m. The totalness from the first condition and the consistency from the second imply equality. □

Note that $\Phi_e(f)$, with $e \in E$, is only required to be total if $f \in \mathcal{F}$, so some of the indices $e \in E$ might belong to operators which are not general recursive. As the problem whether Φ_e is general recursive is Π_1^1-complete, one cannot check for an operator being general recursive with a Π_3^0-condition. Nevertheless, whenever Φ_e is general recursive then $e \in E$ iff Φ_e is $\mathcal{F}$-preserving.

5. Periodic functions

In this section our interest is to consider a special case of $\mathcal{F}$. We restrict the class $\mathcal{F}$ to the class of eventually periodic functions which oscillate, from some point on, with the same period. The reason is that it is often useful to consider specific cases: for example, when considering mapping real numbers to real numbers, one may consider the special case when rational numbers are mapped to rational numbers. Indeed, the eventually periodic functions much resemble the rational numbers as, if they are considered as mappings from positions to digits, they are exactly the eventually periodic real numbers.

From now on, "eventually" will be dropped from "eventually periodic" for the sake of simplicity of the notation.

Definition 26. The class $\mathcal{F}_{per}$ is the union of all $\mathcal{F}_n$ with period n; that is, the union of the classes defined by the condition $f \in \mathcal{F}_n$ iff $\forall^\infty m\ [f(m+n) = f(m)]$.

The class $\mathcal{F}_{per}$ is strongly invertible. Furthermore, it is not coverable as it contains an infinite finitely learnable subclass, namely $\{0^e 1^\infty : e \in \mathbb{N}\}$ (see Proposition 21). Indeed, one can even code very difficult problems into any K'-recursive enumeration of recursive operators covering $\mathcal{F}_{per}$ and the following theorem shows that this class is not coverable.

Theorem 27. *Given any K'-recursive enumeration $\Phi_0, \Phi_1, \ldots$ of recursive operators covering $\mathcal{F}_{per}$, the set $P = \{e :$ Φ_e is $\mathcal{F}_{per}$-preserving$\}$ is not recursively enumerable relative to K'.*

Proof. In the following, let $F(e, x, s)$ be the first nonelement of $W_{e,s}$ which is greater than or equal to x. Now define Ψ_e to be the general recursive operator which maps every function f extending 0^x but not 0^{x+1} to the function

$$0^e 1 0^x 1 0^{F(e,x,0)} 1 0^{F(e,x,1)} 1 0^{F(e,x,2)} 1 \ldots$$

and 0^∞ to $0^e 1 0^\infty$. For every x the function $\Psi_e(0^x 1 0^\infty)$ is periodic iff there is a nonelement of W_e greater than or equal to x.

Assume now by way of contradiction that there is a K'-recursive enumeration $\Phi_0, \Phi_1, \ldots$ of recursive operators covering $\mathcal{F}_{per}$ such that the corresponding set P is recursively enumerable relative to K'. Then, given e, one can find, using oracle K', an x such that one of the following two conditions holds.

(1) $x \in P$ and for all σ and y, if $\Phi_x(\sigma)(y)$ and $\Psi_e(\sigma)(y)$ are both defined then they are equal;
(2) for all $y \geq x$, $y \in W_e$.

If W_e is coinfinite then the search terminates with an x satisfying the first condition since Ψ_e is $\mathcal{F}_{per}$-preserving and there is a general recursive operator Φ_x having the same behaviour on all periodic functions as Ψ_e. In particular, $\Psi_e(\sigma 1 0^\infty)$ and $\Phi_x(\sigma 1 0^\infty)$ must be the same functions and thus the required consistency condition holds. On the other hand, the search obviously cannot terminate according to (2).

If W_e is cofinite then Ψ_e maps some periodic function f (for example, $f = 0^{r+1} 1 0^\infty$, where $r = \max(\overline{W_e})$) to a nonperiodic one. If $x \in P$, then $\Phi_x(f)$ is periodic, and thus there is an n and a y with $\Phi_x(f[n])(y)$ and $\Psi_e(f[n])(y)$ both defined and different. So the search cannot terminate by condition (1), although it terminates by condition (2), with x being the least upper bound of the finitely many nonelements of W_e.

So one gets that $\{e : W_e$ is coinfinite$\}$ is Turing reducible to K', a contradiction to the well-known fact that this set is Π^0_3-complete [9]. □

The above proof produced the family of Ψ_e in a uniform manner, so in the case that $\Phi_0, \Phi_1, \ldots$ is an acceptable numbering, one has a recursive function h with $\Phi_{h(e)} = \Psi_e$. Thus, one can get Π^0_3-completeness in this case.

Corollary 28. *If $\Phi_0, \Phi_1, \ldots$ is an acceptable numbering of all recursive operators then the set $P = \{e : \Phi_e$ is $\mathcal{F}_{per}$-preserving$\}$ is Π^0_3-complete.*

If Ψ strongly inverts Φ, then Ψ produces a finite variant but not necessarily the correct output. One might ask whether this is necessary. Indeed there are only very few classes where one can avoid it. For example, if Ψ is permitted to be partial, then one can invert every general recursive operator on the constant functions by Ψ outputting, on input x^∞, the function y^∞ for the first y found such that $\Phi(y^\infty)(0) = x$. Somehow, if one wants general recursive operators Ψ with this property, one has to go to a sufficiently small subclass. In the case of $\mathcal{F}_{per}$, there are recursive operators Φ where every (even partial) Ψ inverting Φ makes finitely many errors.

Example 29. Let ψ be a partial recursive $\{0, 1\}$-valued function without recursive extension. Then every recursive operator Ψ inverting the following general recursive operator Φ makes errors on some inputs:

$$\Phi(f) = \begin{cases} 0^e 1 0^\infty & \text{if } (f \text{ extends } 0^e 10 \text{ or } 0^e 11) \text{ and } \psi(e) \text{ is undefined;} \\ 0^e 1 0^\infty & \text{if } f \text{ extends } 0^e 1 \psi(e) \text{ and } \psi(e) \text{ is defined;} \\ 0^e 1 0^s 1^\infty & \text{if } f \text{ extends } 0^e 1 \text{ but not } 0^e 1 \psi(e) \text{ and } \psi(e) \text{ halts after exactly } s \text{ steps;} \\ f & \text{otherwise.} \end{cases}$$

If some Ψ would strongly invert Φ without errors then the recursive function $e \mapsto \Psi(0^e 10^\infty)(e+1)$ would be a total extension of ψ in contradiction to its choice.

6. Other notions of inverting

It was already shown (see Theorem 5) that there is a single general recursive operator Φ such that one cannot invert Φ on the class of all recursive functions. As recursive operators preserve recursiveness, it is not very interesting to deal with arbitrary classes for negative results. We now turn our attention to the following question: For every recursive operator Φ and every recursively enumerable class $\mathcal{F}$, is there an operator Ψ which inverts or at least weakly inverts Φ? The next result shows that the technique of inverting by enumeration can be kept as long as the operator to be inverted is total on the whole family $\mathcal{F}$.

Theorem 30. *If $\mathcal{F}$ is recursively enumerable and Φ is $\mathcal{F}$-preserving, although not necessarily general recursive, then there is a general recursive operator Ψ which strongly inverts Φ.*

Proof. Let $f_0, f_1, \ldots$ be a recursively enumerable class and Φ a recursive operator such that $\Phi(f_n)$ is total for all n. Then define Ψ as follows: $\Psi(f)(x)$ is $f_n(x)$ for the least n such that $\Phi(f_n[x-n])$ is consistent with f. Ψ is general recursive as it terminates on all inputs to some $f_n(x)$ with $n \le x$ (as $n = x$ would qualify). Furthermore, if n is the first index with $\Phi(f_n) = f$, then, for all sufficiently large x, every expression $\Phi(f_m[x-m])$, with $m < n$, is inconsistent with f and thus $\Psi(\Phi(f))(x) = f_n(x)$. $\square$

This property is lost if one considers recursive operators which might be partial on functions from the class.

Example 31. Let $\mathcal{F} = \{0^e 10^\infty, 0^e 1^\infty : e \in \mathbb{N}\}$. Furthermore, let ξ^K be a partial K-recursive $\{0, 1\}$-valued function without a total K-recursive extension together with the recursive approximations $\xi_0, \xi_1, \ldots$ as defined in the proof of Theorem 8. One can assume, without loss of generality, that the approximation oscillates between 0 and 1 whenever $\xi^K(e)$ is undefined. Now one defines the following recursive operator Φ: $\Phi(0^e 1 a^\infty)$ is the union of all $0^e 10^s$ such that $\xi_s(e) = a$. So if $\xi^K(e)$ is undefined then both $0^e 10^\infty$ and $0^e 1^\infty$ are mapped to $0^e 10^\infty$; on the other hand if $\xi^K(e)\downarrow = a$, then $\Phi(0^e 1 b^\infty) = 0^e 10^\infty$ iff $b = a$. As a consequence, for any limit-recursive operator Ψ (approximated by $\Psi_0, \Psi_1, \ldots$), which weakly inverts Φ, $\lim_s \Psi_s(0^e 10^\infty)(e+1)$ would exist for all e and coincide with $\xi^K(e)$ whenever $\xi^K(e)$ is defined. Thus, the function $e \mapsto \lim_s \Psi_s(0^e 10^\infty)(e+1)$ would be a total K-recursive extension of ξ^K which by choice does not exist. Thus, no limit-recursive operator weakly inverts Φ.

Another topic is whether, given an enumeration $\Phi_0, \Phi_1, \ldots$ of recursive operators, one can find an operator Ψ which inverts every $\mathcal{F}$-consistent operator Φ_e on at least one function. In the case that all Φ_e are total on $\mathcal{F}$ and $\mathcal{F}$ contains at least one recursive function f, this can be easily achieved: for all functions g, one defines $\Psi(g) = f$. Then one uses that every $\mathcal{F}$-preserving Φ_e satisfies $\Phi_e(f) \in \mathcal{F}$ and, hence, f is the inverse of some function $g \in \mathcal{F}$. The next example shows that this is no longer possible if $\mathcal{F}$ consists of several recursive functions and operators may fail to map all functions in $\mathcal{F}$ to total ones.

Example 32. Let $\mathcal{R}$ be the class of all recursive functions. There is an enumeration $\Phi_0, \Phi_1, \ldots$ of recursive operators which map at least one recursive function to a total one such that no $\Psi = \lim_s \Psi_s$ weakly inverts the operator Φ_e on some total $f \in \Phi_e(\mathcal{R})$, given e and f as input.

Proof. To see this, one defines $\Phi_e(f)(x) = 0$ iff

- either $|W_{e,x}| \le f(0)$;
- or for all $y \le x$, $y \le |W_{e,f(y)}|$.

If these two conditions do not hold then $\Phi_e(f)(x)$ is undefined. Clearly 0^∞ is the only function in $\Phi_e(\mathcal{R})$. Now let F be the index-set of the finite sets. The functions f_e given as

$$f_e(x) = \min(\{s : (e \in F \Rightarrow |W_e| \le s) \wedge (e \notin F \Rightarrow x \le |W_{e,s}|)\})$$

are all recursive, since one needs only to know the cardinality of W_e in order to compute $f_e(x)$ for every x. It is easy to verify that $\Phi_e(f_e) = 0^\infty$ for all e.

But if there were a limit-recursive $\Psi = \lim_s \Psi_s$ which weakly inverts all Φ_e using the parameter e in the limit, then

$$e \in F \Leftrightarrow |W_e| \leq \lim_{s\to\infty} \Psi_s(e, 0^\infty)(0)$$

and $F \leq_T K$ in contradiction to the well-known fact that F is a Σ_2^0-complete set. □

While partial operators might not be invertible, one can easily get the below uniform variant of Proposition 16 (the proof is omitted). This proposition uses dense classes, where a class $\mathcal{F}$ is called *dense* if it contains an extension of every initial segment over $\mathbb{N}$. Note that, for a dense set $\mathcal{F}$ and a general recursive operator Φ, it holds that whenever the range of Φ contains at least k functions so does $\Phi(\mathcal{F})$.

Proposition 33. *Let $\Phi_0, \Phi_1, \ldots$ be a recursive enumeration of all recursive operators and let $\mathcal{F}$ be recursively enumerable and dense. Then there is a recursive enumeration $\Psi_0, \Psi_1, \ldots$ of recursive operators with the following properties.*

- *If Φ_e is general recursive and $\mathcal{F}$-preserving, then Ψ_e is general recursive and strongly inverts Φ_e.*
- *If Φ_e is general recursive and its range at most countable, then the cardinality of the functions $\Phi_e(f)$ such that Ψ_e strongly inverts Φ_e on $\Phi_e(f)$ is the same as the cardinality of the range of Φ_e.*

Proposition 33 depends on the fact that the index e of the operator is supplied. If this index is not known, then there is an enumeration $\Phi_0, \Phi_1, \ldots$ of $\mathcal{F}_{per}$-preserving general recursive operators, all having at least two functions in the range, such that no Ψ inverts every operator Φ_e on at least two functions.

Example 34. Let $\Phi_e(f) = 1^\infty$, if $f(0) = e$ and $\Phi_e(f) = 0^\infty$, otherwise. Given any Ψ, choose e such that $e \neq \Psi(1^\infty)(0)$. Then Ψ does not invert the operator Φ_e on the function 1^∞. Thus, Ψ inverts Φ_e on at most one function, although the range of each Φ_e contains two functions.

Theorem 35. *Let $\mathcal{F} = \{f_0, f_1, \ldots\}$ be a recursively enumerable class. Then there is a Ψ which inverts every general recursive operator Φ on infinitely many members of $\Phi(\mathcal{F})$, whenever Φ is $\mathcal{F}$-preserving and $\Phi(\mathcal{F})$ is infinite.*

Proof. The construction of Ψ needs several auxiliary ingredients. The overall goal is to construct sequences $i_0, i_1, \ldots$ and $j_0, j_1, \ldots$ of indices such that, for every general recursive operator Φ which maps $\mathcal{F}$ to infinitely many functions, there are infinitely many k such that $\Phi(f_{i_k}) = f_{j_k}$ and $\Psi(f_{j_k}) = f_{i_k}$.

Let $\Phi_0, \Phi_1, \ldots$ be an acceptable numbering of all recursive operators. Now one partitions the natural numbers in intervals I such that $|I| > \min(I)$ for each I in the partition. Let I_k denote the interval which contains k. Thus, each member of an interval is also an index of it and the indexing is not one-one. Furthermore, let $e_0, e_1, \ldots$ be a sequence of indices of operators such that

- for all k, k', if $I_k = I_{k'}$ then $e_k = e_{k'}$;
- for all e there are infinitely many k with $e_k = e$.

The mapping $k \mapsto i_k$ is a partial-recursive function such that, for any k, if Φ_{e_k} is total on $\mathcal{F}$ and $|\Phi_e(\mathcal{F})| \geq |I_k|$ then the following holds:

- for all $k' \in I_k$, $i_{k'}$ is defined;
- for all different $k', k'' \in I_k$, there is an x for which $\Phi_{e_k}(f_{i_{k'}})(x)$, $\Phi_{e_k}(f_{i_{k''}})(x)$ are defined and different.

Note that the above partial-recursive function can easily be implemented by a standard search and might also be defined for some e, where Φ_e is not total on $\mathcal{F}$.

The indices j_k are found as limits of the following approximation $j_{k,s}$: if i_k is not yet defined at stage s, then $j_{k,s} = 0$; otherwise $j_{k,s}$ is the least ℓ such that either $\ell = s$ or f_ℓ extends $\Phi_{e_k}(f_{i_k}[s])$. This approximation is recursive and the $j_{k,s}$ converge to the least ℓ for which f_ℓ extends $\Phi_e(f_{i_k})$, whenever such an ℓ exist. Note that $j_{k,s} \leq j_{k,s+1}$, for all s, and $\lim_s j_{k,s} = \infty$, if no f_ℓ extends $\Phi_e(f_{i_k})$.

The operator Ψ is given as the limit of Ψ_s, where $\Psi_s(g)(x)$ is computed by the following algorithm.

1. Let ℓ be the least number such that either $f_\ell[x+s] = g[x+s]$ or $\ell \geq x+s$.
2. Let k be the least number such that either $j_{k,x+s} = \ell$ or $k \geq x+s$.

3. If i_k is defined at step $x+s$, then $\Psi_s(g)(x) = f_{i_k}(x)$, else $\Psi_s(g)(x) = 0$.

It is easy to see that every Ψ_s is a general recursive operator. Furthermore, the algorithm is uniform in s, so one can compute the value $\Psi_s(g)(x)$ from the input s, x effectively.

Assume now that Φ_e is an $\mathcal{F}$-preserving general recursive operator such that $\Phi_e(\mathcal{F})$ is infinite. Then, for every k with $e_k = e$, the index i_k is defined as there are at least $|I_k|$ functions in $\mathcal{F}$ which are mapped to different images. Furthermore, as Φ_e is $\mathcal{F}$-preserving, for such i_k, the f_{i_k} are mapped to some f_{j_k} and the $j_{k,s}$ converge to j_k.

Now select any interval I such that $e_{k'} = e$, for all $k' \in I$. Note that the mapping $k' \mapsto j_{k'}$ is one-one on the domain I. Thus, there is an index $k \in I$ such that $j_k \neq j_{k'}$, for all $k' < \min(I)$. Fix this k and let $x+s$ be so large that the following holds:

– $f_\ell[x+s] \neq f_{j_k}[x+s]$ for all $\ell < j_k$;
– i_k is defined at stage $x+s$ and $j_{k,x+s} = j_k$;
– $j_{k',x+s} > j_k$, for all $k' < k$, where $j_{k',0}, j_{k',1}, \ldots$ converges to a number larger than j_k or to infinity.

Then one can say the following about the algorithm to compute $\Psi_s(f_{j_k})(x)$:

– the ℓ in the algorithm to compute $\Psi_s(f_{j_k})(x)$ is j_k;
– the parameter k from the algorithm has the same value as the k considered here;
– $\Psi_s(f_{j_k})(x) = f_{i_k}(x)$ as i_k is already defined at stage $x+s$.

Thus, every $\Psi_s(f_{j_k})$ is a finite variant of f_{i_k} and almost all $\Psi_s(f_{j_k})$ are equal to f_{i_k}. So Ψ inverts Φ_e on f_{j_k} to f_{i_k}. The function f_{j_k} selected in the interval I was not dealt with in smaller intervals $I_{k'}$ with $e_{k'} = e$. Thus, each such interval contributes a function on which Φ_e is inverted. Therefore Φ_e is correctly inverted on infinitely many functions from $\Phi_e(\mathcal{F})$. □

In the previous proof, Ψ_0 is a general recursive operator which strongly inverts every $\mathcal{F}$-preserving general recursive operator Φ_e, with $|\Phi_e(\mathcal{F})| = \infty$, on infinitely many functions from its range. This is not put into the formulation of the theorem, as in the case that the index e is unknown to the inverting operator, the implication "strongly inverts ⇒ inverts" is no longer clear.

The next result states that, although one can invert infinitely many functions, it can be impossible to invert uncountably many. Thus, only a tiny fraction of the image of the operator can be inverted to its origin.

Proposition 36. *There is a general recursive operator Φ such that the range of Φ is uncountable but every limit-recursive Ψ weakly inverts Φ on at most countably many of the functions in the range of Φ.*

Proof. For a function f, let $O_f = \{x : f(x) \text{ is odd}\}$. Now one defines Φ as follows:

$$\Phi(f)(x) = \begin{cases} 1 & \text{if } x \in O_f \text{ and, for all } e \leq x, \text{ either } W_{e,f(y)}^{O_f} \text{ has at least } y \text{ elements for all} \\ & y \in \{e, e+1, \ldots, x\} \text{ or } W_{e,x}^{O_f} \text{ has at most } f(e) \text{ elements;} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\Phi(f)$ has infinitely many 1s only if $\Phi(f)$ is the characteristic function of O_f. It is easy to see that the range of Φ is a subclass of $\{0, 1\}^\infty$.

On one hand, for every set O, there is a fast growing function f such that (the characteristic function of) O is $\Phi(f)$. Indeed, f can be any function with $f(x)$ being odd iff $x \in O$ and f growing so fast that, for all e and all $y \geq e$,

– if W_e^O is finite then $f(y) \geq |W_e^O|$;
– if W_e^O contains at least y elements so does $W_{e,f(y)}^O$.

Hence, the range of Φ is the full class $\{0, 1\}^\infty$. So the range is uncountable.

On the other hand, any limit-recursive Ψ can only invert Φ on characteristic function of finite sets. To see this, assume by way of contradiction that there are O, f, Ψ such that O is infinite, Ψ is limit recursive, $f = \Psi(O)$ and $\Phi(f) = O$. Note that f has to grow so fast that $f(e) \geq |W_e^O|$, whenever the latter cardinality is finite: otherwise $\Phi(f)$ has only finitely many 1s. Thus,

$$W_e^O \text{ is finite} \Leftrightarrow |W_e^O| \leq f(e).$$

Now one can use the oracle O' to do the following: compute $f(e)$ using $\Psi(O)$, check whether $|W_e^O| \leq f(e)$ and conclude that W_e^O is finite iff this test "$|W_e^O| \leq f(e)$?" turned out to be true. This would show that $\{e : W_e^O$ is finite$\} \leq_T O'$, a contradiction. As the class of $\{0, 1\}$-valued functions with only finitely many 1s is countable, every limit-recursive operator can weakly invert only countably many functions. □

7. Conclusion

In this paper we considered how and when general recursive operators can be inverted. The research was motivated by the fact that, in many situations in real life, one is interested in finding what caused a certain result. We also introduced the notion of coverability, which allows us to find and study simple representative enumerations of operators which satisfy some desired properties.

The main results of the present paper might be summarized as follows. The four presented notions of inversion, as well as the three notions of coverability, form a strict hierarchy. Furthermore, all of the given concepts are shown to contain non-trivial classes.

From a practical point of view, strong inversion is the most interesting type, since it allows us to get a finite variant of the original input uniformly from $\Phi(f)$. Getting the exact input is much harder, as shown at the end of Section 5 about periodic functions. It would be interesting to further explore partial inversion, that is, we might not be able to invert an operator completely, but on sufficiently many outputs. Another interesting topic might be the inversion in other special cases similar to periodicity.

Although we have separated the above notions and given – as we hope – interesting examples, there is more to learn about these concepts. One goal will be to find interesting sufficient and/or necessary conditions for classes to be invertible or coverable. Again from the practical side, the first candidates to look at would be strongly invertible and strongly coverable.

One may consider the generalization of inversion, where operators map $\mathcal{F}$ to $\mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ might be different. Then $(\mathcal{F}, \mathcal{G})$ would be invertible if, for every general recursive operator Φ mapping $\mathcal{F}$ to $\mathcal{G}$, there exists a limit-recursive operator Ψ which maps $\Phi(f)$ to a g such that $\Phi(f) = \Phi(g)$, for all $f \in \mathcal{F}$. For this paper, as our results mainly dealt with $\mathcal{G} = \mathcal{F}$, we decided to keep the simpler version of the definition for notational ease.

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