An inverse problem for a functionally graded elliptical plate with large deflection and slightly disturbed boundary

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Abstract

This paper deals with the inverse problem of a functionally graded material (FGM) elliptical plate with large deflection and disturbed boundary under uniform load. The properties of functionally graded material are assumed to vary continuously through the thickness of the plate, and obey a simple power law expression based on the volume fraction of the constituents. Based on the classical nonlinear von Karman plate theory, the governing equations of a thin plate with large deflection were derived. In order to solve this non-classical problem, a perturbation technique was employed on displacement terms in conjunction with Taylor series expansion of the disturbed boundary conditions. The displacements of in-plane and transverse are obtained in a non-dimensional series expansion form with respect to center deflection of the plate. The approximate solutions of displacements are solved for the first three terms, and the corresponding internal stresses can also be obtained.

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1. Introduction

The influence of structure elements having variable material constituents or stiffness has been an interesting problem and studied extensively in engineering applications. The purpose of varying structural properties is that it can make the structure itself become more efficient and fulfill certain special working conditions. The great advantages of being able to withstand severe temperature gradients and the ability
to maintain structural integrity are the most important reasons for choosing FGM in structural elements, such as spacecraft and nuclear plants. For the structure problems of varying material constituents or stiffness, different methods were proposed and both closed form and approximate solutions were obtained. The inverse problem technique with properly chosen perturbation parameters seems to be one of the best methods that have been widely used for this type of problems.

In a recent paper by Huang and Shen (2004), a theoretical nonlinear vibration and dynamic response of functionally graded plates in thermal environments is presented. Approximate solutions are found by applying the Galerkin procedure and it is found that the volume fraction distribution have a significant effect on dynamic response of the plates. Ma and Wang (2003) have also presented a nonlinear bending and post-buckling of a functionally graded circular plate under mechanical and thermal loadings. Their numerical solution technique involved using the shooting method and nonlinear bending and buckling behavior is discussed in detail. Elishakoff (2001) studied the inverse buckling problem for inhomogeneous columns and derived expressions for the stiffness of the column based on the differential equation that described the buckling of the column under axial load and certain boundary conditions. With a suitable choice of parameters, it was found that the buckling load can be related to a single stiffness coefficient. Romanov et al. (2003) studied an inverse problem for a layered elastic plate where it was assumed that a point pulse force is applied at a boundary of the plate and the displacement vector is measured at the boundary. The authors found that the elasticity modules and density of layers as well as their thickness can be determined by the displacement vector in this inverse problem. Gladwell (1999) studied inverse finite element vibration problems dealing with the reconstruction of a consistent FEM model of an in-line system of 2-dof elements, fixed at one end and free at the other. The author illustrated how to construct an infinite family of such models so that each has a specified undamped frequency response at the free end, and how to construct a system with a damper at the free end so that the system has specified eigenvalues. Exact power series solutions for the axisymmetric vibrations of circular and annular membranes with continuously varying density in the general case have been presented by Willatzen (2002). In their study, a quasi-analytical approach is presented so as to find eigenfrequencies and eigensolutions in the general case where the density can be written as an infinite power series expansion in the radial co-ordinate. A general quasi-analytical model based on the Frobenius power series expansion method is described so as to handle the vibrations of circular and annular membranes with continuously varying density. Natural frequencies are finally computed for three examples of varying membrane density. Borukhov and Vabishchevich (2000) studied the numerical solution of the inverse problem of reconstructing a distributed right-hand side of a parabolic equation. They presented a numerical algorithm based on the transition method to the problem of the loaded parabolic equation. For solving this non-classical problem, they employed a special numerical method similar to the bordering method. Such a general approach can be used both for parabolic equations that are one-dimensional in space and for multidimensional ones. Hassan (2004) has presented a study involving the free transverse vibration of elliptical plates of variable thickness with half of the boundary clamped and the rest free. In this study, first four frequencies were computed by using the Rayleigh–Ritz method. Laura and Rossit (1999) studied the case of an orthotropic material and obtained an exact analytical solution for the thermal bending of clamped, anisotropic, elliptic plates. A solution for the thermomechanical deformations of an isotropic linear thermoelastic functionally graded elliptic plate rigidly clamped at the edges was obtained by Cheng and Batra (2000). A power-law type function was assigned for the through-thickness variation of the volume fraction of the ceramic phase in a metal-ceramic plate. The result of the calculations for the functionally graded plate and classical shear deformation plate theories were found to be slightly different. Muhammad and Singh (2004) used an energy method with displacement fields defined by a shape function of high order polynomials for the bending of plates with various shapes. Plates with eccentric square and circular openings were also analyzed. The transverse vibration of non-homogeneous elliptic and circular plates using the two-dimensional boundary characteristic orthogonal polynomials in the Rayleigh–Ritz
method was studied by Chakraverty and Petyt (1997). The non-homogeneity of the plate is characterized by taking a linear variation of the Young’s modulus and a parabolic variation of the density of the material. Shen (in press) studied the postbuckling of the axially loaded FGM hybrid cylindrical shells in thermal environments for a functionally graded cylindrical shell with piezoelectric actuators subjected to axial compression combined with electric loads in thermal environments. Heat conduction and temperature-dependent material properties were both taken into account. The governing equations were based on a higher order shear deformation theory with a von Karman–Donnel-type of kinematic nonlinearity. A singular perturbation technique was employed to determine the buckling load and postbuckling equilibrium paths. Shen (2005) also studied postbuckling of FGM plates with piezoelectric actuators under thermo-electro-mechanical loadings. A postbuckling analysis was presented for a simply supported, shear deformable functionally graded plate with piezoelectric actuators subjected to the combined action of mechanical, electrical and thermal loads. The temperature field considered was assumed to be uniform over the plate surface and through the plate thickness and the electric field considered only had non-zero-valued component. The governing equations were based on a high order shear deformation plate theory that included thermo-piezoelectric effects. A two-step perturbation technique was employed to determine buckling loads and postbuckling equilibrium paths. Relationships between axisymmetric bending and buckling solutions of FGM circular plates based on third-order plate theory and classical plate theory were presented by Ma and Wang (2004). The third-order shear deformation plate theory (TPT) was employed to solve the axisymmetric bending and buckling problems of functionally graded circular plates. Relationships between the TPT solutions of axisymmetric bending and buckling of functionally graded circular plates and those of isotropic circular plates based on the classical plate theory were presented, from which one can easily obtain the TPT solutions for the axisymmetric bending and buckling of FGM plates.

This study is devoted to obtaining the analytical solutions of functionally graded elliptic plate under uniform load, which are posed as inverse problems. Based on the classical nonlinear von Karman plate theory, the governing equation is derived in a non-dimensional form. By choosing the center deflection of the plate as a perturbation parameter, the governing equation can be solved and the solutions can be expressed as power series that consist of shape functions and perturbation parameters. Since the center deflection can be measured in the inverse problem, the complicated governing equation and computation algorithm can then be simplified. The resulting solutions are important in solving the functionally graded material problem of elliptical plates with respect to different boundary conditions.

2. Formulation

We consider a functionally graded plate, which is made from a mixture of ceramics and metals. It is assumed that the composition properties of FGM vary through the thickness of the plate and the Young’s modulus $E(z)$ can be expressed as

$$E(z) = (E_m - E_c)V_m + E_c$$

(1)

where the subscripts ‘m’ and ‘c’ denote the metallic and ceramic constituents, respectively and $V_m$ denotes the volume fraction of metal and follows a simple power law as

$$V_m = \left(\frac{h + 2z}{2h}\right)^n$$

(2)

where $z$ is the thickness coordinate ($-h/2 \leq z \leq h/2$), and $n$ is a material constant. According to this power law type, the top surface ($z = -h/2$) of the FGM plate is pure ceramics, the bottom surface ($z = h/2$) is pure metal, and the different values of $n$ give the different volume fractions of metal.
Based on the classical nonlinear von Karman plate theory, the equilibrium equations of a thin plate subjected to a uniformly distributed transverse load \( q \) are as follows:

\[
\begin{align*}
N_{xx} + N_{xy,y} &= 0 \quad (3a) \\
N_{yy} + N_{yx,x} &= 0 \quad (3b) \\
M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + N_{x,w,xx} + 2N_{y,w,xy} + N_{y,w,yy} + q &= 0 \quad (3c)
\end{align*}
\]

where the comma denotes differentiation operation and \( w \) is the displacement in the \( z \) direction. The in-plane force and moment components, \( N \) and \( M \), are represented by

\[
\begin{align*}
(N_x, N_y, N_{xy}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) \, dz \quad (4a) \\
(M_x, M_y, M_{xy}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) \, z \, dz \quad (4b)
\end{align*}
\]

The constitutive relations for the FGM plate are given by

\[
\begin{align*}
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} &= \frac{E(z)}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \end{bmatrix} + z \begin{bmatrix} k_x \\ k_y \end{bmatrix} \quad (5a) \\
\tau_{xy} &= G(z) \varepsilon_{xy} \quad (5b)
\end{align*}
\]

where \( G(z) \) is the shear modulus of elasticity. The Poisson’s ratio in Eq. (5a) is assumed to be a constant. In most practical applications, the thickness of a plate is small in comparison with its smallest lateral dimension, and hence Kirchhoff’s hypothesis can be assumed to be valid. Under this assumption, the in-plane displacements \( u, v \) and the transverse displacement \( w \) at an arbitrary point of the plate in \( x, y \) and \( z \) directions may be approximated by

\[
\begin{align*}
u(x,y,z) &= u^0(x,y) - zw_x^0 \quad (6a) \\
v(x,y,z) &= v^0(x,y) - zw_y^0 \quad (6b) \\
w(x,y,z) &= w^0(x,y) \quad (6c)
\end{align*}
\]

where \( u^0, v^0, \) and \( w^0 \) are the values of \( u, v, \) and \( w \) at the middle plane of the uniform plate. Based on continuum mechanics theory (Malvern, 1969), the nonlinear constitutive relations are as follows:

\[
\begin{align*}
\varepsilon_x^0 &= u_x^0 + \frac{1}{2} w_x^2 \quad (7a) \\
\varepsilon_y^0 &= u_y^0 + \frac{1}{2} w_y^2 \quad (7b) \\
\varepsilon_{xy}^0 &= u_{xy}^0 + v_x^0 + w_x w_y \quad (7c)
\end{align*}
\]

From Eqs. (3)–(5), (6c) and (7), one can obtain the following equations in terms of the displacements:

\[
\begin{align*}
\begin{bmatrix} N_x \\ N_y \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} u_x^0 + \frac{1}{2} w_x^2 \\ u_y^0 + \frac{1}{2} w_y^2 \end{bmatrix} - \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} w_{xx} \\ w_{yy} \end{bmatrix} \quad (8a) \\
N_{xy} &= \frac{(1-\nu)}{2} A_{11} \left[ u_y^0 + v_x^0 + w_x w_y \right] - (1-\nu)B_{11} w_{xy} \quad (8b)
\end{align*}
\]
\[
\begin{align*}
\begin{pmatrix} M_x \\ M_y \\ M_{xy} \end{pmatrix} &= \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_0^x \\ v_0^y \\ w_0^z \end{bmatrix} - \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{11} & 0 \\ 0 & 0 & 2D_{66} \end{bmatrix} \begin{bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B_{66} & B_{66} & B_{66} \end{bmatrix} \begin{bmatrix} u_0^x \\ v_0^y \\ w_{x}w_y \end{bmatrix}
\end{align*}
\]

\[ A_{ij}, B_{ij} \text{ and } D_{ij} \text{ are the stiffness coefficients of the plate and can be evaluated by} \]
\[ \begin{align*}
(A_{ij}, B_{ij}, D_{ij}) &= \int_{h/2}^{h/2} Q_{ij}(1, z, z^2) \, dz \\
Q_{11} &= Q_{22} = Q_{33} = \frac{E(z)}{1 - v^2}, \quad Q_{12} = Q_{21} = vQ_{11}, Q_{66} = G(z)
\end{align*} \]

Combining Eqs. (8) and (3), one obtains the following governing equations:

\[ \begin{align*}
A_{11}u_0^{x,xx} + A_{66}u_0^{x,yy} + (A_{12} + A_{66})v_0^{y,xy} - B_{11}(w_{xxx} + w_{xxy}) \\
= -w_y(A_{11}w_{xx} + A_{66}w_{xy}) - (A_{12} + A_{66})w_yw_{xy} \\
(A_{12} + A_{66})u_0^{x,xy} + A_{66}v_0^{x,xx} + A_{11}v_0^{y,yy} - B_{11}(w_{xxy} + w_{yyy}) \\
= -(A_{12} + A_{66})w_xw_{xy} - w_y(A_{66}w_{xx} + A_{11}w_{yy}) \\
B_{11}(u_0^{x,xxy} + u_0^{x,yyy} + v_0^{y,xyy} + v_0^{y,yy} - D_{11}(w_{xxx} + 2w_{xxy} + w_{yyy}) \\
= -q \left( u_0^x + \frac{1}{2}w_y^2 \right) (A_{11}w_{xx} + A_{12}w_{yy}) - \left( v_0^y + \frac{1}{2}w_x^2 \right) (A_{12}w_{xx} + A_{11}w_{xy}) \\
- 2A_{66}w_y(u_0^y + v_0^x + w_xw_y) - B_{11}[w_x(w_{xxx} + w_{xxy}) + w_y(w_{xxy} + w_{yyy})] - 2(B_{12} - B_{66})(w_{xxy} - w_{xx}w_{xy})
\end{align*} \]

\[ (10c) \]

### 3. Inverse problem and perturbation technique

Based on the perturbation technique, approximate solutions are presented here for the large deflection of a rigidly clamped plate under uniform transverse load. The non-dimensional parameters are introduced as follows:

\[ \begin{align*}
\lambda &= \frac{a_0}{b_0}, \quad \xi = \frac{x}{a_0}, \quad \eta = \frac{y}{b_0}, \quad Q = \frac{1}{A_{11}} 24\sqrt{3}a_0q \\
U &= \frac{12u_0^x}{a_0}, \quad V = \frac{12v_0^y}{a_0}, \quad W = 2\sqrt{3} \frac{w}{a_0} \\
d_1 &= \frac{A_{66}}{A_{11}}, \quad d_2 = \frac{A_{12} + A_{66}}{A_{11}}, \quad d_3 = \frac{2\sqrt{3}}{a_0} \frac{B_{11}}{A_{11}}, \quad d_4 = \frac{D_{11} 12}{A_{11}} \frac{1}{2}, \quad d_5 = \frac{A_{12}}{A_{11}} \\
d_6 &= \frac{B_{12} - B_{66}}{A_{11}} \frac{\sqrt{3}}{a_0}, \quad d_7 = \frac{2\sqrt{3}}{a_0} \frac{B_{12}}{A_{11}}, \quad d_8 = \frac{2\sqrt{3}}{a_0} \frac{B_{66}}{A_{11}}
\end{align*} \]

\[ (11) \]
Since the elliptical plate under transverse load is rigidly clamped along its edges, the appropriate boundary conditions satisfy $\frac{\partial W}{\partial n} = 0$ and $W = U = V = 0$, where $n$ is the unit normal vector along its edge. This leads to the following equations:

$$
\frac{\partial W}{\partial n} = \nabla W \cdot \vec{n} = \left( \frac{\partial W}{\partial \xi} + \frac{\partial W}{\partial \eta} \right) \cdot \frac{1}{\sqrt{\xi^2 + \eta^2}} (\xi \hat{i} + \eta \hat{j}) = \frac{1}{\sqrt{\xi^2 + \eta^2}} \left( \xi \frac{\partial W}{\partial \xi} + \eta \frac{\partial W}{\partial \eta} \right) = 0 \tag{13a}
$$

and $W = 0$ along the boundary $s$, $\frac{\partial W}{\partial s} = 0$,

$$
\frac{\partial W}{\partial s} = \nabla W \cdot \vec{s} = \left( \frac{\partial W}{\partial \xi} + \frac{\partial W}{\partial \eta} \right) \cdot \frac{1}{\sqrt{\xi^2 + \eta^2}} (\eta \hat{i} - \xi \hat{j}) = \frac{1}{\sqrt{\xi^2 + \eta^2}} \left( \eta \frac{\partial W}{\partial \xi} - \xi \frac{\partial W}{\partial \eta} \right) = 0 \tag{13b}
$$

From Eqs. (13a) and (13b), the boundary condition can be obtained in the following non-dimensional form:

$$
U = V = W = W_\xi = W_\eta = 0 \quad \text{along} \quad \xi^2 + \eta^2 = (1 + \delta)^2 \tag{13c}
$$

where $\delta$ is a small dimensionless parameter.

As a first step for obtaining a uniform first-order expansion, the following straightforward expansion can be made:

$$
U = U_0(\xi, \eta) + \delta \cdot U_1(\xi, \eta) \tag{14a}
$$

$$
V = V_0(\xi, \eta) + \delta \cdot V_1(\xi, \eta) \tag{14b}
$$

$$
W = W_0(\xi, \eta) + \delta \cdot W_1(\xi, \eta) \tag{14c}
$$

$$
Q = Q_0(\xi, \eta) + \delta \cdot Q_1(\xi, \eta) \tag{14d}
$$

With a Taylor series expansion of displacement terms that satisfy the disturbed boundary conditions with respect to the parameter $\delta$, and combining Eqs. (12a)–(12c), (13a)–(13c) and (14a)–(14d), two sets of partial differential equations with respect to orders $\delta^0$ and $\delta^1$ can then be obtained.

It is difficult to obtain an exact solution of the differential equations of order $\delta^0$ that satisfy all boundary conditions. Therefore, approximate solutions are thus formulated by using the perturbation technique. If
we let non-dimensional center deflection $W(0,0)$ be denoted as $W_c$, then the parameters $Q_0, U_0, V_0$ and $W_0$ can be developed into four ascending perturbation series of $W_c$. For the plate having a twofold symmetry, these parameters may be expressed as (detail in Appendix A)

\begin{align}
Q_0 &= \sum_{n=1,3,...}^{\infty} q_n W^n_c, \quad W_0 = \sum_{n=1,3,...}^{\infty} w_n(\xi, \eta) W^n_c \\
U_0 &= \sum_{n=2,4,...}^{\infty} u_n(\xi, \eta) W^n_c, \quad V_0 = \sum_{n=2,4,...}^{\infty} v_n(\xi, \eta) W^n_c
\end{align}

where $Q_0$ and $W_0$ are direction-dependent, $U_0$ and $V_0$ are independent of the sign of $Q_0$, and the definition of $W_c$ requires that

$$w_1(0,0) = 1, \quad w_3(0,0) = w_5(0,0) = \cdots = 0$$

By substituting Eqs. (15a) and (15b) into the differential equations of order $\delta^0$ and equating the terms that have the same powers of $W_c$, one can obtain a series of differential equations with the corresponding boundary conditions. In the first approximation, if the coefficients of the first power of $W_c$ are set equal to zero, one can obtain the following equation corresponding to small deflection as

$$w_{1,\xi\eta\eta} + 2\lambda^2 w_{1,\xi\eta\eta} + \lambda^4 w_{1,\eta\eta\eta} = q_1$$

with the boundary conditions

$$w_1 = w_{1,\xi} = w_{1,\eta} = 0 \quad \text{along } \xi^2 + \eta^2 = 1$$

By setting the coefficients of $W_c^2$ equal to zero, the following set of simultaneous equations for $u_2$ and $v_2$ are also obtained:

\begin{align}
u_{2,\xi\xi} + d_1 \lambda^2 u_{2,\eta\eta} + d_2 \lambda v_{2,\xi\eta} &= -w_{1,\xi}(w_{1,\xi\xi} + d_1 \lambda^2 w_{1,\eta\eta} - d_2 \lambda^2 w_{1,\xi\eta}) \\
\lambda^2 v_{2,\eta\eta} + d_1 v_{2,\xi\xi} + d_2 \lambda u_{2,\xi\eta} &= -d_2 \lambda w_{1,\xi}(w_{1,\xi\eta} + d_1 w_{1,\xi\eta})
\end{align}

The corresponding boundary conditions are given as

$$u_2 = v_2 = 0 \quad \text{along } \xi^2 + \eta^2 = 1$$

Applying the same procedure for the coefficients of $W_c^3$, the following differential equation for $w_3$ is obtained:

\begin{align*}
&-d_4 (w_{3,\xi\xi\xi} + 2\lambda^2 w_{3,\xi\eta\eta} + \lambda^4 w_{3,\eta\eta\eta}) \\
&= -g_3 - \left( u_{2,\xi} w_{1,\xi\xi} + d_s \lambda^2 u_{2,\eta\eta} w_{1,\xi\eta} + \frac{1}{2} \lambda^2 w_{1,\xi\eta} w_{1,\xi\eta} + \frac{1}{2} d_s \lambda^2 w_{1,\xi\eta} w_{1,\xi\eta} \right) \\
&\quad - d_s \left( \lambda v_{2,\eta} w_{1,\xi\eta} + \frac{1}{2} \lambda^2 w_{1,\xi\eta} w_{1,\xi\eta} \right) - \lambda^3 v_{2,\eta} w_{1,\eta\eta} - \frac{1}{2} \lambda^4 w_{1,\eta\eta} w_{1,\eta\eta} - 2d_4 \lambda w_{1,\eta\eta} (\lambda u_{2,\eta} + v_{2,\eta} + \lambda w_{1,\eta\eta})
\end{align*}

with the boundary conditions

$$w_3 = w_{3,\xi} = w_{3,\eta} = 0 \quad \text{along } \xi^2 + \eta^2 = 1$$

If necessary, the fourth or higher-order approximations can also be obtained in the same manner.
The approximate solutions of the differential equations of order $\delta^0$ are sought in the form of polynomials. An approximate solution of Eq. (17) is assumed to be

$$w_1 = (1 - \xi^2 - \eta^2)^2(1 + A_1 \xi^2 + A_2 \eta^2)$$

(23)

The above equation satisfies the boundary conditions and symmetry of the plate.

Introducing Eq. (23) into Eq. (17) and equating the coefficients of $\xi^2$, $\eta^2$ and constant term to $q_1$ yields three linear algebraic equations which can be solved for the three coefficients $A_i$ and $q_1$. The Poisson’s ratio, $v$, material constant $n$ and the ratio of the thickness and the half-length are taken to be $\frac{1}{3}$, 3 and $\frac{1}{2}$, respectively. To satisfy all the boundary conditions, Eqs. (20), and the approximate solutions of Eq. (19) are assumed to be in the following form:

$$u_2 = \xi(1 - \xi^2 - \eta^2)(B_1 + B_2 \xi^2 + B_3 \eta^2)$$

(24)

$$v_2 = \eta(1 - \xi^2 - \eta^2)(C_1 + C_2 \xi^2 + C_3 \eta^2)$$

(25)

By substituting Eqs. (24) and (25) into Eqs. (19a) and (19b), three algebraic equations are generated by equating the same powers of $\xi$ and $\eta$. The algebraic equations are solved simultaneously for $B_i$ and $C_i$.

For a third-order approximate solution, $w_3$ can be chosen as

$$w_3 = (1 - \xi^2 - \eta^2)^3(H_1 \xi^2 + H_2 \eta^2)$$

(26)

The procedures for the determination of $H_i$ and $q_3$ are the same as the first approximation. Higher order approximations can be obtained successively in a similar way, but only the first three approximations are taken to be the solution of the present boundary-value problem. The solutions of Eqs. (15a) and (15b) based on these approximations are as follows:

$$Q_0 = q_1 W_c + q_3 W_c^3, \quad W_0 = w_1(\xi, \eta) W_c + w_3(\xi, \eta) W_c^3$$

(27a)

$$U_0 = u_2(\xi, \eta) W_c^2, \quad V_0 = v_2(\xi, \eta) W_c^2$$

(27b)

Similarly, for solving the differential equations of order $\delta^1$, we can also assume

$$Q_1 = \sum_{n=1,3,\ldots}^{\infty} \tilde{q}_n W_c^n, \quad W_1 = \sum_{n=1,3,\ldots}^{\infty} \tilde{w}_n(\xi, \eta) W_c^n$$

(28a)

$$U_1 = \sum_{n=2,4,\ldots}^{\infty} \tilde{u}_n(\xi, \eta) W_c^n, \quad V_1 = \sum_{n=2,4,\ldots}^{\infty} \tilde{v}_n(\xi, \eta) W_c^n$$

(28b)

$$\tilde{w}_1 = (1 - \xi^2 - \eta^2)^3(1 + \tilde{A}_1 \xi^2 + \tilde{A}_2 \eta^2)$$

(29)

$$u_2 = \xi(1 - \xi^2 - \eta^2)(\tilde{B}_1 + \tilde{B}_2 \xi^2 + \tilde{B}_3 \eta^2)$$

(30)

$$v_2 = \eta(1 - \xi^2 - \eta^2)(\tilde{C}_1 + \tilde{C}_2 \xi^2 + \tilde{C}_3 \eta^2)$$

(31)

$$\tilde{w}_3 = (1 - \xi^2 - \eta^2)^3(\tilde{H}_1 \xi^2 + \tilde{H}_2 \eta^2)$$

(32)

Applying the solution procedure of order $\delta^0$ to the problem of order $\delta^1$, the coefficients $\tilde{A}_i$, $\tilde{B}_i$, $\tilde{D}_i$, $\tilde{H}_i$ and $\tilde{q}_i$ can be obtained.
4. Results and discussion

By considering the plate deformation as an inverse problem, if the measured non-dimensional center deflection is assumed to be 0.05 and $\delta = 0.015 \cos(8\theta)$, the transverse displacements for three different cases of $\tilde{\lambda}$ are depicted in Figs. 1–3.

From Eqs. (4), (5), (8) and (11), the non-dimensional form of average membrane and extreme bending stress can be written as

\[
\begin{align*}
\left(\sigma_{m,x}^0, \sigma_{m,y}^0, \sigma_{m,xy}^0\right) &= \frac{A_{11}}{12h} \left[ \left(\phi_{m,x}, \phi_{m,y}, \phi_{m,xy}\right) + \delta \cdot \left(\gamma_{m,x}, \gamma_{m,y}, \gamma_{m,xy}\right) \right] \quad (33a) \\
\left(\sigma_{b,x}^0, \sigma_{b,y}^0, \sigma_{b,xy}^0\right) &= \frac{A_{11}}{12h} \left[ \left(\phi_{b,x}, \phi_{b,y}, \phi_{b,xy}\right) + \delta \cdot \left(\gamma_{b,x}, \gamma_{b,y}, \gamma_{b,xy}\right) \right] \quad (33b)
\end{align*}
\]

Fig. 1. Transverse displacement, $W(\xi, \eta)$ with $\tilde{\lambda} = 1$.

Fig. 2. Transverse displacement, $W(\xi, \eta)$ with $\tilde{\lambda} = \frac{1}{2}$. 
where $\phi$ and $\bar{\phi}$ are listed in Appendix B and

$$
\alpha = \frac{1}{6} \frac{a_0 h c}{I_{FGM}}
$$

(33c)

where $c$ is the distance between the bottom surface and the neutral axis, and $I_{FGM}$ is the moment of inertia with respect to the neutral axis.

Here membrane stress and bending stress are compared at the origin of the plate for different $\lambda$ and center deflection $W_0$, and the results are listed in Tables C.1–C.3 in Appendix C. It is observed that, if $W_0$ is given, $\frac{\phi^m_{\xi}}{\phi^b_{\xi}}$ decreases as $\lambda$ decreases for all the cases; meanwhile, $\frac{\phi^m_{\eta}}{\phi^b_{\eta}}$ increases as $\lambda$ decreases for all the cases. This is reasonable since membrane is stretched more in a shorter distance along $\eta$ direction, for the same center deflection, and it gives a higher membrane stress. For a square plate with $\lambda = 1$, both $\frac{\phi^m_{\xi}}{\phi^b_{\xi}}$ and $\frac{\phi^m_{\eta}}{\phi^b_{\eta}}$ decrease as $W_0$ increases. When $\lambda = 2/3$, these two ratios decrease faster but $\frac{\phi^m_{\xi}}{\phi^b_{\xi}}$ decreases even faster than $\frac{\phi^m_{\eta}}{\phi^b_{\eta}}$ as $W_0$ increases. The same trend is observed when $\lambda = 1/3$. It is interesting that membrane stresses along $\xi$ and $\eta$ directions change significantly for decreasing $\lambda$.

5. Conclusions

In this article, the FGM problem with large deflection and disturbed boundary is solved by using the inverse and perturbation technique instead of the methods that have been commonly used, such as the Ritz method, the Galerkin method or the finite element method. The non-dimensional center deflection of the plate is selected as a perturbation parameter. With this small parameter, the displacement is expressed in a series form that converges. The governing equation can be decoupled into three algebraic equations. This simplifies the solution of the FGM problem and results in significant time saving when employing the computational procedures. The techniques established here can be used for further research on the inverse problem of the post-buckling behavior, nonlinear flexural vibration, stress resultants for transverse dynamic loading and the thermal stress of structure elements of functionally graded material.
Appendix A

Assume

\[ Q_0 = \sum_{n=1,3,\ldots}^\infty a_n W_c^n = a_1 W_c + a_2 W_c^3 + a_3 W_c^5 + a_4 W_c^7 + a_6 W_c^9 + \cdots \]

\[ = (a_1 W_c + a_2 W_c^3) + (a_3 W_c^5 + a_4 W_c^7) + (a_5 W_c^9 + a_6 W_c^{11}) + \cdots \]

\[ = (a_1 + a_2 W_c) W_c + (a_3 + a_4 W_c) W_c^3 + (a_5 + a_6 W_c) W_c^5 + \cdots \]

and let

\[ q_1 = a_1 + a_2 W_c \]

\[ q_3 = a_3 + a_4 W_c \]

\[ q_5 = a_5 + a_6 W_c \]

then

\[ Q_0 = q_1 W_c + q_3 W_c^3 + q_5 W_c^5 + \cdots \]

\[ = \sum_{n=1,3,\ldots}^\infty q_n W_c^n \]

In the same manner, the following relations can be obtained:

\[ W_0 = \sum_{n=1,3,\ldots}^\infty w_n(\xi, \eta) W_c^n \]

\[ U_0 = \sum_{n=2,4,\ldots}^\infty u_n(\xi, \eta) W_c^n \]

\[ V_0 = \sum_{n=2,4,\ldots}^\infty v_n(\xi, \eta) W_c^n \]

Appendix B

In Eq. (33)

\[ \phi^m_\xi = U_{,\xi} + d_5 \lambda V_{,\eta} - d_3 W_{,\xi\xi} - d_7 \lambda^2 W_{,\eta\eta} + \frac{1}{2} \left( W_{,\xi}^2 + d_5 \lambda^2 W_{,\eta}^2 \right) \]

\[ \phi^m_\eta = d_5 U_{,\xi} + \lambda V_{,\eta} - d_7 W_{,\xi\xi} - d_3 \lambda^2 W_{,\eta\eta} + \frac{1}{2} \left( d_5 W_{,\xi}^2 + \lambda^2 W_{,\eta}^2 \right) \]

\[ \phi^m_{,\eta\eta} = d_1 (\lambda U_{,\eta} + V_{,\xi}) - 2d_8 \lambda W_{,\xi\eta} + d_1 \lambda W_{,\xi} W_{,\eta} \]

\[ \phi^b_\xi = \sqrt{3}d_3 \left( U_{,\xi} + \frac{1}{2} W_{,\xi}^2 \right) - \sqrt{3}d_4 W_{,\xi\xi} + \sqrt{3}v d_3 \left( \lambda V_{,\eta} + \frac{1}{2} \lambda^2 W_{,\eta}^2 \right) - \sqrt{3}v \lambda^2 d_4 W_{,\eta\eta} \]

\[ \phi^b_\eta = \sqrt{3}v d_3 \left( U_{,\xi} + \frac{1}{2} W_{,\xi}^2 \right) - \sqrt{3}v d_4 W_{,\xi\xi} + \sqrt{3}d_3 \left( \lambda V_{,\eta} + \frac{1}{2} \lambda^2 W_{,\eta}^2 \right) - \sqrt{3}v \lambda^2 d_4 W_{,\eta\eta} \]
\[ \phi_{\xi \eta}^{b} = \frac{1 - v}{2} \sqrt{3} d_3 (\lambda U_{1,\eta} + V_{1,\xi} + \lambda W_{1,\xi} W_{1,\eta}) - (1 - v) \lambda d_4 W_{1,\xi \eta} \]

\[ \bar{\phi}_{\xi}^{m} = U_{1,\xi} + d_3 \lambda V_{1,\eta} - d_3 W_{1,\xi} - d_7 \lambda^2 W_{1,\eta} + \frac{1}{2} \left( W_{1,\xi}^2 + d_3 \lambda^2 W_{1,\eta}^2 \right) \]

\[ \bar{\phi}_{\eta}^{m} = d_5 U_{1,\eta} + \lambda V_{1,\eta} - d_7 W_{1,\xi} = d_3 \lambda^2 W_{1,\eta} + \frac{1}{2} \left( d_3 W_{1,\xi}^2 + \lambda^2 W_{1,\eta}^2 \right) \]

\[ \bar{\phi}_{\xi \eta}^{m} = d_1 (\lambda U_{1,\eta} + V_{1,\xi}) - 2 d_6 \lambda W_{1,\xi \eta} + d_4 \lambda W_{1,\xi} W_{1,\eta} \]

\[ \bar{\phi}_{\xi}^{b} = \sqrt{3} d_3 \left( U_{1,\xi} + \frac{1}{2} W_{1,\xi}^2 \right) - \sqrt{3} d_4 W_{1,\xi} + \sqrt{3} d_3 \left( \lambda V_{1,\eta} + d_4 W_{1,\eta} + \frac{1}{2} \lambda^2 W_{1,\eta}^2 \right) - \sqrt{3} V d_3 \lambda W_{1,\eta} \]

\[ \bar{\phi}_{\eta}^{b} = \sqrt{3} d_3 \left( U_{1,\eta} + \frac{1}{2} W_{1,\eta}^2 \right) - \sqrt{3} d_4 W_{1,\eta} + \sqrt{3} d_3 \left( \lambda V_{1,\eta} + \frac{1}{2} \lambda^2 W_{1,\eta}^2 \right) - \sqrt{3} \lambda^2 d_4 W_{1,\eta} \]

\[ \phi_{\xi \eta}^{b} = \frac{1 - v}{2} \sqrt{3} d_3 (\lambda U_{1,\eta} + V_{1,\xi} + \lambda W_{1,\xi} W_{1,\eta}) - (1 - v) \lambda d_4 W_{1,\xi \eta} \]

### Appendix C

See Tables C.1–C.3.

#### Table C.1

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<th>( \lambda = 1 )</th>
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<th>( W_0 = 0.03 )</th>
<th>( W_0 = 0.05 )</th>
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<tr>
<td>( \phi_{\xi \eta}^{m} / \phi_{\eta}^{b} )</td>
<td>6.38501</td>
<td>6.13346</td>
<td>5.86818</td>
</tr>
<tr>
<td>( \phi_{\xi}^{m} / \phi_{\xi}^{b} )</td>
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#### Table C.2

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#### Table C.3

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<td>6.06287</td>
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<td>( \phi_{\xi}^{m} / \phi_{\xi}^{b} )</td>
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<td>6.17743</td>
<td>5.97226</td>
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References


