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Note

Strong edge colouring of subcubic graphs ☆

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ABSTRACT

A strong edge colouring of a graph G is a proper edge colouring such that every path of length 3 uses three colours. In this paper, we prove that every subcubic graph with maximum average degree strictly less than $\frac{15}{7}$ (resp. $\frac{27}{11}$, $\frac{13}{5}$, $\frac{36}{13}$) can be strong edge coloured with six (resp. seven, eight, nine) colours.

1. Introduction

A proper edge colouring of a graph $G = (V, E)$ is an assignment of colours to the edges of the graph such that two adjacent edges do not use the same colour. A strong edge colouring of a graph G is a proper edge colouring of G, such that any edge of a path of length (number of edges) 3 uses three different colours. We denote by $\chi'_s(G)$ the *strong chromatic index* of G which is the smallest integer k such that G can be strong edge coloured with k colours.

Strong edge colouring can be used to represent the conflict-free channel assignment in radio networks. The goal is to assign frequencies to every pair of transceivers communicating between each other. In a model represented by a graph, one can represent the transceivers by the vertex set and the channels by the edge set. Frequencies must be assigned to edges according to interference constraints. The first type of interference to avoid occurs when two transceivers (vertices) transmit information to the same transceiver using the same channel. In other words the two incident edges have the same assigned frequencies. The second type of interference occurs when in a path of length 3 $uvwx$, u transmits to v and w transmits to x. In this case, since w is adjacent also to v, there is an interference in v: it will receive the message from w and u on the same channel. In the case of strong edge colouring, frequencies are colours assigned to edges. For a brief survey, we refer the reader to [4,3]. The other formulation of the problem can be done in terms of induced matchings: a strong edge colouring of a graph is equivalent to a partition of the set of edges into a collection of induced matchings.

Let Δ denote the maximum degree of a graph. It was conjectured by Faudree et al. [1], that every bipartite graph has a strong edge colouring with Δ^2 colours. In 1985, Erdős and Nešetřil, during a seminar in Prague, gave a construction of graphs having a strong chromatic index equal to $\frac{5}{4}\Delta^2$ when Δ is even and $\frac{1}{4}(5\Delta^2 - 2\Delta + 1)$ when Δ is odd. They conjectured that the strong chromatic index is bounded by these values and it was verified for $\Delta \leq 3$ (see Fig. 1).

In [2] it was conjectured that for planar graphs with $\Delta \leq 3$, $\chi'_s(G) \leq 9$, which if true, is the best possible bound (see Fig. 2).

Let $\mathrm{mad}(G)$ be the maximum average degree of the graph G *i.e.* $\mathrm{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|}, H \subseteq G\right\}$, where $V(H)$ and $E(H)$ is the set of vertices and edges of H, respectively. In this note, we prove the following results:

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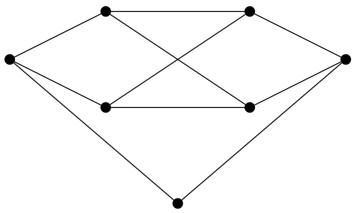

Fig. 1. A graph G proposed by Erdős and Nešetřil with $\chi'_s(G) = 10$.

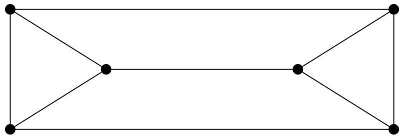

Fig. 2. The prism P with $\chi'_s(P) = 9$.

Theorem 1. *Let G be a subcubic graph:*

(i) *If* $\mathrm{mad}(G) < \frac{15}{7}$, *then* $\chi'_s(G) \le 6$.
(ii) *If* $\mathrm{mad}(G) < \frac{27}{11}$, *then* $\chi'_s(G) \le 7$.
(iii) *If* $\mathrm{mad}(G) < \frac{13}{5}$, *then* $\chi'_s(G) \le 8$.
(iv) *If* $\mathrm{mad}(G) < \frac{36}{13}$, *then* $\chi'_s(G) \le 9$.

The following lemma that belongs to folklore gives the relationship between the maximum average degree and the girth of a planar graph. Recall that the girth of a graph G is the length of a shortest cycle in G.

Lemma 2. *Let G be a planar graph with girth at least g. Then,* $\mathrm{mad}(G) < \frac{2g}{g-2}$.

Proof. Let G be a connected planar graph with girth g. Assume g is finite, otherwise, G is a tree and the result holds. Let H be a subgraph of G. Note that H is planar and has girth at least g. Hence, $g|F(H)| \le 2|E(H)|$, where $F(H)$ is the set of faces of H. According to Euler's Formula, we obtain

$$2g - g|V(H)| + g|E(H)| = g|F(H)| \le 2|E(H)|.$$

Hence,

$$\begin{aligned} &2g + (g-2)|E(H)| \le g|V(H)| \\ &2|E(H)|(2g + (g-2)|E(H)|) \le 2|E(H)|g|V(H)| \\ &\frac{2|E(H)|}{|V(H)|} \le \frac{2g|E(H)|}{2g + (g-2)|E(H)|} < \frac{2g}{g-2} \end{aligned}$$

for every subgraph H of G. $\square$

According to Lemma 2 and Theorem 1, one can derive the following result.

Corollary 3. *Let G be a planar subcubic graph with girth g:*

1. *If* $g \ge 30$, *then* $\chi'_s(G) \le 6$.
2. *If* $g \ge 11$, *then* $\chi'_s(G) \le 7$.
3. *If* $g \ge 9$, *then* $\chi'_s(G) \le 8$.
4. *If* $g \ge 8$, *then* $\chi'_s(G) \le 9$.

Part 1 of this result will be improved later (see Lemma 4).

Notations. Let G be a graph. Let $d(v)$ denote the degree of a vertex v in G. A vertex of degree k (resp. at most k) is called a k-vertex (resp. k^--vertex). A *good* 2-*vertex* is a vertex of degree 2 being adjacent to two 3-vertices, otherwise it is a *bad* 2-*vertex*. A 3_k-vertex is a 3-vertex adjacent to exactly k 2-vertices. Two edges are at distance 1 if they share one of their ends and they are at distance 2 if they are not at distance 1 and there exists an edge adjacent to both of them. We define $N_2(uv)$ as the set of edges at distance at most 2 from the edge uv and we denote by $SC(N_2(uv))$ the set of colours used by edges in $N_2(uv)$. $N(v)$ is the neighbourhood of the vertex v *i.e.* the set of its adjacent vertices. Finally, we use $[\![1; n]\!]$ to denote the set of integers $\{1, 2, \dots, n\}$.

It is easy to see that trees having two adjacent 3-vertices need at least five colours to be strong edge colourable. On the other hand, trees are exactly the class of graphs having the maximum average degree strictly smaller than 2. The graph G of Fig. 3 is exactly six strong edge colourable and $\mathrm{mad}(G) = 2$.

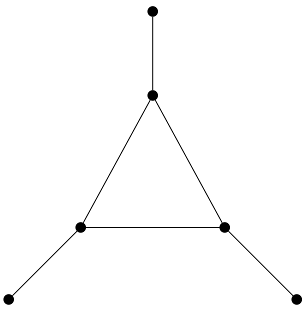

Fig. 3. A graph G with $\mathrm{mad}(G) = 2$ and $\chi'_s(G) = 6$.

In Sections 2–5, we give the proof of Theorem 1 by using the method of reducible configurations and the discharging technique. The proof is done by minimum counterexample. In each of the cases, for the minimum counterexample H, we prove the non-existence of some configurations *i.e.* a set $\mathcal{S}$ of subgraphs which cannot appear in H. We define the weight function $\omega : V(H) \to \mathbb{R}$ with $\omega(x) = d(x) - m$ ($m \in \mathbb{R}$, such that $\mathrm{mad}(H) < m$). It follows from the hypothesis on the maximum average degree that the total sum of weights is strictly negative. In the next step, we define discharging rules to redistribute weights and once the discharging is finished, a new weight function ω^* will be produced. During the discharging process the total sum of weights is kept fixed. Nevertheless, by the non-existence of $\mathcal{S}$, we can show that $\omega^*(x) \geq 0$ for all $x \in V(H)$. This leads to the following contradiction:

$$0 \leq \sum_{x \in V(H)} \omega^*(x) = \sum_{x \in V(H)} \omega(x) < 0$$

and hence, this counterexample cannot exist.

2. Proof of (i) of Theorem 1

Let H be a counterexample to part (i) of Theorem 1 minimizing $|E(H)| + |V(H)|$: H is not strong edge colourable with six colours, $\mathrm{mad}(H) < \frac{15}{7}$ and for any edge e, $\chi'_s(H - e) \leq 6$. Recall that $\omega(x) = d(x) - \frac{15}{7}$. One can assume that H is connected; otherwise, by minimality of H, we can colour independently each connected component. A 3-vertex adjacent to a 1-vertex is a *light* 3-*vertex*. Otherwise it is a *heavy* 3-*vertex*.

Claim 1. *The minimal counterexample H to part* (i) *Theorem* 1 *satisfies the following properties:*

1. *H does not contain a* 1-*vertex adjacent to a* 2-*vertex.*
2. *H does not contain a* 3-*vertex adjacent to a* 1-*vertex and a* 2-*vertex.*
3. *H does not contain a* 3-*vertex adjacent to two* 1-*vertices.*
4. *H does not contain a path uvw where u, v and w are* 2-*vertices.*
5. *H does not contain a path uvw where u, v and w are three light* 3-*vertices.*

Proof. We denote by L the set of colours $L = [\![1; 6]\!]$.

1. Suppose H contains a 1-vertex u adjacent to a 2-vertex v. Let us consider $H' = H \setminus \{uv\}$, which by minimality of H is strong edge colourable with six colours. By counting the number of available colours to extend a colouring of H' to H, it is easy to see that we have at least three colours left for uv.

2–3. Trivial by a counting argument.

4. Suppose H contains a path uvw where u, v and w are 2-vertices. Let us consider $H' = H \setminus \{uv, vw\}$, which by minimality of H is strong edge colourable with six colours. By counting the number of available colours to extend a colouring of H' to H, it is easy to see that we have at least two colours left for uv and at least one colour left for vw (after the colouring of uv).
5. Suppose H contains a path $xuvwy$ where u, v and w are three light 3-vertices. Call u_1 (resp. v_1, w_1) the neighbour of u (resp. v, w) of degree 1. Assume $N(x) = \{u, x_1, x_2\}$, $N(u) = \{x, u_1, v\}$, $N(v) = \{u, v_1, w\}$, $N(w) = \{v, w_1, y\}$, $N(y) = \{w, y_1, y_2\}$ (see Fig. 4). Let us consider $H' = H \setminus \{uu_1, uv, vv_1, vw, ww_1\}$. By minimality of H, there exists a strong edge colouring c of H', using six colours. We will extend this colouring to H. Suppose first, $c(ux) = c(wy)$. We colour uv, vw, uu_1, ww_1 and vv_1 in this order, which is possible by counting for each edge the number of available colours to extend the colouring. Suppose now, $c(ux) \neq c(wy)$. W.l.o.g. we can assume that $c(ux) = 5$ and $c(wy) = 6$. First, we try to colour the edge uu_1 with the colour 6. If it is possible, then we assign the colour 6 to uu_1 and we colour uv, vw, ww_1 and vv_1 in this order, which is possible by counting the number of available colours to extend the colouring. If we cannot colour uu_1 with the colour 6, we are sure that the colour 6 appears in the neighbourhood of x. W.l.o.g. we can assume that $c(xx_1) = 6$. By applying the same reasoning on ww_1, we can assume w.l.o.g. that $c(yy_1) = 5$. We assign now the same colour α to uu_1 and ww_1, with $\alpha \in L \setminus \{c(xx_2), 5, 6, c(yy_2)\}$. Finally, we colour uv, vw and vv_1 in this

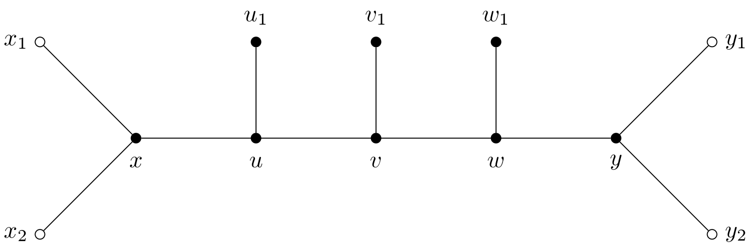

Fig. 4. The configuration of Claim 1.5.

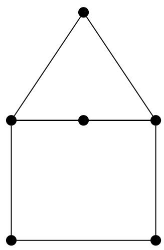

Fig. 5. A graph G with $\mathrm{mad}(G) = \frac{7}{3}$ and $\chi'_s(G) > 6$.

order, which is possible by counting the number of available colours to extend the colouring. In each case the extension of c to H is possible which is a contradiction. □

We carry out the discharging procedure in two steps:

Step 1. Every heavy 3-vertex gives $\frac{2}{7}$ to each adjacent light 3-vertex and $\frac{1}{7}$ to each adjacent 2-vertex.
When Step 1 is finished, a new weight function ω' is produced. We proceed then with Step 2:
Step 2. Every light 3-vertex gives $\frac{8}{7}$ to its unique adjacent 1-vertex.

Let $v \in V(H)$ be a k-vertex. Note that $k \geq 1$.

Case $k = 1$. Observe that $\omega(v) = -\frac{8}{7}$. By Claims 1.1 and 1.2, v is adjacent to a 3-vertex u which is a light 3-vertex by definition. Hence, v receives $\frac{8}{7}$ from u during Step 2. It follows that $\omega^*(v) = -\frac{8}{7} + \frac{8}{7} = 0$.

Case $k = 2$. Observe that $\omega(v) = -\frac{1}{7}$. By Claims 1.1, 1.2 and 1.4, v is adjacent to at least one heavy 3-vertex. Hence, by Step 1, $\omega^*(v) \geq -\frac{1}{7} + \frac{1}{7} = 0$.

Case $k = 3$. Observe that $\omega(v) = \frac{6}{7}$. Suppose v is a heavy 3-vertex. We denote by $n_b(v)$ the number of light 3-vertices in the neighbourhood of v. Note that $0 \leq n_b(v) \leq 3$. Hence, by Step 1, $\omega^*(v) \geq \frac{6}{7} - n_b(v) \times \frac{2}{7} - (3 - n_b(v)) \times \frac{1}{7} \geq 0$, for all $0 \leq n_b(v) \leq 3$. Suppose now that v is a light 3-vertex. By Claim 1.3, v is adjacent to a unique 1-vertex and by Claim 1.2, v is not adjacent to a 2-vertex. Finally, by Claim 1.5, v is adjacent to at least one heavy 3-vertex. Hence, by Steps 1 and 2, $\omega^*(v) \geq \frac{6}{7} + \frac{2}{7} - \frac{8}{7} = 0$.

This completes the proof. An example of graph G with $\mathrm{mad}(G) = \frac{7}{3}$ which is not strong edge colourable with six colours, is given in Fig. 5.

By part 1 of Corollary 3, it follows that every planar subcubic graph with girth at least 30 is strong edge colourable with at most six colours. The following lemma strengthens this result:

Lemma 4. *If G is a planar subcubic graph with girth at least* 16*, then* $\chi'_s(G) \leq 6$.

Proof. It is a folklore fact that every planar graph with girth at least $5d + 1$ and minimum degree at least 2, contains a path with d consecutive 2-vertices.

Suppose H is a planar subcubic graph with girth 16 which is not strong edge colourable with six colours and having the minimum number of edges. Consider H' the graph obtained by removing every 1-vertex from H. By Claims 1.1 and 1.3, H' has minimum degree 2. Since H' is planar with girth 16, it contains a path with at least three consecutives 2-vertices. Let uvw be such a path. By Claims 1.2 and 1.5, neither of u, v, w is a light 3-vertex in H. By Claim 1.4, in H, u, v, w are not all 2-vertices. In both cases we obtain a contradiction. □

3. Proof of (ii) of Theorem 1

Let H be a counterexample to part (ii) of Theorem 1 minimizing $|E(H)| + |V(H)|$: H is not strong edge colourable with seven colours, $\mathrm{mad}(H) < \frac{27}{11}$ and for any edge e, $\chi'_s(H - e) \leq 7$. Recall that $\omega(x) = d(x) - \frac{27}{11}$.

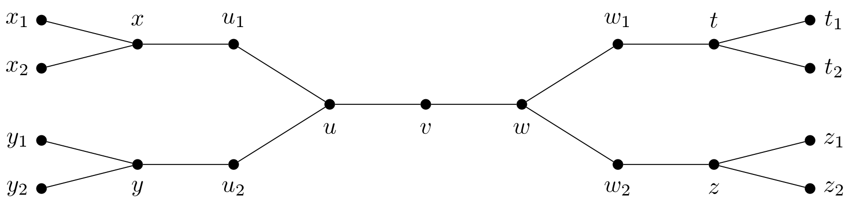

Fig. 6. The configuration of Claim 2.4.

Claim 2. *The minimal counterexample H to part* (ii) *of Theorem 1 satisfies the following properties:*

1. *H does not contain 1^--vertices.*
2. *H does not contain a path uvw where u, v and w are 2-vertices.*
3. *H does not contain a 3-vertex adjacent to two 2-vertices, one of them being bad.*
4. *H does not contain two 3_3-vertices having a 2-vertex as a common neighbour.*

Proof. We denote by L the set of colours $L = [\![1; 7]\!]$.

1–2. Trivial.

3. Suppose H contains a 3_2-vertex u having a bad 2-vertex v as a neighbour. Call w, the bad 2-vertex adjacent to v. Let us consider $H' = H \setminus \{uv, vw\}$, which by minimality of H is strong edge colourable with seven colours. By counting the number of available colours to extend a colouring of H' to H, it is easy to see that we have at least one colour left for uv and at least one colour left for vw (after the colouring of uv).
4. Suppose H contains two 3_3-vertices u and w having a 2-vertex v as a common neighbour. $N(u) = \{u_1, u_2, v\}$, $N(w) = \{w_1, w_2, v\}$, $N(u_1) = \{u, x\}$, $N(u_2) = \{u, y\}$, $N(x) = \{u_1, x_1, x_2\}$, $N(y) = \{u_2, y_1, y_2\}$, $N(w_1) = \{w, t\}$, $N(w_2) = \{w, z\}$, $N(t) = \{w_1, t_1, t_2\}$, $N(z) = \{w_2, z_1, z_2\}$ (see Fig. 6). Let us consider $H' = H \setminus \{uv, vw\}$. Since H is a minimal counterexample, $\chi'_s(H') \leq 7$ and there exists a strong edge colouring of H', c using seven colours. We will extend this colouring to H. First, we want to colour vw. Observe that $|L \setminus SC(N_2(vw))| \geq 1$, so we pick the colour left and we colour vw. Next, if we cannot colour uv, then $|L \setminus SC(N_2(uv))| = 0$ and without loss of generality, we can assume that $c(vw) = 1$, $c(ww_1) = 2$, $c(ww_2) = 3$, $c(uu_1) = 4$, $c(uu_2) = 5$, $c(u_1x) = 6$, $c(u_2y) = 7$. In this case we try to recolour vw. If we cannot, then without loss of generality $c(w_1t) = 6$, $c(w_2z) = 7$, so we try to recolour ww_1. If we cannot, then using the same argument $c(tt_1) = 5$, $c(tt_2) = 4$, and we try to recolour ww_2. We continue to try to recolour in the same manner the remaining edges in the following order: ww_2, uu_1, uu_2. If in one of the steps, the recolouring is possible, then we will have a colour free to use for uv. If it is not possible, then by the end of the procedure, we obtain without loss of generality, the following colours: $c(zz_1) = 4$, $c(zz_2) = 5$, $c(xx_1) = 2$, $c(xx_2) = 3$, $c(yy_1) = 3$, $c(yy_2) = 2$. Next, having this knowledge about the colours of the edges, we can recolour some of the edges: $c(uu_2) = c(ww_1) = 1$, $c(vw) = 5$, $c(ww_2) = 2$; and still have no "conflicts" between the colours. Hence, we have one colour left for uv, which is the colour 3. The extension of c to H is possible which is a contradiction. □

The discharging rules are defined as follows:

(R1) Every 3_3-vertex gives $\frac{2}{11}$ to each adjacent good 2-vertex.

(R2) Every 3_1-vertex and every 3_2-vertex gives $\frac{3}{11}$ to each adjacent good 2-vertex.

(R3) Every 3-vertex gives $\frac{5}{11}$ to its adjacent bad 2-vertex.

Let $v \in V(H)$ be a k-vertex. By Claim 2.1, $k \geq 2$.

Case $k = 2$. Observe that $\omega(v) = -\frac{5}{11}$. Suppose v is a good 2-vertex. By Claim 2.4, v is adjacent to at most one 3_3-vertex. Hence, by (R1) and (R2), $\omega^*(v) \geq -\frac{5}{11} + 1 \times \frac{2}{11} + 1 \times \frac{3}{11} = 0$. Suppose v is bad. By Claim 2.2, v is adjacent to one 3-vertex u. Hence, by (R3), $\omega^*(v) = -\frac{5}{11} + 1 \times \frac{5}{11} = 0$.

Case $k = 3$. Observe that $\omega(v) = \frac{6}{11}$. By Claims 2.3 and 2.4, we have the following cases for v:

- v is adjacent to three good 2-vertices and by (R1), $\omega^*(v) = \frac{6}{11} - 3 \times \frac{2}{11} = 0$.
- v is adjacent to at most two good 2-vertices. Hence, by (R2), $\omega^*(v) \geq \frac{6}{11} - 2 \times \frac{3}{11} = 0$.
- v is adjacent to at most a bad 2-vertex and by (R3), $\omega^*(v) \geq \frac{6}{11} - 1 \times \frac{5}{11} \geq 0$.

This completes the proof. An example of graph G with $\text{mad}(G) = \frac{5}{2}$ which is not strong edge colourable with seven colours, is given in Fig. 7.

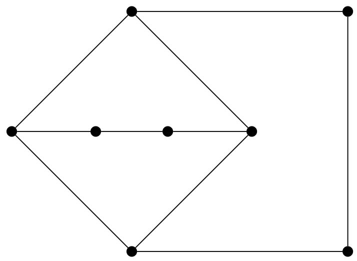

Fig. 7. A graph G with $\mathrm{mad}(G) = \frac{5}{2}$ and $\chi'_s(G) > 7$.

4. Proof of (iii) of Theorem 1

Let H be a counterexample to part (iii) of Theorem 1 minimizing $|E(H)| + |V(H)|$: H is not strong edge colourable with eight colours, $\mathrm{mad}(H) < \frac{13}{5}$ and for any edge e, $\chi'_s(H - e) \le 8$. Recall that $\omega(x) = d(x) - \frac{13}{5}$.

Claim 3. *The minimal counterexample H to part* (iii) *of* Theorem 1 *satisfies the following properties:*

1. *H does not contain 1^--vertices.*
2. *H does not contain two adjacent 2-vertices.*
3. *H does not contain a 3-vertex adjacent to three 2-vertices.*
4. *H does not contain a 2-vertex adjacent to two 3_2-vertices.*

Proof. We denote by L the set of colours $L = [\![1; 8]\!]$.

1. Trivial.
2. Suppose H contains a 2-vertex u adjacent to a 2-vertex v. Let t and w be the other neighbours of u and v respectively. By minimality of H, the graph $H' = H \setminus \{tu, uv, vw\}$ is strong edge colourable with eight colours. Consequently, there exists a strong edge colouring c of H' with eight colours. We show that we can extend this colouring to H. One can observe that $|L \setminus SC(N_2(tu))| \ge 2$, $|L \setminus SC(N_2(uv))| \ge 4$ and $|L \setminus SC(N_2(vw))| \ge 2$. Obviously, we can extend the colouring c to H, which is a contradiction.
3. Suppose H contains a 3-vertex v adjacent to three 2-vertices u, w and t. By minimality of H, there exists a strong edge colouring c of $H' = H \setminus \{vt, vu, vw\}$ with eight colours. We show that we can extend this colouring to H. One can observe that $|L \setminus SC(N_2(vt))| \ge 3$, $|L \setminus SC(N_2(vu))| \ge 3$ and $|L \setminus SC(N_2(vw))| \ge 3$. Obviously, we can extend the colouring c to H, which is a contradiction.
4. Suppose H contains two 3_2-vertices having a 2-vertex as a common neighbour. Hence, there exists a path of five vertices in H, $uvwxy$ such that u, w and y are 2-vertices and v, x are 3_2-vertices. Let us consider $H' = H \setminus \{uv, vw, wx, xy\}$. Since H is a minimum counterexample, $\chi'_s(H') \le 8$ and there exists a strong edge colouring c of H', using eight colours. We extend this colouring to H. Let us first colour the edges uv and xy. Each of these edges has two colours left to use: c^1_{uv}, c^2_{uv} for uv and c^1_{xy}, c^2_{xy} for xy. Suppose, there exists at least one colour in common: $c^1_{uv} = c^1_{xy}$. We choose these colours to colour uv and xy. After the colouring of these edges, vw and wx have each at least two colours left and we can colour them easily. Suppose now that $c^1_{uv}, c^2_{uv}, c^1_{xy}$ and c^2_{xy} are all different. Let us colour uv with c^1_{uv} and xy with c^1_{xy}. Since vw has three colours left to use at the beginning of the process, in the worst case there exists one colour non used, c_{vw}. So, we colour vw with this colour. At the last step we need to colour wx. If it is not possible, then all three colours left to use for this edge at the beginning of the process of extension of c to H, were used by uv, vw and xy. In this case if $c^2_{uv} \ne c_{vw}$, then we change the colour of uv to c^2_{uv}. Otherwise we change the colour of xy to c^2_{xy} (which is possible since $c^1_{uv}, c^2_{uv}, c^1_{xy}$ and c^2_{xy} are all different). Hence, we have a colour left for wx, to complete the colouring of H. □

The discharging rules are defined as follows:

(R1) Every 3_1-vertex gives $\frac{2}{5}$ to its unique adjacent 2-vertex.
(R2) Every 3_2-vertex gives $\frac{1}{5}$ to each adjacent 2-vertex.

Let $v \in V(H)$ be a k-vertex. By Claim 3.1, $k \ge 2$.

Case $k = 2$. Observe that $\omega(v) = -\frac{3}{5}$. By Claims 3.2 and 3.4, v is adjacent to at least one 3_1-vertex. By Claims 3.2 and 3.3, the second neighbour of v is a 3_2-vertex or a 3_1-vertex. Hence, by (R1) and (R2), $\omega^*(v) \ge -\frac{3}{5} + 1 \times \frac{2}{5} + 1 \times \frac{1}{5} = 0$.

Case $k = 3$. Observe that $\omega(v) = \frac{2}{5}$. By Claim 3.3, v is adjacent to at most two 2-vertices. If it is a 3_1-vertex, then by (R1), $\omega^*(v) \ge \frac{2}{5} - 1 \times \frac{2}{5} = 0$. If it is a 3_2-vertex, then by (R2), $\omega^*(v) \ge \frac{2}{5} - 2 \times \frac{1}{5} = 0$.

This completes the proof.

5. Proof of (iv) of Theorem 1

Let H be a counterexample to part (iv) of Theorem 1 minimizing $|E(H)| + |V(H)|$: H is not strong edge colourable with nine colours, $\mathrm{mad}(H) < \frac{36}{13}$ and for any edge e, $\chi'_s(H - e) \leq 9$. Recall that $\omega(x) = d(x) - \frac{36}{13}$.

Claim 4. *The minimal counterexample H to part* (iv) *of Theorem 1 satisfies the following properties:*

1. *H does not contain 1^--vertices.*
2. *H does not contain two adjacent 2-vertices.*
3. *H does not contain a 3-vertex adjacent to two 2-vertices.*
4. *H does not contain two adjacent 3_1-vertices.*

Proof. We denote by L the set of colours $L = [\![1; 9]\!]$.

1. Trivial.
2. Claim 4.2 can be easily checked by using the proof of Claim 3.2.
3. Suppose H contains a 3-vertex v adjacent to two 2-vertices u and w. Call t the third neighbour of v. By minimality of H, there exists a strong edge colouring c of $H' = H \setminus \{vt, vu, vw\}$ with nine colours. We show that we can extend this colouring to H. One can observe that $|L \setminus SC(N_2(vt))| \geq 3$, $|L \setminus SC(N_2(vu))| \geq 3$ and $|L \setminus SC(N_2(vw))| \geq 3$. Obviously, we can extend the colouring c to H, which is a contradiction.
4. Suppose H contains two adjacent 3_1-vertices. Let u and v be these 3_1-vertices and x and y respectively, their adjacent 2-vertices.
   If $x = y$ then, let z be the third adjacent vertex of u. By minimality of H, there exists a strong edge colouring c of $H' = H \setminus \{zu, ux, xv, vu\}$. By counting the number of available colours for each of the edges zu, ux, xv, vu, one can easily extend c to H.
   If $x \neq y$, let t be the 3-vertex adjacent to x. Consider the path $txuvy$. By minimality of H, there exists a strong edge colouring c of $H' = H \setminus \{tx, xu, uv, vy\}$. We will extend c to H. The edges tx and vy have each two colours left to use: c^1_{tx}, c^2_{tx} and c^1_{vy}, c^2_{vy}, respectively. We distinguish two cases:
   4.1 There exists at least one colour in common: $c^1_{tx} = c^1_{vy}$. We colour tx and vy with c^1_{tx} (since these edges are at distance 4, they can have the same colour). Then, we have at least one colour left for uv and we colour this edge with this colour. The edge xu initially had three colours to choose, hence, it has at least one colour left to use and we choose it.
   4.2 All the four colours are different. Let us colour tx and vy with c^1_{tx} and c^1_{vy}, respectively. Next we colour the edge uv, having two possible choices for colours to use. If its colouring is not possible then the two colours left for uv were c^1_{tx} and c^1_{vy} and in this case we change the colour of vy to c^2_{vy} and we colour uv with c^1_{vy}. At the last step we colour the edge xu, having initially three possible choices for colours to use. If its colouring is not possible, then these three colours are: c^1_{tx}, c^1_{vy} and c^2_{vy}. In this case we change the colour of tx to c^2_{tx} and we colour xu with c^1_{tx}. It is possible since all the colours $c^1_{tx}, c^2_{tx}, c^1_{vy}$ and c^2_{vy} are different. □

We carry out the discharging procedure in two steps:

Step 1. Every 3_0-vertex at distance two from a 2-vertex gives $\frac{1}{13}$ to each adjacent 3_1-vertex.
When Step 1 is finished, a new weight function ω' is produced on 3_1-vertices, hence, we proceed with Step 2:

Step 2. Every 3-vertex gives $\frac{5}{13}$ to its unique adjacent 2-vertex.

Let $v \in V(H)$ be a k-vertex. By Claim 4.1, $k \geq 2$.

Case $k = 2$. Observe that $\omega(v) = -\frac{10}{13}$. By Claim 4.2, v is adjacent to two 3_1-vertices. Hence, by Step 2, $\omega^*(v) = -\frac{10}{13} + 2 \times \frac{5}{13} = 0$.

Case $k = 3$. Observe that $\omega(v) = \frac{3}{13}$. By Claim 4.3 v can be a 3_1-vertex or a 3_0-vertex. Suppose, v is a 3_1-vertex. By Claim 4.4 and after Step 1, $\omega'(v) = \frac{3}{13} + 2 \times \frac{1}{13}$, then, by Step 2, $\omega^*(v) \geq 0$. Suppose now that v is a 3_0-vertex. By Step 1, $\omega^*(v) \geq \frac{3}{13} - 3 \times \frac{1}{13} = 0$.

This completes the proof. An example of graph G with $\mathrm{mad}(G) = \frac{20}{7}$ which is not strong edge colourable with nine colours, is given in Fig. 1.

6. Conclusion

In this paper we studied the bounds of the strong chromatic index of subcubic graphs considering their maximum average degree. In order to show the tightness of our result, let us consider the function $f(n) = \inf\{\mathrm{mad}(G) \mid \chi'_s(G) > n\}$. Obviously, $f(5) = 2$, and we proved that for $n = 6$ (7, 8, 9 resp.):

$$\frac{45}{21} = \frac{15}{7} < f(6) \leq \frac{7}{3} = \frac{49}{21}$$
$$\frac{54}{22} = \frac{27}{11} < f(7) \leq \frac{5}{2} = \frac{55}{22}$$
$$\frac{252}{91} = \frac{36}{13} < f(9) \leq \frac{20}{7} = \frac{260}{91}.$$

We did not find a better bound than the one used for $f(9)$ to estimate $f(8)$. This question seems to be more intriguing, since as we remarked so far, the graph having the maximum average degree strictly smaller than $\frac{20}{7}$ and needing nine colours to be strong edge coloured, apparently has more than 12 vertices and sixteen edges—an order which is much bigger than the order of the graphs we found for other values of f.

Speaking about planar graphs, as a corollary, we managed to prove that for a girth $g \geq 8$, the conjecture stated in [2], holds.

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