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Wavelets for iterated function systems

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Abstract

We construct a wavelet and a generalised Fourier basis with respect to some fractal measure given by a one-dimensional iterated function system. In this paper we will not assume that these systems are given by linear contractions generalising in this way some previous work of Dutkay, Jorgensen, and Pedersen to the non-linear setting. As a byproduct we are able to provide a Fourier basis also for such linear fractals like the Middle Third Cantor Set which have been left out by previous approaches.

Keywords: Wavelets; Scaling functions; Fourier basis; Limit sets; Fractals; Iterated function systems

1. Introduction

In this paper we will construct wavelets and generalised Fourier bases on fractal sets constructed via iterated function systems (IFS), which we do not assume to be linear. More precisely, the wavelets under consideration are constructed in the L^2-space with respect to the measure of maximal entropy transported to the so-called enlarged fractal, which is dense in $\mathbb{R}$.

It is a natural approach to consider wavelets in the context of such fractals since both carry a self-similar structure; the fractal inherits it from the prescribed scaling of the IFS while the wavelet satisfies a certain scaling identity. Another interesting aspect is that both wavelets and fractals are used in image compression, where both have advantages and disadvantages like blurring by zooming in or long compression times. Because of these common features it is of interest

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to develop a common mathematical foundation of these objects not least to find out whether it can have an impact on the theory of data or image compression.

The aim of the wavelet analysis is to approximate functions by using superpositions from a wavelet basis. This basis is supposed to be orthonormal and derived from a finite set of functions, the so-called mother wavelets (cf. Proposition 2.9). To obtain such a basis we employ the multiresolution analysis (MRA) (cf. Definition 2.4). Our main goal is therefore to set up an MRA in the non-linear situation. For this we generalise some ideas from [8,3], which are restricted to homogeneous linear cases with respect to the restriction of certain Hausdorff measures. Interesting related results connecting the existence of wavelets and spectral properties both with respect to the Lebesgue measure have been obtained for the higher dimensional linear setting by Wang in [10]. A two dimensional fractal situation has been considered in [1] where the authors developed an MRA for the Sierpiński Gasket.

In Section 3 we are also going to generalise the construction of the Fourier basis in the sense of [9] to our non-linear setting. This will be done in virtue of a homeomorphism conjugating the IFS under consideration with a linear homogeneous IFS. As a consequent of this construction we are able to set up a generalised Fourier basis also for such linear IFS which do not allow a Fourier basis in the sense of [9]. A detailed discussion of this construction for the Middle Third Cantor Set will be given in Example 4.4.

2. Wavelets

2.1. The enlarged limit set and the measure of maximal entropy

The family

$$\mathcal{S} := \big(\sigma_i : [0,1] \to [0,1]\colon\ i \in \underline{p} := \{0, \ldots, p-1\}\big)$$

consisting of $p \in \mathbb{N}$ injective contractions σ_i, which are uniformly Lipschitz with Lipschitz constant $0 < c_{\mathcal{S}} < 1$, i.e. $|\sigma_i(x) - \sigma_i(y)| \leqslant c_{\mathcal{S}}|x - y|$, $x, y \in [0,1]$, $i \in \underline{p}$. We will always assume that all contractions have the same orientation (in fact are increasing) and that the IFS satisfies the open set condition (OSC), i.e. $\bigcup_{i=0}^{p-1} \sigma_i((0,1)) \subset (0,1)$ and $\sigma_i((0,1)) \cap \sigma_j((0,1)) = \varnothing$, $i \neq j$.

It is well known that there exists a unique non-empty compact set $C \subset [0,1]$ such that $C = \bigcup_{i=0}^{p-1} \sigma_i(C)$. This set will be denoted the *limit set* of $\mathcal{S}$. Throughout, we will assume that the IFS $(\sigma_i\colon\ i \in \underline{p})$ is arranged in ascending order, that is $\sigma_i([0,1])$ lies to the left of $\sigma_{i+1}([0,1])$ for all $i = 0, \ldots, p-2$.

It is always possible to extend the IFS $\mathcal{S}$ by linear contractions to obtain the IFS

$$\mathcal{T} = \big(\tau_i : [0,1] \to [0,1]\colon\ i \in \underline{N} := \{0, \ldots, N-1\}\big)$$

which leaves no gaps. More precisely, there exists a number $N \geqslant p$ and a set $A \subset \underline{N}$ such that

1. $\{\tau_j\colon\ j \in A\} = \{\sigma_i\colon\ i \in \underline{p}\}$,
2. $\tau_0(0) = 0$, $\tau_{N-1}(1) = 1$ and $\tau_i(1) = \tau_{i+1}(0)$, $i = 1, \ldots, N-2$,
3. $\forall i \in \underline{N} \setminus A$: $\tau_i : [0,1] \to [0,1]$ is an affine increasing contraction.

In the following the uniform Lipschitz constant for the IFS $\mathcal{T}$ will be denoted by $c_{\mathcal{T}}$.

Remark 2.1. Note that it is not essential to choose the "gap filling functions" τ_i, $i \in \underline{N} \setminus A$, to be affine. Our analysis would work for any set of contracting injections as long as the conditions 1, 2, and 3 above are satisfied. Nevertheless, the particular choice has an influence on the set R and the measure H defined below. Also note that more than one gap filling function can be defined on one gap. Throughout, let

$$\rho_{j,N} : x \mapsto \frac{x+j}{N}.$$

Then for instance, if $\mathcal{S}$ consists of functions $\sigma_i(x) = \frac{x+a_i}{N}$, $a_i \in \underline{N}$, $i \in \underline{p}$, it is a natural choice to extend $\mathcal{S}$ by the functions $\tau_i(x) = \rho_{i,N}(x)$, $i \in \underline{N} \setminus A$, such that $\mathcal{T}$ is equal to $\{\rho_{i,N}\colon\ i \in \underline{N}\}$.

For $\omega := (i_1, \ldots, i_k) \in \underline{N}^k$ let $\tau_\omega := \tau_{i_k} \circ \cdots \circ \tau_{i_1}$ and $\tau_\varnothing = \mathrm{id}$ be the identity on $[0, 1]$. Next we define the *enlarged limit set* R in two steps. Let us first fill the gaps of the limit set C with scaled copies of itself by letting

$$R_{[0,1]} := \bigcup_{k \geqslant 0} \bigcup_{\omega \in \underline{N}^k} \tau_\omega(C),$$

and then set

$$R := R_{[0,1]} + \mathbb{Z} = \bigcup_{l \in \mathbb{Z}} R_{[0,1]} + l.$$

Now let

$$\Sigma := \left\{(i_1, \ldots, i_k) \in \underline{N}^k\colon\ k \in \mathbb{N},\ i_1 \notin A\right\}$$

be the set of finite words over the alphabet $\underline{N}$ such that the initial letter is not from A. Then we can also write $R_{[0,1]}$ as the *disjoint* union

$$R_{[0,1]} = \bigcup_{\omega \in \Sigma \cup \{\varnothing\}} \tau_\omega(C).$$

2.1.1. Fractal measures on the enlarged limit set

In this section we will introduce the appropriate measure H on $\mathbb{R}$ needed for the MRA. The measure will be first defined on $[0, 1]$ and then on $\mathbb{R}$. The construction is analogue to the construction of $R_{[0,1]}$ and R. Let μ be the self-similar Borel probability measure supported on C associated to $\mathcal{S}$ with constant weights, i.e. the unique probability measure μ satisfying $\mu = p^{-1} \sum_{i \in A} \mu \circ \tau_i^{-1}$. This measure has the property that each set of the form $\tau_\omega(C)$, $\omega \in \underline{N}^k$, has measure p^{-k}. Thus, μ is the measure of maximal entropy in the sense of a shift dynamical system.

For $\omega = (i_1, \ldots, i_k) \in \underline{N}^k$ we let $|\omega| = k$ denote the length of ω and $|\varnothing| = 0$.

Fact 2.2. *The function* $\nu : \mathcal{B} \to \overline{\mathbb{R}}_0^+$ *given by*

$$\nu := \sum_{\omega \in \Sigma \cup \{\varnothing\}} p^{-|\omega|} \mu \circ \tau_\omega^{-1}$$

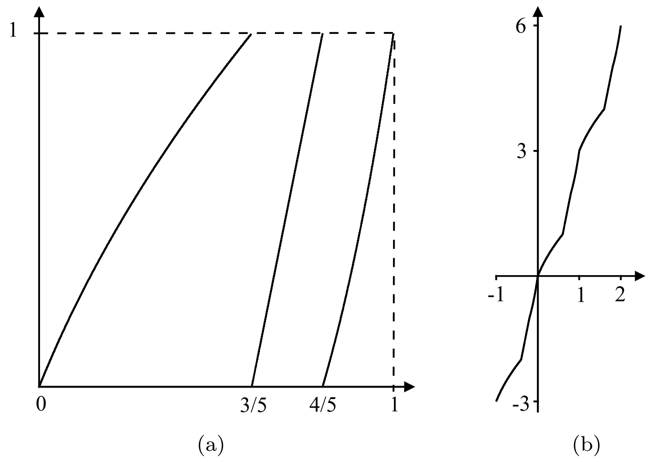

Fig. 1. The expanding inverse branches of an IFS with corresponding scaling function σ.

defines a measure on $[0,1]$. *Also, the sum of its translates*

$$H : \mathcal{B} \to \mathbb{R}_0^+, \qquad B \mapsto \sum_{k\in\mathbb{Z}} \nu(B+k),$$

defines a measure. Its essential support is equal to R.

Throughout, for $x \in \mathbb{R}$, let

$$\sigma(x) := \sum_{k\in\mathbb{Z}} \mathbb{1}_{[k,k+1)}(x)\left(Nk + \sum_{i=0}^{N-1} \mathbb{1}_{[\tau_i(0),\tau_i(1))}(x-k)\big(\tau_i^{-1}(x-k)+i\big)\right)$$

denote the *scaling function* associated to $\mathcal{T}$. Note that σ^{-1} is given, for $x \in \mathbb{R}$, by

$$\sigma^{-1}(x) = \sum_{k\in\mathbb{Z}} \mathbb{1}_{[Nk,N(k+1))}(x)\left(k + \sum_{i=0}^{N-1} \mathbb{1}_{[i,i+1)}(x-Nk)\big(\tau_i(x-i-Nk)\big)\right).$$

Example. Let us give an example for a scaling function σ.

Take the IFS $\mathcal{S} = (\sigma_0, \sigma_1)$ and the extended IFS $\mathcal{T} = (\tau_0, \tau_1, \tau_2)$ with $\sigma_0 = \tau_0 : x \mapsto \frac{x^2}{5} + \frac{2}{5}x$, $\tau_1 : x \mapsto \frac{x}{5} + \frac{3}{5}$ and $\sigma_1 = \tau_2 : x \mapsto \frac{\log(x+1)}{5\cdot\log(2)} + \frac{4}{5}$. The inverse branches of the maps are illustrated in Fig. 1(a). The scaling function σ and the inverse of the scaling function σ^{-1} are then given by:

$$\begin{aligned}\sigma(x) = \sum_{k\in\mathbb{Z}} \mathbb{1}_{[k,k+1)}(x)\big(&\mathbb{1}_{[0,3/5)}(x-k)\big(\sqrt{5(x-k)+1}-1\big)\\ &+ \mathbb{1}_{[3/5,4/5)}(x-k)\big(5(x-k)-2\big) + \mathbb{1}_{[4/5,1)}(x-k)\big(2^{5(x-k)-4}+1\big)+3k\big),\end{aligned}$$

$$\sigma^{-1}(x) = \sum_{k\in\mathbb{Z}} \mathbb{1}_{[3k,3(k+1))}(x)\biggl(\mathbb{1}_{[0,1)}(x-3k)\biggl(\frac{(x-3k)^2}{5}+\frac{2}{5}x-\frac{1}{5}k\biggr) + \mathbb{1}_{[1,2)}(x-3k)\biggl(\frac{x}{5}+\frac{2}{5}+\frac{2}{5}k\biggr)+\mathbb{1}_{[2,3)}(x-3k)\biggl(\frac{\log(x-3k-1)}{5\log(2)}+\frac{4}{5}+k\biggr)\biggr).$$

The graph of σ is shown in Fig. 1(b).

Let us now turn back to the measure H.

Lemma 2.3. *We have $H\circ\sigma = pH$ and in particular, for all $i\in\underline{N}$, $\nu\circ\tau_i = p^{-1}\nu$.*

Proof. For $E\in\mathcal{B}$ we have

$$\begin{aligned} H\bigl(\sigma(E)\bigr) &= \sum_{k\in\mathbb{Z}} \nu\bigl(\sigma(E)+k\bigr) \\ &= \sum_{i=0}^{N-1}\sum_{l\in\mathbb{Z}} \nu\bigl(\sigma(E)-Nl-i\bigr) \\ &= \sum_{l\in\mathbb{Z}} \nu\Biggl(\bigcup_{i=0}^{N-1}\tau_i^{-1}(E-l)\Biggr). \end{aligned}$$

Since

$$\begin{aligned} &\nu\Biggl(\bigcup_{i=0}^{N-1}\tau_i^{-1}(E-l)\Biggr) \\ &\quad= \sum_{\omega\in\Sigma\cup\{\varnothing\}} p^{-|\omega|}\mu\Biggl(\tau_\omega^{-1}\Biggl(\bigcup_{i=0}^{N-1}\tau_i^{-1}(E-l)\Biggr)\Biggr) \\ &\quad= \sum_{i=0}^{N-1}\sum_{\omega\in\Sigma\cup\{\varnothing\}} p^{-|\omega|}\mu\bigl(\tau_\omega^{-1}\bigl(\tau_i^{-1}(E-l)\bigr)\bigr) \\ &\quad= \sum_{i=0}^{N-1}\sum_{\omega\in\Sigma} p^{-|\omega|}\mu\bigl(\tau_{\omega i}^{-1}(E-l)\bigr) + \sum_{i=0}^{N-1}\mu\bigl(\tau_i^{-1}(E-l)\bigr) \\ &\quad= p\sum_{\omega\in\Sigma^*} p^{-|\omega|}\mu\bigl(\tau_\omega^{-1}(E-l)\bigr) + \sum_{i\notin A}\mu\bigl(\tau_i^{-1}(E-l)\bigr) + \sum_{i\in A}\mu\bigl(\tau_i^{-1}(E-l)\bigr) \\ &\quad= p\sum_{\omega\in\Sigma} p^{-|\omega|}\mu\bigl(\tau_\omega^{-1}(E-l)\bigr) + p\mu(E-l) \\ &\quad= p\nu(E-l), \end{aligned}$$

where $\Sigma^* = \{(i_1,\dots,i_k)\in\underline{N}^k\colon\ k\geqslant 2,\ i_1\notin A\}$, we have $H(\sigma(E)) = p\cdot H(E)$. $\square$

2.2. The construction of a wavelet basis

In this section we will show how to find a wavelet basis for $L^2(H)$. This wavelet basis is constructed via an MRA. In our context the definition of the MRA is given as follows.

Definition 2.4. Let $s : \mathbb{R} \to \mathbb{R}$ be a continuous strictly increasing function such that for some fixed $N \in \mathbb{N}$,

$$s(x + k) = s(x) + Nk, \quad x \in [0, 1],\ k \in \mathbb{Z}.$$

Furthermore, let η be a measure on $(\mathbb{R}, \mathcal{B})$ such that $\eta(A) = \eta(A + k)$, $A \in \mathcal{B}$, $k \in \mathbb{Z}$ and $\eta(s(A)) = p\eta(A)$, for some $p \in \mathbb{N}$. We say (s, η) allows a *multiresolution analysis* (MRA) if there exists a family $\{V_j\colon\ j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\eta)$ and a function $\varphi \in L^2(\eta)$ (called the *father wavelet*) such that the following conditions are satisfied.

1. $\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots$,
2. $\mathrm{cl} \bigcup_{j\in\mathbb{Z}} V_j = L^2(\eta)$,
3. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
4. $f \in V_j \Leftrightarrow f \circ s \in V_{j-1}$, $j \in \mathbb{Z}$,
5. $\{x \mapsto \varphi(x - n)\colon\ n \in \mathbb{Z}\}$ is an orthonormal basis in V_0.

Note that for $s : x \mapsto Nx$, $p = N$, and $\eta = \lambda$ chosen to be the Lebesgue measure, this definition coincides with the classical definition of the MRA (see e.g. [2]).

Let us now define the *shift operator* T and the *scaling operator* U on $L^2(\eta)$ by

$$(Tg)(x) = g(x - 1) \quad \text{and} \quad (Ug)(x) = \frac{1}{\sqrt{p}} g\bigl(s^{-1}(x)\bigr), \quad g \in L^2(\eta),\ x \in \mathbb{R}.$$

Remark 2.5. Both operators U and T are unitary.

We complete this section demonstrating that the MRA can be satisfied for (σ, H) if we choose the father wavelet φ to be the characteristic function of the corresponding limit set C, i.e.

$$\varphi := \mathbb{1}_C.$$

First we observe that the function φ satisfies the following scaling equation for H-almost every $x \in \mathbb{R}$,

$$\varphi(x) = \mathbb{1}_{\bigcup_{i\in A} \tau_i(C)}(x) = \sum_{i\in A} \mathbb{1}_{\tau_i(C)}(x) = \sum_{i\in A} \mathbb{1}_C\bigl(\tau_i^{-1}(x)\bigr) = \sum_{i\in A} \varphi\bigl(\tau_i^{-1}(x)\bigr)$$
$$= \sum_{i\in A} \varphi\bigl(\sigma(x) - i\bigr).$$

By virtue of the so-called *low-pass filter* m_0 there exists a relation between the two operators T and U. Let us take the filter m_0 to be given by

$$m_0(z) := \frac{1}{\sqrt{p}} \sum_{i \in A} z^i, \quad z \in \mathbb{T} := \{z \in \mathbb{C}\colon |z| = 1\}.$$

Note that m_0 can also be regarded as a function on the set of unitary operators acting on $L^2(H)$. For this choice the following proposition holds.

Proposition 2.6. *The above defined operators U and T satisfy the following relations.*

1. $\langle T^k \varphi | T^\ell \varphi \rangle_H = \delta_{\ell k}$, $k, \ell \in \mathbb{Z}$,
2. $U\varphi = m_0(T)\varphi$,
3. $UTU^{-1} = T^N$.

Proof. 1. Since $H((C + k) \cap (C + \ell)) = 0$ for $k \neq \ell$ and H is invariant with respect to the mapping $x \mapsto x + 1$, we have

$$\begin{aligned}
\langle T^k \varphi | T^\ell \varphi \rangle_H &= \int T^k \varphi(x) \overline{T^\ell \varphi(x)} \, dH(x) \\
&= \int \mathbb{1}_C(x - k) \overline{\mathbb{1}_C(x - \ell)} \, dH(x) \\
&= \int \mathbb{1}_{C+k}(x) \mathbb{1}_{C+\ell}(x) \, dH(x) = \delta_{kl}.
\end{aligned}$$

2. For $x \in \mathbb{R}$ we have

$$\begin{aligned}
U\varphi(x) &= \frac{1}{\sqrt{p}} \mathbb{1}_C\left(\sigma^{-1}(x)\right) = \frac{1}{\sqrt{p}} \mathbb{1}_{\sigma(C)}(x) \\
&= \frac{1}{\sqrt{p}} \mathbb{1}_{\bigcup_{i \in A} (\tau_i^{-1}(\tau_i(C))) + i}(x) \\
&= \frac{1}{\sqrt{p}} \sum_{i \in A} \mathbb{1}_C(x - i) = m_0(T)\varphi(x).
\end{aligned}$$

3. Let $f \in L^2(H)$ and $x \in \mathbb{R}$. Then

$$\left(UTU^{-1} f\right)(x) = f\left(\sigma\left(\sigma^{-1}(x) - 1\right)\right).$$

For $x \in [Nk, N(k+1))$ we have that $\sigma^{-1}(x) = \tau_j(x - j - Nk) + k$ for some $j \in \{0, \ldots, N-1\}$. Thus, $\sigma^{-1}(x) \in [k, k+1)$ and $\sigma^{-1}(x) - 1 \in [k-1, k)$. Now observe

$$\begin{aligned}\sigma\bigl(\sigma^{-1}(x)-1\bigr) &= \sigma\bigl(\tau_j(x-j-Nk)+k-1\bigr)\\ &= \sum_{i=0}^{N-1} \mathbb{1}_{[\tau_i(0),\tau_i(1))}\bigl(\tau_j(x-j-Nk)+k-1-(k-1)\bigr)\\ &\quad \times \bigl(\tau_i^{-1}\bigl(\tau_j(x-j-Nk)+k-1-(k-1)\bigr)+i\bigr)+N(k-1)\\ &= \tau_j^{-1}\bigl(\tau_j(x-j-Nk)+k-1-(k-1)\bigr)+j+N(k-1)\\ &= x-N.\end{aligned}$$

Consequently, $f(\sigma(\sigma^{-1}(x)-1)) = f(x-N) = T^N f(x)$. $\square$

Remark 2.7. Notice that

$$\begin{aligned}U^jT^k\varphi(x) &= \left(\frac{1}{\sqrt{p}}\right)^j \varphi\bigl(\sigma^{-j}(x)-k\bigr)\\ &= \left(\frac{1}{\sqrt{p}}\right)^j \mathbb{1}_C\bigl(\sigma^{-j}(x)-k\bigr)\\ &= \left(\frac{1}{\sqrt{p}}\right)^j \mathbb{1}_{\sigma^j(C+k)}(x).\end{aligned}$$

Theorem 2.8. *The pair (σ, H) allows an MRA if we set $\varphi := \mathbb{1}_C$ to be the father wavelet and let $V_0 := \operatorname{cl\,span}\{T^k\varphi\colon k \in \mathbb{Z}\}$, $V_j := \operatorname{cl\,span}\{U^jT^k\varphi\colon k \in \mathbb{Z}\}$, $j \in \mathbb{Z}$. In particular, we have*

$$\operatorname{cl\,span}\bigl\{U^nT^k\varphi\colon k, n \in \mathbb{Z}\bigr\} = L^2(H).$$

Proof. To prove that this gives an MRA, we show that the conditions 1 to 5 from Definition 2.4 are satisfied.

1. Recall that $U\varphi = m_0(T)\varphi$ and $UTU^{-1} = T^N$. Consequently,

$$U^{-1}T^k\varphi = U^{-1}T^kU^{-1}m_0(T)\varphi = U^{-2}T^{Nk}m_0(T)\varphi.$$

This shows that $V_{-1} \subset V_{-2}$ and iterating this argument it follows that $\cdots \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots$.

3. Clearly $0 \in \bigcap_{j\in\mathbb{Z}} V_j$. Recall that R is equal to the essential support of H. Now take $f \in \bigcap_{j\in\mathbb{Z}} V_j$. Then $f \in V_j$ for all $j \in \mathbb{Z}$. If we have $0 \neq f \in V_{j_0}$ for some $j_0 \in \mathbb{Z}$, then there exist $k \in \mathbb{Z}$ and $c \neq 0$ such that $f|_{\sigma^{j_0}(C+k)} = c\mathbb{1}_{\sigma^{j_0}(C+k)}$. Since $(V_j)_{j\geqslant j_0}$ is a nested sequence we have for every $j > j_0$ that there exists $k_j \in \mathbb{Z}$ such that $f|_{\sigma^j(C+k_j)} = c\mathbb{1}_{\sigma^j(C+k_j)}$ and $\sigma^j(C+k_j) \supset \sigma^{j-1}(C+k_{j-1})$. Consequently f takes the value c on the nested union $\bigcup_{j\geqslant j_0}\sigma^j(C+k_j)$. Since this union has infinite measure, f must be constantly 0.

4. Let $f \in V_j$, i.e. for some $b_k \in \mathbb{C}$,

$$f(x) = \sum_{k\in\mathbb{Z}} b_kU^jT^k\varphi(x) = \sum_{k\in\mathbb{Z}} c_k\varphi\bigl(\sigma^{-j}(x)-k\bigr)$$

and

$$f\big(\sigma(x)\big)=\sum_{k\in\mathbb{Z}} b_k U^j T^k \varphi\big(\sigma(x)\big)=\sum_{k\in\mathbb{Z}} c_k \varphi\big(\sigma^{-j}\big(\sigma(x)\big)-k\big)$$
$$=\sum_{k\in\mathbb{Z}} c_k \varphi\big(\sigma^{-j+1}(x)-k\big)=\sum_{k\in\mathbb{Z}} d_k U^{j-1} T^k \varphi(x).$$

Thus, $f\circ\sigma\in V_{j-1}$.

5. In Proposition 2.6 it has been shown that $(T^k\varphi)_{k\in\mathbb{Z}}$ is orthonormal. The spanning condition is trivially satisfied.

2. First we will show that $\mathbb{1}_F$ for $F\in\mathcal{B}|_{[k,k+1]}$ mod H, $k\in\mathbb{Z}$, can be approximated by linear combinations of $U^nT^m\varphi=p^{-n/2}\cdot\mathbb{1}_{\sigma^n(C+m)}$, $m,n\in\mathbb{Z}$. For this let us define the set

$$\mathcal{V}_k:=\left\{C_{\omega,k}:=\sigma^{-n}\left(C+\sum_{i=0}^{n-1}a_iN^i+N^nk\right):\ \omega\in\mathcal{C}_n,\ n\geqslant 1\right\},$$

where

$$\mathcal{C}_n:=\big\{(a_1,\dots,a_n):\ a_i\in\underline{N}\big\}.$$

We are going to show that $\mathcal{V}_k\cup\{\varnothing\}$ defines a semiring for $[k,k+1]$. Since

$$C+\sum_{i=0}^{n-1}a_iN^i+N^nk\subset\left[N\left(\sum_{i=1}^{n-1}a_iN^{i-1}+N^{n-1}k\right),N\left(\sum_{i=1}^{n-1}a_iN^{i-1}+N^{n-1}k+1\right)\right]$$

we get inductively

$$\begin{aligned}
&\sigma^{-n}\left(C+\sum_{i=0}^{n-1}a_iN^i+N^{n-1}k\right)\\
&=\sigma^{-n+1}\left(\sum_{j=0}^{N-1}\mathbb{1}_{[j,j+1)}\left(C+\sum_{i=0}^{n-1}a_iN^i+N^nk-N\left(\sum_{i=1}^{n-1}a_iN^{i-1}+N^{n-1}k\right)\right)\right.\\
&\qquad\times\left(\tau_j\left(C+\sum_{i=0}^{n-1}a_iN^i+N^nk-N\left(\sum_{i=1}^{n-1}a_iN^{i-1}+N^{n-1}k\right)-j\right)\right.\\
&\qquad\left.\left.+\sum_{i=1}^{n-1}a_iN^{i-1}+N^{n-1}k\right)\right)\\
&=\sigma^{-n+1}\left(\sum_{j=0}^{N-1}\mathbb{1}_{[j,j+1)}(C+a_0)\left(\tau_j(C+a_0-j)+\sum_{i=1}^{n-1}a_iN^{i-1}+N^{n-1}k\right)\right)\\
&=\sigma^{-n+1}\left(\tau_{a_0}(C)+\sum_{i=1}^{n-1}a_iN^{i-1}+N^{n-1}k\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sigma^{-n+2}\left(\tau_{a_1}\left(\tau_{a_0}(C)\right) + \sum_{i=2}^{n-1} a_i N^{i-2} + N^{n-2}k\right) \\
&\ \vdots \\
&= \sigma^{-1}\left(\tau_{a_{n-2}}\left(\cdots\left(\tau_{a_1}\left(\tau_{a_0}(C)\right)\right)\cdots\right) + a_{n-1} + Nk\right) \\
&= \tau_{a_{n-1}}\left(\tau_{a_{n-2}}\left(\cdots\left(\tau_{a_1}\left(\tau_{a_0}(C)\right)\right)\cdots\right)\right) + k.
\end{aligned}
$$

From this the semiring properties of $\mathcal{V}_k \cup \{\varnothing\}$ follow immediately. Then also

$$
\mathcal{V} := \left\{ \bigcup_{\ell=1}^{m} B_\ell \colon B_\ell \in \bigcup_{k\in\mathbb{Z}} \mathcal{V}_k,\ m \in \mathbb{N} \right\}
$$

defines a semiring. Furthermore, we will show that $\mathcal{B}|_{[k,k+1]} \subset \sigma(\mathcal{V}_k)$ which would also imply $\mathcal{B} \subset \sigma(\mathcal{V})$. In fact $\mathcal{B}|_{[k,k+1]}$ is generated mod H by sets of the form $(a,b) \cap [k,k+1] \cap R$, $a, b \in \mathbb{R}$, which belong obviously to $\sigma(\mathcal{V}_k)$. This shows $\mathcal{B}|_{[k,k+1]} \subset \sigma(\mathcal{V}_k)$. Thus, every set $F \in \mathcal{B}|_{[k,k+1]}$ can be approximated by sets from $\mathcal{V}_k$ and consequently every set $E \in \mathcal{B}$ can be approximated by sets of $\mathcal{V}$. Since

$$
\begin{aligned}
\mathbb{1}_{C_{\omega,k}}(x) &= \mathbb{1}_{\sigma^{-n}(C+\sum_{i=0}^{n-1} a_i N^i + N^n k)}(x) \\
&= \mathbb{1}_C\left(\sigma^n(x) - \sum_{i=0}^{n-1} a_i N^i - N^n k\right) \\
&= c \cdot U^{-n} T^l \varphi(x),
\end{aligned}
$$

where $l = \sum_{i=0}^{n-1} a_i N^i + N^n k$, we find that $\mathbb{1}_E$, $E \in \mathcal{B}$, can be approximated by linear combinations of $U^n T^k \varphi$. Now the claim follows since the simple functions are dense in $L^2(H)$. $\square$

For the construction of the mother wavelets we will introduce further filter functions. Let $A = \{a_0, \ldots, a_{p-1}\}$ and $G := \{0, \ldots N-1\} \backslash \{a_0, \ldots, a_{p-1}\} = \{d_i \colon i = 0, \ldots, N-p-1\}$. Then the first $N-p$ *high-pass filters*, $m_1, \ldots, m_{N-p}$, on $\mathbb{T}$ are defined by

$$
m_{i+1} : z \mapsto z^{d_i}, \quad i \in \underline{N-p-1}.
$$

The remaining $p-1$ filter functions are defined by

$$
m_{N-p+k} : z \mapsto \frac{1}{\sqrt{p}} \sum_{j=0}^{p-1} \eta^{kj} z^{a_j}, \quad \text{for } k \in \{1, \ldots, p-1\},\ \eta = e^{2\pi i/p}.
$$

It has been shown in [3] that the matrix

$$
M(z) := \frac{1}{\sqrt{N}} \left(m_j\left(\rho^l z\right)\right)_{j,l=0}^{N-1},
$$

where $\rho = e^{2\pi i/N}$, is unitary for almost all $z \in \mathbb{T}$ (i.e. $M(z)^*M(z) = I$, where I denotes the identity matrix).

The following proposition shows that

$$\{\psi_i := U^{-1} m_i(T)\varphi,\ i \in \{1, \dots, N-1\}\}$$

defines a set of mother wavelets.

Proposition 2.9. *The set*

$$\{U^n T^k \psi_i\colon\ i \in \{1, \dots, N-1\},\ n, k \in \mathbb{Z}\}$$

is an ONB for $L^2(H)$.

Proof. We have shown in Theorem 2.8 that the father wavelet φ gives rise to an MRA and consequently $\{U^n T^k \varphi\colon\ k, n \in \mathbb{Z}\}$ is dense in $L^2(H)$. This implies that also $\{U^n T^k \psi_i\colon\ i \in \{1, \dots, N-1\},\ n, k \in \mathbb{Z}\}$ is dense in $L^2(H)$. Furthermore, the orthonormality of $\{\psi_i\colon\ i \in \{1, \dots, N-1\}\}$ follows form the unitarity of the filter functions. Finally, since $\{T^k \psi_i\colon\ i \in \{1, \dots, N-1\}, k \in \mathbb{Z}\}$ is an ONB for $V_{-1} \ominus V_0$ and hence $\{U^n T^k \psi_i\colon\ i \in \{1, \dots, N-1\},\ k \in \mathbb{Z}\}$ is an ONB for $V_{n-1} \ominus V_n$, the orthonormality of $\{U^n T^k \psi_i\colon\ i \in \{1, \dots, N-1\},\ n, k \in \mathbb{Z}\}$ follows. $\square$

3. Fourier bases

In this section we will use the set-up of Section 2.1. That is, we consider an arbitrary IFS $\mathcal{S} = (\sigma_i, i \in \underline{p})$ extended to a "gap filling" IFS $\mathcal{T} = (\tau_i, i \in \underline{N})$ consisting of N contractions such that there exists a set $A \subset \underline{N}$ with $\{\sigma_i\colon\ i \in \underline{p}\} = \{\tau_j\colon\ j \in A\}$. Additionally, we consider two corresponding *homogeneous linear* IFSs $\widetilde{\mathcal{S}} = (\widetilde{\sigma}_i, i \in \underline{p})$ and $\widetilde{\mathcal{T}} = (\widetilde{\tau}_i, i \in \underline{N})$, given by the functions $\widetilde{\tau}_i := \rho_{i,N} : x \mapsto \frac{x}{N} + \frac{i}{N}$ such that $\{\widetilde{\sigma}_i\colon\ i \in \underline{p}\} = \{\widetilde{\tau}_j\colon\ j \in A\}$.

3.1. The construction of a conjugating homeomorphism

Let us now investigate the construction of the conjugating homeomorphism from the linear enlarged limit set to the non-linear one. In the first step the construction is given for $[0, 1]$ to be extended to $\mathbb{R}$ in the second step. This homeomorphism on $\mathbb{R}$ can be employed for a different approach to construct the wavelet basis from Section 2.2. See Remark 3.1 for a detailed discussion.

The aim is to find a homeomorphism $\phi : [0, 1] \to [0, 1]$ such that $\phi(\widetilde{R}_{[0,1]}) = R_{[0,1]}$, $\phi(\widetilde{C}) = C$ and $\phi \circ \widetilde{\tau}_i = \tau_i \circ \phi$ for $i \in \underline{N}$, where C, $\widetilde{C}$ are the limit sets corresponding to the IFS $\mathcal{S}$, $\widetilde{\mathcal{S}}$ respectively, and $R_{[0,1]}$, $\widetilde{R}_{[0,1]}$ are the corresponding enlarged limit sets restricted to $[0, 1]$. The idea of the construction can be found e.g. in [6].

Let $D := \{f \in C([0, 1])\colon\ f(0) = 0,\ f(1) = 1,\ f : [0, 1] \to [0, 1]\}$ and let the operator $F : D \to D$ be given by

$$(Ff)(x) = \sum_{i=0}^{N-1} \tau_i \circ f \circ \widetilde{\tau}_i^{-1}(x) \cdot \mathbb{1}_{[\widetilde{\tau}_i(0), \widetilde{\tau}_i(1))}(x) + \mathbb{1}_{\{1\}}(x), \quad x \in [0, 1].$$

Then it is easy to see that F is a contraction and since D is complete, we have by the Banach Fixed Point Theorem, that there exists a fixed point ϕ of F in D. It is not hard to see that the inverse function ϕ^{-1} of ϕ is the unique fixed point of the contractive operator on D given by

$$(Gh)(x) = \sum_{i=0}^{N-1} \widetilde{\tau}_i \circ h \circ \tau_i^{-1}(x) \cdot \mathbb{1}_{[\tau_i(0),\tau_i(1))}(x) + \mathbb{1}_{\{1\}}(x), \quad h \in D,\ x \in [0,1].$$

Consequently $\phi : [0,1] \to [0,1]$ is a homeomorphism and it is straightforward to observe that it has all the desired properties. This homeomorphism may then be extended continuously to $\mathbb{R}$, such that $\phi(\widetilde{R}) = R$. For this notice that any $x \in \mathbb{R}$ can be written uniquely as $x = \{x\} + \lfloor x \rfloor$, where $\lfloor x \rfloor \in \mathbb{Z}$ denotes the largest integer not exceeding x and $\{x\} = x - \lfloor x \rfloor$ the fractional part of x. Then the extended homeomorphism is defined, for $x \in \mathbb{R}$, by

$$\widetilde{\phi}(x) := \phi\big(\{x\}\big) + \lfloor x \rfloor,$$

and consequently, its inverse function $\widetilde{\phi}^{-1} : \mathbb{R} \to \mathbb{R}$ is given by

$$\widetilde{\phi}^{-1}(z) = \phi^{-1}\big(\{z\}\big) + \lfloor z \rfloor, \quad z \in \mathbb{R}.$$

Remark 3.1. (1) We would like to remark that the wavelet basis (constructed in Section 2.2) can also be obtained using the homeomorphisms $\phi : \mathbb{R} \to \mathbb{R}$. In fact, the wavelet basis for the non-linear IFS is just the composition of ϕ with the basis elements of the linear IFS constructed in [3].

(2) For what follows we only need the homeomorphism restricted to $\widetilde{C}$, i.e. $\phi|_{\widetilde{C}}$. Hence, the restricted homeomorphism is in fact independent of the functions defined on the "gaps" of the limit set, i.e. it depends only on $(\widetilde{\tau}_a)_{a\in A}$ and $(\tau_a)_{a\in A}$.

3.2. The appropriate function space

It will be essential to construct first the Fourier basis for the linear Cantor set $\widetilde{C}$ given as the limit set of the IFS $\widetilde{\mathcal{S}} = (\widetilde{\tau}_i\colon\ i \in A)$. The Hausdorff dimension of this set $\widetilde{C}$ is $s = \log p/\log N$. The restriction of the Hausdorff measure H^s to $\widetilde{C}$ will be denoted by $\widetilde{\mu}$, i.e. $\widetilde{\mu} = H^s|_{\widetilde{C}}$. This measure satisfies

$$\widetilde{\mu} = \frac{1}{p} \sum_{i\in A} \widetilde{\mu} \circ \widetilde{\tau}_i^{-1}$$

or equivalently,

$$\int f(x)\, d\mu(x) = \frac{1}{p} \sum_{i\in A} \int f\big(\widetilde{\tau}_i(x)\big)\, d\widetilde{\mu}(x), \quad f \in L^2(\widetilde{\mu}),$$

and, by a theorem of Hutchinson (cf. [5]), it is hence unique with this property. Furthermore, $H^s(\widetilde{C}) = \widetilde{\mu}(\widetilde{C}) = 1$.

We will consider the homeomorphism $\phi : \widetilde{C} \to C$ from Section 3.1, where C is the limit set of $\mathcal{S} = (\sigma_i,\ i \in \underline{p}) = (\tau_i,\ i \in A)$. Notice that the homeomorphism ϕ is measurable with respect to the Borel-σ-algebra. This allows us to consider the space $L^2(\mu)$, where μ is the transported measure, i.e. $\mu = \widetilde{\mu} \circ \phi^{-1}$, which coincides with the unique measure on C satisfying

$$\mu = \frac{1}{p} \sum_{i \in A} \mu \circ \tau_i^{-1}.$$

The above defined homeomorphism ϕ will be used to carry over a Fourier basis in $L^2(\widetilde{\mu})$ to a generalised Fourier basis in $L^2(\mu)$ in Section 3.4.

3.3. The construction of the Fourier basis for homogeneous linear IFSs

In this section we state the main results on Fourier bases for homogeneous linear IFSs from Jorgensen and Pedersen, and Jorgensen and Dutkay [9,3] without proofs. First we consider the construction of the Fourier basis on the Hilbert space $L^2(\widetilde{\mu})$ with the inner product $\langle f|g\rangle_{\widetilde{\mu}} = \int f(x)\overline{g(x)}\,d\widetilde{\mu}(x)$. The *classical Fourier basis* is of the form $\{e_n\colon n \in M\}$ with $M \subset \mathbb{Z}$ and $e_n : x \mapsto e^{i2\pi n x}$. This kind of basis does not exist for all L^2-spaces of functions defined on a limit set. In fact, the existence of appropriate L^2-spaces depends on the underlying algebraic structure of the IFS (cf. [9] and Example 4.4 below).

Recall the definition of the Fourier transform $\widehat{\mu}$ of a measure $\widetilde{\mu}$:

$$\widehat{\mu}(t) := \int e^{i2\pi t x}\, d\widetilde{\mu}(x), \quad t \in \mathbb{R}.$$

Lemma 3.2. *(See [9].) The Fourier transform for the measure $\widetilde{\mu}$ satisfies the following relation*:

$$\widehat{\mu}(t) = \widehat{\mu}\bigl(N^{-1}t\bigr) \cdot \kappa_A(t),$$

where $\kappa_A(t) = p^{-1} \sum_{a \in A} e^{i2\pi t N^{-1} a}$ and $t \in \mathbb{R}$.

The following assumption guaranties the existence of a classical Fourier basis.

Assumption 3.3. We assume that $0 \in A$, and that there exists a set $L \subset \mathbb{Z}$, such that $0 \in L$, $\operatorname{card} L = p$ and

$$H_{AL} := p^{-1/2}\bigl(e^{i2\pi \frac{al}{N}}\bigr)_{a \in A,\, l \in L}$$

is unitary. The set L with this property will be called the *dual set to A*.

Lemma 3.4. *(See [9].) Under Assumption* 3.3 *we have that the set*

$$\{e_\lambda\colon \lambda \in \Lambda\}$$

is orthonormal in $L^2(\widetilde{\mu})$, where $e_\lambda : x \mapsto e^{i2\pi \lambda x}$ and

$$\Lambda := \bigl\{l_0 + Nl_1 + N^2 l_2 + \cdots + N^k l_k\colon l_i \in L,\ k \in \mathbb{N}\bigr\}.$$

We now introduce three conditions under which the set $\{e_\lambda\colon \lambda \in \Lambda\}$ gives an orthonormal basis, i.e. when this set spans the space $L^2(\widetilde{\mu})$. For this we first need the following definitions. Let the dual Ruelle Operator for the above setting be given by

$$(R_L f)(x) := \frac{1}{p}\sum_{l\in L}\left|m_0\left(\frac{x-l}{N}\right)\right|^2 f\left(\frac{x-l}{N}\right), \quad f \in C(\mathbb{R}),$$

where $m_0 : t \mapsto p^{-1/2}\sum_{a\in A} e^{i2\pi ta}$.

Definition 3.5. (See [3].) Let $\widetilde{\sigma}_b := \rho_{-b,N}$ and L and $A := \{a_0, \ldots, a_{p-1}\}$ are given as in Assumption 3.3. Then the family $(z_1, z_2, \ldots, z_k) \in \mathbb{T}^k$ with $z_1 = e^{i2\pi\xi_1}$, $z_2 = e^{i2\pi\xi_2}$, ..., $z_k = e^{i2\pi\xi_k}$ is called an *L-cycle with pairing* $(b_1, b_2, \ldots, b_{k+1}) \in L^{k+1}$, if for $j = 1, \ldots, k$ and $z_{k+1} := z_1$ we have

$$z_j = \exp\bigl(i2\pi\sigma_{b_j}(\xi_{j+1})\bigr),$$

and $|m_0(\xi_j)|^2 = p$, $j = 1, \ldots, k$.

Proposition 3.6. *(See [9,3].) Under Assumption* 3.3 *the following three characterisations of the existence of an orthonormal basis* (*ONB*) *in $L^2(\widetilde{\mu})$ hold.*

1. *The set $\{e_\lambda\colon \lambda \in \Lambda\}$ is an ONB in $L^2(\widetilde{\mu})$, if and only if $Q \equiv 1$, where*

$$Q(t) := \sum_{\lambda\in\Lambda}\bigl|\widehat{\mu}(t-\lambda)\bigr|^2, \quad t \in \mathbb{R}.$$

2. *The set $\{e_\lambda\colon \lambda \in \Lambda\}$ is an ONB in $L^2(\widetilde{\mu})$, if the space*

$$\bigl\{f \in \mathrm{Lip}(\mathbb{R})\colon\ f \geqslant 0,\ f(0) = 1,\ R_L(f) = f\bigr\}$$

   is one-dimensional.
3. *The set $\{e_\lambda\colon \lambda \in \Lambda\}$ is an ONB in $L^2(\widetilde{\mu})$, if the only L-cycle is trivial, i.e. is equal to* (1).

3.4. The Fourier basis on homeomorphic limit sets

In this section the above constructed Fourier basis for homogeneous linear IFSs will be carried over to $L^2(\mu)$ to obtain in this way a generalised Fourier basis. In fact, the following proposition shows that the basis elements obtained in our analysis can again be regarded as characters. Its proof is immediate.

Proposition 3.7. *Let (e_n) be the classical Fourier basis on $\mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ be a homeomorphism. Then $d_n := e_n \circ \phi^{-1}$ define characters on $(\mathbb{R}, \sharp)$, where the addition $\sharp : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is given by $(x, y) \mapsto \phi(\phi^{-1}(x) + \phi^{-1}(y))$.*

Now we are turning to the construction of the generalised Fourier basis on $L^2(\mu)$. It will be crucial that we impose the restriction $0 \in C$. Suppose that $\phi : \widetilde{C} \to C$ is the homeomorphism introduced in Section 3.1. We begin with the analogue statement to Lemma 3.4.

Lemma 3.8. *Let* $\{e_\lambda\colon \lambda \in \Lambda\}$ (*as specified above*) *be orthonormal in* $L^2(\widetilde{\mu})$*, then* $\{e_\lambda \circ \phi^{-1}\colon \lambda \in \Lambda\}$ *is orthonormal in* $L^2(\mu)$*.*

Proof. We have

$$\begin{aligned}\left\langle e_\lambda \circ \phi^{-1} \middle| e_{\lambda'} \circ \phi^{-1}\right\rangle_\mu &= \int \overline{e_\lambda \circ \phi^{-1}} e_{\lambda'} \circ \phi^{-1}\, d\mu \\ &= \int e^{-i2\pi\phi^{-1}(t)\lambda} e^{i2\pi\phi^{-1}(t)\lambda'}\, d\mu(t) \\ &= \int e^{-i2\pi t\lambda} e^{i2\pi t\lambda'}\, d\widetilde{\mu}(t) \\ &= \langle e_\lambda | e_{\lambda'}\rangle_{\widetilde{\mu}} \\ &= \delta_{\lambda\lambda'}. \qquad \square\end{aligned}$$

The existence of an ONB can also be transferred with the help of the homeomorphism ϕ.

Theorem 3.9. *If* $\{e_\lambda\colon \lambda \in \Lambda\}$ *is an ONB in* $L^2(\widetilde{\mu})$*, then* $\{e_\lambda \circ \phi^{-1}\colon \lambda \in \Lambda\}$ *is an ONB in* $L^2(\mu)$*.*

Proof. Only the spanning condition remains to be checked. For this let

$$f \in L^2(\mu) \setminus \operatorname{cl}\operatorname{span}\left\{e_\lambda \circ \phi^{-1}\colon \lambda \in \Lambda\right\}.$$

Then $f \circ \phi \in L^2(\widetilde{\mu}) \setminus \operatorname{cl}\operatorname{span}\{e_\lambda\colon \lambda \in \Lambda\}$ and since $\{e_\lambda\colon \lambda \in \Lambda\}$ is an ONB of $L^2(\widetilde{\mu})$ we have $f \circ \phi = 0$. Since ϕ bijective, it follows that also $f = 0$. $\square$

4. Examples

4.1. Wavelets

Example 4.1 *(1/3-Cantor set).* As an example for the affine case we will determine the wavelet basis for the 1/3-Cantor set C_3 (we refer to [3] for further details). The IFS on [0, 1] for this set is $\mathcal{S} = (\sigma_k\colon k = 0, 1)$ with $\sigma_k = \rho_{a_k,3} : x \mapsto (x + a_k)/3$, $a_0 := 0$ and $a_1 := 2$ and the gap filling IFS is $\mathcal{T} = (\tau_k\colon k = 0, 1, 2)$ with $\tau_k := \rho_{k,3}$. The father wavelet is $\varphi = \mathbb{1}_{C_3}$. The resulting filter functions on $\mathbb{T}$ are

$$\begin{aligned} m_0 &: z \mapsto \frac{1}{\sqrt{2}}\left(1 + z^2\right), \\ m_1 &: z \mapsto z, \\ m_2 &: z \mapsto \frac{1}{\sqrt{2}}\left(1 - z^2\right).\end{aligned}$$

So the mother wavelets are, for $x \in \mathbb{R}$,

$$\begin{aligned}\psi_1(x) &= \sqrt{2}\mathbb{1}_{C_3}(3x - 1), \\ \psi_2(x) &= \mathbb{1}_{C_3}(3x) - \mathbb{1}_{C_3}(3x - 2).\end{aligned}$$

Furthermore, the basis of $L^2(H^s)$, where $s = \log 2/\log 3$ and H^s is the s-Hausdorff measure (cf. [3,4]), is

$$\left\{x \mapsto 2^{-k/2}\psi_i\left(3^k x - l\right):\ i = 1,2,\ k,l \in \mathbb{Z}\right\}.$$

Example 4.2 *(1/4-Cantor set with one gap-filling contraction).* We now consider the IFS $\mathcal{S} := (\sigma_0, \sigma_1)$ and the gap filling IFS $\mathcal{T} := (\tau_0, \tau_1, \tau_2)$ with $\sigma_0 = \tau_0 = \rho_{0,4}$, $\sigma_1 = \tau_2 = \rho_{3,4}$, $\tau_1 : x \mapsto \frac{x}{2} + \frac{1}{4}$. Then the limit set of $\mathcal{S}$ is the 1/4-Cantor set C_4 and let H be the fractal measure constructed in Section 2.1.1. Already this example is not covered by [9] since the system is not homogeneous, i.e. τ_1 has a different scaling than τ_0 and τ_3.

The operators T and U for $f \in L^2(H)$ are then given by $(Tf)(x) := f(x-1)$ and $(Uf)(x) := 2^{-1/2} f(\sigma^{-1}(x))$, where the scaling function σ restricted to [0, 1] is given by

$$\sigma(x) := \mathbb{1}_{[0,\frac{1}{4})}(x)\cdot 4x + \mathbb{1}_{[\frac{1}{4},\frac{3}{4})}(x)\cdot\left(2x + \frac{1}{2}\right) + \mathbb{1}_{[\frac{3}{4},1)}(x)\cdot(4x-1), \quad x \in [0,1].$$

The father wavelet is $\varphi = \mathbb{1}_{C_4}$. The filter functions on $\mathbb{T}$ for the construction of the mother wavelets are the same as for the 1/3-Cantor case (see Example 4.1), because the form of the filter functions depends only on the number and position of the gaps, i.e.

$$\begin{aligned} m_0(z) &= \frac{1}{\sqrt{2}}\left(1 + z^2\right),\\ m_1(z) &= z,\\ m_2(z) &= \frac{1}{\sqrt{2}}\left(1 - z^2\right). \end{aligned}$$

So the mother wavelets are given, for $x \in [0,1]$, by

$$\begin{aligned} \psi_1(x) &= \left(U^{-1}m_1(T)\right)\varphi(x) = \sqrt{2}\varphi\left(\sigma(x) - 1\right)\\ &= \sqrt{2}\varphi\left(\mathbb{1}(x)\cdot 4x + \mathbb{1}_{[\frac{1}{4},\frac{3}{4})}(x)\cdot\left(2x + \frac{1}{2}\right) + \mathbb{1}_{[\frac{3}{4},1)}(x)\cdot(4x-1) - 1\right), \end{aligned}$$

$$\begin{aligned} \psi_2(x) &= \left(U^{-1}m_2(T)\right)\varphi = \varphi\left(\sigma(x)\right) - \varphi\left(\sigma(x) - 2\right)\\ &= \varphi\left(\mathbb{1}_{[0,\frac{1}{4})}(x)\cdot 4x + \mathbb{1}_{[\frac{1}{4},\frac{3}{4})}(x)\cdot\left(2x + \frac{1}{2}\right) + \mathbb{1}_{[\frac{3}{4},1)}(x)\cdot(4x-1)\right)\\ &\quad - \varphi\left(\mathbb{1}_{[0,\frac{1}{4})}(x)\cdot 4x + \mathbb{1}_{[\frac{1}{4},\frac{3}{4})}(x)\cdot\left(2x + \frac{1}{2}\right) + \mathbb{1}_{[\frac{3}{4},1)}(x)\cdot(4x-1) - 2\right). \end{aligned}$$

Thus, the orthonormal basis for $L^2(H)$ is

$$\left\{U^n T^k \psi_i:\ i = 1,2,\ n,k \in \mathbb{Z}\right\}.$$

4.2. Fourier bases

Example 4.3 *(1/4-Cantor set).* (See [9].) Let us recall the standard example for a Fourier basis for the 1/4-Cantor set C_4 supporting the Cantor measure μ_4 and with Hausdorff dimension equal to 1/2. This set is the limit set of the IFS on $[0,1]$ $\mathcal{S}=(\sigma_0,\sigma_1)$ given by $\sigma_0=\rho_{0,4}$ and $\sigma_1=\rho_{3,4}$. Hence, in Assumption 3.3 we have $A=\{0,3\}$. For $L:=\{0,2\}$ the matrix $H_{AL}=2^{-1/2}(e^{i2\pi al/4})_{a\in A,l\in L}$ is unitary and so the set $\{e_\lambda\colon \lambda\in\Lambda\}$ is orthonormal in $L^2(\mu_4)$, where

$$\Lambda=\left\{\sum_{j=0}^{k} l_j 4^j\colon\ l_j\in\{0,2\},\ k\in\mathbb{N}\right\}.$$

To show now that $\{e_\lambda\colon \lambda\in\Lambda\}$ is an ONB we will use the characterisation by L-cycles as stated in Proposition 3.6. We have that $z_\ell=e^{i2\pi\xi_\ell}\in\mathbb{T}$, $\ell=1,\dots,k+1$, is an L-cycle of length $k+1$ for $b_1,\dots,b_{k+1}\in\{0,2\}$ if $z_j=e^{i2\pi\frac{\xi_{j+1}-b_j}{4}}$, $j=1,\dots,k$, and $z_{k+1}=z_1$. Thus, $z_j=e^{i2\pi\frac{\xi_{j+1}}{4}}$ or $z_j=e^{i2\pi\frac{\xi_{j+1}-2}{4}}$ for $j=1,\dots,k$ and $z_1=z_{k+1}$. These conditions can only be satisfied for $k=0$, i.e. for the cycle (1). Hence by Proposition 3.6, $\{e_\lambda\colon \lambda\in\Lambda\}$ gives an ONB basis in $L^2(\mu_4)$.

Example 4.4 *(1/3-Cantor set).* The 1/3-Cantor set is the example for the case where a Fourier basis in the sense of [9] does not exist. The 1/3-Cantor set C_3 is given by the IFS $\mathcal{S}=(\sigma_0,\sigma_1)$ acting on $[0,1]$ with $\sigma_0=\rho_{0,3}$ and $\sigma_1=\rho_{2,3}$. Consequently, in Assumption 3.3 we have $A=\{0,2\}$. To get a Fourier basis, for the orthonormality a set $L\subset\mathbb{Z}$ is needed – such that $\operatorname{card} L=2$ and H_{AL} is unitary. But it is not possible to find such a set L satisfying these conditions (cf. [7,9]). If we would relax the condition $L\subset\mathbb{Z}$, we could choose $L=\{0,\frac{3}{4}\}$ to obtain H_{AL} unitary. If we now set $\Lambda:=\{\frac{3}{4}(l_0+3l_1+3^2l_2+\cdots+3^kl_k)\colon\ l_i\in\{0,1\},\ k\in\mathbb{N}\}$ we will find that $\{e_\lambda\colon \lambda\in\Lambda\}$ is not orthonormal. In fact, if we consider $\lambda=\frac{3}{4}$ and $\lambda'=\frac{9}{4}\in\Lambda$, then

$$\langle e_\lambda|e_{\lambda'}\rangle_\mu=\int e^{i2\pi x(\frac{9}{4}-\frac{3}{4})}\,d\mu(x)=\widehat{\mu}\left(\frac{2}{3}\right).$$

Since (cf. [9])

$$\begin{aligned}\widehat{\mu}(t)&=\int e^{i2\pi tx}\,d\mu(x)\\&=\frac{1}{2}\left(\int e^{i2\pi\frac{t}{3}x}\,d\mu(x)+\int e^{i2\pi\frac{t}{3}x}\cdot e^{i2\pi t\frac{2}{3}}\,d\mu(x)\right)\\&=\frac{1}{2}\left(\widehat{\mu}\left(\frac{t}{3}\right)+e^{i\frac{4}{3}\pi t}\widehat{\mu}\left(\frac{t}{3}\right)\right)\\&=\frac{1}{2}\left(1+e^{i\frac{4}{3}\pi t}\right)\cdot\widehat{\mu}\left(\frac{t}{3}\right),\end{aligned}$$

we find

$$\widehat{\mu}\left(\frac{3}{2}\right)=\underbrace{\frac{1}{2}\left(1+e^{i2\pi}\right)}_{=1}\cdot\widehat{\mu}\left(\frac{1}{2}\right)\neq 0.$$

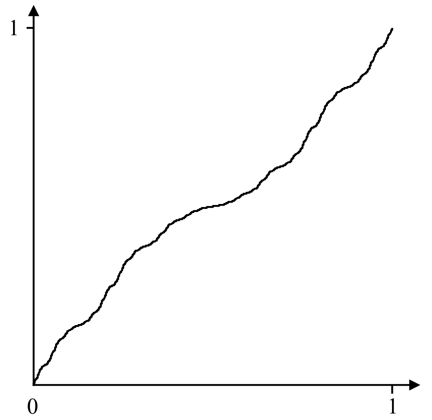

Fig. 2. The graph of the homeomorphism $\phi : [0,1] \to [0,1]$ conjugating the IFSs (τ_0, τ_1, τ_2) and $(\widetilde{\tau}_0, \widetilde{\tau}_1, \widetilde{\tau}_2)$ from Example 4.6.

This shows that the condition $L \subset \mathbb{Z}$ cannot be omitted. In fact by [9,8] there does not exist a set $L \subset \mathbb{R}$ such that $\{e_\lambda\colon \lambda \in \Lambda\}$ is a Fourier basis.

Remark 4.5. (1) From [9] we know that there are no more than two orthogonal functions e_λ (for any $\lambda \in \mathbb{R}$) in the Hilbert space $L^2(\mu)$.

(2) It has been shown in [8] that for an IFS with two branches of the form $\sigma_i(x) = N^{-1}x + b_i$, with $b_i \in \{0, a\}$, $i = 1, 2$, $a \in \mathbb{R} \setminus \{0\}$ and $N \in \mathbb{Z} \setminus \{-1, 0, 1\}$ such that the OSC is satisfied there does not exist a Fourier basis for any $a \in \mathbb{R} \setminus \{0\}$ if N is odd and there exists a basis for all $a \in \mathbb{R} \setminus \{0\}$ if N is even and $|N| \geqslant 4$.

Example 4.6 *(A generalised Fourier basis on the* $1/3$*-Cantor set).* As seen in the last example, it is not possible to construct a classical Fourier basis in the sense of [7,9] on $L^2(\mu_3)$, where μ_3 is the Cantor measure on the $1/3$-Cantor set C_3 given by $\mathcal{S} = (\sigma_0, \sigma_1)$, $\sigma_0 = \rho_{0,3}$, $\sigma_1 = \rho_{2,3}$. In Section 3.1 we have shown that there exists a homeomorphism ϕ conjugating the IFS $\mathcal{T} = (\tau_k\colon k \in \{0,1,2\})$ with $\tau_k = \rho_{k,3}$ and $\widetilde{\mathcal{T}} = (\widetilde{\tau}_0, \widetilde{\tau}_1, \widetilde{\tau}_2)$ with $\widetilde{\tau}_0 = \rho_{0,4}$, $\widetilde{\tau}_1 : x \mapsto \frac{2x+1}{4}$, $\widetilde{\tau}_2 = \rho_{3,4}$ (see Fig. 2). Note that $\mu_3 = \mu_4 \circ \phi^{-1}$ and that the homeomorphism restricted to the $1/4$-Cantor set C_4 is given explicitly by

$$\begin{aligned} &\phi : C_4 \to C_3, \\ &\sum_i \frac{a_i}{4^i} \mapsto \sum_i \frac{2/3 \cdot a_i}{3^i}, \quad a_i \in \{0, 3\}. \end{aligned}$$

Consequently, by Theorem 3.9 the Fourier basis of $L^2(\mu_4)$ can be carried over to $L^2(\mu_3)$. In fact, if we set $\Lambda := \{\sum_{j=0}^{k} l_j 4^j\colon l_j \in \{0,2\},\ k \in \mathbb{N}\}$ then $\{e_\lambda\colon \lambda \in \Lambda\}$ is an ONB in $L^2(\mu_4)$ and hence $\{e_\lambda \circ \phi^{-1}\colon \lambda \in \Lambda\}$ is an ONB in $L^2(\mu_3)$.

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