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Long Cycles through a Linear Forest[1]

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For a graph G and an integer $k \geqslant 1$, let $S(G) = \{x \in V(G) : d_G(x) = 0\}$ and $\sigma_k(G) = \min\{\sum_{i=1}^{k} d_G(v_i) : \{v_1, v_2, ..., v_k\}$ is an independent set of $G\}$. The main result of this paper is as follows. Let $k \geqslant 3$, $m \geqslant 0$, and $0 \leqslant s \leqslant k-3$. Let G be a $(m+k-1)$-connected graph and let F be a subgraph of G with $|E(F)| = m$ and $|S(F)| = s$. If every component of F is a path, then G has a cycle of length $\geqslant \min\{|V(G)|, \frac{2}{k}\sigma_k(G) - m\}$ passing through $E(F) \cup V(F)$. This generalizes three related results known previously. © 2001 Academic Press

Key Words: cycle; linear forest; length; matching.

1. INTRODUCTION

We use Bondy and Murty [3] for terminology and notation not defined here and consider simple graphs only. For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. The cardinality of $V(G)$ is denoted by $|G|$. If A and B are subgraphs of G or subsets of $V(G)$, we define $N_G(A) = \bigcup_{x \in A} N_G(x)$ and $N_B(A) = N_G(A) \cap B$. In particular, when $A = \{x\}$, we set $N_B(x) = N_B(\{x\})$ and $d_B(x) = |N_B(x)|$. For $\emptyset \neq D \subset V(G)$, let $G[D]$ denote the subgraph of G induced by D and define $G - D = G[V(G) - D]$. For $x, y \in V(G)$, $G + xy$ denotes the graph obtained from G by adding the edge xy. ($G + xy = G$ if $xy \in E(G)$.) If C is a cycle of G, we denote by $\vec{C}$ the cycle C with a given orientation. If $u \in V(C)$, then u^+ denotes the successor of u on $\vec{C}$ and u^- its predecessor; $u^{+2} = (u^+)^+$, etc. If $u, v \in V(C)$, we denote by $u\vec{C}v$ the subpath $uu^+ \cdots v^-v$ of C. If $u = v$, we define $u\vec{C}v = \{u\}$. The same subpath, in reverse order, is

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denoted by $v\overleftarrow{C}u$. If X is a cycle or a path of G, the length of X, denoted by $l(X)$, is defined as the number of edges of X. We consider that a single vertex is a path of length 0. The circumference of G, denoted by $c(G)$, is defined as the length of the longest cycle in G. Let $M \subseteq E(G)$ and $S \subseteq V(G)$. We say that a cycle C passes through $M \cup S$ if $M \subseteq E(C)$ and $S \subseteq V(C)$. M is called an m-matching if M is a set of m independent edges of G. (If $m=0$, we let $M=\emptyset$ and call M a 0-matching.)

For a graph G, we denote by $\alpha(G)$ and $\omega(G)$ the number of vertices in a maximum independent set of G and the number of components of G, respectively. Define

$$\sigma_k(G)=\min\left\{\sum_{i=1}^{k} d_G(v_i) : \{v_1, v_2, ..., v_k\} \text{ is an independent set of } G\right\}$$

if $k \leqslant \alpha(G)$; and $\sigma_k(G)=\infty$ if $k>\alpha(G)$.

A graph F is called a linear forest if $V(F)=E(F)=\emptyset$ or every component of F is a path. Define

$$\mathscr{F}_{m,s}=\{F : F \text{ is a linear forest with } |E(F)|=m \text{ and } |S(F)|=s\},$$

where

$$S(F)=\{x \in V(F) : d_F(x)=0\}.$$

There are many results about long cycles in a graph passing through some specified vertices or edges. Among them are the following two theorems.

THEOREM 1 (Enomoto [5]). *Let $m \geqslant 1$ and let G be an $(m+2)$-connected graph. Then G has a cycle of length $\geqslant \min\{|V(G)|, \sigma_2(G)-m\}$ passing through any path of length m.*

THEOREM 2 (Hirohata [8]). *Let $k \geqslant 3$ and $m \geqslant 1$ and let G be an $(m+k-1)$-connected graph. Then G has a cycle of length $\geqslant \min\{|V(G)|, \frac{2}{k}\sigma_k(G)-m\}$ passing through any path of length m.*

The main purpose of this paper is to prove the following result.

THEOREM 3. *Let $k \geqslant 3$, $m \geqslant 0$, and $0 \leqslant s \leqslant k-3$. Let G be a $(m+k-1)$-connected graph and let $F \in \mathscr{F}_{m,s}$ be a subgraph of G. Then G has a cycle of length $\geqslant \min\{|V(G)|, \frac{2}{k}\sigma_k(G)-m\}$ passing through $E(F) \cup V(F)$.*

Since $\frac{2}{k}\sigma_k(G) \geqslant \sigma_2(G)$ $(k \geqslant 3)$ and a path of length m is a special linear forest with m edges, Theorem 3 generalizes both Theorem 1 and Theorem 2. Moreover, by taking $m=s=0$, we get

Corollary 1. *Let $k \geqslant 2$ and let G be a k-connected graph. Then*

$$c(G) \geqslant \min\left\{|V(G)|, \frac{2}{k+1}\sigma_{k+1}(G)\right\}.$$

Corollary 1 was conjectured by Bondy in [1] and proved by Fournier and Fraisse in [6].

Let d be the minimum degree of G. Then, $\frac{1}{k+1}\sigma_{k+1}(G) \geqslant d$. Using Theorem 3 with $m=0$, we have

Corollary 2. *Let G be a k-connected graph, $k \geqslant 2$, with minimum degree d, and with at least $2d$ vertices. Let X be a set of $k-2$ vertices of G. Then, G has a cycle C of length at least $2d$ such that every vertex of X is on C.*

We note that a stronger version of Corollary 2 (the case $|X|=k$) was proved by Egawa *et al.* in [4].

Before beginning the proof of Theorem 3, we will give examples that demonstrate its sharpness.

Example. (a) Let $0 \leqslant t \leqslant m < p$ and let

$$G = (L_{t+1} \cup L_{m-t+1}) + (2K_p \cup \{x\}),$$

where the plus sign denotes the join operation and L_{i+1} $(i = t, m-t)$ denotes a path of length i. Then G is $(m+2)$-connected. It is easy to see that $\frac{2}{3}\sigma_3(G) - m = m + \frac{4p+8}{3}$. On the other hand, the length of a longest cycle passing through $E(L_{t+1} \cup L_{m-t+1}) \cup \{x\}$ is $(m+2)+(p+1) = \frac{2}{3}\sigma_3(G) - m - \frac{p-1}{3}$. Therefore, the condition $0 \leqslant s \leqslant k-3$ in Theorem 3 is best possible.

(b) Let $t \geqslant m+1$ and $G = L_{m+1} + 2K_t$. Then G is $(m+1)$-connected and $\sigma_2(G) = 2(t+m)$. The length of a longest cycle passing through $E(L_{m+1})$ is $m+t+1 = (\sigma_2(G)-m)-(t-1)$. Hence, the condition $k \geqslant 3$ cannot be replaced by $k \geqslant 2$.

(c) Let $p \geqslant 2m+t$ and $k = m+t+1$ and let $G = K_{2m+t} + K_p^c$, where K_p^c is the complement of K_p. Then, G is $(m+k-1)$-connected. Since $\sigma_k(G) = k(2m+t)$, $\frac{2}{k}\sigma_k(G) - m = 3m+2t$. On the other hand, the length of a longest cycle passing through any m-matching of K_{2m+t} is $(2m+t)+(m+t) = 3m+2t$. Hence, the bound $\min\{|V(G)|, \frac{2}{k}\sigma_k(G) - m\}$ is sharp.

2. PRELIMINARIES

In this section, we prove four lemmas. In the proofs of the first two lemmas, we use the following two theorems.

THEOREM 4 (Bondy and Jackson [2]). *Let G be a 2-connected graph on at least four vertices and u, v, w be vertices of G. If each vertex of $V(G)-\{u, v, w\}$ has degree at least d, then G contains a (u, v)-path of length at least d.*

THEOREM 5 (Häggkvist and Thomassen [7]). *Let $k \geqslant 2$ and let G be a k-connected graph. Then every set of $k-1$ independent edges of G is contained in a cycle.*

By Theorem 4, we have

LEMMA 1. *Let G be a 2-connected graph and w be a vertex of G. If each vertex of $V(G)-\{w\}$ has degree at least d, then, for every two distinct vertices u, v of G, G contains a (u, v)-path of length at least d.*

LEMMA 2. *Let m and k be integers with $m \geqslant 0$, $k \geqslant 2$, and $m+k \geqslant 3$. Let G be a $(m+k-1)$-connected graph and M an m-matching of G. If $S \subseteq V(G)-V(M)$ and $|S| \leqslant k-1$, then G contains a cycle passing through $M \cup S$.*

Proof. By $m+k-1 \geqslant \max\{m+1, 2\}$ and Theorem 5, G has a cycle passing through M. Among all of these cycles, we choose one (say C) such that $|V(C) \cap S|$ is maximum. If $S \subseteq V(C)$, then Lemma 2 holds. Assume that $S \nsubseteq V(C)$. We show that a contradiction arises. Let $x \in S-V(C)$ and let $t=\min\{m+k-1, |V(C)|\}$. Then G is t-connected and $|V(C)| \geqslant t$. By Menger's Theorem, there are t distinct paths $P_1, P_2, ..., P_t$ from x to $V(C)$ such that $|V(P_i) \cap V(C)|=1$ and $V(P_i) \cap V(P_j)=\{x\}$, $1 \leqslant i \neq j \leqslant t$. Suppose $V(P_i) \cap V(C)=\{v_i\}$, $1 \leqslant i \leqslant t$, and the vertices $v_1, v_2, ..., v_t$ appear in this order along $\vec{C}$. For $1 \leqslant i \leqslant t$, define $I_i=v_i \vec{C} v_{i+1}$ and set $v_{t+1}=v_1$ and $C_i=v_{i+1} \vec{C} v_i P_i x P_{i+1} v_{i+1}$. We consider two cases.

Case 1: $t=|V(C)|$. In this case, $V(C)=\{v_1, v_2, ..., v_t\}$. Since M is a matching, $E(C)-M \neq \emptyset$. Let $v_i v_{i+1} \in E(C)-M$. Then, C_i is a cycle passing through M and $(V(C) \cap S) \cup \{x\} \subseteq V(C_i)$, which contradicts the choice of C.

Case 2: $t=m+k-1<|V(C)|$. Since $m+k-1>|M|+|S \cap V(C)|$, there exists $i \in \{1, 2, ..., t\}$ such that $E(I_i) \cap M=\emptyset$ and $V(I_i) \cap S \subseteq \{v_i, v_{i+1}\}$. Then $M \subseteq E(C_i)$ and $(V(C) \cap S) \cup \{x\} \subseteq V(C_i)$, contrary to the choice of C. Hence, Lemma 2 is proved. ∎

The following lemma plays a crucial role in the proof of Theorem 3.

LEMMA 3. *Let $m \geqslant 0$ and $k \geqslant 3$. Let G be a $(m+k-1)$-connected graph and M an m-matching of G. Let $S \subseteq V(G) - V(M)$ with $|S| \leqslant k-3$ and let C be a longest cycle passing through $M \cup S$. If $l(C) < \min\{|V(G)|, 2L-m\}$, where L is a constant, then every component of $G - V(C)$ has a vertex x with $d_G(x) < L$.*

Proof. Assume, to the contrary, that $G - V(C)$ has a component H such that $d_G(x) \geqslant L$ for any $x \in V(H)$. Let $N_C(H) = \{v_1, v_2, ..., v_t\}$, and we may assume that the vertices $v_1, v_2, ..., v_t$ appear in this order along $\vec{C}$. For $1 \leqslant i \leqslant t$, define $I_i = v_i \vec{C} v_{i+1}$ and set $v_{t+1} = v_1$. We call I_i a segment of C. By an argument similar to that in the proof of Lemma 2, we can derive that $V(C) \neq N_C(H)$. Since G is $(m+k-1)$-connected, $t \geqslant m+k-1$. Since C is the longest cycle passing through $M \cup S$, $l(I_i) \geqslant 2$ if $E(I_i) \cap M = \emptyset$, and $l(I_i) \geqslant 1$ otherwise. Hence, $2L - m > l(C) \geqslant 2(t-m) + m$, which implies $t < L$. Therefore, we may assume that $|H| > 1$.

CLAIM 2.1. *There exists at most one vertex $x \in V(H)$ such that $d_H(x) = 1$.*

Proof. Assume, to the contrary, that $d_H(x) = d_H(y) = 1$ for some $x \neq y \in V(H)$. Then, $L > t \geqslant d_C(x) = d_G(x) - d_H(x) \geqslant L - 1$. This implies $d_C(x) = t \geqslant L - 1$. Therefore, $N_C(x) = N_C(H)$. Similarly, $N_C(y) = N_C(H)$. Since $t \geqslant m + k - 1 \geqslant |M| + |S| + 2$, there exist $1 \leqslant i < j \leqslant t$ such that, for each $r \in \{i, j\}$, $E(I_r) \cap M = \emptyset$ and $V(I_r) \cap S \subseteq \{v_r, v_{r+1}\}$. Let P be an (x, y)-path in H. Since C is the longest cycle passing through $M \cup S$, $l(I_r) \geqslant l(v_r x P y v_{r+1}) \geqslant 3$ for $r \in \{i, j\}$. Therefore,

$$l(C) \geqslant 3 + 3 + 2(t - m - 2) + m = 2t - m + 2 \geqslant 2L - m.$$

This contradiction completes the proof of Claim 2.1. ∎

It follows from Claim 2.1 that $|H| \geqslant 3$. We consider two cases.

Case 1: H is not 2-connected. In this case, H has at least two distinct end-blocks B_1 and B_2. For $i = 1, 2$, let c_i be the unique cut-vertex of H contained in B_i. (Possibly $c_1 = c_2$.) Without loss of generality, we may assume that $|B_1| \geqslant |B_2|$. By Claim 2.1, we have $|B_1| \geqslant 3$. Define

$$X = \{v_i : d_{B_1 - \{c_1\}}(v_i) \geqslant 2\}.$$

Then,

$$\sum_{u \in V(B_1) - \{c_1\}} d_{V(C) - X}(u) = |N_{V(C) - X}(B_1 - \{c_1\})| \leqslant t - |X|.$$

Hence,

$$\sum_{u \in V(B_1) - \{c_1\}} d_G(u) \leqslant \sum_{u \in V(B_1) - \{c_1\}} (d_{B_1}(u) + d_X(u)) + (t - |X|)$$
$$< (|B_1| - 1)(|B_1| - 1 + |X|) + (L - |X|). \tag{1}$$

On the other hand,

$$\sum_{u \in V(B_1) - \{c_1\}} d_G(u) \geqslant (|B_1| - 1)\, L. \tag{2}$$

By (1) and (2), we have $(|B_1| - 2)\, L < (|B_1| - 2)(|B_1| + |X|) + 1$. Since $|B_1| \geqslant 3$, $|B_1| + |X| > L - 1 > t - 1 \geqslant m + k - 2$. This implies $|B_1| + |X| \geqslant m + k - 1$. Take $X' \subseteq X$ such that $|X'| = \min\{|X|, m + k - 2\}$, and set $r = m + k - 1 - |X'|$. Then, $r \geqslant 1$ and $G - X'$ is r-connected. Note that $|B_1| \geqslant r$ and $|V(C) - X'| > t - |X'| \geqslant m + k - 1 - |X'| = r$. It follows from Menger's Theorem that $G - X'$ contains r pairwise disjoint paths $P_1, P_2, ..., P_r$ from $V(B_1)$ to $V(C) - X'$ such that $|V(P_i) \cap V(B_1)| = |V(P_i) \cap (V(C) - X')| = 1$, $1 \leqslant i \leqslant r$. Suppose $V(P_i) \cap (V(C) - X') = \{y_i\}$ and $u_i y_i \in E(P_i)$, $1 \leqslant i \leqslant r$. Noting that H is a component of $G - V(C)$, we have $V(P_i) \subseteq (V(C) - X') \cup V(H)$. Hence, $y_i \in N_C(H)$ and $u_i \in V(H)$. Since B_1 is an end-block of H, we have, for $1 \leqslant i \leqslant r$, that

$$V(P_i) \cap V(H) \nsubseteq V(B_1) - \{c_1\} \Rightarrow c_1 \in V(P_i).$$

Therefore, there exists at most one path P_i with $V(P_i) \cap V(H) \nsubseteq V(B_1) - \{c_1\}$. Since $u_i \in V(P_i) \cap V(H)$, $1 \leqslant i \leqslant r$, we have

$$|\{u_1, u_2, ..., u_r\} \cap (V(B_1) - \{c_1\})| \geqslant r - 1. \tag{3}$$

Define $X^* = X' \cup Y$, where $Y = \{y_1, y_2, ..., y_r\}$. Suppose $X^* = \{v_{i_1}, v_{i_2}, ..., v_{i_\alpha}\}$. Then, $\alpha = |X^*| = |X'| + |Y| = m + k - 1$ and we may assume that $i_1 < i_2 < \cdots < i_\alpha$. Define $i_{\alpha+1} = i_1$.

CLAIM 2.2. *There exists a triple (i, x, y) such that*

(a) $i_1 \leqslant i < i_2$;

(b) $x \in N_H(v_i)$, $y \in N_H(v_{i+1})$, *and* $x \neq y$;

(c) $\{x, y\} \cap (V(B_1) - \{c_1\}) \neq \varnothing$.

Proof. We first assume that $\{v_{i_1}, v_{i_2}\} \cap X \neq \varnothing$. Say $v_{i_1} \in X$. Then $d_{B_1 - \{c_1\}}(v_{i_1}) \geqslant 2$. Let w_1, w_2 be two distinct neighbors of v_{i_1} in $B_1 - \{c_1\}$ and let $w \in N_H(v_{i_1+1})$. Choose $u \in \{w_1, w_2\} - \{w\}$. Then, $(i, x, y) = (i_1, u, w)$ is as required.

Assume now $\{v_{i_1}, v_{i_2}\} \cap X = \emptyset$. Then, $r \geqslant 2$ and $\{v_{i_1}, v_{i_2}\} \subseteq Y$. Without loss of generality, we may assume that $v_{i_1} = y_1$, and $v_{i_2} = y_2$. By (3), we may assume that $u_1 \in V(B_1) - \{c_1\}$. If $u_1 \in N_H(v_{i_2-1})$, then $(i, x, y) = (i_2 - 1, u_1, u_2)$ is as required. Thus, we may assume that $u_1 \notin N_H(v_{i_2-1})$. Let $j_0 = \min\{j : i_1 \leqslant j < i_2, u_1 \notin N_H(v_j)\}$. Then, since $v_{i_1} = y_1 \in N_C(u_1)$, $j_0 > i_1$. By the choice of j_0, we have $u_1 \in N_H(v_{j_0-1})$. Let $u \in N_H(v_{j_0})$. Then, $u \neq u_1$ and $(i, x, y) = (j_0 - 1, u_1, u)$ is as required. Hence, Claim 2.2 is true. ∎

Claim 2.3. *Suppose $E(v_{i_1}\vec{C}v_{i_2}) \cap M = \emptyset$ and $V(v_{i_1}\vec{C}v_{i_2}) \cap S \subseteq \{v_{i_1}, v_{i_2}\}$. Then, there exists a segment I_i in $v_{i_1}\vec{C}v_{i_2}$ such that $l(I_i) \geqslant L - t + 2$.*

Proof. Let (i, x, y) be a triple that satisfies Claim 2.2. Then, $V(I_i) \subseteq V(v_{i_1}\vec{C}v_{i_2})$, and so $E(I_i) \cap M = \emptyset$ and $V(I_i) \cap S \subseteq \{v_i, v_{i+1}\}$. Let P be a longest (x, y)-path in H. Since C is the longest cycle passing through $M \cup S$,

$$l(I_i) \geqslant l(v_i x P y v_{i+1}) = l(P) + 2. \tag{4}$$

We first assume that $\{x, y\} \subseteq V(B_1)$. Then since B_1 is an end-block of H, we have

$$d_{B_1}(u) = d_H(u) = d_G(u) - d_C(u) \geqslant L - t$$

for each $u \in V(B_1) - \{c_1\}$. By Lemma 1, B_1 has an (x, y)-path Q of length at least $L - t$. By (4), $l(I_i) \geqslant l(P) + 2 \geqslant l(Q) + 2 \geqslant L - t + 2$.

Assume now $\{x, y\} \nsubseteq V(B_1)$. Without loss of generality, assume that $y \notin V(B_1)$. Then, since (i, x, y) satisfies Claim 2.2, we have $x \in V(B_1) - \{c_1\}$ and $y \in V(H) - V(B_1)$. By Lemma 1, B_1 has an (x, c_1)-path P_1 of length at least $L - t$. Since B_1 is an end-block of H, $H' = H - (V(B_1) - \{c_1\})$ is connected. Let P_2 be a (c_1, y)-path in H', then $xP_1c_1P_2y$ is an (x, y)-path in H. By (4), we have $l(I_i) \geqslant l(P) + 2 \geqslant l(xP_1c_1P_2y) + 2 \geqslant L - t + 3$. Hence, Claim 2.3 is true. ∎

Note that $\alpha = m + k - 1 \geqslant |M| + |S| + 2$. There exist p and q with $1 \leqslant p < q \leqslant \alpha$ such that, for each $j \in \{p, q\}$, $E(v_{i_j}\vec{C}v_{i_{j+1}}) \cap M = \emptyset$ and $V(v_{i_j}\vec{C}v_{i_{j+1}}) \cap S \subseteq \{v_{i_j}, v_{i_{j+1}}\}$. By an argument similar to that in the proofs of Claims 2.2 and 2.3, we can deduce that each of $v_{i_p}\vec{C}v_{i_{p+1}}$ and $v_{i_q}\vec{C}v_{i_{q+1}}$ contains a segment of length at least $L - t + 2$. Then

$$l(C) \geqslant 2(L - t + 2) + 2(t - m - 2) + m = 2L - m.$$

This contradiction completes the proof of Case 1 of Lemma 3.

Case 2: H is 2-connected. Let $w \in V(H)$. By replacing $B_1 - \{c_1\}$ with $H - \{w\}$ in the proof of Case 1, we get a similar contradiction. This contradiction completes the proof of Lemma 3. ∎

LEMMA 4. *Let G be a 2-connected graph and C the longest cycle passing through $M \cup S$, where $M \subseteq E(G)$ and $S \subseteq V(G) - V(M)$. Suppose that C is not a hamiltonian cycle and let H be a component of $G - V(C)$. Suppose $\{u, v\} \subseteq N_C(H)$ and $G[V(H) \cup \{u, v\}]$ contains a (u, v)-path P of length at least $r+1$, where r is a positive integer. If $(\{uu^+\} \cup E(v\vec{C}v^{+r})) \cap M = \emptyset$ and $V(v\vec{C}v^{+r}) \cap S \subseteq \{v, v^{+r}\}$, then*

(a) $N_{v^+\vec{C}v^{+r}}(u^+) = \emptyset$, *and*

(b) $d_C(u^+) + d_C(v^{+r}) \leqslant l(C) + |M|$.

Proof. (a) Assume, to the contrary, that there exists some i with $1 \leqslant i \leqslant r$ such that $u^+v^{+i} \in E(G)$. Then

$$u^+v^{+i}\vec{C}uPv\overleftarrow{C}u^+$$

is a cycle longer than C and passing through $M \cup S$, a contradiction.

(b) Let

$$X = \{x : x^+ \in N_{u^+\vec{C}v}(u^+)\},$$
$$Y = \{x : x^- \in N_{v^{+r}\vec{C}u}(u^+)\},$$
$$Z = N_C(v^{+r}).$$

Clearly, $X \cap Y = \{u^+\}$. Suppose $x \in X \cap Z$. Then $\{xv^{+r}, x^+u^+\} \subseteq E(G)$. If $xx^+ \notin M$, then

$$v^{+r}\vec{C}uPv\overleftarrow{C}x^+u^+\vec{C}xv^{+r}$$

is a cycle longer than C and passing through $M \cup S$. This contradiction shows

$$|X \cap Z| \leqslant |E(u^+\vec{C}v) \cap M|. \tag{5}$$

Suppose $x \in Y \cap Z$. Then, $\{x^-u^+, xv^{+r}\} \subseteq E(G)$. If $x^-x \notin M$, then

$$v^{+r}x\vec{C}uPv\overleftarrow{C}u^+x^-\overleftarrow{C}v^{+r}$$

is a cycle longer than C and passing through $M \cup S$. This contradiction shows

$$|Y \cap Z| \leqslant |E(v^{+r}\vec{C}u) \cap M|. \tag{6}$$

By (5) and (6), we get

$$|(X \cup Y) \cap Z| \leqslant |X \cap Z| + |Y \cap Z| \leqslant |M|. \tag{7}$$

Noting that $N_{v^+\vec{C}v^{+r}}(u^+)=\emptyset$, we have $|X|+|Y|=d_C(u^+)$. This together with $|Z|=d_C(v^{+r})$ and (7) implies

$$\begin{aligned}|X\cup Y\cup Z| &= |X\cup Y|+|Z-((X\cup Y)\cap Z)|\\ &= |X|+|Y|-|X\cap Y|+|Z|-|(X\cup Y)\cap Z|\\ &\geqslant d_C(u^+)-1+d_C(v^{+r})-|M|\\ &= d_C(u^+)+d_C(v^{+r})-|M|-1. \end{aligned} \tag{8}$$

Since $v^{+r}\notin X\cup Y\cup Z$,

$$|X\cup Y\cup Z|\leqslant l(C)-1. \tag{9}$$

By (8) and (9),

$$d_C(u^+)+d_C(v^{+r})\leqslant l(C)+|M|.$$

Hence, Lemma 4 is true. ∎

3. PROOF OF THEOREM 3

First, we prove a special case of Theorem 3.

THEOREM 6. *Let $m\geqslant 0$ and $k\geqslant 3$. Let G be a $(m+k-1)$-connected graph and M an m-matching of G. If $S\subseteq V(G)-V(M)$ with $|S|\leqslant k-3$, then G contains a cycle of length $\geqslant \min\{|V(G)|, \frac{2}{k}\sigma_k(G)-m\}$ passing through $M\cup S$.*

Proof. By Lemma 2, G has a cycle passing through $M\cup S$. Among all of these cycles, choose one, say C, with maximum length. By way of contradiction, assume that $l(C)<\min\{|V(G)|, \frac{2}{k}\sigma_k(G)-m\}$.

CLAIM 3.1. $\omega(G-V(C))\leqslant k-1$.

Proof. Assume, to the contrary, that $G-V(C)$ has k components $H_1, H_2, ..., H_k$. Using Lemma 3 with $L=\frac{1}{k}\sigma_k(G)$, we see that H_i $(1\leqslant i\leqslant k)$ has a vertex x_i with $d_G(x_i)<\frac{1}{k}\sigma_k(G)$. Since $\{x_1, x_2, ..., x_k\}$ is an independent set of order k, we have $\sigma_k(G)\leqslant\sum_{i=1}^{k} d_G(x_i)<\sigma_k(G)$, a contradiction. ∎

Let H_1 be any component of $G-V(C)$. Set $N_C(H_1)=\{v_1, v_2, ..., v_t\}$. We may assume that the vertices $v_1, v_2, ..., v_t$ appear in this order along $\vec{C}$. For $1\leqslant i\leqslant t$, define $I_i=v_i\vec{C}v_{i+1}$ and set $v_{t+1}=v_1$. We call I_i a segment of C. By an argument similar to that in the proof of Lemma 2, we have $V(C) \neq N_C(H_1)$. Since G is $(m+k-1)$-connected,

$$|M_C(H_1)| = t \geqslant m+k-1. \tag{10}$$

Define

$$X = \{v_i^+ : 1 \leqslant i \leqslant t,\ E(I_i) \cap M = \emptyset\},$$

$$Y = \{y \in X : N_{G-V(C)}(y) \neq \emptyset\},$$

$$Z = X - Y.$$

For any $y \in Y$, there exists a component H_y of $G - V(C)$ with $N_{H_y}(y) \neq \emptyset$. Note that $y^- \in N_C(H_1)$ and $y^- y \notin M$. Then, since C is the longest cycle passing through $M \cup S$, $H_y \neq H_1$ and $H_y \neq H_{y'}$ if $y \neq y' \in Y$. Therefore, $|Y| \leqslant \omega(G - V(C)) - 1 \leqslant k-2$ by Claim 3.1. By (10), we have

$$k-1-|Z| \leqslant |N_C(H_1)| - m - |Z| \leqslant |X| - |Z| = |Y| \leqslant k-2. \tag{11}$$

Claim 3.2. $|Z| \leqslant 1$.

Proof. Assume, to the contrary, that $|Z| \geqslant 2$. Let $Z = \{u_1^+, u_2^+, ..., u_p^+\}$ $(p \geqslant 2)$. Then, $u_i \in N_C(H_1)$, $u_i u_i^+ \notin M$, and $d_G(u_i^+) = d_C(u_i^+)$ for each i, $1 \leqslant i \leqslant p$. By $l(C) < \frac{2}{k}\sigma_k(G) - m$ and using Lemma 4 with $r = 1$, we have Z is an independent set of G, and

$$d_G(u_1^+) + d_G(u_2^+) < \frac{2}{k}\sigma_k(G),$$

$$d_G(u_2^+) + d_G(u_3^+) < \frac{2}{k}\sigma_k(G),$$

$$\vdots$$

$$d_G(u_p^+) + d_G(u_1^+) < \frac{2}{k}\sigma_k(G).$$

Hence,

$$\sum_{i=1}^{p} d_G(u_i^+) < \frac{p}{k}\sigma_k(G). \tag{12}$$

Using Lemma 3 with $L = \frac{1}{k}\sigma_k(G)$, we see that H_1 has a vertex x with $d_G(x) < \frac{1}{k}\sigma_k(G)$. If $p \geqslant k-1$, then, since $\{x, u_1^+, u_2^+, ..., u_{k-1}^+\}$ is an independent set of G of order k, by an argument similar to that in the proof of (12) we have

$$\sigma_k(G) \leqslant d_G(x) + \sum_{i=1}^{k-1} d_G(u_i^+) < \frac{1}{k}\sigma_k(G) + \frac{k-1}{k}\sigma_k(G).$$

This contradiction shows $p \leqslant k-2$. By (11), $|Y| \geqslant k-1-|Z| = k-p-1 \geqslant 1$. Let Y' be a subset of Y of order $k-p-1$. By the arguments stated before Claim 3.2, for each $y \in Y'$ we can get a component H_y of $G-V(C)$ with $N_{H_y}(y) \neq \emptyset$, which satisfies $H_y \neq H_1$ and $H_y \neq H_{y'}$ if $y \neq y' \in Y'$. Using Lemma 3 with $L = \frac{1}{k}\sigma_k(G)$, we derive that H_y has a vertex w_y with $d_G(w_y) < \frac{1}{k}\sigma_k(G)$. Let $W = \{w_y : y \in Y'\}$. Then, $|W| = k-p-1$ and

$$\sum_{w_y \in W} d_G(w_y) < \frac{k-p-1}{k}\sigma_k(G). \tag{13}$$

Noting that $Z \cup W \cup \{x\}$ is an independent set of order k, by (12) and (13), we have

$$\sigma_k(G) \leqslant \sum_{i=1}^{p} d_G(u_i^+) + \sum_{w_y \in W} d_G(w_y) + d_G(x) < \sigma_k(G).$$

This contradiction completes the proof of Claim 3.2. ▮

By Claim 3.2 and (11), we see that equality holds in Claim 3.2 and (11), and hence also in (10). In particular, we have $|Z| = 1$ and $|Y| = k-2$.

Claim 3.3. *For any i, j $(1 \leqslant i < j \leqslant t)$,*

$$|N_{H_1}(v_i) \cup N_{H_1}(v_j)| \geqslant \min\{|H_1|, 2\}.$$

Proof. Assume, to the contrary, that there are some i, j with $1 \leqslant i < j \leqslant t$ such that $|N_{H_1}(v_i) \cup N_{H_1}(v_j)| < \min\{|H_1|, 2\}$. Then, since $v_i \in N_C(H_1)$, $|N_{H_1}(v_i) \cup N_{H_1}(v_j)| = 1$ and $|H_1| \geqslant 2$. Let $N_{H_1}(v_i) \cup N_{H_1}(v_j) = \{u\}$. Then $G - ((N_C(H_1) - \{v_i, v_j\}) \cup \{u\})$ contains at least two components. Since G is $(m+k-1)$-connected, we have $|N_C(H_1)| - 1 \geqslant m+k-1$. This contradicts the earlier assertion that equality holds in (10). ▮

Recall that $|Z| = 1$ and $|Y| = k-2$. Let $Z = \{u_1^+\}$ and $Y = \{u_2^+, ..., u_{k-1}^+\}$. Then, $\{u_1, u_2, ..., u_{k-1}\} \subseteq N_C(H_1)$ and $N_{G-V(C)}(u_1^+) \neq \emptyset$. For $2 \leqslant i \leqslant k-1$, let H_i be a component of $G - V(C)$ with $N_{H_i}(u_i^+) \neq \emptyset$. By the arguments stated before Claim 3.2, we have $H_i \neq H_j$, $1 \leqslant i < j \leqslant k-1$. It follows from Claim 3.1 that $H_1, H_2, ..., H_{k-1}$ are the only components of

$G-V(C)$. For $1 \leqslant i \leqslant k-1$, let x_i be a vertex of H_i with $d_G(x_i) < \frac{1}{k}\sigma_k(G)$ and define $I_{u_i} = I_j$ if $u_i = v_j$. Since $u_i^+ \in X$, $E(I_{u_i}) \cap M = \emptyset$. Since C is the longest cycle passing through $M \cup S$, $l(I_{u_i}) \geqslant 2$. We consider two cases.

Case 1: $|H_1| \geqslant 2$. Consider $Y = \{u_2^+, ..., u_{k-1}^+\}$. Since $|S| \leqslant k-3$, $Y - S \neq \emptyset$. We may assume that $u_{k-1}^+ \notin S$. By Claim 3.3, $|N_{H_1}(u_1) \cup N_{H_1}(u_{k-1})| \geqslant 2$. There exists a (u_1, u_{k-1})-path of length at least 3 in $G[V(H_1) \cup \{u_1, u_{k-1}\}]$. Since $(\{u_1 u_1^+\} \cup E(u_{k-1}\vec{C}u_{k-1}^{+2})) \cap M = \emptyset$ and $u_{k-1}^+ \notin S$, by Lemma 4, we have that $u_1^+ u_{k-1}^{+2} \notin E(G)$ and

$$d_C(u_1^+) + d_C(u_{k-1}^{+2}) \leqslant l(C) + m < \frac{2}{k}\sigma_k(G). \tag{14}$$

Subcase 1.1: $N_{G-V(C)}(u_{k-1}^{+2}) = \emptyset$. By $u_1^+ \in Z$ and (14), we have

$$d_G(u_1^+) + d_G(u_{k-1}^{+2}) < \frac{2}{k}\sigma_k(G). \tag{15}$$

Since $d_G(x_i) < \frac{1}{k}\sigma_k(G)$ $(i = 1, ..., k-2)$, we get

$$\sum_{i=1}^{k-2} d_G(x_i) < \frac{k-2}{k}\sigma_k(G). \tag{16}$$

Since $\{x_1, ..., x_{k-2}, u_{k-1}^{+2}, u_1^+\}$ is an independent set of order k, by (15) and (16),

$$\sigma_k(G) \leqslant \sum_{i=1}^{k-2} d_G(x_i) + d_G(u_1^+) + d_G(u_{k-1}^{+2}) < \sigma_k(G),$$

a contradiction.

Subcase 1.2: $N_{G-V(C)}(u_{k-1}^{+2}) \neq \emptyset$. Then, there exists some j with $1 \leqslant j \leqslant k-1$ such that $N_{H_j}(u_{k-1}^{+2}) \neq \emptyset$. Since C is the longest cycle passing though $M \cup S$, by $N_{H_{k-1}}(u_{k-1}^+) \neq \emptyset$ and $u_{k-1}^+ u_{k-1}^{+2} \notin M$ we have $j \neq k-1$. Assume first that $j = 1$. Then $u_{k-1}^{+2} \in N_C(H_1)$. Noting that $u_{k-1} \in N_C(H_1)$, we have by Claim 3.3 that $|N_{H_1}(u_{k-1}) \cup N_{H_1}(u_{k-1}^{+2})| \geqslant 2$. Therefore, $G[V(H_1) \cup \{u_{k-1}, u_{k-1}^{+2}\}]$ contains a (u_{k-1}, u_{k-1}^{+2})-path Q of length at least 3. Since $E(u_{k-1}\vec{C}u_{k-1}^{+2}) \cap M = \emptyset$ and $u_{k-1}^+ \notin S$, we get a cycle

$$u_{k-1}^{+2}\vec{C}u_{k-1}Qu_{k-1}^{+2},$$

which is longer than C and passes through $M \cup S$. This contradiction shows $j \neq 1$. Therefore, $1 < j < k-1$. Let $w_1 \in N_{H_1}(u_j)$, $w'_1 \in N_{H_1}(u_{k-1})$, $w_j \in N_{H_j}(u_j^+)$, and $w'_j \in N_{H_j}(u_{k-1}^{+2})$. For $i = 1, j$, let P_i be a (w_i, w'_i)-path in H_i. Then, we get a cycle

$$u_{k-1}^{+2} \vec{C} u_j w_1 P_1 w'_1 u_{k-1} \overleftarrow{C} u_j^+ w_j P_j w'_j u_{k-1}^{+2},$$

which is longer than C and passes through $M \cup S$. This contradiction completes the proof of Case 1 of Theorem 6.

Case 2: $|H_1| = 1$. In this case, we have $V(H_1) = \{x_1\}$. Since equality holds in (10), this implies

$$d_G(x_1) = d_C(x_1) = |N_C(H_1)| = m + k - 1. \tag{17}$$

Noting that $\{u_1, u_2, ..., u_{k-1}\} \subseteq N_C(H_1)$ and $E(I_{u_i}) \cap M = \varnothing$ $(i = 1, 2, ..., k-1)$, we have from Lemma 4(a) that $\{u_1^+, u_2^+, ..., u_{k-1}^+\}$ is an independent set of G. Hence,

$$d_C(u_1^+) \leqslant l(C) - (k-1). \tag{18}$$

Since $u_1^+ \in Z$, by (17) and (18),

$$d_G(u_1^+) + d_G(x_1) \leqslant l(C) + m < \frac{2}{k} \sigma_k(G). \tag{19}$$

Since $\{u_1^+, x_1, x_2, ..., x_{k-1}\}$ is an independent set of order k, by $d_G(x_i) < \frac{1}{k} \sigma_k(G)$ $(i = 2, ..., k-1)$ and (19),

$$\sigma_k(G) \leqslant d_G(u_1^+) + d_G(x_1) + \sum_{i=2}^{k-1} d_G(x_i) < \sigma_k(G).$$

This contradiction completes the proof of Theorem 6. ∎

Finally, we turn to proving Theorem 3.

Proof of Theorem 3. Otherwise, let m be as small as possible such that there exists a graph G satisfying the condition of Theorem 3, but for G and its subgraph $F \in \mathscr{F}_{m,s}$ Theorem 3 does not hold. Then by Theorem 6, we may assume that $m \geqslant 2$ and $E(F)$ is not independent. Suppose $xy, yz \in E(F)$. Let $G' = G - \{y\} + xz$ and $F' = F - \{y\} + xz$. Then, G' is

$(m-1+k-1)$-connected and $F' \in \mathscr{F}_{m-1,s}$ is a subgraph of G'. By the choice of m, there is a cycle C' of length at least

$$\begin{aligned} \min &\left\{|V(G')|, \frac{2}{k}\sigma_k(G')-(m-1)\right\} \\ &\geqslant \min\left\{|V(G)|-1, \frac{2}{k}(\sigma_k(G)-k)-(m-1)\right\} \\ &= \min\left\{|V(G)|-1, \frac{2}{k}\sigma_k(G)-m-1\right\}, \end{aligned}$$

which passes through $E(F') \cup V(F')$ in G'. By replacing the edge xz of C' with xyz, we obtain a cycle of length $\geqslant \min\{|V(G)|, \frac{2}{k}\sigma_k(G)-m\}$ passing through $E(F) \cup V(F)$ in G. This contradiction completes the proof of Theorem 3. ▮

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