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Hopf bifurcation analysis in the 1-D Lengyel–Epstein reaction–diffusion model ☆

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The Lengyel–Epstein model with diffusion and homogeneous Neumann boundary condition is considered in this paper. We give the existence of multiple spatially non-homogeneous periodic solutions though all the parameters of the system are spatially homogeneous.

1. Introduction

In this work, we consider the Lengyel–Epstein reaction–diffusion system:

$$\begin{cases} u_t = \Delta u + a - u - \dfrac{4uv}{1+u^2}, & x \in \Omega,\ t > 0, \\ v_t = c\Delta v + b\left(u - \dfrac{uv}{1+u^2}\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$. Here $u = u(x,t)$ and $v = v(x,t)$ denote the chemical concentration of the activator iodide (I^-) and the inhibitor chlorite (CLO_2^-) respectively, at time $t > 0$ and a point $x \in \Omega$. The parameters a and b are parameters depending on the concentration of the starch, enlarging the effective diffusion ratio to c. We shall assume accordingly that all constants a, b and c are positive.

Problem (1.1) is based on the well-known chlorite–iodide–malonic acid chemical (CIMA) reaction, see [5,6]. A more detailed historical account of the development of CIMA reaction model and experiments can be found in [1]. In the past decade, some mathematical investigations are conducted, see for example [3,4,7–9].

From these papers, we notice that the dynamics of (1.1) is rich because of Turing instability and bifurcation phenomena.

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In [3], Jang, Ni and Tang proved a global bifurcation theorem which gives the existence of non-constant steady states for the c suitably chosen. In [7], Ni and Tang considered the Turing instability and Turing patterns. In [8], Yi, Wei and Shi derived precise conditions for the diffusion-driven instability with respect to the spatially homogeneous equilibrium solutions; they also performed a detailed Hopf bifurcation analysis for both ODE and PDE models, deriving a formula for determining the direction of the Hopf bifurcation and the stability of the bifurcating homogeneous periodic solutions. In [9], Yi, Wei and Shi proved that the constant equilibrium solution is globally asymptotically stable when the parameter a is small, and showed that for small spatial domains, all solutions eventually converge to a spatially homogeneous and time-periodic solution.

The purpose of this work is to find some spatially non-homogeneous periodic solutions, i.e., the periodic solutions caused by diffusion.

Furthermore, the referee comments us that J. Jin et al. [4] considered the same model using the similar methods, but chose the different bifurcation parameter.

2. Main result and the proof

While our calculations can be carried over to higher spatial domains, we restrict ourselves to the case of one-dimensional spatial domain $\Omega = (0, l\pi)$, $l \in \mathbb{R}^+$, that is

$$\begin{cases} u_t = u_{xx} + a - u - \dfrac{4uv}{1+u^2}, & x \in (0, l\pi),\ t > 0, \\ v_t = cv_{xx} + b\left(u - \dfrac{uv}{1+u^2}\right), & x \in (0, l\pi),\ t > 0, \\ u_x(0,t) = v_x(0,t) = u_x(l,t) = v_x(l,t) = 0, & t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (0, l\pi). \end{cases} \tag{2.1}$$

The unique constant steady state is $(u^*, v^*) = (\alpha, 1+\alpha^2)$, here $\alpha = a/5$. We shall maintain the basic hypothesis $3\alpha^2 > 5$ in the rest of this paper. This is important because it is the sufficient and necessary condition to ensure that the system (2.1) is an activator–inhibitor system, see [3] for details. In the following we shall fix a, c and use b as the main bifurcation parameter.

To cast our discussion into the framework of the Hopf bifurcation theorem, we translate (2.1) into the following system by the transition $\hat{u} = u - u^*$ and $\hat{v} = v - v^*$. For sake of convenience, we still let u and v denote $\hat{u}$ and $\hat{v}$ respectively, then we have

$$\begin{cases} u_t - u_{xx} = 4\alpha - u - \dfrac{4(u+\alpha)(v+1+\alpha^2)}{1+(u+\alpha)^2}, & x \in (0, l\pi),\ t > 0, \\ v_t - cv_{xx} = b\left(u + \alpha - \dfrac{(u+\alpha)(v+1+\alpha^2)}{1+(u+\alpha)^2}\right), & x \in (0, l\pi),\ t > 0, \\ u_x(0,t) = v_x(0,t) = u_x(l,t) = v_x(l,t) = 0, & t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (0, l\pi). \end{cases} \tag{2.2}$$

Define

$$f(b,u,v) := 4\alpha - u - \frac{4(u+\alpha)(v+1+\alpha^2)}{1+(u+\alpha)^2},$$
$$g(b,u,v) := b\left(u + \alpha - \frac{(u+\alpha)(v+1+\alpha^2)}{1+(u+\alpha)^2}\right),$$

here $f, g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are C^∞ smooth with $f(b,0,0) = g(b,0,0) = 0$.

Now we define the real-valued Sobolev space

$$\mathbf{X} := \left\{(u,v) \in \left[H^2(0, l\pi)\right]^2 \colon (u_x, v_x)|_{x=0,l\pi} = 0\right\},$$

and the complexification of $\mathbf{X}$:

$$\mathbf{X}_c = \mathbf{X} \oplus \mathrm{i}\mathbf{X} = \{x_1 + \mathrm{i}x_2 \colon x_1, x_2 \in \mathbf{X}\}.$$

The linearized operator of the steady state system of (2.2) evaluated at $(b, 0, 0)$ is

$$\mathcal{L}(b) = \begin{pmatrix} \frac{\partial^2}{\partial x^2} + A(b) & B(b) \\ C(b) & c\frac{\partial^2}{\partial x^2} + D(b) \end{pmatrix}$$

with the domain $D_{\mathcal{L}(b)} = \mathbf{X}_c$, where

$$A(b) = f_u(b,0,0) = \frac{3\alpha^2-5}{1+\alpha^2}, \qquad B(b) = f_v(b,0,0) = -\frac{4\alpha}{1+\alpha^2},$$
$$C(b) = g_u(b,0,0) = \frac{2\alpha^2 b}{1+\alpha^2}, \qquad D(b) = g_v(b,0,0) = -\frac{\alpha b}{1+\alpha^2}.$$

The following condition is essential to guarantee that the Hopf bifurcation occurs:

(H) There exist a number $b^H \in \mathbb{R}$ and a neighborhood O of b^H such that for $b \in O$, $\mathcal{L}(b)$ has a pair of complex, simple, conjugate eigenvalues $\alpha(b) \pm \mathrm{i}\omega(b)$, continuously differentiable in b, with $\alpha(b^H) = 0$, $\omega_0 := \omega(b^H) > 0$, and $\alpha'(b^H) \neq 0$; all other eigenvalues of $\mathcal{L}(b)$ have non-zero real parts for $b \in O$.

Now we can recall the Hopf bifurcation result appeared in [10] and apply them to analysis our model.

It is well known that the eigenvalue problem

$$-\varphi'' = \mu\varphi, \quad x \in (0, l\pi); \qquad \varphi'(0) = \varphi'(l\pi) = 0$$

has eigenvalues $\mu_n = \frac{n^2}{l^2}$ $(n = 0,1,2,\ldots)$, with corresponding eigenfunctions $\varphi_n(x) = \cos\frac{nx}{l}$. Let

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \sum_{n=0}^{\infty}\begin{pmatrix}a_n\\ b_n\end{pmatrix}\cos\frac{nx}{l}$$

be an eigenfunction of $\mathcal{L}(b)$ corresponding to an eigenvalue $\beta(b)$, that is, $\mathcal{L}(b)(\phi,\psi)^T = \beta(b)(\phi,\psi)^T$. Then from a straightforward analysis we obtain the following relation:

$$\mathcal{L}_n(b)\begin{pmatrix}a_n\\ b_n\end{pmatrix} = \beta(b)\begin{pmatrix}a_n\\ b_n\end{pmatrix}, \quad n = 0,1,2,\ldots,$$

where

$$\mathcal{L}_n(b) = \begin{pmatrix} -\frac{n^2}{l^2} + A(b) & B(b)\\ C(b) & -c\frac{n^2}{l^2} + D(b)\end{pmatrix}.$$

It follows that eigenvalues of $\mathcal{L}(b)$ are given by the eigenvalues of $\mathcal{L}_n(b)$ for $n = 0,1,2,\ldots$. The characteristic equation of $\mathcal{L}_n(b)$ is

$$\beta^2 - \beta T_n(b) + D_n(b) = 0, \quad n = 0,1,2,\ldots,$$

where

$$\begin{cases} T_n(b) = \dfrac{3\alpha^2-5}{1+\alpha^2} - \dfrac{\alpha b}{1+\alpha^2} - \dfrac{(1+c)n^2}{l^2},\\[2mm] D_n(b) = \dfrac{5\alpha b}{1+\alpha^2} + \dfrac{\alpha b}{1+\alpha^2}\dfrac{n^2}{l^2} + c\dfrac{n^2}{l^2}\left(\dfrac{n^2}{l^2} - \dfrac{3\alpha^2-5}{1+\alpha^2}\right). \end{cases} \tag{2.3}$$

Therefore, the eigenvalues are determined by

$$\beta(b) = \frac{T_n(b) \pm \sqrt{T_n^2(b) - 4D_n(b)}}{2}, \quad n = 0,1,2,\ldots.$$

If the condition (H) holds, we see that, at $b = b^H$, $\mathcal{L}(b)$ has a pair of simple purely imaginary eigenvalues $\pm\mathrm{i}\omega_0$ if and only if there exists a unique $n \in \mathbb{N}\cup\{0\}$ such that $\pm\mathrm{i}\omega_0$ are the purely imaginary eigenvalues of $\mathcal{L}_n(b)$. In such case, denote the associated eigenvector by $q = q_n = (a_n,b_n)^T\cos\frac{nx}{l}$, with $a_n, b_n \in \mathbb{C}$, such that $\mathcal{L}_n(b^H)(a_n,b_n)^T = \mathrm{i}\omega_0(a_n,b_n)^T$, or $\mathcal{L}(b^H)q = \mathrm{i}\omega_0 q$.

We shall identify the Hopf bifurcation value b^H which satisfies the condition (H), taking the following form now: there exists $n \in \mathbb{N}\cup\{0\}$ such that

$$T_n(b^H) = 0, \qquad D_n(b^H) > 0, \quad \text{and} \quad T_j(b^H) \neq 0, \qquad D_j(b^H) \neq 0 \quad \text{for } j \neq n \tag{2.4}$$

and for the unique pair of complex eigenvalues $\alpha(b) \pm \mathrm{i}\omega(b)$ near the imaginary axis

$$\alpha'(b^H) \neq 0.$$

It is easy to derive from (2.3) that $T_n(b) < 0$ and $D_n(b) > 0$ if

$$b > b_0^* := \frac{3\alpha^2 - 5}{\alpha} > 0$$

and one of the following holds:

$$\text{(i)} \quad \frac{5}{3} \leqslant \alpha^2 \leqslant \frac{1+5l^2}{3l^2-1}, \qquad \text{(ii)} \quad \alpha^2 \geqslant \frac{1+5l^2}{3l^2-1} \quad \text{and} \quad 0 < c < \frac{l^2\alpha b}{\frac{3l^2-1}{1+5l^2}\alpha^2 - 1},$$

which implies that $(0,0)$ is a locally asymptotically stable steady state of system (2.2). Here we discuss the potential bifurcation point in the interval $(0, b_0^*]$. For any Hopf bifurcation point b^H in $(0, b_0^*]$, $\alpha(b^H) \pm i\omega(b^H)$ are the eigenvalues of $\mathcal{L}_n(b^H)$, here

$$\alpha(b^H) = \frac{1}{2}T_n(b^H), \qquad \omega(b^H) = \sqrt{D_n(b^H) - \alpha^2(b^H)},$$

and

$$\alpha'(b^H) = \frac{1}{2}T_n'(b^H) < 0.$$

From the discussion above, the determination of Hopf bifurcation point reduces to describing the set

$$\Lambda := \{b^H \in (0, b_0^*]:\ \text{for some } n \in \mathbb{N} \cup \{0\},\ (2.4) \text{ is satisfied}\},$$

when a set of parameters (c, a) are fixed.

Firstly, $b_0^H := b_0^*$ is always an element of Λ since $T_0(b_0^H) = 0$, $T_j(b_0^H) < 0$ for any $j \geqslant 1$, $D_m(b_0^H) > 0$ for any $m \in \mathbb{N} \cup \{0\}$ and all c suitably large under a rather natural condition. This corresponds to the Hopf bifurcation of spatially homogeneous periodic solutions which have been studied in [8]. Apparently b_0^H is also the unique value for the Hopf bifurcation of the spatially homogeneous periodic solutions for any $l > \sqrt{3}/3$. Hence in the following we look for spatially non-homogeneous Hopf bifurcation points.

Notice that when $b < b_0^*$, $\frac{3\alpha^2-5}{1+\alpha^2} - \frac{\alpha b}{1+\alpha^2} > 0$. It is easy to see that $T_n(b) = 0$ is equivalent to

$$\frac{3\alpha^2 - 5}{1+\alpha^2} - \frac{\alpha b}{1+\alpha^2} = \frac{(1+c)n^2}{l^2},$$

that is

$$b = \left(\frac{3\alpha^2 - 5}{1+\alpha^2} - \frac{(1+c)n^2}{l^2}\right)\frac{1+\alpha^2}{\alpha}.$$

For such value of b, we have

$$\begin{aligned} D_n(b) &= \frac{cn^2}{l^2}\left(\frac{n^2}{l^2} - \frac{3\alpha^2-5}{1+\alpha^2}\right) + \frac{n^2}{l^2}\frac{\alpha b}{1+\alpha^2} + \frac{5\alpha b}{1+\alpha^2} \\ &= \frac{cn^2}{l^2}\left(\frac{n^2}{l^2} - \frac{3\alpha^2-5}{1+\alpha^2}\right) + \frac{n^2}{l^2}\left(\frac{3\alpha^2-5}{1+\alpha^2} - \frac{(1+c)n^2}{l^2}\right) + 5\left(\frac{3\alpha^2-5}{1+\alpha^2} - \frac{(1+c)n^2}{l^2}\right) \\ &= -\frac{n^4}{l^4} - \frac{n^2}{l^2}\left(\frac{3\alpha^2-5}{1+\alpha^2}(c-1) + 5 + 5c\right) + \frac{5(3\alpha^2-5)}{1+\alpha^2}. \end{aligned}$$

Let

$$B_0 := \frac{3\alpha^2 - 5}{1+\alpha^2}(c-1) + 5(1+c),$$

then $D_n(b) > 0$ if and only if

$$\frac{n^2}{l^2} < \frac{-B_0 + \sqrt{B_0^2 + \frac{20(3\alpha^2-5)}{1+\alpha^2}}}{2}.$$

So all the potential bifurcation points can be labeled as $\Lambda = \{b_n^H\}_{n=0}^N$ for some $N \in \mathbb{N} \cup \{0\}$, here

$$b_n^H = \left(\frac{3\alpha^2 - 5}{1+\alpha^2} - \frac{(1+c)n^2}{l^2}\right)\frac{1+\alpha^2}{\alpha}. \tag{2.5}$$

That is

$$0 < b_N^H < b_{N-1}^H < \cdots < b_1^H < b_0^H := b_0^*, \tag{2.6}$$

satisfying

$$0 \leqslant \frac{\frac{3\alpha^2-5}{1+\alpha^2} - \frac{b_n^H \alpha}{1+\alpha^2}}{1+c} < \frac{-B_0 + \sqrt{B_0^2 + \frac{20(3\alpha^2-5)}{1+\alpha^2}}}{2}.$$

Now we only need to verify whether $D_i(b_n^H) \neq 0$ for $i \neq n$. Here we will derive a condition on the parameters so that $D_i(b_n^H) > 0$, for each $i = 0, 1, 2, \ldots$.

Since

$$D_i\left(b_n^H\right) = c\frac{n^4}{l^4} + \frac{i^2}{l^2}\left(\frac{\alpha b_n^H}{1+\alpha^2} - c\frac{3\alpha^2-5}{1+\alpha^2}\right) + \frac{5\alpha b_n^H}{1+\alpha^2},$$

we can choose the diffusion coefficient c as small as possible so that $\frac{\alpha b_n^H}{1+\alpha^2} - c\frac{3\alpha^2-5}{1+\alpha^2} > 0$, i.e., given the fixed N defined by (2.6), for every $0 < n \leqslant N$, $c < M(\alpha, N, l)$, where

$$M(\alpha, N, l) := \frac{\frac{3\alpha^2-5}{1+\alpha^2} - \frac{N^2}{l^2}}{\frac{3\alpha^2-5}{1+\alpha^2} + \frac{N^2}{l^2}} > 0, \tag{2.7}$$

therefore $D_i(b_n^H) > 0$.

To adopt the framework of [2], see also in [10], we rewrite system (2.2) in the abstract form

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \mathcal{L}(b)U + F(b, U), \tag{2.8}$$

where

$$F(b, U) := \begin{pmatrix} f(b, u, v) - A(b)u - B(b)v \\ g(b, u, v) - C(b)u - D(b)v \end{pmatrix},$$

with $U = (u, v)^T \in \mathbf{X}$. At each $b = b_n^H$, $n = 1, 2, 3, \ldots, N$, system (2.8) can be reduced to

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \mathcal{L}\left(b_n^H\right)U + F_n(U), \tag{2.9}$$

where $F_n(U) := F(b, U)|_{b=b_n^H}$. Let $\langle \cdot, \cdot \rangle$ be the complex-valued L^2 inner product on Hilbert space $\mathbf{X}_{\mathbb{C}}$ defined as

$$\langle U_1, U_2 \rangle = \int_0^{l\pi} (\bar{u}_1 u_2 + \bar{v}_1 v_2)\,\mathrm{d}x,$$

with $U_i = (u_i, v_i)^T \in \mathbf{X}_{\mathbb{C}}$ $(i = 1, 2)$. Throughout this paper, we use $\bar{f}$ denote the conjugate of f. Then $\langle bU_1, U_2 \rangle = \bar{b}\langle U_1, U_2 \rangle$. Let $\mathcal{L}^*(b_n^H)$ be the adjoint operator of $\mathcal{L}(b_n^H)$, i.e. $\langle u, \mathcal{L}(b_n^H)v \rangle = \langle \mathcal{L}^*(b_n^H)u, v \rangle$. Then $\mathcal{L}^*(b_n^H)$ is also defined on $\mathbf{X}_{\mathbb{C}}$, and

$$\mathcal{L}^*\left(b_n^H\right) := \begin{pmatrix} \frac{\partial^2}{\partial x^2} + A(b_n^H) & C(b_n^H) \\ B(b_n^H) & c\frac{\partial^2}{\partial x^2} + D(b_n^H) \end{pmatrix}.$$

From (H), we can choose $q := (a_n, b_n)^T \cos\frac{nx}{l}$, $q^* := (a_n^*, b_n^*)^T \cos\frac{nx}{l} \in \mathbf{X}_{\mathbb{C}}$ satisfying

$$\mathcal{L}^*\left(b_n^H\right)q^* = -\mathrm{i}\omega_0^n q^*, \quad \left\langle q^*, q\right\rangle = 1, \quad \left\langle q^*, \bar{q}\right\rangle = 0, \tag{2.10}$$

here ω_0^n, q, q^* will be given concretely by (2.27)–(2.29) in the sequel.

We decompose $\mathbf{X} = \mathbf{X}^c \bigoplus \mathbf{X}^s$, with

$$\mathbf{X}^c := \{zq + \bar{z}\bar{q}\colon z \in \mathbb{C}\}, \qquad \mathbf{X}^s := \left\{u \in \mathbf{X}\colon \left\langle q^*, u\right\rangle = 0\right\}.$$

For any $(u, v) \in \mathbf{X}$, there exists $z \in \mathbb{C}$ and $w = (w_1, w_2) \in \mathbf{X}^s$ such that

$$\begin{pmatrix} u \\ v \end{pmatrix} = zq + \bar{z}\bar{q} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

or

$$\begin{cases} u = za_n \cos\dfrac{nx}{l} + \bar{z}\bar{a}_n \cos\dfrac{nx}{l} + w_1, \\ v = zb_n \cos\dfrac{nx}{l} + \bar{z}\bar{b}_n \cos\dfrac{nx}{l} + w_2. \end{cases} \tag{2.11}$$

Now system (2.9) can be reduced to the following system in (z, w) coordinates:

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{i}\omega_0^n z + \langle q^*, F_n\rangle, \\ \dfrac{\mathrm{d}w}{\mathrm{d}t} = \mathcal{L}\big(b_n^H\big)w + H(z, \bar{z}, w), \end{cases} \tag{2.12}$$

where

$$H(z, \bar{z}, w) := F_n - \langle q^*, F_n\rangle q - \langle \bar{q}^*, F_n\rangle \bar{q}, \qquad F_n := F_n(zq + \bar{z}\bar{q} + w). \tag{2.13}$$

As in [2], we write F_n in the form:

$$F_n(U) := \frac{1}{2}Q(U, U) + \frac{1}{6}C(U, U, U) + O\big(|U|^4\big), \tag{2.14}$$

here Q, C are symmetric multilinear forms. For simplicity, we write $Q_{XY} = Q(X, Y)$, $C_{XYZ} = C(X, Y, Z)$. For later uses, we calculate Q_{qq}, $Q_{q\bar{q}}$ and $C_{qq\bar{q}}$ as follows:

$$Q_{qq} = \begin{pmatrix} c_n \\ d_n \end{pmatrix} \cos^2\frac{nx}{l}, \qquad Q_{q\bar{q}} = \begin{pmatrix} e_n \\ f_n \end{pmatrix} \cos^2\frac{nx}{l}, \qquad C_{qq\bar{q}} = \begin{pmatrix} g_n \\ h_n \end{pmatrix} \cos^3\frac{nx}{l},$$

where (with the partial derivatives evaluated at $(b_n^H, 0, 0)$)

$$\begin{cases} c_n = f_{uu}a_n^2 + 2f_{uv}a_nb_n + f_{vv}b_n^2, \\ d_n = g_{uu}a_n^2 + 2g_{uv}a_nb_n + g_{vv}b_n^2, \\ e_n = f_{uu}|a_n|^2 + f_{uv}(a_n\bar{b}_n + \bar{a}_nb_n) + f_{vv}|b_n|^2, \\ f_n = g_{uu}|a_n|^2 + g_{uv}(a_n\bar{b}_n + \bar{a}_nb_n) + g_{vv}|b_n|^2, \\ g_n = f_{uuu}|a_n|^2a_n + f_{uuv}\big(2|a_n|^2b_n + a_n^2\bar{b}_n\big) + f_{uvv}\big(2|b_n|^2a_n + b_n^2\bar{a}_n\big) + f_{vvv}|b_n|^2b_n, \\ h_n = g_{uuu}|a_n|^2a_n + g_{uuv}\big(2|a_n|^2b_n + a_n^2\bar{b}_n\big) + g_{uvv}\big(2|b_n|^2a_n + b_n^2\bar{a}_n\big) + g_{vvv}|b_n|^2b_n. \end{cases} \tag{2.15}$$

Let

$$H(z, \bar{z}, w) = \frac{H_{20}}{2}z^2 + H_{11}z\bar{z} + \frac{H_{02}}{2}\bar{z}^2 + o\big(|z| \cdot |w|\big), \tag{2.16}$$

then by (2.13) and (2.14), we have

$$\begin{cases} H_{20} = Q_{qq} - \langle q^*, Q_{qq}\rangle q - \langle \bar{q}^*, Q_{qq}\rangle \bar{q}, \\ H_{11} = Q_{q\bar{q}} - \langle q^*, Q_{q\bar{q}}\rangle q - \langle \bar{q}^*, Q_{q\bar{q}}\rangle \bar{q}. \end{cases}$$

It follows from Appendix A of [2] that system (2.12) possesses a center manifold. We can write w in the form:

$$w = \frac{w_{20}}{2}z^2 + w_{11}z\bar{z} + \frac{w_{02}}{2}\bar{z}^2 + o\big(|z|^3\big). \tag{2.17}$$

By (2.16) and (2.17), together with

$$\mathcal{L}\big(b_n^H\big)w + H(z, \bar{z}, w) = \frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial w}{\partial \bar{z}}\frac{d\bar{z}}{dt},$$

we have

$$w_{20} = \big[2\mathrm{i}\omega_0^n I - \mathcal{L}\big(b_n^H\big)\big]^{-1}H_{20}, \qquad w_{11} = -\big[\mathcal{L}\big(b_n^H\big)\big]^{-1}H_{11}.$$

Actually, by [10] we have

$$w_{20} = \begin{cases} \frac{1}{2}[2\mathrm{i}\omega_0^n I - \mathcal{L}(b_n^H)]^{-1}\big[(\cos\frac{2nx}{l} + 1)\binom{c_n}{d_n}\big] & \text{if } n \in \mathbb{N}, \\ [2\mathrm{i}\omega_0^n I - \mathcal{L}(b_n^H)]^{-1}\big[\binom{c_0}{d_0} - \langle q^*, Q_{qq}\rangle\binom{a_0}{b_0} - \langle \bar{q}^*, Q_{qq}\rangle\binom{\bar{a}_0}{\bar{b}_0}\big] & \text{if } n = 0, \end{cases} \tag{2.18}$$

and

$$w_{11} = \begin{cases} -\frac{1}{2}[\mathcal{L}(b_n^H)]^{-1}\left[(\cos\frac{2nx}{l}+1)\binom{e_n}{f_n}\right] & \text{if } n \in \mathbb{N}, \\ -[\mathcal{L}(b_n^H)]^{-1}\left[\binom{e_0}{f_0} - \langle q^*, Q_{q\bar{q}}\rangle\binom{a_0}{b_0} - \langle \bar{q}^*, Q_{q\bar{q}}\rangle\binom{\bar{a}_0}{\bar{b}_0}\right] & \text{if } n = 0. \end{cases} \tag{2.19}$$

Notice that the calculation of $[2\mathrm{i}\omega_0^n I - \mathcal{L}(b_n^H)]^{-1}$ and $[\mathcal{L}(b_n^H)]^{-1}$ in (2.18) and (2.19) are restricted to the subspaces spanned by the eigen-modes 1 and $\cos\frac{2nx}{l}$.

Now the reaction–diffusion system restricted to the center manifold can be given by

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{i}\omega_0^n z + \langle q^*, F_n\rangle = \mathrm{i}\omega_0^n z + \sum_{2\leqslant i+j\leqslant 3}\frac{g_{ij}}{i!j!}z^i\bar{z}^j + O(|z|^4), \tag{2.20}$$

where

$$g_{20} = \langle q^*, Q_{qq}\rangle, \qquad g_{11} = \langle q^*, Q_{q\bar{q}}\rangle, \qquad g_{02} = \langle q^*, Q_{\bar{q}\bar{q}}\rangle,$$
$$g_{21} = 2\langle q^*, Q_{w_{11}q}\rangle + \langle q^*, Q_{w_{20}\bar{q}}\rangle + \langle q^*, C_{qq\bar{q}}\rangle.$$

The dynamics of (2.12) can be determined by the dynamics of (2.20). As in [2], we write the Poincaré normal form of (2.8) (for b in a neighborhood of b_n^H) in the form:

$$\dot{z} = (\alpha(b) + \mathrm{i}\omega(b))z + z\sum_{j=1}^{M} c_j(b)(z\bar{z})^j, \tag{2.21}$$

where z is a complex variable, $M \geqslant 1$ and $c_j(b)$ are complex-valued coefficients. Then following [2], we have

$$c_1(b) = \frac{g_{20}g_{11}(3\alpha(b) + \mathrm{i}\omega(b))}{2(\alpha^2(b) + \omega^2(b))} + \frac{|g_{11}|^2}{\alpha(b) + \mathrm{i}\omega(b)} + \frac{|g_{02}|^2}{2(\alpha(b) + 3\mathrm{i}\omega(b))} + \frac{g_{21}}{2}.$$

Thus

$$\begin{aligned} c_1(b_n^H) &= \frac{\mathrm{i}}{2\omega_0^n}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2} \\ &= \frac{\mathrm{i}}{2\omega_0^n}\langle q^*, Q_{qq}\rangle\cdot\langle q^*, Q_{q\bar{q}}\rangle + \langle q^*, Q_{w_{11}q}\rangle + \frac{1}{2}\langle q^*, Q_{w_{20}\bar{q}}\rangle + \frac{1}{2}\langle q^*, C_{qq\bar{q}}\rangle \end{aligned} \tag{2.22}$$

with w_{20} and w_{11} in the form of (2.18) and (2.19), respectively. From Theorem II and Section 3 in Chapter 1 of [2], under (H), we can establish the main result of our work:

Theorem 1. *For any b_n^H, defined by (2.5), if there exists $M = M(\alpha, N, l)$ defined by (2.7), such that $0 < c < M$, then system (2.2) undergoes Hopf bifurcation at each $b = b_n^H$, $0 \leqslant n \leqslant N$. With s sufficiently small, for $b = b(s)$, $b(0) = b_n^H$, there exist a family of $T(s)$-periodic continuously differentiable solutions $(u(s)(x,t), v(s)(x,t))$, and the bifurcating periodic solutions can be parameterized in the form of*

$$\begin{cases} u(s)(x,t) = s\left(a_n \mathrm{e}^{2\pi\mathrm{i}t/T(s)} + \bar{a}_n \mathrm{e}^{-2\pi\mathrm{i}t/T(s)}\right)\cos\frac{nx}{l} + o(s^2), \\ v(s)(x,t) = s\left(b_n \mathrm{e}^{2\pi\mathrm{i}t/T(s)} + \bar{b}_n \mathrm{e}^{-2\pi\mathrm{i}t/T(s)}\right)\cos\frac{nx}{l} + o(s^2), \end{cases} \tag{2.23}$$

where a_n, b_n will be given concretely by (2.30),

$$T(s) = \frac{2\pi}{\omega_0^n}(1 + \tau_2 s^2) + o(s^4), \quad \tau_2 = -\frac{1}{\omega_0^n}\left(\mathrm{Im}(c_1(b_n^H)) - \frac{\mathrm{Re}(c_1(b_n^H))}{\alpha'(b_n^H)}\omega'(b_n^H)\right),$$

and

$$T''(0) = \frac{4\pi}{\omega_0^n}\tau_2 = -\frac{4\pi}{(\omega_0^n)^2}\left(\mathrm{Im}(c_1(b_n^H)) - \frac{\mathrm{Re}(c_1(b_n^H))}{\alpha'(b_n^H)}\omega'(b_n^H)\right).$$

If all eigenvalues (except $\pm\mathrm{i}\omega_0^n$) of $\mathcal{L}(b_n^H)$ have negative real parts, then the bifurcating periodic solutions are stable (resp. unstable) if $\mathrm{Re}(c_1(b_n^H)) < 0$ (resp. > 0). The bifurcation is supercritical (resp. subcritical) if $-\frac{1}{\alpha'(b_n^H)}\mathrm{Re}(c_1(b_n^H)) < 0$ (resp. > 0).

Moreover:

(1) *The bifurcating periodic solutions from b_0^H are spatially homogeneous, which coincides with the periodic solutions of the corresponding ODE system.*
(2) *The bifurcating periodic solutions from b_n^H, $n > 0$, are spatially non-homogeneous.*

Proof. Following the bifurcation formula (pp. 28–32, [2]), we observe that (2.21) is rotationally invariant: if z is a solution then so is $ze^{i\phi}$ for any real number ϕ, and the trajectories of (2.21) are circles with centers at $z = 0$. This simple geometry is reflected in efficient computation of the Maclaurin expansions of $b(s)$ and $T(s)$. Forming $\bar{z}\frac{dz}{dt} + z\frac{d\bar{z}}{dt}$ from (2.21), we obtain

$$\frac{d}{dt}(z\bar{z}) = 2z\bar{z}\left(\alpha(b) + \sum_{j=1}^{M} \operatorname{Re}\big(c_j(b)\big)(z\bar{z})^j\right). \tag{2.24}$$

The right-hand side of (2.24) is zero if and only if $z = 0$ or

$$\alpha(b) + \sum_{j=1}^{M} \operatorname{Re}\big(c_j(b)\big)(z\bar{z})^j = 0. \tag{2.25}$$

Since $z(t) \not\equiv 0$, (2.25) is a proper condition to ensure that the right-hand side of (2.24) is zero. In such case, $z(t)\bar{z}(t)$ is a non-negative constant, denoted by s^2 for some $s \geqslant 0$. Thus, (2.21) can be written as

$$\dot{z} = iz\left[\omega(b) + \operatorname{Im}\left(\sum_{j=1}^{M} c_j(b)s^{2j}\right)\right]. \tag{2.26}$$

It follows from (2.26) that $z(t) = se^{2\pi it/T(s)}$, and by expansion, we have

$$T(s) = \frac{2\pi}{\omega_0^n}\big(1 + \tau_2 s^2\big) + o\big(s^4\big), \quad \tau_2 = -\frac{1}{\omega_0^n}\left(\operatorname{Im}\big(c_1\big(b_n^H\big)\big) - \frac{\operatorname{Re}(c_1(b_n^H))}{\alpha'(b_n^H)}\omega'\big(b_n^H\big)\right).$$

From (2.11) we get solutions of (2.2) in the form of (2.23).

Since q and q^* satisfy (2.10), it is easy to get

$$q = (a_n, b_n)^T \cos\frac{nx}{l} = \left(1, \frac{3\alpha^2 - 5}{4\alpha} - \frac{(1+\alpha^2)n^2}{4\alpha l^2} - \frac{\omega_0^n(1+\alpha^2)}{4\alpha}i\right)^T \cos\frac{nx}{l}, \tag{2.27}$$

$$q^* = \big(a_n^*, b_n^*\big)^T \cos\frac{nx}{l} = \frac{4\alpha}{\omega_0^n(1+\alpha^2)l\pi}\left(\frac{\omega_0^n(1+\alpha^2)}{4\alpha} + \left(\frac{3\alpha^2 - 5}{4\alpha} - \frac{(1+\alpha^2)n^2}{4\alpha l^2}\right)i, -i\right)^T \cos\frac{nx}{l}, \tag{2.28}$$

where

$$\omega_0^n = \left[\frac{8\alpha^3 b_n^H}{(1+\alpha^2)^2} - \left(\frac{n^2}{l^2} - \frac{3\alpha^2 - 5}{1+\alpha^2}\right)^2\right]^{1/2}. \tag{2.29}$$

This implies that

$$\begin{aligned} &a_n = 1, \qquad b_n = \frac{3\alpha^2 - 5}{4\alpha} - \frac{(1+\alpha^2)n^2}{4\alpha l^2} - \frac{\omega_0^n(1+\alpha^2)}{4\alpha}i, \\ &a_n^* = \frac{4\alpha}{\omega_0^n(1+\alpha^2)l\pi}\left[\frac{\omega_0^n(1+\alpha^2)}{4\alpha} + \left(\frac{3\alpha^2 - 5}{4\alpha} - \frac{(1+\alpha^2)n^2}{4\alpha l^2}\right)i\right], \qquad b_n^* = -i\frac{4\alpha}{\omega_0^n(1+\alpha^2)l\pi}. \end{aligned} \tag{2.30}$$

Now we consider the stability and bifurcation direction. Since $\alpha'(b_n^H) < 0$, we only need to compute the sign of $\operatorname{Re}(c_1(b_n^H))$.

By (2.22), we have

$$\operatorname{Re}\big(c_1\big(b_n^H\big)\big) = \operatorname{Re}\langle q^*, Q_{w_{11}q}\rangle + \frac{1}{2}\operatorname{Re}\langle q^*, Q_{w_{20}\bar{q}}\rangle + \frac{1}{2}\operatorname{Re}\langle q^*, C_{qq\bar{q}}\rangle. \tag{2.31}$$

From [10], we get $\langle q^*, Q_{qq}\rangle = \langle q^*, Q_{q\bar{q}}\rangle = 0$. In order to calculate $\operatorname{Re}(c_1(b_n^H))$, it remains to calculate

$$\langle q^*, Q_{w_{11}q}\rangle, \qquad \langle q^*, Q_{w_{20}\bar{q}}\rangle \quad \text{and} \quad \langle q^*, C_{qq\bar{q}}\rangle.$$

It is straightforward to compute that

$$\left[2\mathrm{i}\omega_0^n I-\mathcal{L}_{2n}\left(b_n^H\right)\right]^{-1}=\begin{pmatrix}2\mathrm{i}\omega_0^n-(\frac{3\alpha^2-5}{1+\alpha^2}-\frac{4n^2}{l^2}) & \frac{4\alpha}{1+\alpha^2}\\ -\frac{2\alpha^2 b_n^H}{1+\alpha^2} & 2\mathrm{i}\omega_0^n+\frac{\alpha b_n^H}{1+\alpha^2}+\frac{4cn^2}{l^2}\end{pmatrix}^{-1}$$
$$=(\alpha_1+\alpha_2\mathrm{i})^{-1}\begin{pmatrix}2\mathrm{i}\omega_0^n+\frac{\alpha b_n^H}{1+\alpha^2}+\frac{4cn^2}{l^2} & -\frac{4\alpha}{1+\alpha^2}\\ \frac{2\alpha^2 b_n^H}{1+\alpha^2} & 2\mathrm{i}\omega_0^n-\frac{3\alpha^2-5}{1+\alpha^2}+\frac{4n^2}{l^2}\end{pmatrix},$$

$$\left[2\mathrm{i}\omega_0^n I-\mathcal{L}_{0}\left(b_n^H\right)\right]^{-1}=\begin{pmatrix}2\mathrm{i}\omega_0^n-\frac{3\alpha^2-5}{1+\alpha^2} & \frac{4\alpha}{1+\alpha^2}\\ -\frac{2\alpha^2 b_n^H}{1+\alpha^2} & 2\mathrm{i}\omega_0^n+\frac{\alpha b_n^H}{1+\alpha^2}\end{pmatrix}^{-1}$$
$$=(\alpha_3+\alpha_4\mathrm{i})^{-1}\begin{pmatrix}2\mathrm{i}\omega_0^n+\frac{\alpha b_n^H}{1+\alpha^2} & -\frac{4\alpha}{1+\alpha^2}\\ \frac{2\alpha^2 b_n^H}{1+\alpha^2} & 2\mathrm{i}\omega_0^n-\frac{3\alpha^2-5}{1+\alpha^2}\end{pmatrix},$$

where

$$\alpha_1=-4\left(\omega_0^n\right)^2-\left(\frac{\alpha b_n^H}{1+\alpha^2}+\frac{4cn^2}{l^2}\right)\left(\frac{3\alpha^2-5}{1+\alpha^2}-\frac{4n^2}{l^2}\right)+\frac{4\alpha}{1+\alpha^2}\frac{2\alpha^2 b_n^H}{1+\alpha^2},$$
$$\alpha_2=2\omega_0^n\left(\frac{\alpha b_n^H}{1+\alpha^2}+\frac{4cn^2}{l^2}-\frac{3\alpha^2-5}{1+\alpha^2}+\frac{4n^2}{l^2}\right),$$
$$\alpha_3=-4\left(\omega_0^n\right)^2-\frac{\alpha b_n^H}{1+\alpha^2}\frac{3\alpha^2-5}{1+\alpha^2}+\frac{4\alpha}{1+\alpha^2}\frac{2\alpha^2 b_n^H}{1+\alpha^2},$$
$$\alpha_4=2\omega_0^n\left(\frac{\alpha b_n^H}{1+\alpha^2}-\frac{3\alpha^2-5}{1+\alpha^2}\right).$$

By (2.18), we have that, when $n\in\mathbb{N}$,

$$w_{20}=\left(\frac{[2\mathrm{i}\omega_0^n I-\mathcal{L}_{2n}(b_n^H)]^{-1}}{2}\cos\frac{2nx}{l}+\frac{[2\mathrm{i}\omega_0^n I-\mathcal{L}_{0}(b_n^H)]^{-1}}{2}\right)\begin{pmatrix}c_n\\ d_n\end{pmatrix}$$
$$=\frac{[\alpha_1+\alpha_2\mathrm{i}]^{-1}}{2}\begin{pmatrix}(2\mathrm{i}\omega_0^n I+\frac{\alpha b_n^H}{1+\alpha^2}+\frac{4cn^2}{l^2})c_n-\frac{4\alpha}{1+\alpha^2}d_n\\ \frac{2\alpha^2 b_n^H}{1+\alpha^2}c_n+(2\mathrm{i}\omega_0^n-\frac{3\alpha^2-5}{1+\alpha^2}+\frac{4n^2}{l^2})d_n\end{pmatrix}\cos\frac{2n}{l}x$$
$$+\frac{(\alpha_3+\alpha_4\mathrm{i})^{-1}}{2}\begin{pmatrix}(2\mathrm{i}\omega_0^n+\frac{\alpha b_n^H}{1+\alpha^2})c_n-\frac{4\alpha}{1+\alpha^2}d_n\\ \frac{2\alpha^2 b_n^H}{1+\alpha^2}c_n+(2\mathrm{i}\omega_0^n-\frac{3\alpha^2-5}{1+\alpha^2})d_n\end{pmatrix}.$$

Likewise, when $n\in\mathbb{N}$, we get

$$w_{11}=-\frac{\alpha_5^{-1}}{2}\begin{pmatrix}(-\frac{\alpha b_n^H}{1+\alpha^2}-\frac{4cn^2}{l^2})e_n+\frac{4\alpha}{1+\alpha^2}f_n\\ -\frac{2\alpha^2 b_n^H}{1+\alpha^2}e_n+(\frac{3\alpha^2-5}{1+\alpha^2}-\frac{4n^2}{l^2})f_n\end{pmatrix}-\frac{\alpha_6^{-1}}{2}\begin{pmatrix}-\frac{\alpha b_n^H}{1+\alpha^2}e_n+\frac{4\alpha}{1+\alpha^2}f_n\\ -\frac{2\alpha^2 b_n^H}{1+\alpha^2}e_n+\frac{3\alpha^2-5}{1+\alpha^2}f_n\end{pmatrix},$$

where

$$\alpha_5=\frac{4\alpha}{1+\alpha^2}\frac{2\alpha^2 b_n^H}{1+\alpha^2}-\left(\frac{\alpha b_n^H}{1+\alpha^2}+\frac{4cn^2}{l^2}\right)\left(\frac{3\alpha^2-5}{1+\alpha^2}-\frac{4n^2}{l^2}\right),$$
$$\alpha_6=\frac{5\alpha b_n^H}{1+\alpha^2}.$$

The direct computation yields

$$\begin{cases}f_{uu}=\dfrac{8\alpha(3-\alpha^2)}{(1+\alpha^2)^2}, \quad f_{uv}=-\dfrac{4(1-\alpha^2)}{(1+\alpha^2)^2}, \quad f_{vv}=f_{vvv}=f_{uvv}=0,\\ f_{uuu}=\dfrac{24(\alpha^4-6\alpha^2+1)}{(1+\alpha^2)^3}, \quad f_{uuv}=\dfrac{8\alpha(3-\alpha^2)}{(1+\alpha^2)^3}, \quad g_{uu}=\dfrac{1}{4}f_{uu}, \quad g_{vv}=g_{uvv}=0,\\ g_{uv}=\dfrac{1}{4}f_{uv}, \quad g_{uuu}=\dfrac{1}{4}f_{uuu}, \quad g_{uuv}=\dfrac{1}{4}f_{uuv}, \quad g_{vvv}=0.\end{cases}\tag{2.32}$$

Here and in the following we always assume that all the partial derivatives of f and g are evaluated at $(b_n^H, 0, 0)$. It is easy to get

$$Q_{w_{20}\bar q} = \begin{pmatrix} f_{uu}\xi + f_{uv}\eta + f_{uv}\bar b_n \xi \\ g_{uu}\xi + g_{uv}\eta + g_{uv}\bar b_n \xi \end{pmatrix} \cos\frac{2nx}{l}\cos\frac{nx}{l} + \begin{pmatrix} f_{uu}\tau + f_{uv}\zeta + f_{uv}\bar b_n \tau \\ g_{uu}\tau + g_{uv}\zeta + g_{uv}\bar b_n \tau \end{pmatrix} \cos\frac{nx}{l},$$

$$Q_{w_{11}q} = \begin{pmatrix} f_{uu}\tilde\xi + f_{uv}\tilde\eta + f_{uv} b_n \tilde\xi \\ g_{uu}\tilde\xi + g_{uv}\tilde\eta + g_{uv} b_n \tilde\xi \end{pmatrix} \cos\frac{2nx}{l}\cos\frac{nx}{l} + \begin{pmatrix} f_{uu}\tilde\tau + f_{uv}\tilde\zeta + f_{uv} b_n \tilde\tau \\ g_{uu}\tilde\tau + g_{uv}\tilde\zeta + g_{uv} b_n \tilde\tau \end{pmatrix} \cos\frac{nx}{l},$$

where

$$\xi = \frac{(\alpha_1+\alpha_2 \mathrm{i})^{-1}}{2}\left[\left(2\mathrm{i}\omega_0^n + \frac{\alpha b_n^H}{1+\alpha^2} + \frac{4cn^2}{l^2}\right)c_n - \frac{4\alpha}{1+\alpha^2} d_n\right],$$

$$\eta = \frac{(\alpha_1+\alpha_2 \mathrm{i})^{-1}}{2}\left[\frac{2\alpha^2 b_n^H}{1+\alpha^2} c_n + \left(2\mathrm{i}\omega_0^n - \frac{3\alpha^2-5}{1+\alpha^2} + \frac{4n^2}{l^2}\right)d_n\right],$$

$$\tau = \frac{(\alpha_3+\alpha_4 \mathrm{i})^{-1}}{2}\left[\left(2\mathrm{i}\omega_0^n + \frac{\alpha b_n^H}{1+\alpha^2}\right)c_n - \frac{4\alpha}{1+\alpha^2} d_n\right],$$

$$\zeta = \frac{(\alpha_3+\alpha_4 \mathrm{i})^{-1}}{2}\left[\frac{2\alpha^2 b_n^H}{1+\alpha^2} c_n + \left(2\mathrm{i}\omega_0^n - \frac{3\alpha^2-5}{1+\alpha^2}\right)d_n\right],$$

$$\tilde\xi = -\frac{1}{2\alpha_5}\left[\left(-\frac{\alpha b_n^H}{1+\alpha^2} - \frac{4cn^2}{l^2}\right)e_n + \frac{4\alpha}{1+\alpha^2} f_n\right],$$

$$\tilde\eta = -\frac{1}{2\alpha_5}\left[-\frac{2\alpha^2 b_n^H}{1+\alpha^2} e_n + \left(\frac{3\alpha^2-5}{1+\alpha^2} - \frac{4n^2}{l^2}\right)f_n\right],$$

$$\tilde\tau = -\frac{1}{2\alpha_6}\left(-\frac{\alpha b_n^H}{1+\alpha^2} e_n + \frac{4\alpha}{1+\alpha^2} f_n\right),$$

$$\tilde\zeta = -\frac{1}{2\alpha_6}\left(-\frac{2\alpha^2 b_n^H}{1+\alpha^2} e_n + \frac{3\alpha^2-5}{1+\alpha^2} f_n\right).$$

By (2.15) and (2.32), we have

$$c_n = f_{uu} + 2f_{uv}b_n, \qquad d_n = g_{uu} + 2g_{uv}b_n, \qquad e_n = f_{uu} + 2f_{uv}\operatorname{Re} b_n, \qquad f_n = g_{uu} + 2g_{uv}\operatorname{Re} b_n,$$
$$g_n = f_{uuu} + (3\operatorname{Re} b_n + \operatorname{Im} b_n \mathrm{i}) f_{uuv}, \qquad h_n = g_{uuu} + (3\operatorname{Re} b_n + \operatorname{Im} b_n \mathrm{i}) g_{uuv}.$$

Notice that for any $n \in \mathbb{N}$,

$$\int_0^{l\pi} \cos^2\frac{nx}{l}\,\mathrm{d}x = \frac{1}{2}l\pi, \qquad \int_0^{l\pi} \cos\frac{2nx}{l}\cos^2\frac{nx}{l}\,\mathrm{d}x = \frac{1}{4}l\pi, \qquad \int_0^{l\pi} \cos^4\frac{nx}{l}\,\mathrm{d}x = \frac{3}{8}l\pi,$$

so

$$\begin{aligned}\langle q^*, Q_{w_{20}\bar q}\rangle = {} & \frac{l\pi}{4}\left[\bar a_n^*(f_{uu}\xi + f_{uv}\eta + f_{uv}\xi\bar b_n) + \bar b_n^*(g_{uu}\xi + g_{uv}\eta + g_{uv}\xi\bar b_n)\right] \\ & + \frac{l\pi}{2}\left[\bar a_n^*(f_{uu}\tau + f_{uv}\zeta + f_{uv}\tau\bar b_n) + \bar b_n^*(g_{uu}\tau + g_{uv}\zeta + g_{uv}\tau\bar b_n)\right],\end{aligned}$$

$$\begin{aligned}\langle q^*, Q_{w_{11}q}\rangle = {} & \frac{l\pi}{4}\left[\bar a_n^*(f_{uu}\tilde\xi + f_{uv}\tilde\eta + f_{uv}\tilde\xi b_n) + \bar b_n^*(g_{uu}\tilde\xi + g_{uv}\tilde\eta + g_{uv}\tilde\xi b_n)\right] \\ & + \frac{l\pi}{2}\left[\bar a_n^*(f_{uu}\tilde\tau + f_{uv}\tilde\zeta + f_{uv}\tilde\tau b_n) + \bar b_n^*(g_{uu}\tilde\tau + g_{uv}\tilde\zeta + g_{uv}\tilde\tau b_n)\right],\end{aligned}$$

$$\langle q^*, C_{qq\bar q}\rangle = \frac{3}{8}l\pi\left(\bar a_n^* g_n + \bar b_n^* h_n\right).$$

Since

$$l\pi\bar a_n^* = 1 - \left(\frac{3\alpha^2-5}{\omega_0^n(1+\alpha^2)} - \frac{n^2}{\omega_0^n l^2}\right)\mathrm{i}, \qquad l\pi\bar b_n^* = \frac{4\alpha}{\omega_0^n(1+\alpha^2)}\mathrm{i},$$

it follows that

$$\operatorname{Re}\langle q^*, C_{qq\bar{q}}\rangle = \frac{3}{8} f_{uuu} + \frac{9}{8}\left(\frac{3\alpha^2-5}{4\alpha} - \frac{3}{8}\frac{(1+\alpha^2)n^2}{4\alpha l^2}\right) f_{uuv} - \frac{3}{8}\left(\frac{3\alpha^2-5}{\omega_0^n(1+\alpha^2)} - \frac{n^2}{\omega_0^n l^2}\right)\frac{\omega_0^n(1+\alpha^2)}{4\alpha} f_{uuv} + \frac{3}{8} g_{uuv}, \tag{2.33}$$

$$\begin{aligned}\operatorname{Re}\langle q^*, Q_{w_{20}\bar{q}}\rangle = {} & \frac{1}{4}\left[f_{uu}(\xi_R + 2\tau_R) + f_{uv}\left(\xi_R + 2\zeta_R - \frac{\omega_0^n(1+\alpha^2)}{4\alpha}(\xi_I + 2\tau_I)\right) + \left(\frac{3\alpha^2-5}{4\alpha} - \frac{(1+\alpha^2)n^2}{4\alpha l^2}\right) f_{uv}(\xi_R + 2\tau_R)\right] \\ & + \frac{1}{4}\left(\frac{3\alpha^2-5}{\omega_0^n(1+\alpha^2)} - \frac{n^2}{\omega_0^n l^2}\right)\left[f_{uu}(\xi_I + 2\tau_I) + f_{uv}(\eta_I + 2\zeta_I) + \frac{\omega_0^n(1+\alpha)}{4\alpha} f_{uv}(\xi_R + 2\tau_R) + f_{uv}\left(\frac{3\alpha^2-5}{4\alpha} - \frac{(1+\alpha^2)n^2}{4\alpha l^2}\right)(\xi_I + 2\tau_I)\right] \\ & - \frac{\alpha}{\omega_0^n(1+\alpha^2)}\left\{ g_{uu}(\xi_I + 2\tau_I) + g_{uv}\left[\eta_I + 2\zeta_I + (\xi_I + 2\tau_I)\left(\frac{3\alpha^2-5}{4\alpha} - \frac{(1+\alpha^2)n^2}{4\alpha l^2}\right) + (\xi_R + 2\tau_R)\frac{\omega_0^n(1+\alpha^2)}{4\alpha}\right]\right\},\end{aligned} \tag{2.34}$$

$$\operatorname{Re}\langle q^*, Q_{w_{11}q}\rangle = \frac{1}{4}\big(f_{uu}(\tilde{\xi} + 2\tilde{\tau}) + f_{uv}(\tilde{\xi} + 2\tilde{\zeta}) + g_{uv}(\tilde{\xi} + 2\tilde{\tau})\big), \tag{2.35}$$

where we have denoted $\Gamma_R = \operatorname{Re}\Gamma$ and $\Gamma_I = \operatorname{Im}\Gamma$ for $\Gamma = \xi, \eta, \tau, \zeta$. More precisely,

$$\xi_R = \frac{\alpha_1}{2(\alpha_1^2+\alpha_2^2)}\left[\left(\frac{\alpha b_n^H}{1+\alpha^2} + \frac{4cn^2}{l^2}\right)(f_{uu} + 2f_{uv}b_n) - \frac{4\alpha}{1+\alpha^2}(g_{uu} + 2g_{uv}b_n)\right] + \frac{\alpha_2\omega_0^n}{\alpha_1^2+\alpha_2^2}(f_{uu} + 2f_{uv}b_n),$$

$$\xi_I = -\frac{\alpha_2}{2(\alpha_1^2+\alpha_2^2)}\left[\left(\frac{\alpha b_n^H}{1+\alpha^2} + \frac{4cn^2}{l^2}\right)(f_{uu} + 2f_{uv}b_n) - \frac{4\alpha}{1+\alpha^2}(g_{uu} + 2g_{uv}b_n)\right] + \frac{\alpha_1}{\alpha_1^2+\alpha_2^2}\omega_0^n(f_{uu} + 2f_{uv}b_n),$$

$$\eta_R = \frac{\alpha_1}{2(\alpha_1^2+\alpha_2^2)}\left[\frac{2\alpha^2 b_n^H}{1+\alpha^2}(f_{uu} + 2f_{uv}b_n) + \left(\frac{4n^2}{l^2} - \frac{3\alpha^2-5}{1+\alpha^2}\right)(g_{uu} + 2g_{uv}b_n)\right] + \frac{\alpha_2}{\alpha_1^2+\alpha_2^2}\omega_0^n(g_{uu} + 2g_{uv}b_n),$$

$$\eta_I = \frac{\alpha_1}{\alpha_1^2+\alpha_2^2}\omega_0^n(g_{uu} + 2g_{uv}b_n) - \frac{\alpha_2}{2(\alpha_1^2+\alpha_2^2)}\left[\frac{2\alpha^2 b_n^H}{1+\alpha^2}(f_{uu} + 2f_{uv}b_n) + \left(\frac{4n^2}{l^2} - \frac{3\alpha^2-5}{1+\alpha^2}\right)(g_{uu} + 2g_{uv}b_n)\right],$$

$$\tau_R = \frac{\alpha_3}{2(\alpha_3^2+\alpha_4^2)}\left(\frac{\alpha b_n^H}{1+\alpha^2}(f_{uu} + 2f_{uv}b_n) - \frac{4\alpha}{1+\alpha^2}(g_{uu} + 2g_{uv}b_n)\right) + \frac{\alpha_4}{\alpha_3^2+\alpha_4^2}\omega_0^n(f_{uu} + 2f_{uv}b_n),$$

$$\tau_I = \frac{\alpha_3}{\alpha_3^2+\alpha_3^4}\omega_0^n(f_{uu} + 2f_{uv}b_n) - \frac{\alpha_4}{2(\alpha_3^2+\alpha_4^2)}\left(\frac{\alpha b_n^H}{1+\alpha^2}(f_{uu} + 2f_{uv}b_n) - \frac{4\alpha}{1+\alpha^2}(g_{uu} + 2g_{uv}b_n)\right),$$

$$\zeta_R = \frac{\alpha_3}{\alpha_3^2+\alpha_4^2}\left(\frac{2\alpha^2 b_n^H}{1+\alpha^2}(f_{uu} + 2f_{uv}b_n) - \frac{3\alpha^2-5}{1+\alpha^2}(g_{uu} + 2g_{uv}b_n)\right) + \frac{\alpha_4}{\alpha_3^2+\alpha_4^2}\omega_0^n(g_{uu} + 2g_{uv}b_n),$$

$$\zeta_I = \frac{\alpha_3}{\alpha_3^2+\alpha_4^2}\omega_0^n(g_{uu} + 2g_{uv}b_n) - \frac{\alpha_4}{2(\alpha_3^2+\alpha_4^2)}\left(\frac{2\alpha^2 b_n^H}{1+\alpha^2}(f_{uu} + 2f_{uv}b_n) - \frac{3\alpha^2-5}{1+\alpha^2}(g_{uu} + 2g_{uv}b_n)\right).$$

Finally, substituting (2.33)–(2.35) into (2.31), we get the expression of $\operatorname{Re}(c_1(b_n^H))$. The proof is completed. □

3. Discussion

In this section, we give some discussion on the system (2.1).

1. The sign of $\mathrm{Re}(c_1(b_n^H))$ is important for determining the stability of the bifurcating periodic solutions and bifurcation direction. Although we have given the expression of $\mathrm{Re}(c_1(b_n^H))$, since this expression is very complicated, it is not easy to judge the sign of $\mathrm{Re}(c_1(b_n^H))$.

2. We summarize some known dynamics of system (2.1). Note that the system (2.1) is an activator–inhibitor system if and only if $3\alpha^2 > 5$. So we assume that $3\alpha^2 - 5 > 0$. Since our model is a little different from [3,7–9], to cast our discussion, we change their results into the uniform expressions by some computation.

Conclusion 1. (See [8].) When $b = b_0^*$, the system undergoes a Hopf bifurcation at $(\alpha, 1+\alpha^2)$. Moreover,

(a) if

$$\alpha^2 \geqslant \frac{1+5l^2}{3l^2-1} \quad \text{and} \quad c > \frac{l^2\alpha b}{\frac{3l^3-1}{1+5l^2}\alpha^2 - 1},$$

the bifurcating homogeneous periodic solutions are unstable;

(b) if

$$\alpha^2 \geqslant \frac{1+5l^2}{3l^2-1} \quad \text{and} \quad 0 < c < \frac{l^2\alpha b}{\frac{3l^2-1}{1+5l^2}\alpha^2 - 1},$$

or

$$\frac{5}{3} \leqslant \alpha^2 \leqslant \frac{1+5l^2}{3l^2-1},$$

the bifurcating homogeneous periodic solutions are stable.

Conclusion 2. (See [3,7–9].) When $b > b_0^*$, the equilibrium $(\alpha, 1+\alpha^2)$ is local asymptotically stable for ODE version. Moreover,

(a) if

$$\alpha^2 > \frac{1+5l^2}{3l^2-1} \quad \text{and} \quad c > \frac{l^2\alpha b}{\frac{3l^2-1}{1+5l^2}\alpha^2 - 1},$$

then the Turing instability happens;

(b) if

$$\alpha^2 \geqslant \frac{1+5l^2}{3l^2} \quad \text{and} \quad 0 < c < \frac{l^2\alpha b}{\frac{3l^2-1}{1+5l^2}\alpha^2 - 1},$$

or

$$\frac{5}{3} \leqslant \alpha^2 \leqslant \frac{1+5l^2}{3l^2-1},$$

then $(\alpha, 1+\alpha^2)$ is local asymptotically stable for the system (2.1).

The final conclusion is the main result of the present paper:

Conclusion 3. When $b < b_0^*$ and c is suitable small, multiple spatially non-homogeneous periodic orbits occur while the system parameters are all spatially homogeneous.

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