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Holomorphic extension from the sphere to the ball

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ABSTRACT

Real analytic functions on the boundary of the sphere which have separate holomorphic extension along the complex lines through a boundary point have holomorphic extension to the ball. This was proved in Baracco (2009) [4] by an argument of CR geometry. We give here an elementary proof based on the expansion in holomorphic and antiholomorphic powers.

1. Main result – statement and proof

The characterization of boundary values of holomorphic functions by the extension along complex lines has a long history and many contributions in the context of harmonic analysis: as main references we quote Agranovsky and Semenov [2], Agranovsky and Valsky [3], Globevnik and Stout [6], Nagel and Rudin [8], and Stout [10]. More recently, the problem has been brought by Tumanov [11] in the framework of the CR geometry. From this point of view, we have obtained in [4] a principle of holomorphic extension from a convex boundary of $\mathbb{C}^n$ for functions which have separate extension along generic $(2n-2)$-parameter families of discs. In particular, the discs which pass through a fixed boundary point. It is well known that if we move the "center" of the system of discs to the interior, the conclusion fails; in this case, two interior points are needed according to Agranovsky [1]. The results of [4] apply to general convex sets but use stationary discs instead of straight lines (for the theory of stationary discs see [5] and [7]). What we wish to show here is that for the straight lines through a boundary point of the sphere the proof is much more direct and simple. In this specific problem, the theory of lifts of discs developed in [4] is not needed. Using Taylor expansions we can see that the moment condition forces the coefficients of the antiholomorphic powers to vanish.

Denote by C^ω the class of real analytic functions. Let $\mathbb{B}^n$ be the unit ball of $\mathbb{C}^n$ and let z_o be a boundary point.

Theorem 1.1. *Let f be a function in $C^\omega(\partial\mathbb{B}^n)$ and suppose that f extends holomorphically from $\partial\mathbb{B}^n$ along each line passing through z_o. Then f extends holomorphically to $\mathbb{B}^n$.*

Proof. (a) We first prove the result for $\mathbb{B}^2$ in $\mathbb{C}^2$. It is not restrictive to assume that z_o is the pole $(0,1)$. The straight discs through $(0,1)$ can be parametrized over $a \in \mathbb{C}$ as the sets D_a described by

$$D_a(\tau) = \left(\frac{\tau-1}{1+|a|^2}a, \frac{\tau-1}{1+|a|^2}+1\right) \quad \forall \tau \in \bar{\Delta}.$$

Note that when $|a| \gg 1$ the disc D_a gets close to the complex tangent line to the sphere at the point z_o, and moreover D_a lies in a neighborhood of z_o.

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Since $f \in C^{\omega}(\partial\mathbb{B}^2)$, and $\bar{\partial}_{z_2}$ is transverse to $\partial\mathbb{B}^2$ at z_o, f can be extended in a neighborhood of z_o holomorphically in z_2; this is an immediate consequence of Cauchy–Kovalewsky Theorem. We denote again by f this extension. We consider the power series expansion of f at z_o

$$f(z_1, \bar{z}_1, z_2) = \sum_{l=0}^{+\infty} \sum_{h+k+2m=l} b_{h,k,m} z_1^h \bar{z}_1^k (z_2 - 1)^m$$

note that we reordered the terms in a weighted degree (giving weight 2 to z_2). Taking $|a|$ sufficiently big we consider the N-momentum on the disc D_a:

$$\begin{aligned} G(a, N) &= \int_{\partial\Delta} \tau^N f\big(D_a(\tau)\big)\, d\tau \\ &= \int_{\partial\Delta} \tau^N \sum_{l=0}^{+\infty} \sum_{h+k+2m=l} b_{h,k,m} \left(\frac{\tau - 1}{1 + |a|^2} a\right)^h \left(\overline{\frac{\tau - 1}{1 + |a|^2} a}\right)^k \left(\frac{\tau - 1}{1 + |a|^2}\right)^m d\tau. \end{aligned} \tag{1.1}$$

We want to prove that $b_{h,k,m} = 0$ whenever $k > 0$. To this end, let l_o be the lowest weighted degree such that $b_{h,k,m} \neq 0$ for some $k > 0$ and let k_o be the highest degree in $\bar{z}_1$ for which this happens. We get $G(a, N) = 0$ for any N and any a, in particular, for ta with $|a| = 1$ and $t \to +\infty$. Consider the limit

$$\begin{aligned} \lim_{t\to+\infty} G(ta, N)t^{l_o} &= \lim_{t\to+\infty} \sum_{l=l_o}^{+\infty} \sum_{h+k+2m=l} 2\pi i(-1)^{k+h+m+N+1} \\ &\quad \times \binom{h+k+m}{k-N-1} a^h \bar{a}^k \left(\left(\frac{1}{t^2} + |a|^2\right)^m\right) t^{l_o - l} \frac{1}{(\frac{1}{t^2} + |a|^2)^l} b_{h,k,m} \\ &= \sum_{h+k+2m=l_o} (2\pi i)(-1)^{h+k+m+N+1} \binom{h+k+m}{k-N-1} b_{h,k,m} \frac{a^h \bar{a}^k |a|^{2m}}{|a|^{2l_o}} = 0, \end{aligned} \tag{1.2}$$

where we have used the fact that $\int_{\partial\Delta} \tau^N (\tau - 1)^h (\bar{\tau} - 1)^k (\tau - 1)^m\, d\tau = (-1)^k \int_{\partial\Delta} \frac{(\tau-1)^{h+k+m}}{\tau^{(k-N)}}\, d\tau = (-1)^{h+m+k+N+1} \binom{h+k+m}{k-1-N}$. We first notice that in the above summations, we can take $k > N$; in fact, since $\bar{\tau}\tau = 1$, then $\overline{(\tau - 1)^k} = \tau^{-k}(1 - \tau)^k$ and the factor τ^{N-k} and Cauchy Theorem imply the vanishing of the terms with $N - k \geqslant 0$. Next, choosing $N = k_o - 1$, we get the following relation on the coefficients b's:

$$\sum_{h+k_o+2m=l_o} (-1)^{h+m} \binom{h+k_o+m}{0} b_{h,k_o,m} a^{h+m} \bar{a}^{k_o+m} = 0.$$

Writing $a = e^{i\theta}$, we get

$$\sum_{h+k_o+2m=l_o} (-1)^{h+m+k_o} b_{h,k_o,m} e^{i\theta(h-k_o)} = 0,$$

which implies $b_{h,k_o,m} = 0$ for $h + k_o + 2m = l_o$. Therefore, when $k \geqslant 1$, we have $b_{h,k,m} = 0$ for any weighted degree l. This concludes the proof in dimension 2.

(b) We pass from $\mathbb{B}^2$ to $\mathbb{B}^n$. We still suppose that z_o is the point $(0, \ldots, 1)$. By (a) we know that f extends holomorphically along the slices of $\mathbb{B}^n$ with the 2-dimensional planes through z_o. All these extensions to different slices glue together to a single well-defined function. In fact, two slices have intersection which is a line through z_o unless it is empty; we still call f this extension. It is clear that $f \in C^{\omega}(\mathbb{B}^n)$ because f is given by the integral Cauchy formula for a real analytic function on a real analytic family of circles. We note that f is holomorphic on any straight line through 0 (since either this pass through z_o or it is contained in a single 2-dimensional slice through z_o). Application of Forelli's Theorem (see [9]) yields the conclusion. □

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