For a positive integer \( k \geq 2 \), the \( k \)-Fibonacci sequence \( \{g^{(k)}_n\} \) is defined as:

\[
g^{(k)}_1 = 0, \quad g^{(k)}_2 = 1, \quad g^{(k)}_n = g^{(k)}_{n-1} + g^{(k)}_{n-2} + \cdots + g^{(k)}_{n-k}
\]

Moreover, the \( k \)-Lucas sequence \( \{l^{(k)}_n\} \) is defined as:

\[
l^{(k)}_n = g^{(k)}_{n-1} + g^{(k)}_{n+k-1}
\]

In this paper, we consider the relationship between \( g^{(k)}_n \) and \( l^{(k)}_n \) and 1-factors of a bipartite graph. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: \( k \)-Fibonacci sequence; \( k \)-Lucas sequence; 1-factor; Permanent

1. Introduction

The well-known Fibonacci sequence \( \{F_n\} \) is defined as

\[
F_1 = F_2 = 1 \quad \text{and for } n > 2, \quad F_n = F_{n-1} + F_{n-2}.
\]

We call \( F_n \) the \( n \)th Fibonacci number. The Fibonacci sequence is

\[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots\]

We consider a generalization of the Fibonacci sequence which is called the \( k \)-Fibonacci sequence for positive integer \( k \geq 2 \). The \( k \)-Fibonacci sequence \( \{g^{(k)}_n\} \) is defined as

\[
g^{(k)}_1 = \cdots = g^{(k)}_{k-2} = 0, \quad g^{(k)}_{k-1} = g^{(k)}_k = 1
\]
and for \( n > k \geq 2 \),
\[
S_n^{(k)} = S_{n-1}^{(k)} + S_{n-2}^{(k)} + \ldots + S_{n-k}^{(k)}.
\]
We call \( S_n^{(k)} \) the \( n \)th \( k \)-Fibonacci number. By the definition of the \( k \)-Fibonacci sequence, we know that
\[
\begin{align*}
S_1^{(k)} &= S_1^{(k)} + S_{1-1}^{(k)} = 1 + 1 = 2, \\
S_2^{(k)} &= S_2^{(k)} + S_{2-1}^{(k)} = 2 + 1 + 1 = 2^2, \\
S_3^{(k)} &= S_3^{(k)} + S_{3-1}^{(k)} + S_{3-2}^{(k)} + S_{3-3}^{(k)} = 2^2 + 2 + 1 + 1 = 2^3, \\
& \vdots \\
S_{2k-2}^{(k)} &= S_{2k-2}^{(k)} + \ldots + S_k^{(k)} + S_{k-1}^{(k)} = 2^{k-3} + \ldots + 2 + 1 + 1 = 2^{k-2}, \\
S_{2k-1}^{(k)} &= S_{2k-1}^{(k)} + \ldots + S_k^{(k)} + S_{k-1}^{(k)} = 2^{k-2} + 2^{k-3} + \ldots + 2 + 1 + 1 = 2^{k-1}.
\end{align*}
\]
Thus, we have that \( g_j^{(k)} = 2^{j-k} \) for \( j = k, k+1, \ldots, 2k-1 \). For example, if \( k = 2 \), then \( \{g_n^{(2)}\} \) is the Fibonacci sequence. If \( k = 5 \), then the 5-Fibonacci sequence is
\[
0, 0, 1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3535, 6930, \ldots
\]
Let \( E \) be the \( 1 \times (k-1) \) matrix all of whose entries are ones and let \( I_k \) be the identity matrix of order \( n \). For any \( k \geq 2 \), the fundamental recurrence relation, \( n > k \),
\[
S_n^{(k)} = S_{n-1}^{(k)} + S_{n-2}^{(k)} + \ldots + S_{n-k}^{(k)}
\]
can be defined by the vector recurrence relation
\[
\begin{bmatrix}
S_{n+1}^{(k)} \\
S_{n+2}^{(k)} \\
\vdots \\
S_{n+k}^{(k)}
\end{bmatrix} = Q_k^n \begin{bmatrix}
S_n^{(k)} \\
S_{n+1}^{(k)} \\
\vdots \\
S_{n+k-1}^{(k)}
\end{bmatrix},
\]
where
\[
Q_k = \begin{bmatrix}
0 & I_{k-1} \\
1 & E
\end{bmatrix}.
\]
By applying (1), we have
\[
\begin{bmatrix}
S_{n+1}^{(k)} \\
S_{n+2}^{(k)} \\
\vdots \\
S_{n+k}^{(k)}
\end{bmatrix} = Q_k^n \begin{bmatrix}
S_1^{(k)} \\
S_2^{(k)} \\
\vdots \\
S_k^{(k)}
\end{bmatrix}.
\]
In [4–6], relationships between the \( k \)-Fibonacci numbers and their associated matrices can be found.
We now introduce \( k \)-Lucas sequences.

We let \( L_n \) represent the \( n \)th Lucas number, that is, for \( n \geq 1 \),
\[
L_n = F_{n-1} + F_{n+1},
\]
where \( F_0 = 0 \). The Lucas sequence \( \{L_n\} \) is
\[
1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, \ldots
\]
One can find applications and properties of the Lucas number in [3]. Let \( g_0 = 0 \).

The \( k \)-Lucas sequence \( \{l_n^{(k)}\} \) is defined by
\[
l_n^{(k)} = g_{n-1}^{(k)} + g_{n+k-1}^{(k)}.
\]
We call \( l_n^{(k)} \) the \( n \)th \( k \)-Lucas number. Then we have
\[
l_j^{(j)} = 2^{j-1}, j = 1, 2, \ldots, k - 1, \quad \text{and} \quad l_k^{(k)} = 1 + 2^{k-1}.
\]
If \( k = 2 \), then \( l_n^{(2)} = L_n \). For example, if \( k = 5 \), then the 5-Lucas sequence is
\[
1, 2, 4, 8, 17, 32, 63, 124, 244, 480, 943, 1854, 3645, 7166, \ldots
\]
The permanent of an \( n \)-square matrix \( A = [a_{ij}] \) is defined by
\[
\text{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},
\]
where the summation extends over all permutations \( \sigma \) in the symmetric group \( S_n \).

For any square matrix \( A \) and any permutation matrices \( P \) and \( Q \), \( \text{per} A = \text{per} P A Q \).

A bipartite graph \( G \) is a graph whose vertex set \( V \) can be partitioned into two subsets \( V_1 \) and \( V_2 \) such that every edge of \( G \) joins a vertex in \( V_1 \) and a vertex in \( V_2 \). A 1-factor (or perfect matching) of a graph with \( 2n \) vertices is a spanning subgraph of \( G \) in which every vertex has degree 1. The enumeration or actual construction of 1-factors of a bipartite graph has many applications, for example, in maximal flow problems and in assignment and scheduling problems. Let \( A(G) \) be the adjacency matrix of the bipartite graph \( G \), and let \( \mu(G) \) denote the number of 1-factors of \( G \).

Then, one can find the following fact in [7]: \( \mu(G) = \sqrt{\text{per} A(G)} \). Also, one can find more applications of permanents in [7].

Let \( G \) be a bipartite graph whose vertex set \( V \) is partitioned into two subsets \( V_1 \) and \( V_2 \) such that \( |V_1| = |V_2| = n \). We construct the bipartite adjacent matrix \( B(G) = [b_{ij}] \) of \( G \) as following: \( b_{ij} = 1 \) if and only if \( G \) contains an edge from \( v_i \in V_1 \) to \( v_j \in V_2 \), and 0 otherwise. Then, in [7], the number of 1-factors of bipartite graph \( G \) equals the permanent of its bipartite adjacency matrix.

Let \( G \) and \( G' \) denote two general graphs of order \( n \), and let the adjacency matrices of these graphs be denoted by \( A \) and \( A' \), respectively. Then \( G \) and \( G' \) are isomorphic if and only if \( A \) is transformable into \( A' \) by simultaneous permutations of the lines of \( A \). Thus, \( G \) and \( G' \) are isomorphic if and only if there exists a permutation matrix \( P \) of order \( n \) such that \( P^T A P = A' \).

Let \( A = [a_{ij}] \) be an \( m \times n \) real matrix with row vectors \( \alpha_1, \alpha_2, \ldots, \alpha_m \). We say \( A \) is contractible on column (resp. row) \( k \) if column (resp. row) \( k \) contains exactly two nonzero entries. Suppose \( A \) is contractible on column \( k \) with \( a_{ik} \neq 0 \neq a_{jk} \) and
Then the matrix obtained from $A$ by replacing row $i$ with $a_{ik}i$ and deleting row $j$ and column $k$ is called the contraction of $A$ on column $k$ relative to rows $i$ and $j$. If $A$ is contractible on row $k$ with $a_{ik} = 0 = a_{kj}$ and $i \neq j$, then the matrix $A_{k} = [A^T_{ij}]$ is called the contraction of $A$ on row $k$ relative to columns $i$ and $j$. Every contraction used in this paper will be on the first column using the first and second rows. We say that $A$ can be contracted to a matrix $B$ if either $B = A$ or there exist matrices $A_0, A_1, \ldots, A_t$ such that $A_0 = A$, $A_t = B$ and $A_r$ is a contraction of $A_{r-1}$ for $r = 1, \ldots, t$. One can find the following fact in [2]: let $A$ be a nonnegative integral matrix of order $n > 1$ and let $B$ be a contraction of $A$.


defiition

2. $k$-Lucas numbers

In this section, we determine a class of bipartite graphs whose number of 1-factors is $l^{(k)}_n$.

A matrix is said to be a $(0,1)$-matrix if each of its entries is either 0 or 1.

Let $k$ and $n$ be positive integers. Let $T_n = [t_{ij}]$ be the $n \times n$ tridiagonal $(0,1)$-matrix such that $t_{ij} = 1$ if and only if $|j - i| \leq 1$. Let $U_n^{(k)} = [u_{ij}]$ be the $n \times n$ $(0,1)$-matrix defined by $u_{ij} = 1$ if and only if $2 \leq j - i \leq k - 1$. It should be observed that $U_n^{(k)} = U_n^{(n)}$ for $k \geq n$.

In [4,5], the author found a class of bipartite graphs whose number of 1-factors is the $n$th $k$-Fibonacci number and the following result was proven:

**Theorem 2.1.** Let $G(\mathcal{P}(n,k))$ be the bipartite graph with bipartite adjacency matrix $\mathcal{P}(n,k) = [f_{ij}] = T_n + U_n^{(k)}$. Then the number of 1-factors of $G(\mathcal{P}(n,k))$ is $g^{(k)}_n$.

Since the Fibonacci numbers are connected by the fundamental recursion $F_n = F_{n-1} + F_{n-2}$, it follows immediately that the Lucas numbers are likewise related by $L_n = L_{n-1} + L_{n-2}$ for $n > 2$. We now consider the $k$-Lucas numbers.

**Lemma 2.2.** For $n > k$,

\[
\begin{align*}
\ell_n^{(k)} &= \ell_{n-1}^{(k)} + \ell_{n-2}^{(k)} + \cdots + \ell_{n-k}^{(k)}.
\end{align*}
\]

**Proof.** Since, by definition of $\ell_n^{(k)}$,

\[
\begin{align*}
\ell_{n-(k-1)}^{(k)} &= g^{(k)}_{n-(k-1)-1} + g^{(k)}_n \\
& \vdots \\
\ell_{n-1}^{(k)} &= g^{(k)}_{n-2} + g^{(k)}_{n+k-2}
\end{align*}
\]
By applying (1), we have

\[
\begin{bmatrix}
I_{n-(k-1)}^{(k)} \\
\vdots \\
I_{n-1}^{(k)} \\
I_n^{(k)}
\end{bmatrix} =
\begin{bmatrix}
g_{n-(k-1)-1}^{(k)} + g_n^{(k)} \\
\vdots \\
g_{n-2}^{(k)} + g_{n+k-2}^{(k)} \\
g_{n-1}^{(k)} + g_{n+k-1}^{(k)}
\end{bmatrix} = Q_k
\begin{bmatrix}
l_{n-k}^{(k)} \\
\vdots \\
l_{n-2}^{(k)} \\
l_n^{(k)}
\end{bmatrix}.
\]

From (2), the proof is completed. \(\square\)

Let \(S_n^{(k)} = [s_{ij}]\) be the \(n \times n\) (0,1)-matrix defined by \(s_{ij} = 1\) if and only if \(-1 \leq j - i \leq k - 1\). For \(k < n\), let \(\gamma^{(n,k)} = s_n^{(k)} = \sum_{j=2}^{k} E_{1j} + E_{1k+1}\), where \(E_{ij}\) denotes the \(n \times n\) matrix with 1 in the \((i, j)\) position and zeros elsewhere. If \(k \leq n\), then the matrix \(E_{1j+1}, j \geq k\), is not defined, and hence we let \(\gamma^{(n,k)} = S_n^{(k)} - \sum_{j=2}^{n} E_{1j}\) for \(n \leq k\). Then we have the following theorem.

**Theorem 2.3.** Let \(G(\gamma^{(n,2)})\) be the bipartite graph with bipartite adjacency matrix \(\gamma^{(n,2)}, n \geq 3\). Then the number of 1-factors of \(G(\gamma^{(n,2)})\) is \(L_{n-1}\).

**Proof.** If \(n = 3\), then we have

\[
\gamma^{(3,2)} = \begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix}
\]

and hence per \(\gamma^{(3,2)} = 3 = L_2\).
Let $\mathcal{C}_p^{(n,2)}$ be the $p$th contraction of $\mathcal{C}^{(n,2)}$, $1 \leq p \leq n - 2$. Since

$$
\phi^{(n,2)} = \begin{bmatrix}
1 & 0 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 1 & 1 \\
0 & 0 & \cdots & 0 & 1 & 1 \\
\end{bmatrix},
$$

the matrix $\phi^{(n,2)}$ can be contracted on column 1 so that

$$
\phi_1^{(n,2)} = \begin{bmatrix}
1 & 2 & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 1 & 1 \\
0 & 0 & \cdots & 0 & 1 & 1 \\
\end{bmatrix},
$$

Since the matrix $\phi_1^{(n,2)}$ can be contracted on column 1 and $L_1 = 1$, $L_2 = 3$,

$$
\phi_2^{(n,2)} = \begin{bmatrix}
3 & 1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 1 & 1 \\
0 & 0 & \cdots & 0 & 1 & 1 \\
\end{bmatrix},
$$

$$
\begin{bmatrix}
L_2 & L_1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 1 & 1 \\
0 & 0 & \cdots & 0 & 1 & 1 \\
\end{bmatrix}
$$

Furthermore, the matrix $\phi_2^{(n,2)}$ can be contracted on column 1 so that
Continuing this process, we have
\[
\begin{bmatrix}
L_p & L_{p-1} & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 1 & 1 \\
0 & 0 & \cdots & 0 & 1 & 1
\end{bmatrix}
\]
for \(3 \leq p \leq n-4\). Hence,
\[
\begin{bmatrix}
L_{n-3} & L_{n-4} & 0 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix}
\]
which, by contraction of \(\psi^{(n,2)}_{n-4}\) on column 1, gives
\[
\begin{bmatrix}
L_{n-3} + L_{n-4} & L_{n-3} \\
1 & 1 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
L_{n-2} & L_{n-3} \\
1 & 1
\end{bmatrix}.
\]
By applying (3), we have per \(\psi^{(n,2)}_{n-2}\) = per \(\psi^{(n,2)}_{n-2} = L_{n-1}\). \(\square\)

Let \(A^{(n)} = [a_{ij}]\) be the \(n \times n\) \((0,1)\)-matrix defined by \(a_{ij} = 1\) if and only if \(-1 \leq j - i\). Brualdi [1] gave the following result: for \(n \geq 2\),
\[
\text{per } A^{(n)} = 2^{n-1}.
\]
(4)

From the above result, we have the following lemma.

**Lemma 2.4.** For \(2 \leq n \leq k\),
\[
\text{per } \psi^{(n,k)} = 2^{n-2} = j^{(k)}_{n-1}.
\]

**Proof.** Since \(n \leq k\),
Then
\[ \text{per } C_{(n;k)} = \text{per } C_{(n;k)}^1 = \text{per } A^{(n-1)}. \]

By (4), the lemma is proved. \( \square \)

In the following theorem, we have a class of bipartite graphs whose number of 1-factors equals the \((n-1)\)st \(k\)-Lucas number.

**Theorem 2.5.** Let \(k\) and \(n\) be positive integers. Let \(G(C_{(n;k)})\) be the bipartite graph with bipartite adjacency matrix \(C_{(n;k)}\), \(n \geq 3\). Then, for \(n > k\), the number of 1-factors of \(G(C_{(n;k)})\) is the \((n-1)\)st \(k\)-Lucas number \(l_{n-1}^{(k)}\). In particular, if \(n \leq k\), then the number of 1-factors equals \(l_{n-1}^{(k)} = 2^{n-2}\).

**Proof.** If \(n \leq k\), then we are done, by Lemma 2.4. If \(k = 2\), then we are also done, by Lemma 2.3.

Now, suppose that \(n > k \geq 3\). Let \(G(C_{p}) = [c_{ij}^{(p)}] \) be the \(p\)th contraction of \(G(C_{(n;k)})\) = \([c_{ij}]\) for \(1 \leq p \leq n-2\). We only consider the first row because the entries of other rows of \(G(C_{p})\) are 0 or 1.

For \(q \geq k + 1\), we can easily verify that if \(p = 1\), then
\[ c_{11}^{(1)} = \cdots = c_{1k}^{(1)} = l_{1}^{(k)} = 1, \quad c_{1k}^{(1)} = l_{2}^{(k)} = 2, \quad c_{1q}^{(1)} = 0. \]
And, for \(q \geq k + 1\), if \(p = 2\), then
\[ c_{11}^{(2)} = \cdots = c_{1k-2}^{(2)} = l_{2}^{(k)} = 2, \quad c_{1k-1}^{(2)} = l_{1}^{(k)} = 2, \quad c_{1q}^{(2)} = 0. \]

By repeated contractions, for \(p \leq k - 1\), we have
\[ c_{11}^{(p)} = \cdots = c_{1k-p}^{(p)} = 2^{p-1}, \quad c_{1k-p+1} = 2^{p-1} + 1, \]
\[ c_{1k-p+s} = 2^{p-1} - 2^{s-2}, \quad 2 \leq s \leq p. \]

Since
\[ l_{j}^{(k)} = 2^{j-1}, \quad j = 1, 2, \ldots, k - 1 \quad \text{and} \quad l_{k}^{(k)} = 1 + 2^{k-1} \quad \text{for} \quad q \geq k + 1, \]
we have
\[ c_{11}^{(k-1)} = 2^{k-2} = l_{k-1}^{(k)}, \quad c_{12}^{(k-1)} = 1 + 2^{k-2} = l_{1}^{(k)} + l_{k-1}^{(k)}. \]
\[ c_{13}^{(k-1)} = 2^{k-2} - 1 = \sum_{q=1}^{k-2} l_q^{(k)}, \quad c_{14}^{(k-1)} = 2^{k-2} - 2 = \sum_{q=2}^{k-2} l_q^{(k)}, \]

\[ \vdots \]

\[ c_{1k-1}^{(k-1)} = 2^{k-2} - 2^k = \sum_{q=k-3}^{k-2} l_q^{(k)}, \quad c_{1k}^{(k-1)} = 2^{k-2} - 2^k = l_{k-2}^{(k)}, \]

\[ c_{1k}^{(k-1)} = 2^k = 0. \]

That is, the matrix \( C_{n;k} \) is as follows:

\[
\begin{bmatrix}
  l_{k-1} & l_{k-1} & \sum_{q=1}^{k-2} l_q^{(k)} & \sum_{q=2}^{k-2} l_q^{(k)} & \ldots & l_{k-2}^{(k)} & 0 & \ldots & 0 \\
  1 & 1 & 1 & \ldots & 1 & 1 & \ldots & 0 \\
  0 & 1 & 1 & \ldots & 1 & 1 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 \\
\end{bmatrix}
\]

By the contraction of \( C_{k-1} \) for \( k \leq p \leq n - k \), we have

\[ c_{11}^{(p)} = l_p^{(k)}, \quad c_{12}^{(p)} = \sum_{j=p-k+1}^{p-1} l_j^{(k)}, \quad c_{13}^{(p)} = \sum_{j=p-k+2}^{p-1} l_j^{(k)}, \ldots, \quad c_{1k}^{(p)} = \sum_{j=k}^{p-1} l_j^{(k)}. \]

and \( c_{1q}^{(p)} = 0 \) for \( q \geq k + 1 \). That is, the matrix \( C_p \) for \( k \leq p \leq n - k \), is the following:

\[
\begin{bmatrix}
  l_p & \sum_{q=p-k+1}^{p-1} l_q^{(k)} & \sum_{q=p-k+2}^{p-1} l_q^{(k)} & \ldots & l_{p-1}^{(k)} & 0 & \ldots & 0 \\
  1 & 1 & 1 & \ldots & 1 & 1 & \ldots & 0 \\
  0 & 1 & 1 & \ldots & 1 & 1 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 \\
\end{bmatrix}
\]

Since \( p \leq n - 2 \), \( p + k + 1 \leq n - k - 1 \). By the definition of \( C^{(p,k)} = [c_{ij}] \), we know that \( c_{in} = 0 \) for \( i = 1, 2, \ldots, n - k \) and \( c_{in} = 1 \) for \( i = n - k + 1, \ldots, n \). From (5), by repeated contractions, we have

\[ c_{11}^{(p)} = l_p^{(k)}, \quad c_{12}^{(p)} = \sum_{j=p-k+1}^{p-1} l_j^{(k)}, \]

\[ c_{13}^{(p)} = \sum_{j=p-k+2}^{p-1} l_j^{(k)}, \ldots, \quad c_{1n-p}^{(p)} = \sum_{j=n-k-1}^{n-1} l_j^{(k)}. \]

Thus, for \( n - k + 1 \leq p \), we have
Let 

\[ B(n) = T_n + E_{13} - E_{23} + E_{24} - E_{34} \]

and let \( G(B(n)) \) be the bipartite graph with bipartite adjacency matrix \( B(n) \).

**Lemma 2.6.** Let \( G(B(n)) \) be the bipartite graph with bipartite adjacency matrix \( B(n) \). Then the bipartite graph \( G(B(n)) \) is not isomorphic to the bipartite graph \( G(\mathcal{L}(n,2)) \).

**Proof.** From the definition of \( \mathcal{L}(n,k) \), we have \( \mathcal{L}(n,2) = S^{(2)}_n = E_{12} + E_{13} \). If \( k = 2 \), then \( S^{(2)}_n = T_n \) and hence \( \mathcal{L}(n,2) = T_n - E_{12} + E_{13} \). Then the third column of \( \mathcal{L}(n,2) \) has exactly four ones. But the matrix \( B(n) \) has no column that has exactly four ones. That is, there are no permutation matrices \( P \) and \( Q \) such that \( PB(n)Q = \mathcal{L}(n,2) \). Thus, the bipartite graph \( G(B(n)) \) is not isomorphic to \( G(\mathcal{L}(n,2)) \).

The next theorem shows that we can find a bipartite graph whose number of 1-factors is the \((n - 1)\)th Lucas number \( L_{n-1} \).

**Theorem 2.7.** For \( n \geq 4 \), let \( G(B(n)) \) be a bipartite graph with bipartite adjacency matrix \( B(n) \). Then the number of 1-factors of \( G(B(n)) \) is the \((n - 1)\)th Lucas number \( L_{n-1} \).

**Proof.** If \( n = 4 \), then

\[
\begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix}
= 4
\]

and \( L_3 = 4 \). By induction on \( n \), we assume that
per $B^{(n+1)} = L_{n-1}$

and we consider $n + 1$. Let $D$ be an $n \times 1$ matrix such that

$$D = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$  

Then

$$B^{(n+1)} = \begin{bmatrix} B^{(n)} \\ D^T \\ 1 \end{bmatrix}.$$  

Let $B_p^{(n)}$ be the $p$th contraction of $B^{(n)}$. Since

$$B_{n-2}^{(n)} = \begin{bmatrix} L_{n-2} & L_{n-3} \\ 1 & 1 \end{bmatrix},$$  

we have

$$\per B^{(n+1)} = \per B_{n-2}^{(n+1)} = \per \begin{bmatrix} L_{n-2} & L_{n-3} & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = L_{n-1} + L_{n-2} = L_n.$$  

Therefore, per $B^{(n)} = L_{n-1}$. \qed

Theorem 2.7 gives us a class of bipartite graphs that are not isomorphic to the bipartite graph $G(\mathcal{G}(n, 2))$.

References