Quasidirect Decompositions of Hankel and Toeplitz Matrices

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ABSTRACT

Decompositions, over an algebraically closed field, of a Hankel matrix into a sum of Hankel matrices the sum of the ranks of which is equal to the rank of the original matrix, are completely described. Similar results hold for Toeplitz matrices.

1. INTRODUCTION

Hankel matrices, i.e. square matrices $A=(a_{ik})$, $i,k=0,1,\dots,n-1$, such that $a_{ik}=\alpha_{i+k}$ for some $\alpha_0,\alpha_1,\dots,\alpha_{2n-2}$, play an important role in various problems (moment problem, Padé approximations, etc.) of mathematical analysis. We intend to investigate algebraic properties of these matrices and, in particular, their quasidirect decompositions into sums of Hankel matrices of smaller rank.

As is well known, Toeplitz matrices of order n, i.e. matrices $A=(a_{ik})$, $i,k=0,\dots,n-1$, for which $a_{ik}=\tau_{i-k}$ for some $\tau_{-(n-1)},\tau_{-(n-2)},\dots,\tau_0,\dots,\tau_{n-2},\tau_{n-1}$, are easily obtained from Hankel matrices by postmultiplication by a constant $n\times n$ matrix

$$J_n=\begin{pmatrix}0&0&\cdots&0&1\\0&0&\cdots&1&0\\ \cdots&\cdots&\cdots&\cdots&\cdots\\0&1&\cdots&0&0\\1&0&\cdots&0&0\end{pmatrix}$$

and vice versa. Therefore, all results about quasidirect decompositions of Hankel matrices correspond to results about quasidirect decompositions of Toeplitz matrices and conversely.

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As in [1], we call a matrix A a quasidirect sum of the matrices B and C if $A = B + C$ and there exist nonsingular matrices P, Q and matrices B_0, C_0 of appropriate sizes such that

$$B = P\begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix}Q,$$

$$C = P\begin{pmatrix} 0 & 0 \\ 0 & C_0 \end{pmatrix}Q$$

with conformal partitions. We recall:

THEOREM 1.1. *A is a quasidirect sum of B and C iff for the ranks $r(A) = r(B) + r(C)$, or iff the column space of A is a direct sum of the column spaces of B and C.*

An important example of a quasidirect sum is the following theorem:

THEOREM 1.2. *Let the submatrix A_{11} in*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be nonsingular of order k. Then the sum

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{21}A_{11}^{-1}A_{12} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

is quasidirect and the first summand has rank k.

REMARK. The matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$ has been called [3] the Schur complement of A_{11} in A.

If $A = (a_{ik})$ is a matrix, we shall call the square upper-left-corner submatrices leading submatrices of A.

We shall call a Hankel matrix A proper if its leading submatrix of order $r(A)$ is nonsingular.

An $n \times n$ Hankel matrix A of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix},$$

where Z is a square matrix of order k, $0 \leqslant k < n$, will be called degenerate.

REMARK 1.3. It is easily seen that a degenerate $n \times n$ Hankel matrix has always the form

$$\begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix}$$

where Y is a "lower triangular" Hankel matrix

$$Y = \begin{pmatrix} 0 & 0 & \cdots & 0 & \xi \\ 0 & 0 & \cdots & \xi & \eta \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \xi & \eta & \cdots & & \zeta \end{pmatrix}$$

of order k', $0 \leqslant k' < n$, $\xi \neq 0$.

2. RESULTS

We shall prove first a lemma which is essentially proved in [2].

LEMMA 2.1. *Let $A = (\alpha_{i+k})$ be a singular Hankel matrix of order n. Let r be a nonnegative integer. Then the following are equivalent:*

(a) *the first r columns of A are linearly independent but the first $r+1$ columns of A are linearly dependent;*

(b) *among the leading nonsingular submatrices of A the one of maximum order has order r.*

In the case that A is proper, r in conditions (a) *and* (b) *is equal to the rank of A, and there exist numbers $\lambda_0, \dots, \lambda_{r-1}$ such that*

$$\alpha_{s+r} = \lambda_0 \alpha_s + \lambda_1 \alpha_{s+1} + \cdots + \lambda_{r-1} \alpha_{s+r-1}, \qquad s = 0, 1, \dots, 2n - r - 2. \quad (1)$$

REMARK 2.2. If A is proper, the numbers $\lambda_0, \dots, \lambda_{r-1}$ are uniquely determined by the first r conditions in (1), and the polynomial

$$x^r - \lambda_{r-1} x^{r-1} - \cdots - \lambda_0$$

will be called the Hankel polynomial, or briefly H-polynomial, of the (proper) Hankel matrix A.

Proof. Denote, for a moment, by r' the number defined in (a). Clearly $r \geqslant r'$. To prove that $r \leqslant r'$, let $A_0, \dots, A_{n-1}$ be columns of the matrix A, and $\tilde{A}_0, \dots, \tilde{A}_{n-1}$ the columns of the submatrix $\tilde{A}$ of A consisting of the first r rows of A. Define the numbers $\lambda_0, \dots, \lambda_{r-1}$ by

$$A_r = \lambda_0 A_0 + \lambda_1 A_1 + \cdots + \lambda_{r-1} A_{r-1}. \tag{2}$$

Consequently, (1) is satisfied for $s = 0, 1, \dots, n-1$, so that also

$$\tilde{A}_{r+k} = \lambda_0 \tilde{A}_k + \lambda_1 \tilde{A}_{k+1} + \cdots + \lambda_{r-1} \tilde{A}_{r+k-1}, \qquad k = 0, 1, \dots, n-r-1. \tag{3}$$

Since $\tilde{A}$ has rank r, this can happen only if $\tilde{A}_0, \dots, \tilde{A}_{r-1}$ are linearly independent. Thus the leading submatrix of A of order r is nonsingular, and $r \leqslant r'$ as well.

The last part follows from the fact that, A having the same rank as $\tilde{A}$, its columns satisfy the same linear dependence relations as the columns of $\tilde{A}$, i.e. (3). ■

THEOREM 2.3. *A quasidirect sum of proper Hankel matrices is again a proper Hankel matrix. The H-polynomials of the summands are mutually relatively prime, and their product is equal to the H-polynomial of the sum (if it exists).*

Proof. Let $B + C$ be quasidirect where B, C are nonzero proper Hankel matrices of order n with ranks r, s respectively and H-polynomials

$$\varphi_B(x) = x^r - \lambda_{r-1} x^{r-1} - \cdots - \lambda_0,$$

$$\varphi_C(x) = x^s - \mu_{s-1} x^{s-1} - \cdots - \mu_0.$$

By Lemma 2.1, the columns $B_0, \dots, B_{n-1}$ of B and $C_0, \dots, C_{n-1}$ of C satisfy

$$B_j = \lambda_0 B_{j-r} + \lambda_1 B_{j-r+1} + \cdots + \lambda_{r-1} B_{j-1}, \qquad j = r, r+1, \dots, n-1,$$

$$C_k = \mu_0 C_{k-s} + \mu_1 C_{k-s+1} + \cdots + \mu_{s-1} C_{k-1}, \qquad k = s, s+1, \dots, n-1.$$

Let

$$d(x) = x^t - \nu_{t-1}x^{t-1} - \cdots - \nu_0$$

be the least common multiple of the polynomials $\varphi_B(x)$, $\varphi_C(x)$; for its degree t we have

$$t \leqslant r + s. \tag{4}$$

Let us show that since $\varphi_B(x)$ divides $d(x)$:

$$B_m = \nu_0 B_{m-t} + \nu_1 B_{m-t+1} + \cdots + \nu_{t-1}B_{m-1}, \qquad m = t, t+1,\dots,n-1.$$

This is clear if $d(x) = x\varphi_B(x)$ or if $d(x) = c\varphi_B(x)$ for some constant c. The general case then follows easily. Similarly,

$$C_m = \nu_0 C_{m-t} + \nu_1 C_{m-t+1} + \cdots + \nu_{t-1}C_{m-1}, \qquad m = t, t+1,\dots,n-1.$$

This means, however, that the columns $A_0,\dots,A_{n-1}$ of the matrix $A = B + C$ also satisfy such linear dependence relations. Thus A has rank at most t. Since $B + C$ is a quasidirect sum, this can be written as

$$r + s \leqslant t.$$

It follows from (4) that $r + s = t$, $B + C$ is proper, and for singular A, its H-polynomial is the product of the H-polynomials of B and C, which are thus relatively prime.

For more than two summands, the result follows by an easy induction. ■

LEMMA 2.4. *Let n, r be integers, $n > r \geqslant 1$. If A is a proper Hankel matrix of order $r+1$ and rank r, then there exists a unique Hankel matrix of order n and rank r with A as the leading submatrix.*

Proof. Let $A = (\alpha_{i+k})$, $i, k = 0,\dots,r$. The last column A_r of A can be written as in Lemma 2.1 in the form

$$A_r = \lambda_0 A_0 + \lambda_1 A_1 + \cdots + \lambda_{r-1}A_{r-1}. \tag{5}$$

The numbers $\alpha_0,\dots,\alpha_{2r}$ satisfy thus (1) for $s = 0,\dots,r$. Defining numbers $\alpha_{2r+1},\dots,\alpha_{2n-2}$ recurrently by (1), we obtain an $n \times n$ Hankel matrix satisfying the condition of the lemma.

To prove uniqueness, note that any Hankel matrix satisfying the condition of Lemma 2.4 must be proper. By Lemma 2.1, the entries of such a matrix satisfy (1), whose coefficients λ_i are determined by A uniquely. ■

THEOREM 2.5. *Every Hankel matrix A can be expressed as a quasidirect sum of a proper Hankel matrix and a degenerate Hankel matrix. If A is singular, this decomposition is unique. Consequently, the sum of the rank r of the "proper part" of A and of the rank k of the "degenerate part" of A is equal to the rank m of A:*

$$r + k = m.$$

Proof. Let A be a Hankel matrix of order n, let r be the number from (a) and (b) in Lemma 2.1 if A is singular, and $r = n$ if A is nonsingular. In the second case, the first assertion is true, the degenerate Hankel matrix being the zero matrix. If $r = 0$, the first column is a zero column and A itself is degenerate.

Suppose thus that $0 < r < n$. Then for A_{11} of order r,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

By Lemma 2.4, there exists a uniquely determined Hankel matrix B of the form

$$B = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & B_{22} \end{pmatrix}$$

which has rank r. However, $B_{22} = A_{21}A_{11}^{-1}A_{12}$ by Theorem 1.2, and by the same theorem

$$A = B + C$$

is a quasidirect sum, where

$$C = \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}.$$

The first summand and A being Hankel matrices, the second summand is a

Hankel matrix as well. Consequently, C is a degenerate Hankel matrix and the first assertion is proved.

Assume now that A is singular and that

$$A = B' + C'$$

is a quasidirect sum, B' being a proper Hankel matrix of rank r' and C' a degenerate matrix.

The first $r'+1$ rows of B' are identical with those of A, since C' has rank $\leqslant n-1-r'$ and has no nonzero entry in the first $r'+1$ rows. Thus $r'=r$ by (a) of Lemma 2.1. Consequently,

$$B' = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & B'_{22} \end{pmatrix};$$

B' being of rank r, we have $B'_{22} = A_{21}A_{11}^{-1}A_{12} = B_{22}$ by Theorem 1.2. Thus $B' = B$ and uniqueness is proved. ■

Theorem 2.5 enables us to define the H-polynomial for any singular Hankel matrix A as the H-polynomial of the proper part of A. In addition, the rank of the degenerate part of A will be called the improper degree of A. Thus a degenerate Hankel matrix is characterized by the fact that its H-polynomial is 1. The zero matrix has H-polynomial 1, and its improper degree is zero.

Let us show now that Theorem 2.5 has, in a sense, a converse:

THEOREM 2.6. *The sum of a proper Hankel matrix and a degenerate Hankel matrix of the same order n is quasidirect whenever the sum of the ranks of these matrices does not exceed n.*

Proof. Let A be a proper Hankel matrix of rank r, let D be a degenerate Hankel matrix of rank d, and let $r+d=n$. Write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is $r \times r$ and nonsingular. Since A has rank r, we know $A_{22} = A_{21}A_{11}^{-1}A_{12}$. Since rank $D = d \leqslant n-r$, we can write

$$D = \begin{pmatrix} 0 & 0 \\ 0 & \overline{D} \end{pmatrix},$$

where $\overline{D}$ is a lower triangular Hankel matrix of size $(n-r)\times(n-r)$ and rank d. Then

$$A+D=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}+\overline{D} \end{pmatrix}$$

and

$$\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix}(A+D)=\begin{pmatrix} A_{11} & A_{21} \\ 0 & \overline{D} \end{pmatrix},$$

so $\operatorname{rank}(A+D)=r+\operatorname{rank}\overline{D}=r+d$. ■

In the sequel, we shall call a Hankel matrix H-indecomposable if it cannot be expressed as a quasidirect sum of nonzero Hankel matrices.

THEOREM 2.7. *A degenerate Hankel matrix is always H-indecomposable.*

Proof. Let A be degenerate, and let $A=B+C$ with Hankel matrices B,C. Let $B=B_0+P$, B_0 proper, P degenerate, and $C=C_0+Q$, C_0 proper, Q degenerate. Thus

$$A=B_0+C_0+P+Q$$

is a quasidirect decomposition. Since A is singular and $P+Q$ degenerate, it follows from Theorems 2.3 and 2.5 that

$$P+Q=A,$$

and

$$B_0+C_0=0,$$

i.e.,

$$B_0=C_0=0.$$

Now, we have for the ranks

$$r(P)+r(Q)=r(A)\leqslant\max(r(P),r(Q)).$$

Thus $r(P)=0$ or $r(Q)=0$, i.e. A is H-indecomposable. ■

Let us formulate now a trivial observation. We shall denote by J_n the matrix

$$J_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 0 & 0 \end{pmatrix} \tag{7}$$

of order n.

Lemma 2.8. *If A is an $n \times n$ Hankel matrix, then $B = J_n A J_n$ is also a Hankel matrix. If A is H-indecomposable, then so is B.*

Corollary 2.9. *Every Hankel matrix of the form*

$$\begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \tag{8}$$

is H-indecomposable.

We shall call a Hankel matrix of the form (8) simple.

Remark 2.10. The assertion of Corollary 2.9 is true only for matrices of the form (8) for which the zero blocks are nonvoid. For instance, the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

is not H-indecomposable, since it can be written in the form

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let t be an indeterminate, m, n integers, $0 < m \leqslant n$. We denote by $P_{mn}(t)$ the $m \times n$ matrix

$$P_{mn}(t) = \begin{pmatrix} 1 & t & t^2 & \cdots & t^{n-1} \\ 0 & 1 & 2t & \cdots & (n-1)t^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \binom{n-1}{m-1} t^{n-m} \end{pmatrix} = (p_{ij}),$$

$$p_{ij} = \binom{j}{i} t^{j-i}, \qquad i = 0,\dots,m-1, \quad j = 0,\dots,n-1. \tag{9}$$

The following is easily proved:

$$P_{mn}(t_1)P_{np}(t_2) = P_{mp}(t_1 + t_2), \qquad m \leqslant n \leqslant p, \tag{10}$$

$$[P_{nn}(t)]^{-1} = P_{nn}(-t). \tag{11}$$

THEOREM 2.11. *Let A be a Hankel matrix of order n; then for any t,*

$$B = [P_{nn}(t)]^T A P_{nn}(t)$$

is again a Hankel matrix.

If $A = (\alpha_{i+j})$ then $B = (\beta_{i+j})$, where

$$\beta_r = \sum_{s=0}^{r} \binom{r}{s} t^{r-s} \alpha_s. \tag{12}$$

In other words, for the column vectors

$$\alpha = (\alpha_0, \ldots, \alpha_{2n-2})^T, \qquad \beta = (\beta_0, \ldots, \beta_{2n-2})^T,$$

$$\beta = (P_{2n-1,2n-1}(t))^T \alpha. \tag{13}$$

If A is proper then B is proper. In the case that A is singular then so is B and the Hankel polynomials φ_A, φ_B of A and B satisfy

$$\varphi_B(x) = \varphi_A(x - t). \tag{14}$$

If A is degenerate then so is B.

Proof. Let $A = (\alpha_{i+j})$. Then for $r, s = 0, \ldots, n-1$ we have $B = (b_{rs})$,

$$\begin{aligned} b_{rs} &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \binom{r}{i} t^{r-i} \alpha_{i+j} \binom{s}{j} t^{s-j} \\ &= \sum_{i,j=0}^{n-1} \binom{r}{i}\binom{s}{j} \alpha_{i+j} t^{r+s-(i+j)} \\ &= \sum_{k=0}^{r+s} \binom{r+s}{k} \alpha_k t^{r+s-k} = \beta_{r+s}, \end{aligned}$$

so that (12) and (13) are satisfied.

Since $P_{nn}(t)$ is an upper triangular matrix, it is easily seen that any leading submatrix of B of order $m = n$ is a product

$$[P_{mm}(t)]^T A_m P_{mm}(t),$$

where A_m is the leading submatrix of A of order m. Consequently, if A is proper then B is proper.

Let A be singular with the Hankel polynomial φ_A of order r. Then r is the rank of the proper part of A and

$$B_{r+1} = [P_{r+1,r+1}(t)]^T A_{r+1} P_{r+1,r+1}(t). \tag{15}$$

Let $\varphi_A(x) = x^r - \lambda_{r-1}x^{r-1} - \cdots - \lambda_0$, so that for $u = (-\lambda_0, -\lambda_1, \ldots, -\lambda_{r-1}, 1)^T$, $A_{r+1}u = 0$. Denote by v the vector $v = P_{r+1,r+1}(-t)u$. By (10) and (15),

$$B_{r+1}v = [P_{r+1,r+1}(t)]^T A_{r+1}u = 0.$$

Therefore,

$$\begin{aligned}\varphi_B(x) &= P_{1,r+1}(x)v = P_{1,r+1}(x)P_{r+1,r+1}(-t)u\\ &= P_{1,r+1}(x-t)u = \varphi_A(x-t).\end{aligned}$$

If A is degenerate, i.e. $\alpha_k = 0$ for $k = 0, \ldots, n-1$ then the same is true for β_k by (12). ■

In the following theorem, we use the notion of the Smith canonical form [4]:

THEOREM 2.12. *Let A be a singular $n \times n$ Hankel matrix. Denote, for an indeterminate λ, by Λ the column $(1, \lambda, \ldots, \lambda^{n-1})^T$. Then the Smith canonical form of the λ-matrix*

$$(A, \Lambda)$$

is

$$\begin{pmatrix} I_r & 0 & 0 \\ 0 & \varphi_A(\lambda) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{16}$$

where $r = r(A)$ and $\varphi_A(\lambda)$ is the H-polynomial of A.

Proof. By Theorem 2.5, A is a uniquely determined quasidirect sum

$$A = B + D,$$

where B is a proper Hankel matrix of rank, say, s, and D a degenerate Hankel matrix of rank d:

$$D = \begin{pmatrix} 0 & 0 \\ 0 & \hat{D} \end{pmatrix},$$

where $\hat{D}$ is nonsingular, $d \times d$. Let

$$B = \begin{pmatrix} B_{00} & B_{01} & B_{02} \\ B_{10} & B_{11} & B_{12} \\ B_{20} & B_{21} & B_{22} \end{pmatrix},$$

where B_{00} is $s \times s$ and B_{22} $d \times d$. Let

$$\Lambda = \begin{pmatrix} \Lambda_0 \\ \Lambda_1 \\ \Lambda_2 \end{pmatrix}$$

be the partitioning of Λ corresponding to that of B. Then, B being proper, (A, Λ) is equivalent (in the sense of equivalence of λ-matrices) to

$$\begin{pmatrix} B_{00} & 0 & 0 & \Lambda_0 \\ B_{10} & 0 & 0 & \Lambda_1 \\ B_{20} & 0 & D & \Lambda_2 \end{pmatrix} \tag{17}$$

by column operations using (1).

Expressing the last row of B_{10} as a linear combination of the previous s rows [the coefficients correspond to those of the polynomial $\varphi_A(\lambda)$] and subtracting the corresponding multiples from this last row, (17) will be equivalent to

$$\begin{pmatrix} B_{00} & 0 & 0 & \Lambda_0 \\ \hat{B}_{10} & 0 & 0 & \hat{\Lambda}_1 \\ 0 & 0 & 0 & \varphi_A(\lambda)\lambda^{n-s-d-1} \\ B_{20} & 0 & \hat{D} & \Lambda_2 \end{pmatrix}.$$

Similarly, doing the same with the last row of $\hat{B}_{10}$, etc., we obtain that (A, Λ)

is equivalent to

$$\begin{pmatrix} B_{00} & 0 & 0 & \Lambda_0 \\ 0 & 0 & 0 & \varphi_A(\lambda)\hat{\Lambda} \\ B_{20} & 0 & \hat{D} & \Lambda_2 \end{pmatrix},$$

where $\hat{\Lambda} = (1, \lambda, \dots, \lambda^{n-s-d-1})^T$.

Since $\hat{D}$ as well as B_{00} is nonsingular and $s + d = r$, (A, Λ) is equivalent to

$$\begin{pmatrix} I_r & 0 & 0 \\ 0 & 0 & \varphi_A(\lambda)\hat{\Lambda} \end{pmatrix}$$

and thus to (16). ∎

REMARK 2.13. If we use instead of Λ the homogeneous form $\Lambda_h = (\lambda_0^{n-1}, \lambda_0^{n-2}\lambda_1, \dots, \lambda_1^{n-1})^T$ and the homogeneous Smith canonical form [this is possible because each subdeterminant of (A, Λ_h) is homogeneous], we obtain by the same argument that (A, Λ_h) is equivalent to

$$\begin{pmatrix} I_r & 0 & 0 \\ 0 & \psi_A(\lambda_0, \lambda_1) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\psi_A(\lambda_0, \lambda_1)$, the "homogeneous H-polynomial" of A, is related to $\varphi_A(\lambda)$ by

$$\psi_A(\lambda_0, \lambda_1) = K\lambda_0^r\varphi_A(\lambda_1/\lambda_0) \tag{18}$$

(K is some nonzero constant). Thus $\psi_A(\lambda_0, \lambda_1)$ is divisible exactly by λ_0^d (and not by λ_0^{d+1}).

We are now able to strengthen Lemma 2.8.

THEOREM 2.14. *If $A = (\alpha_{i+j})$ is a Hankel matrix and J_n is the matrix from* (7), *then $B = J_nAJ_n$ is again a Hankel matrix $B = (\beta_{i+j})$, where*

$$\beta_k = \alpha_{2n-2-k}. \tag{19}$$

For the column vectors $\alpha = (\alpha_0, \dots, \alpha_{2n-2})$, $\beta = (\beta_0, \dots, \beta_{2n-2})$,

$$\beta = J_{2n-1}\alpha. \tag{20}$$

If A is singular, then so is B, and for the H-polynomials,

$$\varphi_B(x) = kx^r\varphi_A\left(\frac{1}{x}\right), \tag{21}$$

where $r = r(A) = r(B)$.

Proof. It suffices to prove (21). Using the homogeneous H-polynomials, we have clearly

$$\psi_B(\lambda_0, \lambda_1) = k\psi_A(\lambda_1, \lambda_0).$$

By a repeated use of (18), we obtain

$$\varphi_B(x) = K_1\psi_B(1, x) = K_2\psi_A(x, 1) = K_3x^r\varphi_A\left(\frac{1}{x}\right),$$

where K_1, K_2, K_3 are nonzero constants. ■

REMARK 2.15. It follows from Theorems 2.11 and 2.14 that the set of $n \times n$ Hankel matrices is invariant with respect to transformations generated by those in both theorems. It is easily seen that these transformations form the group of all complex nonsingular linear transformations in a projective one-dimensional space, and the homogeneous H-polynomials corresponding to all matrices B which arise that way from one fixed matrix A are obtained as $\psi_B(\lambda_0, \lambda_1) = \psi_A(c_{00}\lambda_0 + c_{01}\lambda_1, c_{10}\lambda_0 + c_{11}\lambda_1)$, where $\det(c_{ik}) \neq 0$.

THEOREM 2.16. *A singular $n \times n$ Hankel matrix has the H-polynomial $(x - t)^r$ and the improper degree zero iff it has the form*

$$[P_{rn}(t)]^T SP_{rn}(t), \tag{22}$$

where S is an upper triangular nonsingular $r \times r$ Hankel matrix or equivalently, the form

$$A = [P_{nn}(t)]^T\tilde{S}P_{nn}(t),$$

where

$$\tilde{S} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \tag{23}$$

is a simple Hankel matrix of rank r. This matrix is then H-indecomposable.

Proof. If a singular matrix A has the form (22) or (23), then S has H-polynomial x^r, and A, by Theorem 2.11, has H-polynomial $(x-t)^r$. By Corollary 2.9, $\tilde{S}$ is H-indecomposable, so that the same is true for A: Any H-decomposition of A would lead to an H-decomposition of $\tilde{S}$ by Theorem 2.11, since

$$\tilde{S} = [P_{nn}(-t)]^T A P_{nn}(-t).$$

Conversely, let A be a proper singular $n \times n$ Hankel matrix having the H-polynomial $(x-t)^r$. By Theorem 2.11, the matrix

$$\tilde{A} = [P_{nn}(-t)]^T A P_{nn}(-t)$$

has H-polynomial x^r and is a proper Hankel matrix of rank r. By Lemma 2.1 and (1), A has the form (23) and A the form (22). ∎

THEOREM 2.17. *A sum of two nonzero Hankel matrices of the same order n is quasidirect iff both these matrices are singular and their homogeneous H-polynomials are relatively prime with degrees whose sum does not exceed n.*

Proof. By Theorems 2.3 and 2.5, the "only if" part follows immediately. To prove the "if" part, let B and C be nonzero singular Hankel matrices whose homogeneous H-polynomials $\gamma_B(\lambda_0, \lambda_1)$ and $\psi_C(\lambda_0, \lambda_1)$ are relatively prime. Let their degrees $r = r(B)$ and $s = r(C)$ satisfy $r + s = n$. We shall use induction with respect to the number q of distinct "roots" of ψ_C (i.e. linearly independent linear factors dividing ψ_C). Let first $q = 1$. If $\psi_C(\lambda_0, \lambda_1) = k\lambda_0^s$, k constant, then C is degenerate. Since ψ_B and ψ_C are relatively prime, λ_0 does not divide ψ_B. Thus B is proper by the last sentence of Remark 2.13, and the sum is quasidirect by Theorem 2.6. If this is not the case, $\varphi_C(x) = (x-t)^s$. Using Theorem 2.11, we can assume without loss of generality that $t = 0$. The matrix C is then simple by Theorem 2.16, and using the transformation from Theorem 2.14, the matrix $\tilde{C} = J_n C J_n$ is degenerate and $\tilde{B} = J_n B J_n$ proper, since otherwise the homogeneous H-polynomials of B and C would not be relatively prime. By Theorem 2.6, the sum $\tilde{B} + \tilde{C}$ is quasidirect, and the same is clearly true for $B + C$.

Let now $q > 1$, and suppose the assertion is true if the number of distinct roots of ψ_C is less than q. If C is not proper, $C = C_0 + D$ with C_0 proper and D degenerate. Since B and C_0 have relatively prime H-polynomials and C_0 has less than q roots, $B + C_0$ is quasidirect and hence proper by Theorem 2.3. It follows that $B + C = (B + C_0) + D$ is quasidirect as well by Theorem 2.6.

It remains to settle the case that both B and C are proper. In view of Theorem 2.11, we can assume that φ_C has the root zero. Then J_nCJ_n is not proper, while J_nBJ_n is proper and their sum is quasidirect by the previous case. Thus $B+C$ is quasidirect and the proof is complete. ■

REMARK 2.18. This theorem can easily be generalized to more than two summands.

Now we are able to prove the main theorem on decomposition of singular Hankel matrices.

THEOREM 2.19. *An $n \times n$ Hankel matrix A is singular and has the H-polynomial*

$$\varphi(x) = \prod_{i=1}^{s} (x - t_i)^{r_i}, \qquad t_i \text{ mutually distinct},$$

and the improper degree d iff

$$\sum_{i=1}^{s} r_i + d < n \tag{24}$$

and A has the form

$$A = \sum_{i=1}^{s} \left[P_{r_i n}(t_i) \right]^T S_i P_{r_i n}(t_i) + D, \tag{25}$$

where D is a degenerate Hankel matrix of rank d and S_i is a nonsingular upper triangular $r_i \times r_i$ Hankel matrix for $i = 1, \ldots, s$.

The sum (25) is then quasidirect, all summands are H-indecomposable Hankel matrices, and the decomposition into H-indecomposable Hankel matrices is (up to the order) unique.

Proof. First let A have the form (25) and satisfy (24). Then clearly A is singular, and the sum (25) is quasidirect by Theorem 2.17 and Remark 2.18. Since the summands have H-polynomials $(x-t_i)^{r_i}$, $i = 1, \ldots, s$ (D has H-polynomial 1), A has H-polynomial φ and improper degree d as asserted.

To prove the converse, assume that A is a singular $n \times n$ Hankel matrix with the H-polynomial

$$\varphi(x) = \prod_{i=1}^{s} (x - t_i)^{r_i}$$

with t_i distinct and the improper degree d. Since the left-hand side in (24) is $r(A)$, (24) is satisfied.

By Theorem 2.5, $\hat{A} = A + D$, where $\hat{A}$ is proper and D is degenerate of rank d, the sum being quasidirect. Furthermore, A has H-polynomial $\varphi(x)$ again.

We shall prove by induction with respect to s that there exists the quasidirect decomposition

$$\hat{A} = \sum_{i=1}^{s} \left[P_{r_i n}(t_i)\right]^T S_i P_{r_i n}(t_i) \tag{26}$$

for nonsingular upper triangular Hankel matrices S_i of order r_i.

This assertion is true for $s = 1$ by Theorem 2.16. Thus let $s > 1$, and assume the assertion true if the number of distinct roots of the H-polynomial is less than s.

The matrix

$$B = \left[P_{nn}(-t_s)\right]^T \hat{A} P_{nn}(-t_s)$$

is by Theorem 2.11 a Hankel matrix whose H-polynomial has r_s-tuple root zero. Consequently,

$$C = J_n B J_n$$

is not proper and has a quasidirect decomposition

$$C = C_0 + D_0$$

where C_0 is proper and D_0 degenerate of rank r_s. Consequently,

$$B = J_n C_0 J_n + J_n D_0 J_n,$$

where $\tilde{S}_s = J_n D_0 J_n$ is a simple Hankel matrix of rank r_s. Moreover,

$$\hat{A} = \tilde{A} + \left[P_{nn}(t_s)\right]^T \tilde{S}_s P_{nn}(t_s) = \tilde{A} + \left[P_{r_s n}(t_s)\right]^T S_s P_{r_s n}(t_s),$$

where S_s is an upper triangular nonsingular Hankel matrix of order r_s, $\tilde{A}$ is a Hankel matrix with H-polynomial

$$\prod_{i=1}^{s-1} (x - t_i)^{r_i},$$

and the sum is quasidirect. By the induction hypothesis, $\tilde{A}$ has the corresponding form (26), so that the same is true for A. The proof is complete. ∎

Before we turn to the nonsingular case, we shall present a lemma.

LEMMA 2.20. *The H-polynomial of a proper singular matrix $A = (\alpha_{i+j})$ of rank r can be written in the form*

$$\det\begin{pmatrix} \alpha_0 & \cdots & \alpha_{r-1} & 1 \\ \alpha_1 & \cdots & \alpha_r & x \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{r-1} & \cdots & \alpha_{2r-2} & x^{r-1} \\ \alpha_r & \cdots & \alpha_{2r-1} & x^r \end{pmatrix}$$

$$\div \det\begin{pmatrix} \alpha_0 & \cdots & \alpha_{r-1} \\ \cdots & \cdots & \cdots \\ \alpha_{r-1} & \cdots & \alpha_{2r-2} \end{pmatrix}. \tag{27}$$

Proof. Clear. ∎

LEMMA 2.21. *Let A be a nonsingular Hankel matrix of order $n > 1$ over an algebraically closed field. Then there exists a one-parameter system of proper singular Hankel matrices of order $n+1$ which have A as their common leading submatrix. For at least one value of the parameter the corresponding matrix of order $n+1$ is a quasidirect sum of at least two nontrivial Hankel matrices.*

Proof. Let $A = (\alpha_{i+j})$. Put $\alpha_{2n-1} = \xi$, a parameter. Then there exists a uniquely determined number α_{2n} such that the matrix $\tilde{A}(\xi) = (\alpha_{i+j})$ of order $n+1$ is singular.

By Lemma 2.20, the H-polynomial of $\tilde{A}(\xi)$ is (up to a nonzero factor)

$$\det\begin{pmatrix} \alpha_0 & \cdots & \alpha_{n-2} & \alpha_{n-1} & 1 \\ \alpha_1 & \cdots & \alpha_{n-1} & \alpha_n & x \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-1} & \cdots & \alpha_{2n-3} & \alpha_{2n-2} & x^{n-1} \\ \alpha_n & \cdots & \alpha_{2n-2} & \xi & x^n \end{pmatrix}$$

and is thus of the form

$$\omega(x)+\xi\psi(x) \tag{28}$$

where $\omega(x)$ is of degree n and ψ is a nonzero polynomial (otherwise, A would be singular) of degree at most $n-1$, independent of ξ.

Let us show that for some ξ, the polynomial (28) has at least two distinct roots. Suppose this is not the case. Then $\omega(x)=\sigma(x-a)^n$ for some $\sigma \neq 0$ and some a. Let $\psi(x)=\hat{\psi}(x-a)$. If the expansion of $\hat{\psi}(x)$ as polynomial in x has a single nonzero term, we get a contradiction. If $\hat{\psi}$ has at least two nonzero terms, (28) is of the form $\tau(x-a)$ where

$$\tau(x)=\sigma x^n+\xi u x^m+\xi v x^p+\cdots, \qquad n>m>p, \quad u\neq 0, \quad v\neq 0.$$

Assume that $\tau(x)=\sigma(x+t)^n$. Then

$$\xi u=\sigma\binom{n}{m}t^{n-m}, \qquad \xi v=\sigma\binom{n}{p}t^{n-p},$$

so that

$$\binom{n}{p}^{n-m}\left(\frac{\xi}{\sigma}\right)^{m-p}u^{n-p}-\binom{n}{m}^{n-p}v^{n-m}=0. \tag{29}$$

It follows that if ξ does not satisfy (29), $\tau(x)$ has at least two distinct roots, a contradiction. Now, if (28) has at least two distinct roots, the corresponding matrix $A(\xi)$ cannot then be H-indecomposable by Theorem 2.19. ■

COROLLARY 2.22. *A nonsingular Hankel matrix of order greater than one is never H-indecomposable over an algebraically closed field; it can be decomposed in infinitely many ways.*

THEOREM 2.23. *A Hankel matrix of order greater than one is H-indecomposable over an algebraically closed field iff it is singular and either degenerate or of the form* (22).

Every other Hankel matrix can be expressed as a quasidirect sum of (at least two) H-indecomposable matrices. This sum is (up to the ordering) unique iff the matrix is nonsingular.

Proof. Follows from Corollary 2.22 and Theorems 2.5, 2.6, and 2.19. ■

REMARK 2.24. As we pointed out in the introduction, a corresponding theorem for Toeplitz matrices follows immediately.

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