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Bivariate orthogonal polynomials in the Lyskova class ☆

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ABSTRACT

Classical orthogonal polynomials in two variables can be characterized as the polynomial solutions of a matrix second-order partial differential equation involving matrix polynomial coefficients. In this work, we study classical orthogonal polynomials in two variables whose partial derivatives satisfy again a second-order partial differential equation of the same type.

1. Preliminaries

In 1991, Lyskova [1] studied partial differential equations in n variables

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial v}{\partial x_i} + \lambda\, v = 0, \tag{1}$$

satisfying that all the partial derivatives of the solutions are again solutions of a similar equation; then, Eq. (1) is said to belong to the *basic class*. On the basis of this fact, he constructed systems of orthogonal polynomial eigenfunctions of the differential operator. This kind of equation has been studied by several authors (see, for instance, [2,3]).

In this paper, we focus our attention on the two-variable case. For a complete description of this and another related subjects see, for instance, [4].

We give an extended definition of classical orthogonal polynomials in two variables, using the vector notation for orthogonal polynomials introduced in [5].

A *polynomial system* is a sequence of vectors $\{\mathbb{P}_n\}_{n\geq 0}$ such that

$$\mathbb{P}_n = \left(P_{n,0}, P_{n-1,1}, \ldots, P_{0,n}\right)^t,$$

where $\left\{P_{n,0}, P_{n-1,1}, \ldots, P_{0,n}\right\}$ are independent polynomials of total degree n.

A polynomial system is called *monic* if

$$P_{h,k}(x,y) = x^h y^k + R(x,y), \quad h+k=n,$$

where $R(x,y)$ is a polynomial of total degree at most $n-1$.

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Let u be a linear moment functional. We will say that a polynomial system $\{\mathbb{P}_n\}_{n\geq 0}$ is a *weak orthogonal polynomial system* (WOPS) with respect to u if

$$\begin{aligned} \langle u, \mathbb{P}_n\mathbb{P}_m^t\rangle &= 0, \quad m \neq n,\\ \langle u, \mathbb{P}_n\mathbb{P}_n^t\rangle &= H_n, \end{aligned} \tag{2}$$

where H_n is a $(n+1)\times(n+1)$ nonsingular matrix. In the particular case in which H_n is a diagonal matrix, the polynomial system is said to be an *orthogonal polynomial system* (OPS).

A linear moment functional u is said to be *quasi-definite* if there is an orthogonal polynomial system with respect to u. In this case, there is a *unique* monic WOPS with respect to u (see [6]).

Definition 1. Let u be a quasi-definite moment functional, and let $\{\mathbb{P}_n\}_{n\geq 0}$ be the monic WOPS associated with u. Then, u is *classical* if and only if for all $n \geq 0$, there exist nonsingular $(n+1)\times(n+1)$ matrices Λ_n with constant entries such that

$$L[\mathbb{P}_n] \equiv a\,\partial_{xx}\mathbb{P}_n + 2b\,\partial_{xy}\mathbb{P}_n + c\,\partial_{yy}\mathbb{P}_n + d\,\partial_x\mathbb{P}_n + e\,\partial_y\mathbb{P}_n = \Lambda_n\,\mathbb{P}_n, \tag{3}$$

where a, b, and c are polynomials of total degree less than or equal to 2, and d, e are polynomials of degree 1.

Remark 2. Note that the definition of *classical* does not depend on the particular choice of the monic WOPS. In fact, a matrix partial differential equation equivalent to (3) is satisfied by every WOPS associated with a classical moment functional u. Let $\{\mathbb{P}_n\}_{n\geq 0}$ be the monic WOPS associated with a classical moment functional u, and let $\{\mathbb{Q}_n\}_{n\geq 0}$ be a different WOPS associated with u. For $n \geq 0$, let A_n be the nonsingular matrix corresponding to the change of basis $\mathbb{Q}_n = A_n\,\mathbb{P}_n$, $n \geq 0$. Then,

$$L[\mathbb{Q}_n] = \tilde{\Lambda}_n\,\mathbb{Q}_n,$$

where $\tilde{\Lambda}_n = A_n^{-1}\Lambda_n A_n$, that is, Λ_n and $\tilde{\Lambda}_n$ are similar matrices [7].

With this definition, we proved in [7] that the gradients of the vector polynomials satisfy

$$\langle u, (\nabla\mathbb{P}_n^t)^t\,\Phi\,\nabla\mathbb{P}_m^t\rangle = 0, \quad m \neq n,$$

where

$$\Phi = \begin{pmatrix} a & b\\ b & c\end{pmatrix}.$$

In [8], the authors proved a second-order partial differential equation for these gradients. However, in general, the partial derivatives of the vector polynomials, do not satisfy an orthogonality condition or a partial differential equation.

2. Orthogonal polynomials in the extended Lyskova class

From now on, $\{\mathbb{P}_n\}_{n\geq 0}$ will denote the monic WOPS associated with a classical moment functional u. The next definition is a natural extension of the *basic class* as given by Kim et al. in [2].

Definition 3. The matrix partial differential equation (3) belongs to the *extended Lyskova class* if $a_y = b_{xy} = c_x = d_y = e_x = 0$, that is,

$$\begin{aligned} a(x,y) &= a(x) = a_{20}x^2 + a_{10}x + a_0,\\ b(x,y) &= b_{11}xy + b_{10}x + b_{01}y + b_0,\\ c(x,y) &= c(y) = c_{02}y^2 + c_{01}y + c_0,\\ d(x,y) &= d(x) = d_{10}x + d_0,\\ e(x,y) &= e(y) = e_{01}y + e_0. \end{aligned} \tag{4}$$

From this definition, we recover an extended version of the original definition of *basic class* given by Lyskova in [1].

Theorem 4. *If the matrix partial differential equation (3) for monic WOPS belongs to the extended Lyskova class, then Λ_n, for $n \geq 0$, are diagonal matrices whose diagonal elements $\Lambda_n = \mathrm{diag}[\lambda_{n,0}, \lambda_{n-1,1}, \ldots, \lambda_{0,n}]$ are given by*

$$\lambda_{n-i,i} = (n-i)[a_{20}\,(n-i-1) + b_{11}\,i + d_{10}] + i[b_{11}\,(n-i) + c_{02}\,(i-1) + e_{01}],$$

for $0 \leq i \leq n$. Moreover, for fixed nonnegative integers $i, j \geq 0$, and for all $n \geq i+j$, there exist nonsingular $(n+1)\times(n+1)$ matrices $\Lambda_n^{(i,j)}$ with constant entries, such that $\partial_x^i\,\partial_y^j\,\mathbb{P}_n$ is solution of the partial differential equation

$$a\,\partial_{xx}(\partial_x^i\,\partial_y^j\,\mathbb{P}_n) + 2\,b\,\partial_{xy}(\partial_x^i\,\partial_y^j\,\mathbb{P}_n) + c\,\partial_{yy}(\partial_x^i\,\partial_y^j\,\mathbb{P}_n) + d^{(i,j)}\,\partial_x(\partial_x^i\,\partial_y^j\,\mathbb{P}_n) + e^{(i,j)}\partial_y(\partial_x^i\,\partial_y^j\,\mathbb{P}_n) = \Lambda_n^{(i,j)}\,(\partial_x^i\,\partial_y^j\,\mathbb{P}_n), \tag{5}$$

where

$$d^{(i,j)} = d + i\,a_x + 2\,j\,b_y, \qquad e^{(i,j)} = e + 2\,i\,b_x + j\,c_y, \qquad \Lambda_n^{(i,j)} = \Lambda_n - \nu_{i,j}\,I_{n+1},$$

with

$$\nu_{i,j} = \frac{i(i-1)}{2} a_{xx} + 2\, i\, j\, b_{xy} + \frac{j(j-1)}{2} c_{yy} + i\, d_x + j\, e_y,$$

and I_{n+1} represents the identity matrix of size $n+1$.

Proof. Using Proposition 15 in [7], the special shape of the polynomial coefficients gives the diagonal character of the matrix Λ_n, for $n \geq 1$. The explicit expression for the diagonal elements of the matrix Λ_n can be easily obtained from Eq. (3) by considering the highest total order terms.

On the other hand, differentiating (3) i times with respect to x, and j times with respect to y, $i, j \geq 0$, and using the explicit expressions for the polynomial coefficients, we get (5). □

From this result, we remark that if a matrix partial differential equation such as (3) for the monic polynomials belongs to the Lyskova class, then the matrix Λ_n is diagonal.

There are many examples of partial differential equations belonging to the extended Lyskova class.

For instance, classical orthogonal polynomials on the unit disk, $B_2 = \{(x, y) : x^2 + y^2 \leq 1\}$, are associated with the weight function

$$\omega(x, y) = (1 - x^2 - y^2)^{\mu - 1/2}, \qquad \mu > -1/2,$$

and they satisfy the partial differential equation

$$(x^2 - 1)\, v_{xx} + 2\, x\, y\, v_{xy} + (y^2 - 1)\, v_{yy} + g\, x\, v_x + g\, y\, v_y = \lambda_n\, v.$$

The polynomial coefficients of this partial differential equation satisfy (4), and then, the partial differential equation belongs to the *extended Lyskova class*.

The Appell polynomials are orthogonal on the simplex $T = \{(x, y) :\ x, y \geq 0,\ 1 - x - y \geq 0\}$ with respect to the weight function

$$\omega(x, y) = x^\alpha y^\beta (1 - x - y)^\gamma, \qquad \alpha, \beta, \gamma > -1.$$

They satisfy the partial differential equation

$$(x^2 - x)\, v_{xx} + 2\, x\, y\, v_{xy} + (y^2 - y)\, v_{yy} + [(\alpha + \beta + \gamma + 2)x - (\alpha + 1)]v_x + [(\alpha + \beta + \gamma + 2)y - (\beta + 1)]v_y = \lambda_n\, v,$$

whose polynomial coefficients satisfy (4). Then, the above partial differential equation belongs to the extended Lyskova class.

Nevertheless, all the partial differential equations for classical polynomials in two variables do not belong to the Lyskova class. In [6], Krall and Sheffer showed that the partial differential equation

$$3\, y\, v_{xx} + 2\, v_{xy} - x\, v_x - y\, v_y = -n\, v,$$

has a WOPS as solution. These polynomials are considered classical in the Krall and Sheffer classification. However, this equation is not in the extended Lyskova class.

Using induction on n, and the special shape of the coefficients (4), we prove the following theorem.

Theorem 5. *Let u be a classical moment functional, and let $\{\mathbb{P}_n\}_{n\geq 0}$ be the monic WOPS associated with u satisfying (3). Assume that (3) belongs to the extended Lyskova class. Then, for $n \geq 1$, the following statements hold:*

(i) *If for all $k < n$, and for $i = 0, \ldots, k$, $\lambda_{n,0} \neq \lambda_{k-i,i}$, then*

$$\partial_y P_{n,0}(x, y) = 0.$$

(ii) *If for all $k < n$, and for $i = 0, \ldots, k$, $\lambda_{0,n} \neq \lambda_{k-i,i}$, then*

$$\partial_x P_{0,n}(x, y) = 0.$$

Moreover, if we define

$$p_n(x) = P_{n,0}(x, y), \quad \text{and} \quad q_n(y) = P_{0,n}(x, y),$$

then $\{p_n(x)\}_{n\geq 0}$ and $\{q_n(y)\}_{n\geq 0}$ are monic orthogonal polynomial sequences in one variable satisfying

$$a(x)\, p_n''(x) + d(x)\, p_n'(x) = \lambda_{n,0}\, p_n(x),$$
$$c(y)\, q_n''(y) + e(y)\, q_n'(y) = \lambda_{0,n}\, q_n(y),$$

and therefore, they are classical in one variable.

Proof. We will show the result by induction on $n \geq 0$. Observe that $P_{n,0}(x, y)$ only depends on x, for $n = 0, 1$. Suppose that, for $0 \leq k \leq n$, the polynomials $P_{k,0}(x, y)$ only depend on x, that is, $P_{k,0}(x, y) = P_{k,0}(x)$. Then, we can express the monic polynomial $P_{n+1,0}(x,y)$ as

$$P_{n+1,0}(x,y) = x^{n+1} + \sum_{k=0}^{n}\sum_{i=0}^{k} m_{k-i,i}\, P_{k-i,i}(x,y),$$

where $m_{k-i,i}$ are constants. By linearity of L, we have

$$L[P_{n+1,0}(x,y)] = L[x^{n+1}] + \sum_{k=0}^{n}\sum_{i=0}^{k} m_{k-i,i}\, L[P_{k-i,i}(x,y)].$$

Using the special shape of the coefficients (4), we get

$$L[x^{n+1}] = (n+1)\left((n\,a_{20}+d_{10})x^{n+1} + (n\,a_{10}+d_0)x^n + n\,a_0\,x^{n-1}\right).$$

On the other hand, since the polynomials $P_{n+1,0}(x,y)$, and $P_{k-i,i}(x,y)$, $0 \le i \le k \le n$, satisfy the partial differential equation (3), we obtain

$$(n+1)(n\,a_{10}+d_0)x^n + (n+1)n\,a_0\,x^{n-1} + \sum_{k=0}^{n}\sum_{i=0}^{k}(\lambda_{k-i,i}-\lambda_{n+1,0})\,m_{k-i,i}\,P_{k-i,i}(x,y) = 0, \tag{6}$$

simplifying the terms of degree $n+1$. By the induction hypothesis, $P_{k,0}(x,y) \equiv P_{k,0}(x)$ only depends on the variable x, so we can write

$$(n+1)(n\,a_{10}+d_0)x^n + (n+1)\,n\,a_0\,x^{n-1} = \sum_{k=0}^{n}\alpha_k\,P_{k,0}(x),$$

and therefore, the highest term degree of (6)

$$(\alpha_n + (\lambda_{n,0}-\lambda_{n+1,0})m_{n,0})P_{n,0}(x) + \sum_{k=1}^{n}(\lambda_{n-k,k}-\lambda_{n+1,0})m_{n-k,k}\,P_{n-k,k}(x,y),$$

must be zero since the polynomials $\{P_{n-k,k}(x,y)\}_{k=0}^{n}$ are linearly independent modulo $\mathcal{P}_{n-1}$, the linear space of polynomials in two variables of total degree less than or equal to $n-1$. In this way,

$$(\lambda_{n-k,k}-\lambda_{n+1,0})m_{n-k,k} = 0, \quad 1 \le k \le n,$$

and then, $m_{n-k,k} = 0$, $1 \le k \le n$, using the hypothesis

$$\lambda_{n+1,0} \ne \lambda_{k-i,i}, \quad k < n+1,\ 0 \le i \le k.$$

Therefore, using the same reasoning, expression (6) provides that

$$(n+1)(n\,a_{10}+d_0)x^n + (n+1)n\,a_0\,x^{n-1} + (\lambda_{n,0}-\lambda_{n+1,0})m_{n,0}P_{n,0}(x) + (\lambda_{n-1,0}-\lambda_{n+1,0})m_{n-1,0}P_{n-1,0}(x)$$
$$+ \sum_{k=1}^{n-1}(\lambda_{n-1-k,k}-\lambda_{n+1,0})\,m_{n-1-k,k}\,P_{n-1-k,k}(x,y),$$

must be zero modulo $\mathcal{P}_{n-2}$. The first four terms of the above expression only depend on x, and then, they can be expressed as a linear combination of $\{P_{k,0}(x)\}_{k=0}^{n}$. Hence,

$$\beta_n\,P_{n,0}(x) + \beta_{n-1}\,P_{n-1,0}(x) + \sum_{k=1}^{n-1}(\lambda_{n-1-k,k}-\lambda_{n+1,0})m_{n-1-k,k}\,P_{n-1-k,k}(x,y) = 0,$$

modulo $\mathcal{P}_{n-2}$. Since $P_{n,0}(x)$ and $\{P_{n-1-k,k}(x,y)\}_{k=0}^{n-1}$ are linearly independent modulo $\mathcal{P}_{n-2}$, we get $m_{n-1-k,k} = 0$, $1 \le k \le n-1$. The same reasoning shows that

$$m_{k-i,i} = 0, \quad 1 \le k \le n,\ 1 \le i \le k,$$

and then

$$P_{n+1,0}(x,y) = x^{n+1} + \sum_{k=0}^{n} m_{k,0}\,P_{k,0}(x). \quad \square$$

Other examples of orthogonal polynomials satisfying partial differential equations in the extended Lyskova class can be constructed using the tensor product of classical orthogonal polynomials in one variable. Let $\{R_h\}_{h\ge0}$ and $\{S_k\}_{k\ge0}$ be two families of classical orthogonal polynomials in one variable. The family of polynomials in two variables defined by the tensor product of these two families, that is,

$$P_{h,k}(x,y) = R_h(x)S_k(y), \quad h,k \ge 0,$$

satisfies a partial differential equation in the Lyskova class with the condition $b \equiv 0$. In fact, if the polynomials $\{R_h\}_{h\geq 0}$ and $\{S_k\}_{k\geq 0}$ are solutions of the differential equations

$$a(x)\,R_h''(x) + d(x)\,R_h'(x) = \lambda_h\,R_h(x),$$
$$c(y)\,S_k''(y) + e(y)\,S_k'(y) = \lambda_k'\,S_k(y),$$

respectively, then $\mathbb{P}_n = \left(P_{n,0}, P_{n-1,1}, \dots, P_{0,n}\right)^t$ is a solution of the partial differential equation

$$a(x)\partial_{xx}\mathbb{P}_n + c(y)\partial_{yy}\mathbb{P}_n + d(x)\partial_x\mathbb{P}_n + e(y)\partial_y\mathbb{P}_n = \Lambda_n\,\mathbb{P}_n.$$

In this case, the polynomial coefficients satisfy (4), and the matrix Λ_n is diagonal with elements $\lambda_{n-i,i} = \lambda_{n-i} + \lambda_i'$, for $0 \leq i \leq n$.

In the following proposition, we prove the reciprocal of this property for the polynomials in the extended Lyskova class, as a consequence of Theorem 5.

Proposition 6. *Let u be a classical moment functional, and let $\{\mathbb{P}_n\}_{n\geq 0}$ be the monic WOPS associated with u satisfying Eq. (3). Assume that (3) belongs to the extended Lyskova class. If $b \equiv 0$, then*

(i) $\lambda_{h,k} = \lambda_{h,0} + \lambda_{0,k}$.

(ii) *If $\lambda_{n,0} \neq \lambda_{k-i,i}$, and $\lambda_{0,n} \neq \lambda_{k-i,i}$, for $i = 0, \dots, k$, and $\forall k < n$, then $\{p_n(x)\}_{n\geq 0} = \{P_{n,0}(x,y)\}_{n\geq 0}$ and $\{q_n(y)\}_{n\geq 0} = \{P_{0,n}(x,y)\}_{n\geq 0}$ are classical monic orthogonal polynomial sequences in one variable, and then $\{\mathbb{P}_n\}_{n\geq 0}$ is the tensor product of two classical families in one variable.*

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References

[1] A.S. Lyskova, Orthogonal polynomials in several variables, Soviet. Math. Dokl. 43 (1991) 264–268.
[2] Y.J. Kim, K.H. Kwon, J.K. Lee, Orthogonal polynomials in two variables and second-order partial differential equations, J. Comput. Appl. Math. 82 (1997) 239–260.
[3] J.K. Lee, L.L. Littlejohn, B.H. Yoo, Orthogonal polynomials satisfying partial differential equations belonging to the basic class, J. Korean Math. Soc. 41 (2004) 1049–1070.
[4] C.F. Dunkl, Y. Xu, Orthogonal Polynomials of Several Variables, in: Encyclopedia of Mathematics and its Applications, vol. 81, Cambridge University Press, 2001.
[5] M.A. Kowalski, The recursion formulas for orthogonal polynomials in n variables, SIAM J. Math. Anal. 13 (1982) 309–315.
[6] H.L. Krall, I.M. Sheffer, Orthogonal polynomials in two variables, Ann. Mat. Pura Appl. (4) 76 (1967) 325–376.
[7] L. Fernández, T.E. Pérez, M.A. Piñar, Weak classical orthogonal polynomials in two variables, J. Comput. Appl. Math. 178 (2005) 191–203.
[8] L. Fernández, T.E. Pérez, M.A. Piñar, Second order partial differential equations for gradients of orthogonal polynomials in two variables, J. Comput. Appl. Math. 199 (2007) 113–121.