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The matrix-valued H^p corona problem in the disk and polydisk

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Abstract

In this paper, we consider the matrix-valued H^p corona problem in the disk and polydisk. The result for the disk is rather well-known, and is usually obtained from the classical Carleson Corona Theorem by linear algebra. Our proof provides a streamlined way of obtaining this result and allows one to get a better estimate on the norm of the solution. In particular, we were able to improve the estimate found in the recent work of Trent in [J. Funct. Anal. 189 (2002) 267–282]. Note that, the solution of the H^∞ matrix corona problem in the disk can be easily obtained from the H^2 corona problem either by factorization, or by the Commutant Lifting Theorem. The H^p corona problem in the polydisk was originally solved by Lin in [Bull. Sci. Math. 110(2) (1986) 69–84, Trans. Amer. Math. Soc. 341 (1994) 371–375]. The solution used Koszul complexes and was rather complicated because one had to consider higher order $\bar{\partial}$-equations. Our proof is more transparent and it improves upon Lin's result in several ways. First, we deal with the more general matrix corona problem. Second, we were able to show that the norm of the solution is independent of the number of generators. Additionally, we illustrate that the norm of the solution of the H^2 corona problem in the polydisk $\mathbb{D}^n$ grows at most proportionally to $\sqrt{n}$. Our approach is based on one that was originated by Andersson in [Math. Z. 201 (1989) 121–130]. In the disk it essentially depends on Green's Theorem and duality to obtain the estimate. In the polydisk we use Riesz projections to reduce the problem to the disk case.

Keywords: H^p Corona Theorem; Matrix Corona Theorem; Toeplitz Corona Theorem; Polydisk

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Notation

$:=$	equal by definition
$\mathbb{C}$	the complex plane
$\mathbb{D}$	the unit disk, $\mathbb{D} := \{z \in \mathbb{C} : \lvert z\rvert < 1\}$
$\mathbb{T}$	the unit circle, $\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} : \lvert z\rvert = 1\}$
$d\mu$	measure on $\mathbb{D}$ with $d\mu = \frac{2}{\pi}\log\frac{1}{\lvert z\rvert}\,\mathrm{d}x\,\mathrm{d}y$
dm	normalized Lebesgue measure on $\mathbb{T}$, $m(\mathbb{T}) = 1$
$\langle\cdot,\cdot\rangle$	inner product
$\\|\cdot\\|$	norm; since we are dealing with matrix and operator-valued functions this symbol is a bit overloaded, but we hope it will not cause any confusion. The norm in the function spaces can be always distinguished by subscript. Thus for a vector-valued function f the symbol $\\|f\\|_2$ denotes its L^2-norm, but the symbol $\\|f\\|$ stands for the scalar valued function whose value at a point z is the norm of the vector $f(z)$
$\operatorname{tr} A$	Trace of the operator A
$H^\infty(\mathbb{D})$	space of bounded analytic functions on $\mathbb{D}$ with the supremum norm
$L^p(\mathbb{D}^n; E)$	vector-valued Lebesgue spaces
$H^p(\mathbb{D}^n; E)$	vector-valued Hardy classes
$H^\infty(\mathbb{D}; E \to E_*)$	operator Hardy class of bounded analytic functions from the disk whose values are bounded operators from E to E_*, $\\|F\\|_\infty := \sup_{z\in\mathbb{D}} \\|F(z)\\|$
$\partial, \bar\partial$	derivatives with respect to z and $\bar z$, respectively: $\partial := \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$, $\bar\partial := \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$
$\partial_j, \bar\partial_j$	derivatives with respect to the variables z_j and $\bar z_j$ respectively
$\mathbf{z}$	point in $\mathbb{C}^n$

$\mathbf{z}_j$	$\mathbf{z}$ with the coordinate z_j omitted; slightly abusing notation we will write $\mathbf{z} = (z_j, \mathbf{z}_j)$ or $\mathbf{z} = (\mathbf{z}_j, z_j)$
$\widetilde{\Delta}$	"normalized" Laplacian, $\widetilde{\Delta} := \frac{1}{4}\Delta = \partial\bar{\partial}$

Throughout the paper all Hilbert spaces are assumed to be separable. We always assume that in any Hilbert space an orthonormal basis is fixed, so an operator $A : E \to E_*$ can be identified with its matrix. Thus besides the usual involution $A \mapsto A^*$ (A^* is the adjoint of A), we have two more: $A \mapsto A^T$ (transpose of the matrix) and $A \mapsto \overline{A}$ (complex conjugation of the matrix), so $A^* = (\overline{A})^T = \overline{A^T}$. Although everything in the paper can be presented in an invariant, "coordinate-free", form, use of transposition and complex conjugation makes the notation easier and more transparent.

0. Introduction and main result

The classical Carleson Corona Theorem, see [3], states that if functions $f_j \in H^\infty(\mathbb{D})$ are such that $\sum_{j=1}^{\infty} |f_j|^2 \geqslant \delta^2 > 0$ then there exist functions $g_j \in H^\infty(\mathbb{D})$ such that $\sum_{j=1}^{\infty} g_j f_j = 1$. This is equivalent to the fact that the unit disk $\mathbb{D}$ is dense in the maximal ideal space of the algebra H^∞, but the importance of the Corona Theorem goes much beyond the theory of maximal ideals of H^∞.

The Corona Theorem, and especially its generalization, the so called Matrix (Operator) Corona Theorem play an important role in operator theory (such as the angles between invariant subspaces, unconditionally convergent spectral decompositions, computation of spectrum, etc.). The Matrix Corona Theorem says that if $F \in H^\infty(\mathbb{D}; E_* \to E)$ is a bounded analytic function whose values are operators from a Hilbert space E_*, $\dim E_* < +\infty$, to another Hilbert space E such that

$$F^*(z)F(z) \geqslant \delta^2 I > 0, \quad \forall z \in \mathbb{D}, \tag{C}$$

then F has a bounded analytic left inverse $G \in H^\infty(\mathbb{D}; E_* \to E)$, $GF \equiv I$. We should emphasize that the requirement $\dim E_* < +\infty$ is essential here. It was shown in [13], see also [14] or [15], that the Operator Corona Theorem fails if $\dim E_* = +\infty$. Note also that the above condition (C) is necessary for the existence of a bounded left inverse.

The classical Carleson Corona Theorem is a particular case of the matrix one: one just needs to consider F being the column $F = (f_1, f_2, \dots, f_n)^T$. It also worth noticing that the Matrix Corona Theorem follows from the classical one. Using a simple linear algebra argument Fuhrmann, see [4], was able to get the matrix version ($\dim E_*, \dim E < +\infty$) of the theorem from the classical result of Carleson. Later, using ideas from Wolff's proof of the Corona Theorem, M. Rosenblum, V. Tolokonnikov and A. Uchiyama, see [11,12,17], independently extended the Corona Theorem to infinitely many functions f_k. Using their result, V. Vasyunin was able to get the Operator Corona Theorem in the case $\dim E_* < +\infty$, $\dim E = +\infty$.

Since the Corona Theorem turns out to be very important in operator theory, there were some attempts to prove it using operator methods. While these attempts were not completely successful, some interesting relations were discovered. In particular, it was shown that a function $F \in H^\infty = H^\infty(\mathbb{D}; E_* \to E)$ is left invertible in H^∞ if and only if the Toeplitz operator $T_{\overline{F}}$ is left invertible; here $\overline{F}$ denotes the complex conjugate of the matrix F.

Let us recall that given an operator function $\Phi \in L^\infty(\mathbb{T}; E_* \to E)$, the Toeplitz operator $T_\Phi : H^2(E_*) \to H^2(E)$ with symbol Φ is defined by

$$T_\Phi f := P_+(\Phi f),$$

where P_+ is the Riesz Projection (orthogonal projection onto H^2).

Considering the adjoint operator $(T_{\overline{F}})^* = T_{\overline{F}^*} = T_{F^T}$ one can conclude from here that F is left invertible in H^∞ if and only if the Toeplitz operator $T_{F^T} : H^2(E) \to H^2(E_*)$ is *right* invertible. Since F^T is an analytic function

$$T_{F^T} f = F^T f, \quad \forall f \in H^2(E),$$

and F is left invertible in H^∞ if and only if for any $g \in H^2(E_*)$ the equation

$$F^T f = g \tag{0.1}$$

has a solution $g \in H^2(E)$ satisfying the uniform estimate $\|f\|_2 \leqslant C\|g\|_2$.

The result that condition (C) implies (if $\dim E_* < +\infty$) left invertibility of the Toeplitz operator $T_{\overline{F}}$, or equivalently the solvability of Eq. (0.1), is called the *Toeplitz Corona Theorem*. In the case of the unit disk $\mathbb{D}$ one can easily deduce the Matrix Corona Theorem from the Toeplitz Corona Theorem by using the Commutant Lifting Theorem.

The main result of this paper is the Toeplitz Corona Theorem for the polydisk, see Theorem 0.2 below. To simplify the notation we used F instead of F^T, so the condition (C) is replaced by the condition $FF^* \geqslant \delta^2 I$. While in the polydisk it is not known how to get the Corona Theorem from the Toeplitz Corona Theorem (the Commutant Lifting Theorem for the polydisk is currently not known) the result seems to be of independent interest. In a particular case when F from Theorem 0.2 is a row vector (a $1 \times n$ matrix) this theorem was proved by Lin, see [8] or [7]. His approach involved using the Koszul complex to write down the $\bar\partial$-equations. Unfortunately, in several variables, unlike the one-dimensional case, higher order equations appear in addition to the $\bar\partial$-equation so the computation become quite messy. Moreover, it is not clear how to use his technique to get the result in the matrix case we are treating here since the Fuhrmann–Vasyunin trick of getting the matrix result from the result for a column (row) vector does not work to solve the Toeplitz Corona Theorem.

To prove the main result we use tools from complex differential geometry to solve $\bar\partial$-equations on holomorphic vector bundles. In doing this we are following the ideas of Andersson, see [1] or [2], which in turn go back to Berndtsson.

While our approach is quite similar to the one used by Andersson, there are some essential differences. To solve the $\bar\partial$-equation he uses a Hörmander type approach with weights and a modification of a Bochner–Kodaira–Nakano–Hörmander identity from complex geometry. While our approach is more along the lines of T. Wolff's proof and does not require anything more advanced than Green's formula.

We first use our technique to get an estimate in the Toeplitz Corona Theorem in the disk:

Theorem 0.1. *Let $F \in H^\infty(\mathbb{D}; E \to E_*)$, $\dim E_* = r < +\infty$, such that $\delta^2 I \leqslant FF^* \leqslant I$ for some $0 < \delta^2 \leqslant \frac{1}{e}$. For $1 \leqslant p \leqslant \infty$ if $g \in H^p(\mathbb{D}; E_*)$ then the equation*

$$Ff = g$$

has an analytic solution $f \in H^p(\mathbb{D}; E)$ with the estimate

$$\|f\|_p \leqslant \left(\frac{C}{\delta^{r+1}} \log \frac{1}{\delta^{2r}} + \frac{1}{\delta}\right) \|g\|_p \tag{0.2}$$

with $C = \sqrt{1+e^2} + \sqrt{e} + \sqrt{2}e \approx 8.38934$.

For the $p = 2$ case the above result with a different constant C was obtained recently using a different method by Trent [16]. The constant he obtained was $C = 2\sqrt{e} + 2\sqrt{2}e \approx 10.9859$.

The result for all p can be obtained from the case $p = 2$ via the Commutant Lifting Theorem, but we present here a simple direct proof.

Remark. Note, that we do not assume $\dim E < +\infty$ here.

Using a simple modification of our proof in one dimension we are also able to get the following result in the polydisk:

Theorem 0.2. *Let $F \in H^\infty(\mathbb{D}^n; E \to E_*)$, $\dim E_* = r < +\infty$, such that $\delta^2 I \leqslant FF^* \leqslant I$ for some $0 < \delta^2 \leqslant \frac{1}{e}$. For $1 < p < \infty$ if $g \in H^p(\mathbb{D}^n; E_*)$ then the equation*

$$Ff = g$$

has an analytic solution $f \in H^p(\mathbb{D}^n; E)$ with the estimate

$$\|f\|_p \leqslant \left(\frac{nCC(p)^n}{\delta^{r+1}} \log \frac{1}{\delta^{2r}} + \frac{1}{\delta}\right) \|g\|_p, \tag{0.3}$$

where $C = \sqrt{1+e^2} + \sqrt{e} + \sqrt{2}e \approx 8.38934$, *and* $C(p) = 1/\sin(\pi/p)$ *the norm of the* (*scalar*) *Riesz projection from* $L^p(\mathbb{T})$ *onto* $H^p(\mathbb{D})$. *For* $p = 2$ *the estimate can be improved to*

$$\|f\|_2 \leqslant \left(\frac{\sqrt{n}C}{\delta^{r+1}} \log \frac{1}{\delta^{2r}} + \frac{1}{\delta}\right) \|g\|_2 \tag{0.4}$$

with $C = \sqrt{1+e^2} + \sqrt{e} + \sqrt{2}e \approx 8.38934$.

0.1. Plan of the paper

We will start with proving Theorem 0.1 for $p = 2$.

In Section 1 we set up the main estimate needed to prove the theorem. Section 2 is devoted to a version of the Carleson Embedding Theorem and its analogue for functions defined on holomorphic vector bundles, which will be later used to prove the main estimates.

In Section 3 we perform computation of some derivatives and Laplacians that will be used in the estimates. We also construct there subharmonic functions to be used in the embedding theorems.

Section 4 deals with the main estimate for $p = 2$; Section 5 explains how to use the construction for other p. In Section 6 we treat the case of the polydisk for $p = 2$ and in Section 7 we treat the case of general p.

1. Reduction to the main estimate

To prove Theorem 0.1 for $p = 2$, for a given $g \in H^2 := H^2(E_*)$ with $\|g\|_2 = 1$, we need to solve the equation

$$Ff = g, \quad f \in H^2(E) \tag{1.1}$$

with the estimate $\|f\|_2 \leqslant C = C(\delta, r)$. By a normal families argument it is enough to suppose that F and g are analytic in a neighborhood of $\mathbb{D}$. Any estimate obtained in this case can be used to find an estimate when F is only analytic on $\mathbb{D}$. Since $\delta^2 I \leqslant FF^* \leqslant I$, it is easy to find a non-analytic solution f_0 of (1.1),

$$f_0 := \Phi g := F^*(FF^*)^{-1} g.$$

To make f_0 into an analytic solution, we need to find $v \in L^2(E)$ such that $f := f_0 - v \in H^2$ and $v(z) \in \ker F(z)$ a.e. on $\mathbb{T}$. Then

$$Ff = F(f_0 - v) = Ff_0 - Fv = g,$$

and we are done. The standard way to find such v is to solve a $\bar{\partial}$-equation with the condition $v(z) \in \ker F(z)$ insured by a clever algebraic trick. This trick also admits a "scientific" explanation, for one can get the desired formulas by writing a Koszul complex. What we do in this paper essentially amounts to solving the $\bar{\partial}$-equation $\bar{\partial} v = \bar{\partial} f_0$ on the holomorphic vector bundle $\ker F(z)$. We mostly follow the ideas of Andersson found in [1]. He used ideas from complex differential geometry to solve the corona problem by finding solutions to the $\bar{\partial}$-equation on holomorphic vector bundles.

Since our target audience consists of analysts, all differential geometry will be well hidden. Our main technical tool will be Green's formula

$$\int_{\mathbb{T}} u\,dm - u(0) = \frac{1}{2\pi} \int_{\mathbb{D}} \Delta u \log \frac{1}{|z|}\,dx\,dy. \tag{1.2}$$

Instead of the usual Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ it is more convenient for us to use the "normalized" one $\widetilde{\Delta} := \frac{1}{4}\Delta = \bar{\partial}\partial = \partial\bar{\partial}$. If we denote by μ the measure defined by

$$d\mu = \frac{2}{\pi} \log \frac{1}{|z|}\,dx\,dy,$$

then Green's formula can be rewritten as

$$\int_{\mathbb{T}} u\,dm - u(0) = \int_{\mathbb{D}} \widetilde{\Delta} u\,d\mu. \tag{1.3}$$

1.1. Set-up

To find the function v we will use duality. We want $f_0 - v \in H^2(E)$, therefore the equality

$$\int_{\mathbb{T}} \langle f_0, h\rangle\,dm = \int_{\mathbb{T}} \langle v, h\rangle\,dm$$

must hold for all $h \in (H^2)^{\perp}$. Using Green's formula we get

$$\int_{\mathbb{T}} \langle f_0, h\rangle\,dm = \int_{\mathbb{T}} \langle \Phi g, h\rangle\,dm = \int_{\mathbb{D}} \partial\bar{\partial}\left[\langle \Phi g, h\rangle\right] d\mu = \int_{\mathbb{D}} \partial\left[\langle \bar{\partial}\Phi g, h\rangle\right] d\mu.$$

Here we used the harmonic extension of h, so h is anti-analytic and $h(0) = 0$. The functions $\Phi := F^*(FF^*)^{-1}$ and g are already defined in the unit disk $\mathbb{D}$.

Now the critical moment: let $\Pi(z) := P_{\ker F(z)}$ be the orthogonal projection onto $\ker F(z)$, $\Pi = I - F^*(FF^*)^{-1}F$. Direct computation shows that $\bar\partial\Phi = \Pi(\partial\Phi)^*(FF^*)^{-1}$, so $\Pi\bar\partial\Phi = \bar\partial\Phi$. Therefore, if we define a vector-valued function ξ on $\mathbb{D}$ by $\xi(z) := \Pi(z)h(z)$, then

$$\int_{\mathbb{D}} \partial[\langle\bar\partial\Phi g, h\rangle]\,d\mu = \int_{\mathbb{D}} \partial[\langle\bar\partial\Phi g, \Pi h\rangle]\,d\mu = \int_{\mathbb{D}} \partial[\langle\bar\partial\Phi g, \xi\rangle]\,d\mu =: L(\xi)$$
$$= L_g(\xi). \tag{1.4}$$

Note, that $L = L_g$ is a conjugate linear functional, i.e. $\overline{L}$ (defined by $\overline{L}(\xi) := \overline{L(\xi)}$) is a linear functional. Suppose we are able to prove the estimate

$$|L(\xi)| \leqslant C(r,\delta)\|\xi\|_2, \quad \forall\xi = \Pi h, \quad h \in H^2(E)^\perp. \tag{1.5}$$

Then (by a Hilbert space version of the Hahn–Banach Theorem, which is trivial) L can be extended to a bounded linear functional on $L^2(E)$, so there exists a function $v \in L^2(E)$, $\|v\|_2 \leqslant C$, such that

$$L(\xi) = \int_{\mathbb{T}} \langle v, \xi\rangle\,dm, \quad \forall\xi = \Pi h, \quad h \in H^2(E)^\perp.$$

Replacing v by Πv we can always assume without loss of generality that $v(z) \in \ker F(z)$ a.e. on $\mathbb{T}$, so $Fv = 0$. By the construction

$$\int_{\mathbb{T}} \langle v, h\rangle\,dm = \int_{\mathbb{T}} \langle v, \Pi h\rangle\,dm = L(\Pi h) = \int_{\mathbb{T}} \langle \Phi g, h\rangle\,dm \quad \forall h \in H^2(E)^\perp,$$

so $f := f_0 - v := \Phi g - v \in H^2(E)$ is the analytic solution we want to find. It satisfies the estimate

$$\|f\|_2 \leqslant \|f_0\|_2 + \|v\|_2 \leqslant \frac{1}{\delta}\|g\|_2 + C(r,\delta)\|g\|_2.$$

Therefore, Theorem 0.1 would follow from the following proposition

Proposition 1.1. *Under the assumptions of Theorem* 0.1 *the linear functional L defined by* (1.4) *satisfies the estimate*

$$|L(\xi)| \leqslant C(r,\delta)\|\xi\|_2, \quad \forall\xi = \Pi h, \quad h \in H^2(E)^\perp$$

with

$$C(r,\delta) = \tfrac{C}{\delta^{r+1}} \log \tfrac{1}{\delta^{2r}},$$

where $C = \sqrt{1+e^2} + \sqrt{e} + \sqrt{2}e$.

In what follows we will need the following simple technical lemma that is proved by direct computation.

Lemma 1.2. *For* Π *and* Φ *defined above we have*

$$\partial\Pi = -F^*\left(FF^*\right)^{-1}F'\Pi,$$
$$\bar{\partial}\Phi = \Pi\left(F'\right)^*\left(FF^*\right)^{-1},$$
$$\text{and } \partial\bar{\partial}\Phi = \partial\Pi(F')^*(FF^*)^{-1} - (\bar{\partial}\Phi)F'\Phi = \partial\Pi\bar{\partial}\Phi + (\partial\Pi)^*\Phi F'\Phi.$$

Corollary 1.3. *For the projection* Π *defined above we have*

$$\Pi\partial\Pi = 0, \quad (\partial\Pi)\Pi = \partial\Pi, \quad (\bar{\partial}\Pi)\Pi = 0, \quad \Pi\bar{\partial}\Pi = \bar{\partial}\Pi.$$

The above identities are well-known in complex differential geometry, but we can easily get them from Lemma 1.2. Namely, since Π is the orthogonal projection onto $\ker F$ we have $F\Pi = 0$. Taking the adjoint we get $\Pi F^* = 0$ which implies $\Pi\partial\Pi = 0$. The second identity is trivial, and the last two are obtained from the first two by taking adjoints.

2. Embedding theorems and Carleson measures

As is well known, Carleson measures play a prominent role in the proof of the Corona theorem, both in Carleson's original proof and in T. Wolff's proof and subsequent modifications. It is also known to the specialists, that essentially all [1] Carleson measures can be obtained from the Laplacian of a bounded subharmonic function. We will need the following well-known theorem, see [9], which was probably first proved by Uchiyama.

[1] By "essentially all" we mean here that a Carleson measure should first be mollified, to make it smooth, and then it can be obtained from the Laplacian of a subharmonic function.

Theorem 2.1 (*Carleson Embedding Theorem*). *Let φ be a non-negative, bounded, subharmonic function. Then for any $f \in H^2(E)$*

$$\int_{\mathbb{D}} \widetilde{\Delta}\varphi(z)\|f(z)\|^2\,d\mu(z) \leqslant e\|\varphi\|_\infty \|f\|_2^2.$$

Here $d\mu = \frac{2}{\pi}\log\frac{1}{|z|}\,dx\,dy$, and $\widetilde{\Delta} = \frac{1}{4}\Delta = \partial\bar{\partial}$.

Proof. Because of homogeneity, we can assume without loss of generality that $\|\varphi\|_\infty = 1$. Direct computation shows that

$$\widetilde{\Delta}(e^{\varphi(z)}\|f(z)\|^2) = e^{\varphi}\widetilde{\Delta}\varphi\|f\|^2 + e^{\varphi}\|\partial\varphi f + \partial f\|^2 \geqslant \widetilde{\Delta}\varphi\|f\|^2.$$

Then Green's formula implies

$$\begin{aligned}\int_{\mathbb{D}} \widetilde{\Delta}\varphi\,\|f\|^2\,d\mu \leqslant \int_{\mathbb{D}} \widetilde{\Delta}(e^{\varphi}\|f\|^2)\,d\mu = \int_{\mathbb{T}} e^{\varphi}\|f\|^2\,dm - e^{\varphi(0)}\|f(0)\|^2 \\ \leqslant e\int_{\mathbb{T}}\|f\|^2\,dm = e\,\|f\|_2^2. \quad \square\end{aligned}$$

Remark 2.2. It is easy to see, that the above Lemma implies the embedding $\int_{\mathbb{D}}\|f\|^2\,d\mu \leqslant C\int_{\mathbb{T}}\|f\|^2\,dm$ (with $C = e$) for all analytic functions f. Using the function $4/(2-\varphi)$ instead of e^{φ} it is possible to get the embedding for harmonic functions with the constant $C = 4$. We suspect the constants e and 4 are the best possible for the analytic and harmonic embedding respectively. We cannot prove that, but it is known that 4 is the best constant in the dyadic (martingale) Carleson Embedding Theorem.

We will need a similar embedding theorem for functions of form $\xi = \Pi h$, $h \in H^2(E)^{\perp}$. Such functions are not analytic or harmonic,[2] so the classical Carleson Embedding Theorem does not apply. As a result, the proof is more complicated, and the constant is significantly worse.

We will need several formulas. Recall that $\Pi(z) = P_{\ker F(z)}$ is the orthogonal projection onto $\ker F(z)$, $\Pi = I - F^*(FF^*)^{-1}F$, and that $d\mu = \frac{2}{\pi}\log\frac{1}{|z|}\,dx\,dy$.

Lemma 2.3. *Let φ be a non-negative, bounded, subharmonic function in $\mathbb{D}$ satisfying*

$$\widetilde{\Delta}\varphi(z) \geqslant \|\partial\Pi(z)\|^2, \quad \forall z \in \mathbb{D},$$

[2] To be precise, such functions are anti-holomorphic functions (with respect to the metric connection) on the holomorphic hermitian vector bundle $\ker F(z)$.

and let $K = \|\varphi\|_\infty$. *Then for all* ξ *of the form* $\xi = \Pi h$, $h \in H^2(E)^\perp$

$$\int_{\mathbb{D}} \widetilde{\Delta}\varphi(z)\|\xi(z)\|^2\, d\mu(z) \leqslant eKe^K\|\xi\|_2^2$$

and

$$\int_{\mathbb{D}} \|\bar{\partial}\xi\|^2\, d\mu \leqslant (1 + eKe^K)\|\xi\|_2^2.$$

Proof. Let us take an arbitrary non-negative bounded subharmonic function φ and compute $\widetilde{\Delta}(e^\varphi\|\xi\|^2)$. Corollary 1.3 implies that $\Pi\partial\Pi = 0$ and $\partial\Pi\Pi = \partial\Pi$. Therefore, using $\partial h = 0$ we get $\partial\xi = \partial(\Pi h) = \partial\Pi h + \Pi\partial h = \partial\Pi h = \partial\Pi\xi$, and so

$$\langle \partial\xi, \xi\rangle = \langle \partial\xi, \Pi\xi\rangle = \langle \partial\Pi\xi, \Pi\xi\rangle = 0.$$

Therefore

$$\partial(e^\varphi\|\xi\|^2) = e^\varphi\partial\varphi\|\xi\|^2 + e^\varphi\langle\partial\xi, \xi\rangle + e^\varphi\langle\xi, \bar{\partial}\xi\rangle = e^\varphi\partial\varphi\|\xi\|^2 + e^\varphi\langle\xi, \bar{\partial}\xi\rangle.$$

Taking $\bar{\partial}$ of this equality (and again using $\langle\xi, \partial\xi\rangle = 0$) we get

$$\widetilde{\Delta}(e^\varphi\|\xi\|^2) = e^\varphi(\widetilde{\Delta}\varphi\|\xi\|^2 + \|\bar{\partial}\varphi\xi + \bar{\partial}\xi\|^2 + \langle\xi, \widetilde{\Delta}\xi\rangle).$$

To handle $\langle\xi, \widetilde{\Delta}\xi\rangle$ we take the ∂ derivative of the equation $\langle\xi, \partial\xi\rangle = 0$ to get

$$\langle\partial\xi, \partial\xi\rangle + \langle\xi, \bar{\partial}\partial\xi\rangle = 0,$$

and therefore $\langle\xi, \widetilde{\Delta}\xi\rangle = -\|\partial\xi\|^2 = -\|(\partial\Pi)\xi\|^2$. Since $\varphi \geqslant 0$

$$\begin{aligned} &\int_{\mathbb{D}} (\widetilde{\Delta}\varphi\|\xi\|^2 - \|(\partial\Pi)\xi\|^2)\, d\mu \\ &\quad \leqslant \int_{\mathbb{D}} (\widetilde{\Delta}\varphi\|\xi\|^2 - \|(\partial\Pi)\xi\|^2 + \|\bar{\partial}\varphi\xi + \bar{\partial}\xi\|^2)e^\varphi\, d\mu = \int_{\mathbb{T}} e^\varphi\|\xi\|^2\, dm; \end{aligned} \tag{2.1}$$

the equality is just Green's formula (recall that $\xi(0) = 0$). In the last inequality replacing φ by $t\varphi$, $t > 1$ we get

$$\int_{\mathbb{D}} (t\widetilde{\Delta}\varphi\|\xi\|^2 - \|(\partial\Pi)\xi\|^2)\, d\mu \leqslant \int_{\mathbb{T}} e^{t\varphi}\|\xi\|^2\, dm \leqslant e^{tK}\|\xi\|_2^2.$$

Now we use the inequality $\widetilde{\Delta}\varphi \geqslant \|\partial\Pi\|^2$. It implies $\widetilde{\Delta}\varphi\,\|\xi\|^2 - \|\partial\Pi\xi\|^2 \geqslant 0$, and therefore

$$(t-1)\int_{\mathbb{D}} \widetilde{\Delta}\varphi\|\xi\|^2\,d\mu \leqslant e^{tK}\,\|\xi\|_2^2.$$

Hence

$$\int_{\mathbb{D}} \widetilde{\Delta}\varphi\|\xi\|^2\,d\mu \leqslant \min_{t>1} \frac{e^{tK}}{t-1}\|\xi\|_2^2 = eKe^K\,\|\xi\|_2^2$$

(minimum is attained at $t = 1 + 1/K$), and thus the first statement of the lemma is proved.

To prove the second statement, put $\varphi \equiv 0$ in (2.1) (we do not use any properties of φ except that $\varphi \geqslant 0$ in (2.1)) to get

$$\int_{\mathbb{D}} (\|\bar{\partial}\xi\|^2 - \|(\partial\Pi)\xi\|^2)\,d\mu = \int_{\mathbb{T}} \|\xi\|^2\,dm = \|\xi\|_2^2.$$

But the second term can be estimated as

$$\int_{\mathbb{D}} \|(\partial\Pi)\xi\|^2\,d\mu \leqslant \int_{\mathbb{D}} \widetilde{\Delta}\varphi\|\xi\|^2\,d\mu \leqslant eKe^K\,\|\xi\|_2^2,$$

and therefore $\int_{\mathbb{D}} \|\bar{\partial}\xi\|^2\,d\mu \leqslant (1 + eKe^K)\|\xi\|_2^2$. $\square$

3. Finding the correct subharmonic functions

There will be points in the proof where we would like to invoke Carleson's Embedding Theorem. To do so we will need a non-negative, bounded, subharmonic function. In this section we construct the necessary subharmonic functions so they will be available when we finally estimate the integral in question. With this in mind we define the two functions used and collect their relevant properties. First, we recall a basic fact that will aid in showing that the functions we construct are subharmonic.

Lemma 3.1. *Let $A(t)$ be a differentiable $n \times n$ matrix-valued function. Define the function $f(t) = \det(A(t))$. Then*

$$f'(t) = \det(A(t))\mathrm{tr}(A^{-1}(t)A'(t)).$$

Proof. Fix a point t and for brevity of notation let us use A instead of $A(t)$. Since $A(\cdot)$ is differentiable

$$\begin{aligned}\det(A(t+h)) = \det(A + A'h + o(h)) &= \det A \det(I + A^{-1}A'h + o(h)) \\ &= \det A \prod(1 + h\mu_k + o(h)),\end{aligned}$$

where μ_k are the eigenvalues of $A^{-1}(t)A'(t)$. Expanding this product we have

$$\prod(1 + h\mu_k + o(h)) = 1 + h\sum \mu_k + o(h) = 1 + h\operatorname{tr}(A^{-1}A') + o(h).$$

Then

$$\det(A(t+h)) = \det(A) + h\det(A)\operatorname{tr}(A^{-1}A') + o(h),$$

which implies the desired formula for the derivative. $\square$

Define the function $\varphi = \operatorname{tr}(\log(\delta^{-2}FF^*)) = \log(\delta^{-2n}\det(FF^*))$. Then a straight forward application of the above lemma gives

$$\begin{aligned}\widetilde{\Delta}\varphi &= \partial\bar{\partial}\varphi \\ &= \partial[\operatorname{tr}((FF^*)^{-1}F(F')^*)] \\ &= \operatorname{tr}[(FF^*)^{-1}F'\Pi(F')^*]\end{aligned}$$

with the last line following by substitution of Π. For another approach to this computation see [16]. Using the identities $\Pi^2 = \Pi$, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, and recalling that

$$\partial\Pi = -F^*(FF^*)^{-1}F'\Pi$$

we get

$$\begin{aligned}\widetilde{\Delta}\varphi &= \operatorname{tr}[(FF^*)^{-1}F'\Pi(F')^*] \\ &= \operatorname{tr}[F^*(FF^*)^{-1}F'\Pi\Pi(F')^*(FF^*)^{-1}F] \\ &= \operatorname{tr}[\partial\Pi(\partial\Pi)^*] \\ &\geqslant \|\partial\Pi\|^2\end{aligned}$$

with the last inequality following since $\operatorname{tr}[AA^*] \geqslant \|A\|^2$. This function will play a prominent role in the estimation of certain integrals. We should also note that

$$0 \leqslant \varphi \leqslant K := \log \tfrac{1}{\delta^{2n}}.$$

We will also need another function to help in the estimation of the linear functional L in question. Let $\lambda = \mathrm{tr}((FF^*)^{-1})$. A simple computation gives,

$$\begin{aligned}\widetilde{\Delta}\lambda &= \mathrm{tr}[\Phi^*(F')^*(FF^*)^{-1}F'\Phi] \\ &\quad -\mathrm{tr}[(FF^*)^{-1}F'\Pi(F')^*(FF^*)^{-1}] \\ &\geqslant \mathrm{tr}[\Phi^*(F')^*(FF^*)^{-1}F'\Phi^*] - \delta^{-2}\mathrm{tr}[\partial\Pi(\partial\Pi)^*].\end{aligned}$$

Now we define the function $\psi = \lambda + \delta^{-2}\varphi$. Then, recalling that $\Phi = F^*(FF^*)^{-1}$ we get

$$\begin{aligned}\widetilde{\Delta}\psi &\geqslant \mathrm{tr}[\Phi^*(F')^*(FF^*)^{-1}F'\Phi] \\ &= \mathrm{tr}[\Phi F'\Phi(\Phi F'\Phi)^*] \\ &\geqslant \|\Phi F'\Phi\|^2.\end{aligned}$$

So ψ is subharmonic and $0 \leqslant \psi \leqslant \frac{n}{\delta^2} + \frac{1}{\delta^2}\log\frac{1}{\delta^{2n}}$. We should note that the assumption $0 < \delta^2 \leqslant \frac{1}{e}$ implies $\log\delta^{-2} \geqslant 1$. This gives

$$0 \leqslant \psi \leqslant L := \frac{2}{\delta^2}\log\frac{1}{\delta^{2n}}.$$

4. Estimating the integral

Now we need to estimate $L(\xi)$. Computing ∂ of the inner product we get

$$\begin{aligned}L(\xi) &= \int_{\mathbb{D}} \partial\left[\langle\bar{\partial}\Phi g, \xi\rangle\right] d\mu \\ &= \int_{\mathbb{D}} \langle\partial\bar{\partial}\Phi g, \xi\rangle\, d\mu + \int_{\mathbb{D}} \langle\bar{\partial}\Phi g', \xi\rangle\, d\mu + \int_{\mathbb{D}} \langle\bar{\partial}\Phi g, \bar{\partial}\xi\rangle\, d\mu \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

We need to estimate each of the above integrals as closely as possible. Each integral has a term involving derivatives of Π, g and ξ. The idea is to separate the integrals using Cauchy–Schwarz, giving one derivative to each term.

We now estimate the first integral. Recalling that $\partial\bar{\partial}\Phi = \partial\Pi\bar{\partial}\Phi + (\partial\Pi)^*\Phi F'\Phi$ we get

$$\mathrm{I} = \int_{\mathbb{D}} \langle\partial\bar{\partial}\Phi g, \xi\rangle\, d\mu = \int_{\mathbb{D}} \{\langle\partial\Pi\bar{\partial}\Phi g, \xi\rangle + \langle(\partial\Pi)^*\Phi F'\Phi g, \xi\rangle\}\, d\mu.$$

Since $(\partial\Pi)^*\Pi = 0$ we have $(\partial\Pi)^*\xi = 0$, and so $\langle \partial\Pi\bar{\partial}\Phi g, \xi\rangle = 0$. Therefore

$$\mathrm{I} = \int_{\mathbb{D}} \langle (\partial\Pi)^* \Phi F' \Phi g, \xi\rangle \, d\mu = \int_{\mathbb{D}} \langle \Phi F' \Phi g, (\partial\Pi)\xi\rangle \, d\mu,$$

and the Cauchy–Schwarz inequality implies

$$|\mathrm{I}| \leqslant \left(\int_{\mathbb{D}} \|\Phi F' \Phi g\|^2 \, d\mu \right)^{1/2} \left(\int_{\mathbb{D}} \|(\partial\Pi)\xi\|^2 \, d\mu \right)^{1/2}.$$

To estimate the second factor we use Lemma 2.3. Recall that the function

$$\varphi = \log\left(\delta^{-2n} \det(FF^*) \right),$$

constructed in Section 3 satisfies the inequalities

$$\widetilde{\Delta}\varphi \geqslant \|\partial\Pi\|^2, \quad \text{and} \quad 0 \leqslant \varphi \leqslant K := \log \delta^{-2n}. \tag{4.1}$$

Therefore, Lemma 2.3 implies

$$\int_{\mathbb{D}} \|(\partial\Pi)\xi\|^2 \, d\mu \leqslant eKe^K \|\xi\|_2^2 = e\delta^{-2n} \log \delta^{-2n} \|\xi\|_2^2.$$

To estimate the first factor, notice that the function ψ constructed in Section 3 satisfies

$$\widetilde{\Delta}\psi \geqslant \|\Phi F' \Phi\|^2, \quad \text{and} \quad 0 \leqslant \psi \leqslant L := 2\delta^{-2} \log \delta^{-2n}.$$

Then the Carleson Embedding Theorem (Theorem 2.1) implies

$$\int_{\mathbb{D}} \|\Phi F' \Phi g\|^2 \, d\mu \leqslant eL \|g\|_2^2 = 2e\delta^{-2} \log \delta^{-2n} \|g\|_2^2,$$

and thus

$$|\mathrm{I}| \leqslant \sqrt{KL} \|\xi\|_2 \|g\|_2 = \frac{\sqrt{2}e}{\delta^{n+1}} \log \delta^{-2n} \|\xi\|_2 \|g\|_2.$$

Now we estimate II. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |\mathrm{II}| &\leqslant \int_{\mathbb{D}} |\langle \bar{\partial}\Phi g', \xi\rangle| \, d\mu \\ &\leqslant \left(\int_{\mathbb{D}} \|\bar{\partial}\Phi\|^2 \|\xi\|^2 \, d\mu \right)^{1/2} \left(\int_{\mathbb{D}} \|g'\|^2 \, d\mu \right)^{1/2}. \end{aligned}$$

Observe that $\widetilde{\Delta}\|g\|^2 = \|g'\|^2$ since g is holomorphic. So, applying Green's Theorem to the second factor we get

$$\int_{\mathbb{D}} \|g'\|^2 \, d\mu = \int_{\mathbb{T}} \|g\|^2 \, dm - \|g(0)\|^2 \leqslant \|g\|_2^2.$$

To estimate the first integral, notice, that

$$\|\Phi\|^2 = \|\Phi^*\Phi\| = \|(FF^*)^{-1}\| \leqslant \delta^{-2}$$

(recall that $\Phi = F^*(FF^*)^{-1}$). Since $\bar{\partial}\Phi = -(\partial\Pi)^*\Phi$, we can estimate

$$\|\bar{\partial}\Phi\|^2 = \|(\bar{\partial}\Phi)^*\bar{\partial}\Phi\| = \|\Phi^*\partial\Pi(\partial\Pi)^*\Phi\| \leqslant \|\partial\Pi(\partial\Pi)^*\| \cdot \|\Phi\|^2 \leqslant \delta^{-2}\|\partial\Pi\|^2.$$

Therefore (see (4.1)), $\|\bar{\partial}\Phi\|^2 \leqslant \delta^{-2}\widetilde{\Delta}\varphi$, where $\varphi = \log(\delta^{-2n}\det(FF^*))$ is the subharmonic function constructed in Section 3. Applying Lemma 2.3 we get

$$\int_{\mathbb{D}} \|\bar{\partial}\Phi\|^2 \|\xi\|^2 \, d\mu \leqslant \delta^{-2} \int_{\mathbb{D}} \widetilde{\Delta}\varphi \|\xi\|^2 \, d\mu \leqslant \delta^{-2} eKe^K \|\xi\|_2^2,$$

where $K = \log \delta^{-2n}$, see (4.1). Joining the estimates together, we get

$$|\mathrm{II}| \leqslant \delta^{-1}\sqrt{eK}e^{K/2}\|g\|_2\|\xi\|_2 \leqslant \delta^{-1}\sqrt{e}Ke^{K/2}\|g\|_2\|\xi\|_2 = \frac{\sqrt{e}}{\delta^{n+1}} \log \delta^{-2n} \|g\|_2\|\xi\|_2$$

(since $\delta^2 \leqslant \frac{1}{e}$, the value of K satisfies $K^{1/2} \leqslant K$).

Finally moving on to integral III. Using Cauchy–Schwarz, we have

$$\begin{aligned} |\mathrm{III}| &\leqslant \int_{\mathbb{D}} |\langle \bar{\partial}\Phi g, \bar{\partial}\xi \rangle| \, d\mu \\ &\leqslant \left(\int_{\mathbb{D}} \|\bar{\partial}\Phi\|^2 \|g\|^2 \, d\mu \right)^{1/2} \left(\int_{\mathbb{D}} \|\bar{\partial}\xi\|^2 \, d\mu \right)^{1/2}. \end{aligned}$$

As we already have shown above, $\|\bar{\partial}\Phi\|^2 \leqslant \delta^{-2}\widetilde{\Delta}\varphi$. The Carleson Embedding Theorem (Theorem 2.1) implies

$$\int_{\mathbb{D}} \|\bar{\partial}\Phi\|^2 \|g\|^2 \, d\mu \leqslant \delta^{-2} \int_{\mathbb{D}} \widetilde{\Delta}\varphi \|g\|^2 \, d\mu \leqslant \delta^{-2} eK \|g\|_2^2.$$

Using Lemma 2.3 we can estimate

$$\int_{\mathbb{D}} \|\bar{\partial}\xi\|^2 \, d\mu \leqslant (1 + eKe^K)\|\xi\|_2^2 \leqslant (e^{-1} + e)Ke^K \|\xi\|_2^2.$$

Here we are using the fact that $K \geqslant 1$ for $\delta^2 \leqslant 1/e$. Combining the estimates, we get

$$|\mathrm{III}| \leqslant \sqrt{1+e^2} K e^{K/2} \|g\|_2 \|\xi\|_2 \ = \frac{\sqrt{1+e^2}}{\delta^{n+1}} \log \delta^{-2n} \|g\|_2 \|\xi\|_2.$$

Joining the estimates for I, II, III we get

Proposition 4.1. *Under the assumptions of Theorem* 0.1 *the linear functional L defined by* (1.4) *satisfies the estimate*

$$|L(\xi)| \leqslant C(r,\delta)\|\xi\|_2, \quad \forall \xi = \Pi h, \quad h \in H^2(E)^{\perp},$$

with

$$C(r,\delta) = \tfrac{C}{\delta^{r+1}} \log \tfrac{1}{\delta^{2r}},$$

where $C = \sqrt{1+e^2} + \sqrt{e} + \sqrt{2}e$.

Proposition 4.1 is just a restatement of Proposition 1.1, and this then proves Theorem 0.1 for the case of $p = 2$. Note, that the constant C is a bit better than the constant $2\sqrt{2}e + 2\sqrt{e} \approx 10.9859$ obtained by Trent in [16].

5. The H^p corona problem in the disk

Now we indicate how we can use the H^2 result to figure out the H^p result. We can use much of the same approach as in the $H^2(E)$ case. Our goal is to solve the equation

$$Ff = g, \quad f \in H^p(E)$$

for the given $g \in H^p(E_*)$, with $\|g\|_p = 1$, and furthermore we want the estimate $\|f\|_p \leqslant C$. Again we will have the obvious non-analytic solution to the problem

$$f_0 := \Phi g := F^*(FF^*)^{-1} g.$$

To make this into an analytic solution we will need to find a function $v \in L^p(E)$ such that $f_0 - v \in H^p$ and $v(z) \in \ker F(z)$. This will be accomplished by duality. As in the $H^2(E)$ case we need

$$\int_{\mathbb{T}} \langle f_0, h\rangle \, dm = \int_{\mathbb{T}} \langle v, h\rangle \, dm$$

to hold for all $h \in H^p(E)^\perp = H_0^q(E)$ (this uses the standard duality of H^p spaces see [5] or [9]). Again we can ensure that $v \in \ker F(z)$ since $\overline{\partial}\Phi = \Pi\overline{\partial}\Phi$. So we need to get an estimate on the linear functional

$$L(\xi) = L_g(\xi) = \int_{\mathbb{D}} \partial[\langle\overline{\partial}\Phi g, \xi\rangle]\, d\mu$$

with $\xi = \Pi h$ and $h \in H^p(E)^\perp$. If we can then prove that

$$|L(\xi)| \leqslant C\|\xi\|_q$$

then by the Hahn–Banach Theorem and duality in L^p spaces with values in a Hilbert space we would have the existence of a function $v \in L^p(E)$ with $\|v\|_p \leqslant C$, such that

$$L(\xi) = \int_{\mathbb{T}} \langle v, \xi\rangle\, dm, \quad \forall \xi = \Pi h, \quad h \in H^p(E)^\perp.$$

Then replacing v by Πv we can assume without loss of generality that $v(z) \in \ker F(z)$ a.e. on $\mathbb{T}$. But then the construction would give,

$$\int_{\mathbb{T}} \langle v, h\rangle\, dm = \int_{\mathbb{T}} \langle v, \Pi h\rangle\, dm = L(\Pi h) = \int_{\mathbb{T}} \langle \Phi g, h\rangle\, dm, \quad \forall h \in H^p(E)^\perp,$$

so $v - f_0 \in H^p(E)$. So we only need to show how to prove the estimate

$$|L(\xi)| \leqslant C\|\xi\|_q.$$

The main idea is to use the L^2 result we just proved. Namely, if we replace g by $\widetilde{g} = \varphi^{-1}g$ and ξ by $\widetilde{\xi} = \overline{\varphi}\xi$, where φ is an appropriate (scalar) outer function, then

$$L_g(\xi) = L_{\widetilde{g}}(\widetilde{\xi}).$$

Suppose we are able to find the outer function φ such that $\|\widetilde{g}\|_2\|\widetilde{\xi}\|_2 \leqslant \|g\|_p\|\xi\|_q$. Then, since φ is analytic, $\widetilde{g} \in H^2(E)$ and

$$\widetilde{\xi} \in K := \operatorname{clos}_{L^2}\{\Pi h : h \in H^2(E)^\perp\}.$$

Therefore we can apply the L^2 result we have proved before to get

$$|L_g(\xi)| = |L_{\widetilde{g}}(\widetilde{\xi})| \leqslant \tfrac{C}{\delta^{n+1}} \log \delta^{-2n} \|\tilde{g}\|_2\|\tilde{\xi}\|_2 \leqslant \tfrac{C}{\delta^{n+1}} \log \delta^{-2n} \|g\|_p\|\xi\|_q. \tag{5.1}$$

To find the function φ we need to consider the cases $p < 2$ and $p > 2$ separately.

First look at the case $p < 2$. Consider the outer part of g, i.e. a scalar-valued outer function g_{out} such that

$$|g_{\text{out}}(z)| = \|g(z)\| \quad \text{a.e. on } \mathbb{T}.$$

Define

$$\tilde{g}(z) = (g_{\text{out}})^{p/2-1}(z)g(z) \quad \text{and}$$
$$\tilde{\xi}(z) = (\overline{g}_{\text{out}})^{1-p/2}(z)\xi(z).$$

Then $\|\tilde{g}\|_2 = \|g\|_p^{p/2}$, and computation using Hölder's Inequality gives and $\|\tilde{\xi}\|_2 \leqslant \|\xi\|_q \|g\|_p^{1-p/2}$, where $1/p + 1/q = 1$. Therefore $\|\widetilde{g}\|_2 \|\widetilde{\xi}\|_2 \leqslant \|g\|_p \|\xi\|_q$ and the main inequality (5.1) is proved.

The case when $p > 2$ is analogous, except in this case we need to construct a scalar outer function ξ_{out} such that

$$|\xi_{\text{out}}(z)| = \|\xi(z)\| \quad \text{a.e. on } \mathbb{T}.$$

Note, that here we cannot say that ξ_{out} is the outer part of ξ, because ξ is neither holomorphic nor antiholomorphic. So, a little more explanation is needed.

First of all recall that we assumed in Section 1 (without loss of generality) that F is an analytic function in a slightly bigger disk than $\mathbb{D}$, so the projection $\Pi = I - F^*(FF^*)^{-1}F$ is real analytic on the unit circle $\mathbb{T}$. Second, we only need to estimate the functional L on a dense set, so we can assume that the test function h is a trigonometric polynomial in $(H^2)^{\perp}$. Therefore the function $\xi = \Pi h$ is real analytic on $\mathbb{T}$, and so $\int_{\mathbb{T}} \log \|\xi(z)\| dm(z) > -\infty$ which guarantees existence of the outer function ξ_{out}.

Similarly to the above reasoning for the case $p < 2$ define for our case $p > 2$ $(q < 2)$,

$$\widetilde{\xi} := (\overline{\xi}_{\text{out}})^{q/2-1}\xi \quad \text{and}$$
$$\widetilde{g} := (\xi_{\text{out}})^{1-q/2}g,$$

where $1/p + 1/q = 1$. Then $\|\widetilde{\xi}\|_2 = \|\xi\|_q^{q/2}$ and applying Hölder inequality to $\widetilde{g}$ we get $\|\widetilde{g}\|_2 \leqslant \|g\|_p \|\xi\|_q^{1-q/2}$ (note, that the computations are the same as in the case $1 < p < 2$ if we interchange p with q and g with ξ). Then again $\|\widetilde{g}\|_2 \|\widetilde{\xi}\|_2 \leqslant \|g\|_p \|\xi\|_q$, so (5.1) holds.

As we discussed in the beginning of this section, the main estimate (5.1) implies (via duality) the solution of the H^p corona problem for $1 < p \leqslant \infty$.

The case $p = 1$ requires just a little more work since L^1 is not the dual of L^∞, and a bounded linear functional on L^∞ is generally a measure. Namely, the main estimate

(5.1) implies that L is a bounded conjugate-linear functional, and by Hahn–Banach Theorem it can be extended to a bounded conjugate-linear functional on $L^\infty(E)$. Since any bounded linear functional on L^∞ is a bounded linear functional on the space of continuous functions on the unit circle, there exists a vector-valued measure v such that

$$L(\xi) = \int_{\mathbb{T}} \langle dv, \xi \rangle.$$

Without loss of generality one can replace v with Πv, then

$$\int_{\mathbb{T}} \langle dv, h \rangle = \int_{\mathbb{T}} \langle dv, \xi \rangle = L(\Pi h) = \int_{\mathbb{T}} \langle f_0, h \rangle \, dm.$$

Then rewriting this, and treating $f_0 \, dm$ as a vector-valued measure we have

$$\int_{\mathbb{T}} \langle (f_0 \, dm - dv), h \rangle = 0$$

for any anti-analytic polynomial h. Then applying the F. & M. Riesz Theorem, see [9], we can conclude that the measure $f_0 \, dm - dv$ is absolutely continuous with respect to Lebesgue measure, and moreover it is an analytic measure meaning $f_0 \, dm - dv = (f_0 - v) \, dm$ with $f_0 - v \in H^1(E)$ (Of course, the F. & M. Riesz Theorem is usually stated for scalar measures, but applying it to the "coordinate" of the measure with respect to some orthonormal basis, one can easily see that it holds for measures with values in a separable Hilbert space as well). $\square$

6. The H^2 corona problem in the polydisk

In the following sections we will be considering operator- and vector-valued functions on the polydisk $\mathbb{D}^n$. We begin with the $H^2(E)$ case. The general goal from previous sections has not changed. We want, for a given $F \in H^\infty(\mathbb{D}^n; E \to E^*)$ and $g \in H^2 := H^2(\mathbb{D}^n; E_*)$ with $\|g\|_2 = 1$, to solve the equation

$$Ff = g, \quad f \in H^2(\mathbb{D}^n; E) \tag{6.1}$$

with the estimate $\|f\|_2 \leqslant C$. Again by a normal families argument it is enough to suppose that F and g are analytic in a neighborhood of $\mathbb{D}^n$ because any estimate obtained can be used to get an estimate when F is only analytic in $\mathbb{D}^n$. It is still easy to find a non-analytic solution f_0 of (6.1),

$$f_0 := \Phi g := F^*(FF^*)^{-1} g,$$

because we have $\delta^2 I \leqslant FF^* \leqslant I$. We will again need to find a $v \in L^2(\mathbb{T}^n; E)$ such that $f := f_0 - v \in H^2(\mathbb{D}^n; E)$ with $v(z) \in \ker F(z)$ a.e. on $\mathbb{T}^n$. Our approach is straightforward reduction to the one variable case, unfortunately this approach will not yield a proof of the H^∞ Corona problem on the polydisk since the projections are not bounded when $p = \infty$.

We will denote a point in $\mathbb{D}^n$ or $\mathbb{T}^n$ by $\mathbf{z} = (z_1, z_2, \dots, z_n)$. We will use the symbol $\mathbf{z}_j$ for $\mathbf{z}$ without the coordinate z_j and, slightly abusing notation, we can then write $\mathbf{z} = (\mathbf{z}_j, z_j) = (z_j, \mathbf{z}_j)$.

Let $H_j^p = H_j^p(\mathbb{D}^n; E)$ be a subspace of $L^p(\mathbb{T}^n; E)$ consisting of all functions analytic in z_j, i.e.

$$H_j^p(\mathbb{D}^n; E) := \{f \in L^p(\mathbb{T}^n, E) : f(\mathbf{z}_j, \cdot) \in H^p(\mathbb{D}; E) \text{ for almost all } \mathbf{z}_j \in \mathbb{T}^{n-1}\}. \tag{6.2}$$

6.1. Lemmas about decompositions

Lemma 6.1. *Any $h \in H^2(\mathbb{D}^n; E)^\perp$ can be written as $h = \sum_{j=1}^n h_j$ with $h_j \in H_j^2(\mathbb{D}^n; E)^\perp$.*

Proof. Let $P_j := P_{H_j^2}$ be the orthogonal projection onto $H_j^2 := H_j^2(\mathbb{D}^n; E)$. We can decompose h in the following way:

$$h = P_1 h + (I - P_1)h = h^1 + h_1 \quad h_1 \in H_1^2(\mathbb{D}^n; E)^\perp, \ h^1 = P_1 h.$$

Similarly,

$$h^1 = P_2 h^1 + (I - P_2)h^1 = h^2 + h_2 \quad h_2 \in H_2^2(\mathbb{D}^n; E)^\perp, \ h^2 = P_2 P_1 h.$$

Continuing the procedure we get

$$h^{k-1} = P_k h^{k-1} + (I - P_k)h^{k-1} = h^k + h_k \quad h_k \in H_k^2(\mathbb{D}^n; E)^\perp, \ h^k = P_k \cdots P_2 P_1 h.$$

Combining everything we get

$$h = h_1 + h_2 + \cdots + h_n + h^n, \quad h^n = P_n P_{n-1} \cdots P_1 h$$

which proves the lemma, because the assumption $h \in H^2(\mathbb{D}^n; E)^\perp$ implies that $h^n = P_n \cdots P_2 P_1 h = 0$ $\square$

We also are going to need an analogue of Lemma 6.1 dealing with the decomposition of functions on the holomorphic vector bundle ΠH^2, i.e. for the functions of the form $\xi = \Pi h$, $h \in H^2(\mathbb{D}^n; E)^\perp$. To state this lemma we need some auxiliary definitions. Let

$$K(\mathbb{D}^n; E) := \operatorname{clos}(\Pi(H^2(\mathbb{D}^n)^\perp)), \tag{6.3}$$

and

$$K_j(\mathbb{D}^n; E) := \operatorname{clos}(\Pi(H^2_j(\mathbb{D}^n)^\perp)), \quad \forall j = 1, \dots, n, \tag{6.4}$$

Lemma 6.2. *Let $\xi \in K$, then $\xi = \sum_{j=1}^n \xi_j$ with $\xi_j \in K_j$ for $j = 1, \dots, n$ and*

$$\|\xi\|_2^2 = \sum_{j=1}^n \|\xi_j\|_2^2.$$

To prove Lemma 6.2 we will need a few other lemmas. The first one is a simple fact about the geometry of a Hilbert space.

Lemma 6.3. *Let X be a subspace of a Hilbert space H, and let Π be some orthogonal projection in H. Then $\operatorname{Ran}\Pi = \Pi H$ is decomposed into the orthogonal sum*

$$\Pi H = \operatorname{clos}(\Pi X) \oplus (X^\perp \cap \Pi H).$$

Proof. The proof is a simple exercise in functional analysis, and we leave it to the reader. □

Define the subspaces

$$Q(\mathbb{D}^n; E) := H^2 \cap \Pi L^2, \quad Q_j(\mathbb{D}^n; E) := H^2_j \cap \Pi L^2. \tag{6.5}$$

Applying the above lemma to $H = L^2$ and $X = (H^2)^\perp$ or $X = (H^2_j)^\perp$ we get the following result.

Corollary 6.4. *The subspace $\Pi L^2 = \Pi L^2(\mathbb{D}^n; E)$, $n = 1, 2, 3, \dots$ admits the orthogonal decompositions*

$$\Pi L^2 = K \oplus Q, \quad \Pi L^2 = K_j \oplus Q_j,$$

with the subspaces $K := K(\mathbb{D}^n; E)$, $K_j := K_j(\mathbb{D}^n; E)$, $Q := Q(\mathbb{D}^n; E)$ and $Q_j := Q_j(\mathbb{D}^n; E)$ defined by (6.3), (6.4) *and* (6.5), *respectively*.

Remark 6.5. Note, that the orthogonal projections P_{K_j} and P_{Q_j} are essentially "one-variable" operators. Namely, to perform the projection P_{Q_j} on the function $\xi \in \Pi L^2$ we simply need to perform for each $\mathbf{z}_j \in \mathbb{T}^{n-1}$ (recall that $\mathbf{z} = (z_j, \mathbf{z}_j)$) the "one-variable" projection $P_{Q(\mathbf{z}_j)}$ onto the subspace

$$Q(\mathbf{z}_j) := H^2(\mathbb{D}; E) \cap \Pi(\cdot, \mathbf{z}_j) L^2(\mathbb{D}; E) \subset H^2 = H^2(\mathbb{D}; E),$$

and similarly for the projection P_{K_j}.

Indeed, if

$$\xi^1(\cdot, \mathbf{z}_j) := P_{Q(\mathbf{z}_j)}\xi(\cdot, \mathbf{z}_j) \quad \text{for almost all } \mathbf{z}_j \in \mathbb{T}^{n-1},$$

then clearly

$$\xi^1(\cdot, \mathbf{z}_j) \in H^2(\mathbb{D}; E) \cap \Pi(\cdot, \mathbf{z}_j)L^2(\mathbb{T}) \quad \text{for almost all } \mathbf{z}_j \in \mathbb{T}^{n-1},$$

so $\xi^1 \in H^2(\mathbb{D}^n; E) \cap \Pi L^2(\mathbb{D}^n; E)$. Moreover, for $\xi_1 := \xi - \xi^1$ and any $\eta \in H^2(\mathbb{D}^n; E) \cap \Pi L^2$

$$\int_{\mathbb{T}} \langle \xi_1(z_j, \mathbf{z}_j), \eta(z_j, \mathbf{z}_j) \rangle \, dm(z_j) = 0 \quad \text{for almost all } \mathbf{z}_j \in \mathbb{T}^{n-1},$$

and integrating over other variables $\mathbf{z}_k$ we get that $\xi_1 \perp \eta$.

The following two lemmas says that in many respects the projection P_{Q_j} behaves like the projection $I - P_j$ from Lemma 6.1.

Lemma 6.6. *Let $H^2 = H^2(\mathbb{D}^2; E)$ and let Q and Q_j, $j = 1, 2$, be the subspaces as defined above in (6.5). Then for the orthogonal projections P_{Q_j} onto the subspaces Q_j we have*

$$P_{Q_1} P_{Q_2} = P_{Q_2} P_{Q_1} = P_Q.$$

Proof. It follows from the definition of Q and Q_j and from the inclusion $H^2 \subset H_j^2$ that

$$Q = \Pi L^2 \cap H^2 \subset \Pi L^2 \cap H_j^2 = Q_j$$

we can conclude that for $\xi \in Q$ we have $P_{Q_j}\xi = \xi$, $j = 1, 2$.

Since by Corollary 6.4 we have the orthogonal decomposition $\Pi L^2 = K \oplus Q$, to prove the lemma we need to show that the equalities $P_{Q_2} P_{Q_1} \xi = 0$, $P_{Q_1} P_{Q_2} \xi = 0$ hold for all $\xi \in K$. Clearly, it is sufficient to prove only one, say the first as the second can be obtained by interchanging indices.

Consider the orthogonal decomposition of $\xi \in K$,

$$\xi = P_{K_1}\xi + P_{Q_1}\xi =: \xi_1 + \xi^1.$$

To prove that $P_{Q_2} P_{Q_1} \xi = 0$ we need to show that $\xi^1 \in K_2$.

By definition $\xi^1 \perp K_1 := \operatorname{clos}(\Pi((H_1^2)^\perp))$, and since $\Pi((H_1^2)^\perp) \supset (H_1^2)^\perp \cap \Pi L^2$, we can conclude that

$$\xi^1 \perp (H_1^2)^\perp \cap \Pi L^2.$$

We know that $\xi, \xi_1 \in K$ ($\xi_1 \in K$ because $K_1 \subset K$), so $\xi^1 \in K$. By Corollary 6.4,

$$\xi^1 \perp Q := H^2 \cap \Pi L^2.$$

Combining the above two orthogonality relations we get

$$\xi^1 \perp \Big((H_1^2)^\perp + H^2\Big) \cap \Pi L^2,$$

and since in the bidisk $H_2^2 \subset (H_1^2)^\perp + H^2$, we get that

$$\xi^1 \perp \Pi L^2 \cap H_2^2 =: Q_2$$

i.e. that $\xi^1 \in K_2$. $\square$

As an important corollary we get the following lemma.

Lemma 6.7. *On $\Pi L^2 := \Pi L^2(\mathbb{T}^n; E)$ we have*

$$P_{Q_k} P_{Q_j} = P_{Q_j} P_{Q_k} = P_{Q_k \cap Q_j} = P_{H^2_{jk} \cap \Pi L^2} \quad \forall 1 \leqslant k, j \leqslant n,$$

where $H^2_{jk}(\mathbb{D}^n) := H^2_j(\mathbb{D}^n) \cap H^2_k(\mathbb{D}^n)$. Furthermore, this implies

$$P_{Q_1} \dots P_{Q_n} = P_{Q_n} \dots P_{Q_1} = P_{H^2 \cap \Pi L^2}.$$

One can think of the space $H^2_{jk}(\mathbb{D}^n)$ as the space of functions in $L^2(\mathbb{T}^n)$ which are, upon fixing the other variables, holomorphic in both the jth and kth variable.

Proof. The first part of the lemma follows immediately from Lemma 6.6, because we can just “freeze” all variables except z_j and z_k. Namely, to perform the projection P_{Q_j} on the function $\xi \in \Pi L^2$ we simply need to perform for each $\mathbf{z}_j \in \mathbb{T}^{n-1}$ (recall that $\mathbf{z} = (z_j, \mathbf{z}_j)$) the “one variable” projection $P_{Q(\mathbf{z}_j)}$ onto the subspace

$$Q(\mathbf{z}_j) := H^2(\mathbb{D}; E) \cap \Pi(\cdot, \mathbf{z}_j) L^2(\mathbb{D}; E) \subset H^2 = H^2(\mathbb{D}; E),$$

see Remark 6.5.

To prove the second statement of the lemma let us notice that a product of commuting orthogonal projections is an orthogonal projection. Therefore $P = P_{Q_1} P_{Q_2} \dots P_{Q_n}$ is an orthogonal projection.

Since for $\xi \in H^2(\mathbb{D}^n; E) \cap \Pi L^2 = Q \subset Q_j$

$$P_{Q_j} \xi = \xi \quad \forall j = 1, 2, \dots, n,$$

we can conclude that

$$Q = H^2(\mathbb{D}^n; E) \cap \Pi L^2 \subset \operatorname{Ran} P.$$

On the other hand, since the projections P_{Q_j} commute and $\operatorname{Ran} P_{Q_j} = H_j^2 \cap \Pi L^2$

$$\operatorname{Ran} P \subset H_j^2 \cap \Pi L^2 = Q_j \quad \forall j = 1, 2, \dots, n,$$

so

$$\operatorname{Ran} P \subset \bigcap_{j=1}^{n} Q_j = \bigcap_{j=1}^{n} H_j^2 \cap \Pi L^2 = H^2 \cap \Pi L^2 = Q.$$

Therefore $\operatorname{Ran} P = Q$, i.e. P is the orthogonal projection onto Q. $\square$

We can now move onto proving Lemma 6.2.

Proof of Lemma 6.2. We will follow the argument in Lemma 6.1. For $\xi \in K$ consider the orthogonal decomposition

$$\xi = P_{K_1} \xi + P_{Q_1} \xi =: \xi_1 + \xi^1, \quad \xi_1 \in K_1(\mathbb{D}^n; E).$$

Since $\xi_1 \perp \xi^1$,

$$\|\xi\|_2^2 = \|\xi_1\|_2^2 + \|\xi^1\|_2^2.$$

Decomposing ξ^1 as

$$\xi^1 = P_{K_2} \xi^1 + P_{Q_2} \xi^1 =: \xi_2 + \xi^2, \quad \|\xi^1\|_2^2 = \|\xi_2\|_2^2 + \|\xi^2\|_2^2$$

we get the decomposition of ξ

$$\xi = \xi_1 + \xi_2 + \xi^2, \quad \xi_j \in K_j, \quad \xi^2 = P_{Q_2} P_{Q_1} \xi,$$

and

$$\|\xi\|_2^2 = \|\xi_1\|_2^2 + \|\xi_2\|_2^2 + \|\xi^2\|_2^2.$$

Repeating the procedure of decomposing on each step ξ^k using $P_{K_{k+1}}$ we finally obtain

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n + \xi^n, \quad \xi_j \in K_j, \ \ j = 1, 2, \ldots, n, \quad \xi^n = P_{Q_n} \ldots P_{Q_2} P_{Q_1} \xi,$$

and

$$\|\xi\|_2^2 = \|\xi_1\|_2^2 + \|\xi_2\|_2^2 + \cdots + \|\xi_n\|_2^2 + \|\xi^n\|_2^2.$$

But according Lemma 6.7 $\xi^n = 0$, so the lemma is proved. □

6.2. Proof of the H^2 corona for the polydisk

The idea of the proof is quite simple, we want to reduce everything to one-variable estimates. In the one-variable case we defined the functional L on functions of the form Πh where $h \in (H^2(\mathbb{D}))^\perp$ by

$$L(\xi) = \int_{\mathbb{D}} \partial[\langle \bar{\partial}\Phi g, \xi\rangle]\, d\mu,$$

where $d\mu = \frac{2}{\pi} \log \frac{1}{|z|}\, dx\, dy$, see (1.4). We have also proved (see Proposition 1.1) that the functional L is bounded in the L^2 norm on $\operatorname{clos}\{\Pi h : h \in (H^2(\mathbb{D}))^\perp\}$ (this is the one-variable analogue of the space K defined for the polydisk).

For the polydisk, define (conjugate linear) functionals L_j on K_j by

$$L_j(\xi) := \int_{\mathbb{T}^{n-1}} L_{g(\cdot, \mathbf{z}_j)}(\xi(\cdot, \mathbf{z}_j))\, dm_{n-1}(\mathbf{z}_j).$$

Since $\xi(\cdot, \mathbf{z}_j) \in K$ for almost all $\mathbf{z}_j \in \mathbb{T}^{n-1}$ if $\xi \in K_j$ (see Remark 6.5) the functionals L_j are well defined and bounded, $\|L_j\| = \|L\|$. Note also, that on a dense set of ξ of the form $\xi = \Pi h$, $h \in (H_j^2)^\perp$ we can represent

$$L_j(\xi) = \int_{\mathbb{T}^{n-1}} \int_{\mathbb{D}} \partial_j [\langle \bar{\partial}_j \Phi g, \xi\rangle]\, d\mu(z_j)\, dm_{n-1}(\mathbf{z}_j).$$

Define a conjugate linear functional $\mathbf{L}$ on K by decomposing $\xi \in K$ as

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n, \quad \xi_j \in K_j, \quad j = 1, 2, \ldots, n \tag{6.6}$$

and putting

$$\mathbf{L}(\xi) := \sum_{j=1}^{n} L_j(\xi_j).$$

We will show later that the functional $\mathbf{L}$ is well defined, i.e. that it does not depend on the choice of decomposition of ξ (note that by Lemma 6.2 one can always find at least one such decomposition).

Assuming for now that $\mathbf{L}$ is well defined, let us prove Theorem 0.2 for $p = 2$. First of all, by Lemma 6.2 any function $\xi \in K$ can be decomposed as

$$\xi = \sum_{j=1}^{n} \xi_j, \quad \text{where } \xi_j \in K_j, \quad \text{and } \sum_{j=1}^{n} \|\xi_j\|^2 = \|\xi\|^2.$$

Therefore, using the fact that $\|L_j\| = \|L\|$ we get for $\xi \in K$

$$|\mathbf{L}(\xi)| \leqslant \sum_{j=1}^{n} \|L_j\| \cdot \|\xi_j\| = \|L\| \sum_{j=1}^{n} \|\xi_j\| \leqslant \|L\|\sqrt{n}\left(\sum_{k=1}^{n} \|\xi_j\|^2\right)^{1/2} = \sqrt{n}\,\|L\| \cdot \|\xi\|,$$

so

$$\|\mathbf{L}\| \leqslant \sqrt{n}\,\|L\| \leqslant \tfrac{\sqrt{n}C}{\delta^{r+1}} \log \tfrac{1}{\delta^{2r}},$$

where $C = \sqrt{1+e^2} + \sqrt{e} + \sqrt{2}e \approx 8.38934$ is the constant from Theorem 0.1.

Take $h \in (H^2)^{\perp}$, and decompose it according to Lemma 6.1 as

$$h = \sum_{j=1}^{n} h_j, \quad h_j \in (H_j^2)^{\perp}.$$

Denote

$$\xi := \Pi h, \quad \xi_j = \Pi h_j.$$

Repeating the reasoning with the Green's Formula from the one-variable case we can easily show that

$$\int_{\mathbb{T}^n} \langle \Phi g, h_j \rangle \, dm_n(\mathbf{z}) = \int_{\mathbb{T}^{n-1}} \int_{\mathbb{D}} \partial_j [\langle \overline{\partial}_j \Phi g, \xi_j \rangle] \, d\mu(z_j) \, dm_{n-1}(\mathbf{z}_j) = L_j(\xi_j),$$

so

$$\int_{\mathbb{T}^n} \langle \Phi g, h \rangle \, dm_n(\mathbf{z}) = \mathbf{L}(\Pi h) = \mathbf{L}(\xi).$$

By the Hilbert space version of the Hahn–Banach Theorem the linear functional $\overline{\mathbf{L}}$ can be extended to a bounded functional on all of L^2, i.e., we can find $v \in L^2 = L^2(\mathbb{T}^n; E)$ such that

$$\mathbf{L}(\xi) = \int_{\mathbb{T}^n} \langle v, \xi \rangle \, dm_n(\mathbf{z}) \quad \forall \xi \in K.$$

Replacing v by Πv if necessary, one can assume without loss of generality that $v(\mathbf{z}) \in \operatorname{Ran} \Pi(\mathbf{z}) = \ker F(\mathbf{z})$ a.e. on $\mathbb{T}^n$, so $Fv \equiv 0$ on $\mathbb{T}^n$. Since by the construction

$$\begin{aligned} \int_{\mathbb{T}^n} \langle v, h \rangle \, dm_n(\mathbf{z}) &= \int_{\mathbb{T}^n} \langle v, \Pi h \rangle \, dm_n(\mathbf{z}) = \mathbf{L}(\Pi h) \\ &= \int_{\mathbb{T}^n} \langle \Phi g, h \rangle \, dm_n(\mathbf{z}) \quad \forall h \in H^2(\mathbb{D}^n; E)^\perp, \end{aligned}$$

the function $f := f_0 - v := \Phi g - v$ is analytic. Since $Fv = 0$, it satisfies $Ff = Ff_0 = g$, so f is the analytic solution we want to find. $\square$

6.3. The functional **L** *is well defined*

Let us consider first the case of the bidisk $\mathbb{D}^2$. To show that **L** is well defined in this case, it is sufficient to show that if

$$0 = \xi_1 + \xi_2, \quad \xi_j \in K_j$$

then $L_1(\xi_1) + L_2(\xi_2) = 0$ (simply take the difference of two representations of the same function in K). This holds if and only if

$$L_1(\xi) = L_2(\xi) \quad \forall \xi \in K_1 \cap K_2.$$

Thus, the following lemma shows that **L** is well defined in the case of bidisk $\mathbb{D}^2$.

Lemma 6.8. *Let $\xi \in K_1 \cap K_2 \subset \Pi L^2(\mathbb{T}^2; E)$. Then*

$$L_1(\xi) = L_2(\xi)$$

Proof. The proof of this lemma is really nothing more than repeated applications of Green's Formula, and using that $K_1 \cap K_2 = \operatorname{clos}(\Pi \overline{H^2})$ where $\overline{H^2}$ are the functions which are anti-holomorphic in both variables. To see that $K_1 \cap K_2 = \operatorname{clos}(\overline{\Pi H^2})$ we use Lemma 6.3. Since $(K_1 \cap K_2)^\perp = Q_1 + Q_2 = K_1^\perp + K_2^\perp = (H_1^2 + H_2^2) \cap \Pi L^2$, then by Lemma 6.3 we have the result.

By density we can work with ξ of the form $\xi = \Pi h$ with h anti-holomorphic in both variables. So applying Green's Formula twice gives

$$\begin{aligned} L_1(\xi) &= \int_{\mathbb{T}} \int_{\mathbb{D}} \partial_1 \langle \bar{\partial}_1 \Phi g, \xi \rangle \, d\mu(z_1) \, dm(z_2) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} \langle \Phi g, h \rangle \, dm(z_1) \, dm(z_2) \\ &= \int_{\mathbb{T}} \int_{\mathbb{D}} \partial_2 \langle \bar{\partial}_2 \Phi g, \xi \rangle \, d\mu(z_2) \, dm(z_1) \\ &= L_2(\xi). \end{aligned}$$

Since this result holds on a dense set of ξ, and the functionals L_1 and L_2 are continuous we have the result for all $\xi \in K_1 \cap K_2$. $\square$

For the polydisk the lemma has the following important corollary

Corollary 6.9. *Let* $\xi \in K_j \cap K_k \subset L^2(\mathbb{T}^n; E)$. *Then*

$$L_j(\xi) = L_k(\xi).$$

Proof. To prove the corollary one needs to apply Lemma 6.8 to the bidisk in variables z_j and z_k and then integrate the obtained equality over $\mathbb{T}^{n-2}$ (with respect to Lebesgue measure in all other variables). $\square$

Now we are ready to prove that $\mathbf{L}$ is well defined. To prove this it is sufficient to show for any representation of 0

$$0 = \sum_{j=1}^{n} \xi_j, \quad \xi_j \in K_j \tag{6.7}$$

the equality

$$\sum_{j=1}^{n} L_j(\xi_j) = 0$$

holds.

We will use induction in n. The case $n = 2$ is already settled, so let us assume the functional $\mathbf{L}$ is well defined for the polydisk $\mathbb{D}^{n-1}$. It follows from (6.7) that

$$\xi_n \in K_n \cap (K_1 + K_2 + \cdots + K_{n-1}) = (K_1 \cap K_n) + (K_2 \cap K_n) + \cdots + (K_{n-1} \cap K_n),$$

so ξ_n can be represented as

$$\xi_n = \sum_{j=1}^{n-1} \eta_j, \quad \eta_j \in K_j \cap K_n, \quad j = 1, 2, \ldots, n-1.$$

On the other hand we know that $\xi_n = -\sum_{j=1}^{n-1} \xi_j$. Using the induction hypothesis and integrating it over $\mathbb{T}$ with respect to $dm(z_n)$ we obtain that

$$\sum_{j=1}^{n-1} L_j(\eta_j) = -\sum_{j=1}^{n-1} L_j(\xi_j).$$

Since $\eta_j \in K_j \cap K_n$, Corollary 6.9 implies that $L_j(\eta_j) = L_n(\eta_j)$. Therefore

$$L_n(\xi_n) = \sum_{j=1}^{n-1} L_n(\eta_j) = \sum_{j=1}^{n-1} L_j(\eta_j) = -\sum_{j=1}^{n-1} L_j(\xi_j),$$

and so $\sum_{j=1}^{n} L_j(\xi_j) = 0$. $\square$

7. The H^p corona problem in the polydisk

A simple idea of proving the H^p corona problem in the polydisk is to try to mimic the proof of the H^2 case. However, there is a much easier way: just use objects which are already defined, and modify the crucial estimates.

First of all notice, that replacing the Corona data F and g by $F(r\mathbf{z})$ and $g(r\mathbf{z})$, $r < 1$ and using the standard normal families argument one can assume without loss of generality (as long as we are getting the same uniform estimates on the norm of the solution) that both F and G are holomorphic in a slightly bigger polydisk. So we can always assume that, for example, the right hand side g is not only in H^p, but is also bounded, smooth, etc.

As in the H^2 case we first construct a smooth solution $f_0 := \Phi g$, where $\Phi := F^*(FF^*)^{-1}$, of the equation $Ff = g$ and then correct it to be analytic. To do that it is sufficient to show that the conjugate linear functional $\mathbf{L}$ introduced in the previous section is L^q bounded, $1/p + 1/q = 1$, i.e. that

$$|\mathbf{L}(\xi)| \leqslant C\|\xi\|_q$$

for all ξ of form $\xi = \Pi h$, where h is a trigonometric polynomial in $H^2(\mathbb{D}^n; E)^\perp$.

If this estimate is proved, the linear functional $\overline{\mathbf{L}}$ can be extended by the Hahn–Banach Theorem to a linear functional on L^q, so there will exist a function $v \in L^p(\mathbb{T}^n; E)$, $\|v\|_p = \|\mathbf{L}\|_p$ such that

$$\mathbf{L}(\xi) = \int_{\mathbb{T}^n} \langle v, \xi \rangle \, dm_n(\mathbf{z}) \quad \forall \xi = \Pi h, \quad h \in H^2(\mathbb{D}^n; E)^\perp \cap \mathrm{Pol}.$$

Again, replacing v by Πv we can always assume without loss of generality that $v(\mathbf{z}) \in \operatorname{Ran}\Pi(z) = \ker F(z)$ a.e. on $\mathbb{T}^n$. As in the previous section, decomposing h as

$$h = \sum_{j=1}^{n} \xi_j, \quad \xi_j \in H_j^2$$

(h is a trigonometric polynomial, so we can use Lemma 6.1 here), we can show that $\int_{\mathbb{T}^n} \langle \Phi g, h\rangle \, dm_n(\mathbf{z}) = \mathbf{L}(\Pi h) = \mathbf{L}(\xi)$ so

$$\int_{\mathbb{T}^n} \langle v, h\rangle \, dm_n(\mathbf{z}) = \int_{\mathbb{T}^n} \langle v, \Pi h\rangle \, dm_n(\mathbf{z}) = \mathbf{L}(\Pi h) = \int_{\mathbb{T}^n} \langle \Phi g, h\rangle \, dm_n(\mathbf{z}),$$

for all $h \in H^2(\mathbb{D}^n; E)^{\perp} \cap \mathrm{Pol}$. Therefore, the function $f = f_0 - v = \Phi g - v$ is analytic, and it clearly solves the equation $Ff = g$ (on $\mathbb{T}^n$, and therefore on $\mathbb{D}^n$).

7.1. Main estimates

Let us introduce some notation. Denote

$$K^q := \operatorname{clos}(\Pi((H^p)^{\perp})) \subset \Pi L^q, \quad Q^q := H^q \cap \Pi L^q,$$

so for K and Q introduced in the previous section $K = K^2$ and $Q = Q^2$. Let also

$$H_j^q = H_j^q(\mathbb{D}^n; E) := \{f \in L^q(\mathbb{T}^n; E) : f(\cdot, \mathbf{z}^j) \in H^q(\mathbb{D}; E)\}$$

be the spaces of functions analytic in variable z_j, and let

$$K_j^q := \operatorname{clos}(\Pi(H_j^p(\mathbb{D}^n; E)^{\perp})) \subset \Pi L^q(\mathbb{T}^n; E), \quad Q_j^q := H_j^q(\mathbb{D}^n; E) \cap \Pi L^q(\mathbb{T}^n; E).$$

To estimate the functional $\mathbf{L}$ we need the following analogue of Lemma 6.2

Lemma 7.1. *Any function $\xi \in K^q$ can be decomposed as*

$$\xi = \sum_{j=1}^{n} \xi_j, \quad \xi_j \in K_j^q, \quad \|\xi_j\|_q \leqslant C(q)^j \|\xi\|_q,$$

where $C(q) = 1/\sin(\pi/q)$ is the norm of the scalar Riesz Projection P_+ from $L^q(\mathbb{T})$ onto $H^q(\mathbb{D})$ (note that $C(p) = C(q)$ for $1/p + 1/q = 1$).

Let us show how this lemma implies the estimate for $\mathbf{L}$. In Section 5 we have proved the L^p bound for the functional L (in the one-variable case),

$$|L(\xi)| \leqslant C(r, \delta)\|\xi\|_p, \quad C(r, \delta) = C\frac{1}{\delta^{r+1}} \log \frac{1}{\delta^{2r}},$$

where $C = \sqrt{1+e^2} + \sqrt{e} + \sqrt{2}e$. That would imply the same estimates for the functionals L_j on $L^q(\mathbb{T}^n; E)$, so applying Lemma 7.1 we get

$$|L(\xi)| \leqslant C(r,\delta) \sum_{j=1}^{n} \|\xi_j\|_q \leqslant C(r,\delta)\|\xi\|_q \sum_{j=1}^{n} C(q)^j \leqslant C(r,\delta) n C(q)^n \|\xi\|_q.$$

Recalling that $C(p) = C(q)$ we get the desired estimate of the solution.

There is a little detail here as the functional $\mathbf{L}$ was defined initially only on K^2. So formally, if $q < 2$ (i.e. if $p > 2$) the functional is not defined on K^q. However this is not a big problem and the simplest way of dealing with it is to use the standard approximation arguments. Since the polynomials in $(H_j^2)^\perp \cap \mathrm{Pol}$ are dense in $(H_j^p)^\perp$, the functions of form Πh, $h \in H_j^2 \cap \mathrm{Pol}$ are dense in K_j^q. So, approximating functions ξ_j from Lemma 7.1 by functions of this form, we will get the desired estimate. Note, that we are estimating $\mathbf{L}(\xi)$ on a dense set of functions $\xi = \Pi h$, $h \in (H^2)^\perp \cap \mathrm{Pol}$, so we do not need it be formally defined on K^q.

The main step in proving Lemma 7.1 is the following result that states that in the one-variable case the norm of the orthogonal projections P_K and P_Q in L^q is the same as the norm of the Riesz projection P_+ in L^q. See [6] for the norms of P_+ in L^p.

Lemma 7.2. *Let $H^2 = H^2(\mathbb{D}; E)$ and let $K, Q \subset H^2$ be the subspaces defined above in* (6.3) *and* (6.5). *Then for $1 < q < \infty$*

$$\|P_K \xi\|_q \leqslant C(q)\|\xi\|_q, \quad \|P_Q \xi\|_q \leqslant C(q)\|\xi\|_q \quad \forall \xi \in \Pi L^2 \cap \Pi L^q,$$

where $C(q) = 1/\sin(\pi/q)$ is the norm of the Riesz Projection P_+ in L^q (*or in L^p*, $1/p + 1/q = 1$).

Note that since $\Pi L^2 \cap \Pi L^q$ is dense in ΠL^q, the projections P_K and P_Q extend to bounded operators on ΠL^q.

Proof. Take $\xi \in \Pi L^2 \cap \Pi L^q$ and decompose it as

$$\xi = P_K \xi + P_Q \xi =: \xi_K + \xi_Q.$$

Since Q is a z-invariant subspace of $H^2(\mathbb{D}, E)$, by the Beurling–Lax theorem, see [10], it can be represented as $Q = \Theta H^2(\mathbb{D}; E_*)$, where $\Theta \in H^\infty(E_* \to E)$ is an inner function (i.e. $\Theta(z)$ is an isometry a.e. on $\mathbb{T}$) and E_* is an auxiliary Hilbert space. So ξ_Q can be represented as

$$\xi_Q = \Theta \eta, \quad \eta \in H^2(E_*) \cap H^q(E_*).$$

By duality

$$\|\xi_Q\|_q = \|\eta\|_q = \sup_{\substack{h \in L^p \cap L^2: \\ \|h\|_q = 1}} \left| \int_{\mathbb{T}} \langle \eta, h \rangle \, dm \right|.$$

Let $h_+ = P_+ h$. Since $\eta \in H^2$

$$\begin{aligned}\int_{\mathbb{T}} \langle \eta, h \rangle \, dm = \int_{\mathbb{T}} \langle \eta, h_+ \rangle \, dm = \int_{\mathbb{T}} \langle \Theta\eta, \Theta h_+ \rangle \, dm \\ = \int_{\mathbb{T}} \langle \xi_Q, \Theta h_+ \rangle \, dm = \int_{\mathbb{T}} \langle \xi, \Theta h_+ \rangle \, dm;\end{aligned}$$

the second equality holds because Θ is an isometry a.e. on $\mathbb{T}$, and the last one holds because $\xi_K \in K \perp \Theta h_+$. Therefore, since $\|h_+\|_p \leqslant C(p)\|h\|_p$, we can conclude

$$\left| \int_{\mathbb{T}} \langle \eta, h \rangle \, dm \right| \leqslant \left| \int_{\mathbb{T}} \langle \xi, \Theta h_+ \rangle \, dm \right| \leqslant \|\xi\|_q \|h_+\|_p \leqslant C(p)\|\xi\|_q \|h\|_p$$

so $\|\xi_Q\|_q \leqslant C(p)\|\xi\|_q$. Thus we get the desired estimate for the norm of P_Q.

Since $P_K + P_Q = I$ we can estimate the norm of P_K by $C(p) + 1$ for free. Note, that unlike the case of Hilbert spaces, complementary projections in Banach spaces do not necessarily have equal norms. So, to get rid of the 1 some extra work is needed.

It is easy to see that $\cap_{n>0} \bar{z} K = \{0\}$, so the decomposition $\Pi L^2 = K \oplus Q$ implies that the set

$$\bigcup_{n>0} \bar{z}^n Q = \bigcup_{n>0} \bar{z}^n \Theta H^2(E_*)$$

is dense in ΠL^2. Thus $\Pi L^2 = \Theta L^2$, and since Θ is an isometry a.e. on $\mathbb{T}$ we can conclude that $K = \Theta(H^2(E)^\perp)$. Therefore we can represent ξ_K as

$$\xi_K = \Theta\eta, \quad \eta \in H^2(E_*)^\perp \cap L^q(E_*).$$

Performing the same calculations as in the case of ξ_Q, only using $h_- = P_- h$, $P_- = I - P_+$ instead of h_+ we get the estimate $\|P_K\|_{L^q} \leqslant \|P_-\|_{L^q}$. But the isometry τ,

$$\tau(z^k) = z^{-k-1}, \quad k \in \mathbb{Z}$$

interchanges H^2 and $(H^2)^\perp$, and since τ is an isometry in all L^p, we conclude the $\|P_-\|_{L^q} = \|P_+\|_{L^q}$. $\square$

Corollary 7.3. *Let $H^2 = H^2(\mathbb{D}^n; E)$ and let $K_j, Q_j \subset H^2$ be the subspaces defined in* (6.4) *and* (6.5). *Then for $1 < q < \infty$ and $1 \leqslant j \leqslant n$ we have*

$$\|P_{K_j}\xi\|_q \leqslant C(q)\|\xi\|_q, \quad \|P_{Q_j}\xi\|_q \leqslant C(q)\|\xi\|_q \quad \forall \xi \in \Pi L^2 \cap \Pi L^q,$$

where $C(q) = 1/\sin(\pi/q)$ is the norm of the (*one-dimensional*) *Riesz Projection P_+ in L^q* (*or in L^p*, $1/p + 1/q = 1$).

Proof. This corollary follows directly from Lemma 7.2. Since by Remark 6.5 we can view P_{K_j} and P_{Q_j} as "one-variable" operators. Then we "freeze" all variables except the z_j variable and apply Lemma 7.2 and then integrate in the "frozen" variables. $\square$

It only remains to prove Lemma 7.1.

Proof of Lemma 7.1. The proof is almost the same as the proof of Lemma 6.2, only here we cannot use the fact that the P_{k_j} are orthogonal projections. However, according to Corollary 7.3 the projections P_{K_j} are bounded, and this allows the proof to go through.

Take $\xi \in K^q$. Repeating the proof of Lemma 6.2 we can write

$$\xi = P_{K_1}\xi + P_{Q_1}\xi =: \xi_1 + \xi^1.$$

By Corollary 7.3 we have that $\xi_1 \in K_1^q$ with $\|\xi_1\|_q \leqslant C(q)\|\xi\|_q$ and $\|\xi^1\|_q \leqslant C(q)\|\xi\|_q$. Decomposing ξ^1 in the same manner we have

$$\xi^1 = P_{K_2}\xi^1 + P_{Q_2}\xi^1 =: \xi_2 + \xi^2,$$

so

$$\xi = \xi_1 + \xi_2 + \xi^2, \quad \xi_j \in K_j^q, \quad \xi^2 = P_{Q_2}P_{Q_1}\xi.$$

Corollary 7.3 applied twice gives $\|\xi_2\|_q \leqslant C(q)\|\xi^1\|_q \leqslant C(q)^2\|\xi\|_q$, and thus $\|\xi_j\|_q \leqslant C(q)^j\|\xi\|_q$. Continuing this decomposition at each step we find

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n + \xi^n, \quad \xi_j \in K_j^q, \quad \xi^n = P_{Q_n} \ldots P_{Q_2}P_{Q_1}\xi,$$

and $\|\xi_j\|_q \leqslant C(q)^j\|\xi\|_q$ by applying Corollary 7.3 j times. Finally, by Lemma 6.7 $P_{Q_n} \cdots P_{Q_1} = 0$ on the dense set $K^q \cap K^2$. $\square$

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