Abstract

Let \( D \) be a Noetherian domain of Krull dimension 2, and let \( H \subseteq R \) be integrally closed overrings of \( D \). We examine when \( H \) can be represented in the form \( H = (\bigcap_{V \in \Sigma} V) \cap R \), with \( \Sigma \) a Noetherian subspace of the Zariski–Riemann space of the quotient field of \( D \). We characterize also the special case in which \( \Sigma \) can be chosen to be a finite character collection of valuation overrings of \( D \).

MSC: Primary: 13A18; 13E05; 13B22

1. Introduction

If \( D \) is an integral domain with quotient field \( F \), then an overring of \( D \) is a ring \( R \) such that \( D \subseteq R \subseteq F \). If \( D \) is a Noetherian domain of Krull dimension 1, then every integrally closed overring of \( D \) is a Dedekind domain. However, if \( D \) has Krull dimension 2, then no comprehensive description of the integrally closed overrings of \( D \) has yet been obtained. It is easy to find non-Noetherian overrings of any two-dimensional Noetherian domain, and one encounters such non-Noetherian overrings in applications such as those that involve the affineness of open sets of projective schemes, rings of invariants, holomorphy rings and direct limits of blowup algebras. In fact, a recent article by Loper and Tartarone which classifies the integrally closed rings between \( \mathbb{Z}[X] \) and \( \mathbb{Q}[X] \) suggests that a description of integrally closed overrings of two-dimensional Noetherian domains should be quite complex [12].

In this article we examine integrally closed overrings of two-dimensional Noetherian domains \( D \) that arise as the intersection of an arbitrary integrally closed overring and a collection of valuation overrings of \( D \) from a Noetherian subspace of the Zariski–Riemann space of the quotient field of \( D \). Since an integral domain is integrally closed if and only if it is an intersection of its valuation overrings, it is natural in seeking to describe the integrally closed overrings of a given domain to examine the intersections of its valuation overrings.

Let \( A \) be a subring of a field \( F \), and denote by \( \text{Zar}(A|F) \) the collection of all valuation rings containing \( A \) and having quotient field \( F \). When \( F \) is the quotient field of \( A \), we write \( \text{Zar}(A) \) for \( \text{Zar}(A|F) \). The Zariski–Riemann
space of the domain $A$ is the set $\text{Zar}(A)$ endowed with the topology whose basic open sets are of the form

$$U(x_1, \ldots, x_n) := \{ V \in \text{Zar}(A) : x_1, \ldots, x_n \in V \},$$

where $x_1, \ldots, x_n \in F$. A topological space is $\text{Noetherian}$ if its open sets satisfy the ascending chain condition. Our main focus in this article is the question: If $D$ is a Noetherian domain of Krull dimension 2, $\Sigma$ is a Noetherian subspace of $\text{Zar}(D)$ and $R$ is an integrally closed overring of $D$, what is the structure of $H := (\bigcap_{V \in \Sigma} V) \cap R$? We say that such a ring $H$ has a Noetherian $R$-representation. (This terminology is introduced more formally in Section 2.)

If $\Sigma$ is a Noetherian subspace of $\text{Zar}(H)$, then for each prime ideal $P$ of $H$, $H_P = (\bigcap_{V \in \Sigma} V_P) \cap R_P$ (Proposition 2.4(1)). Moreover, the set $\{ V_P : V \in \Sigma \}$ is also a Noetherian subspace of $\text{Zar}(H)$ (Proposition 2.4(5)). Thus it is of interest to describe the structure of $H$ when $H$ is a quasilocal domain. We do this in Section 6, and obtain in Corollary 6.7 the following classification. Let $H \subseteq R$ be an integrally closed overrings of $D$, where $H$ is a quasilocal domain with maximal ideal $M$ that is not a valuation domain. Set $E = \{ r \in R : rM \subseteq M \}$. Then these statements are equivalent:

1. There exists a Noetherian subspace $\Sigma$ of $\text{Zar}(H)$ such that $H = (\bigcap_{V \in \Sigma} V) \cap R$.
2. $E/M$ is a Noetherian ring and $E = A \cap B \cap R$, where $B$ is a Noetherian integrally closed overring of $H$ and $A$ is either the quotient field of $H$ or a finite intersection of valuation overrings of $H$, each having irrational value group.

We are also interested in the special case in which $\Sigma$ is a finite character collection. Recall that a collection $\Sigma$ of overrings of a domain $H$ has finite character if every nonzero element of $H$ is a unit in all but finitely many members of $\Sigma$. Heinzer showed in [8] that if $\Sigma$ is a finite character collection of valuation overrings of $D$ and each member of $\Sigma$ is a rank one discrete valuation ring (DVR), then $H$ is a Noetherian domain. Thus, in more concise terminology, Heinzer’s theorem is:

**Theorem 1.1 (Heinzer [8]).** Every Krull overring of a two-dimensional Noetherian domain is a Noetherian domain.

This raises the question: What is the structure of $H$ when the members of $\Sigma$ are not restricted to DVRs? Since a finite character collection of valuation overrings of a domain is a Noetherian space (see Proposition 2.4), an answer to the main question above includes also a solution to the latter question. Specifically, we see in Corollary 6.7 that $\Sigma$ in statement (1) above can be chosen to be a finite character collection if and only if (2) holds, where it is assumed additionally that $E/M$ is a finitely generated $H/M$-algebra.

These classifications help clarify when such overrings $H$ of $D$ arise. We show in fact in Corollary 4.9 that these rings arise in the following natural way. Let $R$ be an integrally closed overring of $D$, and suppose that $J$ is an ideal of $R$ such that $D \cap J$ is a maximal ideal of $D$ and $R/J$ is a reduced indecomposable ring having finitely many minimal prime ideals (e.g., choose $J$ to be a prime ideal of $R$). Then every integrally closed overring $H$ of $D$ such that $J \subseteq H \subseteq R$ has a Noetherian $R$-representation. If in addition we have that $R/J$ is a finitely generated $H/(J \cap H)$-algebra (e.g., $R$ is a finitely generated $H$-algebra), then every integrally closed overring $H$ of $D$ such that $J \subseteq H \subseteq R$ has a finite character $R$-representation (Corollary 5.4).

Thus, phrased geometrically, the rings $H$ we are interested in for this last case arise from connected affine pieces of exceptional curves of blowups of $D$. In more detail, let $I$ be an ideal of $D$ such that $D/I$ has Krull dimension 0, and let $0 \neq x \in I$. Let $R = D[x^{-1}]$, so that $\text{Spec}(R)$ is an affine open subspace of the blowup of $D$ along $I$ given by $\text{Proj}(D[I])$, where $t$ is an indeterminate for $D$. Let $J$ denote the radical of $xR$ in $R$. Then $\text{Spec}(R/J)$ is an affine open subspace of the exceptional curve of the blowup. Moreover, $R/J$ is an indecomposable ring if and only if $\text{Spec}(R/J)$ is connected. (For example, when $D$ is a regular local ring and $I$ is the maximal ideal of $D$, then $J$ is in fact a prime ideal, so that $\text{Spec}(R/J)$ is necessarily connected.) So, as discussed above, if $\text{Spec}(R/J)$ is connected, then every integrally closed overring $H$ of $D$ with $J \subseteq H \subseteq R$ has a finite character $R$-representation.

2. Preliminaries

**Standing hypotheses.** To economize on notation and avoid repeating hypotheses, we fix for the rest of this article the following objects.
• $D$ denotes a Noetherian domain of Krull dimension 2 with quotient field $F$.
• $H \subseteq R$ are arbitrary integrally closed overrings of $D$. The ring $H$ is our primary focus, and ultimately we seek to describe the structure of $H$ in terms of $R$ and the valuation rings used to represent $H$. The case where $H = F$ will be vacuous in our arguments, and hence not of interest. However, in our framework the case $R = F$ is important, and is considered explicitly a number of times, especially in Section 8.
• $M$ denotes an arbitrary maximal ideal of $H$. Sometimes we will consider the case where $H$ is quasilocal, and in this case, $M$ is of course the unique maximal ideal of $H$.
• $E := \{ r \in R : rM \subseteq M \}$. Although the ring $E$ depends on $R$ and $M$, these latter two objects are fixed throughout the article, so our notation does not reflect this dependency. Note that $M$ is an ideal of $E$; in fact, $E$ is the largest overring of $H$ contained in $R$ in which $M$ is an ideal. One of our main objects of study is then the ring $E/M$.
• $X_1 := \{ P \in \text{Spec}(H) : H_P$ is a DVR and $R \not\subseteq H_P \}$. (We review the notion of a DVR below in (2.1).)
• $H_1 := (\bigcap_{P \in X_1} H_P) \cap R$. It is easy to see that since each $H_P$ in this intersection is a DVR, $E \subseteq H_1 \subseteq R$. In later sections, for technical reasons we sometimes replace a given $R$-representation with a smaller $H_1$-representation.

To summarize, we have the following containment of rings:

$$D \subseteq H \subseteq E \subseteq H_1 \subseteq R \subseteq F.$$ 

Our primary focus is the structure of the ring $H$ in terms of $R$. The rings $E$ and $H_1$ are auxiliary objects that arise in the analysis of $H$, and the ring $E$ in particular plays an especially key role in our approach.

Notation. Throughout the article, we use the following notation.

• For every valuation overring $V$ of $D$, we write $m_V$ for the maximal ideal of $V$. If $V$ is a field, we define $p_V = 0$; otherwise, we define $p_V$ to be the height 1 prime ideal of $V$. (Since $D$ is a Noetherian domain of Krull dimension 2, every valuation overring of $D$ has Krull dimension $\leq 2$; see (2.1) below.) Thus $\text{Spec}(V) = \{ 0, p_V, m_V \}$, where possibly there is some redundancy in this list.
• If $\Sigma$ is a collection of valuation overrings of $D$, we let $\Sigma_1 = \{ V_{p_V} : V \in \Sigma \}$. Then every member of $\Sigma_1$ has Krull dimension $\leq 1$.
• Given a collection $\Sigma$ of valuation overrings of $D$, we sometimes want to single out the members $V$ of $\Sigma$ that dominate the quasilocal ring $H_M$; i.e., $MV \neq V$. To do this, we set $\Sigma^d = \{ V \in \Sigma : MV \neq V \}$.
• If $A \subseteq B \subseteq K$ are rings, with $K$ a field, then we set $\text{Zar}_B(A|K) = \{ V \in \text{Zar}(A|K) : B \not\subseteq V \}$, and $\text{Zar}_B(A) = \{ V \in \text{Zar}(A) : B \not\subseteq V \}$. We are particularly interested in the space $\text{Zar}_E(H)$, where $E$ and $H$ are as above.

We review next some properties of valuation overrings of $D$.

(2.1) A valuation domain is rational if its value group is isomorphic as a totally ordered abelian group to a nonzero subgroup of the rational numbers. A valuation domain is irrational if it is not rational and its value group is isomorphic as a totally ordered abelian group to a nonzero subgroup of the real numbers. A valuation overring of $D$ that is not a field is either rational, irrational or has Krull dimension 2 [1, Theorem 1]. If $V$ is a valuation overring of $D$ that has Krull dimension 2, then $V$ is discrete, so that $V_{p_V}$ and $V/p_V$ are DVRs [1, Remark 2], where, as usual, a DVR is a rank one discrete valuation ring; i.e., a DVR is a Noetherian valuation domain that is not a field.

(2.2) Among the DVRs, the essential prime divisors of $D$ are those of the form $\overline{D}_p$, where $\overline{D}$ is the integral closure of $D$ in its quotient field and $P$ is a height one prime ideal of $\overline{D}$. The hidden prime divisors are those valuation overrings $V$ that are DVRs having maximal ideals contracting to a maximal ideal of $D$ and such that the residue field of $V$ has transcendence degree 1 over the residue field of its center in $D$. A hidden prime divisor has the property that its residue field is a finitely generated extension of the residue field of its center $m_V \cap D$ in $D$ [1, Theorem 1(4)]. Moreover, a hidden prime divisor cannot be an essential prime divisor. (This can be seen by applying Abhyankar’s inequality which states that for a valuation overring $V$ of $D$, the sum of the rank of $V$ and the transcendence degree of the residue field of $V$ over its center $p$ in $D$ is at most the height of $p$ [1, Theorem 1]). The classes of essential and hidden prime divisors do not account for all the DVR overrings of $D$. There remains the class of DVRs having maximal ideals contracting to a height 2 maximal ideal of $D$ and such that the residue field of $V$ is algebraic over the residue field of its center in $D$. See [20, p. 102] for explicit examples of such DVRs.

In Proposition 2.3 we collect a few properties of integrally closed overrings of $D$. Statement (1) is well-known, as is may be the rest of the proposition, but for lack of a reference we include a proof.
Proposition 2.3. Let \( P \) be a nonzero prime ideal of \( H \). Then:

1. \( H \) has Krull dimension \( \leq 2 \).
2. If \( Q \in \text{Spec}(H) \), then \( \text{ht}(Q) \leq \text{ht}(Q \cap D) \).
3. \( H/P \) is a Noetherian domain.
4. If \( P \) is not a maximal ideal of \( H \), then \( H \setminus Q \) is a DVR.
5. If \( P \) is an invertible ideal of \( H \), then \( H \setminus P \) is a valuation domain.

Proof. (1) By (2.1) every valuation overring of \( D \) has Krull dimension \( \leq 2 \), so by [6, Theorem 25.8] or [13, Theorem 15.5] every overring of \( D \) has Krull dimension \( \leq 2 \).

(2) By (1) it suffices to show that if \( \text{ht}(Q) = 2 \), then \( \text{ht}(Q \cap D) = 2 \). Suppose that \( P \) is a height 1 prime ideal of \( H \) properly contained in \( Q \). Then there exists \( x \in Q \setminus P \), so that \( D[x] \) is a Noetherian domain for which \( P \cap D[x] \) is properly contained in \( Q \cap D[x] \). Since \( D \) and \( D[x] \) are Noetherian rings, we may apply the dimension inequality to obtain that \( \text{ht}(Q \cap D) = 2 \) [13, Theorem 15.5].

(3) and (4). If \( P \) is a maximal ideal of \( H \), then (3) is clear, so we suppose that \( P \) is a nonmaximal ideal of \( H \). Let \( M \) be a maximal ideal of \( H \) properly containing \( P \), and let \( x \in M \setminus P \). Then by the Krull–Akizuki Theorem the integral closure \( B \) of \( D[x] \) is a Noetherian domain, and since \( H \) is integrally closed, \( B \subseteq H \). Moreover, \( x \in (M \cap B) \setminus (P \cap B) \), so that \( P \cap B \) is a nonmaximal prime ideal of \( B \). Thus \( B \cap P = B \setminus P \) is a DVR since by (1), \( P \cap B \) must have height 1. Also, \( B \cap P = B \setminus P \), since \( B \cap P \) is a DVR, these two rings are equal. Hence \( H \setminus P \) is a DVR, which proves (4). It follows that \( H \setminus P \) is isomorphic to an overring of the one-dimensional Noetherian domain \( B/(P \cap B) \), so that \( H \setminus P \) is a Noetherian domain; see also [8, Corollary 2].

(5) Without loss of generality we may assume that \( H \) is quasilocal with principal maximal ideal \( P \). Then \( Q := \bigcap_{k \geq 0} P^k \) is the unique largest nonmaximal prime ideal of \( H \), \( Q = HQ \) and \( H/Q \) is a DVR. If \( Q = 0 \), then \( H \) is a DVR. Otherwise, if \( Q \neq 0 \) we have by (4) that \( H/Q \) is a DVR. In this case, since \( Q = HQ \), it follows that \( H \) is a valuation domain; see for example [4, Proposition 1.1.8].

We discuss next the notion of strongly irredundant representations of a domain, as developed in [16,17]. Let \( A \) be a domain, and let \( B \) be an overring of \( A \). A collection \( \Sigma \) of valuation overrings of \( A \) is a \( B \)-representation of \( A \) if \( A = (\bigcap_{V \in \Sigma} V) \cap B \). In case \( B \) is the quotient field of \( A \), we omit \( B \) and say simply that \( \Sigma \) is a representation of \( A \). In any case, if \( A \) is an integrally closed domain, then \( \text{Zar}(A) \) is a \( B \)-representation of \( A \), regardless of the choice of \( B \). Thus, rather than the existence of a \( B \)-representation, the interesting issue instead is the existence of “nice” \( B \)-representations for a given integrally closed domain. The main case we consider is when \( \Sigma \) can be chosen to be a Noetherian subspace of \( \text{Zar}(A) \) (as defined in the introduction).

A crucial property of Noetherian representations, and one that we rely on throughout this article, is that they can always be refined into “strongly irredundant” Noetherian \( B \)-representations (see Proposition 2.4). A \( B \)-representation \( \Sigma \) of \( A \) is irredundant if no proper subset of \( \Sigma \) is a \( B \)-representation of \( A \). A \( B \)-representation \( \Sigma \) of \( A \) is strongly irredundant if no member \( V \) of \( \Sigma \) can be replaced with a proper overring \( V_1 \) of \( V \). More precisely, \( \Sigma \) is a strongly irredundant \( B \)-representation of \( A \) if for every \( V \in \Sigma \) and proper overring \( V_1 \) of \( V \), \( \{V_1\} \cup (\Sigma \setminus \{V\}) \) is not a \( B \)-representation of \( A \).

It is not hard to find examples of irredundant representations of a domain that are not strongly irredundant. For example, let \( A \) be a two-dimensional integrally closed local Noetherian domain, and let \( p \) be a height 1 prime ideal of \( A \). Then there exists a valuation overring \( V \) of \( A \) such that \( V \subseteq A_p \). Now \( \{V\} \cup \{A_q : q \text{ is a height 1 prime ideal distinct from } p\} \) is a representation of \( A \), and \( V \) is irredundant in this representation, but not strongly irredundant.

Let \( V \) be a valuation overring of \( A \). As in [16] we say that \( V \) is a strongly irredundant \( B \)-representative of \( A \) if there exists a \( B \)-representation \( \Sigma \) of \( A \) such that \( V \in \Sigma \) and \( V \) is strongly irredundant in this representation. Thus, since every integrally closed domain is an intersection of its valuation overrings, \( V \) is a strongly irredundant \( B \)-representative of \( A \) if and only if \( A = V \cap B_1 \), where \( B_1 \) is an integrally closed overring of \( A \) contained in \( B \). In the case where \( B \) is the quotient field of \( A \), we simply say that \( V \) is a strongly irredundant representative of \( A \). We use the following notation:

- \( \text{Rep}(A) \) is the set of strongly irredundant representatives of \( A \),
- \( \text{Rep}_B(A) \) is the set of strongly irredundant \( B \)-representatives of \( A \).
In case $A$ is a quasilocal domain with maximal ideal $N$, then we sometimes consider the members $V$ of $\text{Rep}(A)$ that dominate $A$, meaning that $N = m_V \cap A$, where as usual $m_V$ is the maximal ideal of $V$; equivalently, since $N$ is a maximal ideal of $A$, $NV \neq V$. The collections of such strongly irredundant representatives are denoted:

- $\text{Rep}^d(A) = \{V \in \text{Rep}(A) : NV \neq V\}$,
- $\text{Rep}^d_B(A) = \{V \in \text{Rep}_B(A) : NV \neq V\}$.

Returning now to our context where $H \subseteq R$ are overrings of the two-dimensional Noetherian domain $D$, the set $\text{Rep}_R(H)$ plays an important role in the point of view taken by this article. One reason for this is demonstrated by the next proposition, which is taken from [16]. In particular, in our setting of overrings of $D$, if $H$ has a Noetherian $R$-representation, then $\text{Rep}_R(H)$ is the unique strongly irredundant Noetherian $R$-representation of $H$, and hence, as we shall see later, is useful in classifying the ring $H$.

**Proposition 2.4** ([16, Corollary 5.7] and [17, Theorems 3.4, 3.5, 3.7 and 4.2]). If there exists a Noetherian $R$-representation $\Sigma$ of $H$, then:

1. For every multiplicatively closed subset $S$ of $H$, $H_S = (\bigcap_{V \in \Sigma} V_S) \cap R_S$ and $\{V_S : V \in \Sigma\}$ is a Noetherian subspace of $\text{Zar}(H)$.
2. $\text{Rep}_R(H)$ is a Noetherian $R$-representation of $H$.
3. $\text{Rep}_R(H)$ is the unique strongly irredundant $R$-representation of $H$.
4. $\text{Rep}_R(H) \subseteq \Sigma \cup \Sigma_1$.
5. $\Sigma_1$ has finite character, and $\Sigma \cup \Sigma_1$ is a Noetherian space. \(\square\)

One of the advantages of working with finite character collections $\Sigma$ of valuation rings is that the mapping $\Sigma \to \text{Spec}(H)$ that sends a valuation ring to its center on $H$ has finite fibers everywhere off the ideal $(0)$. This is stated more algebraically in the following remark, which we record here for future use.

**Remark 2.5.** If $\Sigma$ is a finite character collection of valuation overrings of the domain $A$, then for any nonzero ideal $I$ of $A$, the set $\{V \in \Sigma : I V \neq V\}$ is finite. For if $0 \neq x \in I$, then since $\Sigma$ has finite character, $x$ is a unit in all but finitely many valuation rings in $\Sigma$.

If $\Sigma$ is a finite character collection of valuation overrings of the domain $A$, then $\Sigma$ is a Noetherian subspace of $\text{Zar}(A)$ [17, Proposition 3.2]. In general the converse is not true, even in our circumstance in which we consider only overrings of the two-dimensional Noetherian domain $D$ (see [16]). However, a partial converse, and one that we rely on heavily in this article, is given by:

**Proposition 2.6** ([16, Theorem 4.5]). Let $\Gamma = \{V \in \text{Zar}_R(H) : R \subseteq V_{p_V}\}$. If there is an $R$-representation $\Sigma \subseteq \Gamma$ of $H$ such that $\Sigma_1$ has finite character and there at most finitely many essential prime divisors of $D$ contained in $\Sigma_1$, then $\Sigma = \Gamma$ and $\Sigma$ is a strongly irredundant Noetherian $R$-representation of $H$. \(\square\)

In the final proposition of this section we collect some technical properties of strongly irredundant representatives that we need later. These facts are contained in Lemma 3.1, Proposition 3.2, Corollary 3.5, Propositions 3.4 and 3.9 of [16].

**Proposition 2.7.** Suppose there exists a valuation overring $V$ of $H$ such that $H = V \cap R$ and $M = m_V \cap H$. Let $P = p_V \cap H$.

- (a) If $H \neq R$ and $V$ is a rational valuation ring, then $V = H_M$.
- (b) $V$ is strongly irredundant in $H = V \cap R$ if and only if $H \neq R$, and $V = H_M$ or $P = M$; if and only if $V$ is strongly irredundant in $H_M = V \cap R_M$.
- (c) If $V$ is strongly irredundant in $H = V \cap R$ and $P$ is a nonmaximal prime ideal of $H$, then $V = H_M$.
- (d) Suppose that $V$ has Krull dimension 2. Then $V = H_M$ if and only if $V \subseteq H_P$; if and only if $\{q \in F : q MV \subseteq MV\} \neq V_{p_V}$.
- (e) $W \in \text{Zar}_R(D)$ such that $R \subseteq W_{p_W}$, then $W_{p_W} = R_{p_W \cap R}$. \(\square\)
3. The ring $E$

Recall our standing hypotheses from Section 2. For a nonzero ideal $I$ of a domain $A$, we define $\text{End}(I)$ to be the overring of $A$ consisting of all elements $q$ of the quotient field of $A$ such that $qI \subseteq I$. Thus $\text{End}(I)$ can be identified with the usual ring of endomorphisms, $\text{End}_R(I)$. Using this notation, the ring $E$ (as defined in Section 2) can be expressed also as $E = \text{End}(M) \cap R$. A domain $A$ is *completely integrally closed* if every $x$ in the quotient field of $A$ such that the powers $x^n$ ($n \geq 0$) are contained in a finitely generated $A$-submodule of the quotient field of $A$, belongs to $A$. Examples of completely integrally closed rings include integrally closed Noetherian domains, valuation rings of Krull dimension 1, and intersections of the members of any collection of such rings. In working with the ring $E$ in this and later sections, we occasionally refer to the following observation regarding completely integrally closed rings.

**Remark 3.1.** If $I$ is a nonzero ideal of the completely integrally closed domain $A$, then $\text{End}(I) = A$; see for example [6]. Thus in our context, it follows that if $B$ is an integrally closed overring of $H$, then $E \subseteq B$. For we have $E \subseteq \text{End}(M) \subseteq \text{End}(MB) = B$. In particular, if $R$ is completely integrally closed, then $E = \text{End}(M)$.

The next proposition, which we will refer to often, shows that $\text{Zar}_E(H)$ is a small subspace of $\text{Zar}(H)$. Recall the notation introduced in Section 2.

**Proposition 3.2.** If $V \in \text{Zar}_E(H)$, then $V$ has Krull dimension 2, $M \subseteq p_V$ and $E \subseteq \text{End}(M) \subseteq V_{p_V}$.

**Proof.** Let $V \in \text{Zar}_E(H)$. If $V$ has Krull dimension 1, then by Remark 3.1, $E \subseteq V$, a contradiction. Hence by (2.1), $V$ has Krull dimension 2. If $M \nsubseteq p_V$, then it is impossible that $V_{p_V} \subseteq \text{End}(M)$, since this would force $V_{p_V} = MV_{p_V} \subseteq V$, a contradiction to the fact that $V$ has Krull dimension 2. Thus $V \subseteq \text{End}(MV) \subseteq V_{p_V}$, and we conclude from the fact that $p_V$ has height 1 that $\text{End}(MV) = V$. But then again we have $E \subseteq \text{End}(M) \subseteq V$, a contradiction. Hence $M \subseteq p_V$. Finally, since $V_{p_V}$ is a valuation ring of Krull dimension 1, Remark 3.1 implies that $E \subseteq \text{End}(M) \subseteq V_{p_V}$. □

**Corollary 3.3.** If $H$ has an $R$-representation consisting only of valuation rings of Krull dimension 1, then $H = E$.

**Proof.** If $\Sigma$ is an $R$-representation of $H$ and every member of $\Sigma$ has Krull dimension 1, then $\text{Zar}_E(H)$ is empty since by Proposition 3.2, $\text{Zar}_E(H)$ consists only of valuation rings having Krull dimension 2. Therefore, since $H$ is integrally closed, $H = E$. □

Before proving the next two propositions of this section, we consider in the following simple lemma the general situation of maximal ideals of integrally closed domains. In particular, we do not assume in the lemma that the domain in question be an overring of our two-dimensional Noetherian domain $D$.

**Lemma 3.4.** If $N$ is a maximal ideal of an integrally closed domain $A$, then $\text{End}(N)$ is an integrally closed domain and $\text{End}(N)/N$ is a reduced indecomposable ring.

**Proof.** If $\text{End}(N) = A$, then the claim is clear, so suppose that $\text{End}(N) \neq A$. Then since $N$ is a maximal ideal of $A$, $NN^{-1} = N$, so that $\text{End}(N) = N^{-1}$. Let $x$ be an element of the quotient field of $A$ that is integral over $\text{End}(N)$. Then there exists a monic polynomial $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in \text{End}(N)[X]$ such that $f(x) = 0$. Let $b \in N$, and define $g(X) = X^n + a_{n-1}bX^{n-1} + a_{n-2}b^2X^{n-2} + \cdots + a_1b^{n-1}X + a_0b^n$. Since each $a_i \in \text{End}(N)$, it follows that $g(X) \in A[X]$. Moreover, $g(xb) = b^nf(x) = 0$, so since $g$ is a monic polynomial in $A[X]$ and $A$ is integrally closed, we conclude that $xb \in A$. Since the choice of $b \in N$ was arbitrary, we have $x \in N^{-1} = \text{End}(N)$. Hence $\text{End}(N)$ is an integrally closed domain.

Now suppose that there exists $e \in \text{End}(N)$ such that $e^2 - e \in N$. Since $A$ is integrally closed, $e \in A$; hence $1 - e \in A$. Thus since $e(e - 1) \in N$, we have that either $e \in N$ or $1 - e \in N$. Consequently, the ring $\text{End}(N)/N$ has no nontrivial idempotents. Moreover, $\text{End}(N)/N$ is reduced since for any $x \in \text{End}(N)$ with $x^n \in N$ for some $n > 0$, it must be that $x \in A$, so that $x \in N$. □

**Proposition 3.5.** If $H \neq E$, then $H_M$ is not a valuation domain, $E$ is an integrally closed domain and $E/M$ is a reduced indecomposable ring.
Proposition 3.2. First we claim that Proposition 2.3 that
Lemma 3.4. Proposition 2.7.

Proof. By Proposition 2.3(5), $M$ is not an invertible ideal of $H$. Consequently, since $M$ is a maximal ideal of $H$, $MM^{-1} = M$, so that $M^{-1} = \text{End}(M)$. Now for each $V \in \Sigma^d$, we may apply Proposition 2.7(d) to obtain $\text{End}(MV) = V_{p_V}$. Moreover, for each $V \in \Sigma \setminus \Sigma^d$, it is clear that since $MV = V$, $\text{End}(MV) = V$. Using the convention that when $\Sigma^d$ is empty, $F = \bigcap_{V \in \Sigma^d} \text{End}(MV), \bigcap_{V \in \Sigma \setminus \Sigma^d} \text{End}(MV)$, we have:

$$\text{End}(M) \subseteq \left( \bigcap_{V \in \Sigma^d} \text{End}(MV) \right) \cap \left( \bigcap_{V \in \Sigma \setminus \Sigma^d} \text{End}(MV) \right) \cap \text{End}(MR)$$

$$= \left( \bigcap_{V \in \Sigma^d} V_{p_V} \right) \cap \left( \bigcap_{V \in \Sigma \setminus \Sigma^d} V \right) \cap \text{End}(MR)$$

$$\subseteq \left( \bigcap_{V \in \Sigma} (V :_F MV) \right) \cap (R :_F MR) = M^{-1}.$$  

Therefore, all inclusions are equalities, and hence:

$$\text{End}(M) = \left( \bigcap_{V \in \Sigma^d} V_{p_V} \right) \cap \left( \bigcap_{V \in \Sigma \setminus \Sigma^d} V \right) \cap \text{End}(MR).$$

Thus $E = \text{End}(M) \cap R = (\bigcap_{V \in \Sigma^d} V_{p_V}) \cap (\bigcap_{V \in \Sigma \setminus \Sigma^d} V) \cap R$. □

4. Noetherian $E$-representations

As a step along the way to describing when $H$ has a Noetherian $R$-representation, in this section we consider the situation in which $D$ has a Noetherian $E$-representation, where $E$ as usual is as in Section 2. The main purpose of this section and the next is to prepare for the classification of quasilocal overrings of $D$ having Noetherian $R$-representations, which is treated in Section 6.

We prove a series of lemmas, the first of which does not require us to restrict to overrings of the two-dimensional Noetherian domain $D$.

Lemma 4.1. Let $A$ be an integrally closed domain, and let $N$ be a maximal ideal of $A$. Suppose that there exists an integrally closed overring $C$ of $A$ such that $A \subseteq C \subseteq \text{End}(N)$. If $N$ is a prime ideal of $C$, then there exists an integrally closed overring $B$ of $A$ with $A \subseteq B \subseteq C$ such that $N$ is a nonmaximal prime ideal of $B$. 

The next proposition, which we make use of in Section 4, shows that $E$ can be computed in terms of strongly irredundant $R$-representatives centered on $M$, assuming that there are sufficiently many such representatives of $H$. Recall from Section 2 that $\Sigma^d = \{ V \in \Sigma : MV \neq V \}$ and $\Sigma_1 = \{ V_{p_V} : V \in \Sigma \}$.

Proposition 3.6. Suppose that $H_M$ is not a valuation domain. If $\Sigma$ is an $R$-representation of $H$ and $\Sigma^d$ is empty, then $E = H$. Otherwise, if $\Sigma^d$ is nonempty and each member of $\Sigma^d$ is a strongly irredundant $R$-representative of $H$, then

$$E = \left( \bigcap_{V \in \Sigma^d} V_{p_V} \right) \cap \left( \bigcap_{V \in \Sigma \setminus \Sigma^d} V \right) \cap R.$$ 

Proof. By Proposition 2.3(5), $M$ is not an invertible ideal of $H$. Consequently, since $M$ is a maximal ideal of $H$, $MM^{-1} = M$, so that $M^{-1} = \text{End}(M)$. Now for each $V \in \Sigma^d$, we may apply Proposition 2.7(d) to obtain $\text{End}(MV) = V_{p_V}$. Moreover, for each $V \in \Sigma \setminus \Sigma^d$, it is clear that since $MV = V$, $\text{End}(MV) = V$. Using the convention that when $\Sigma^d$ is empty, $F = \bigcap_{V \in \Sigma^d} \text{End}(MV) = \bigcap_{V \in \Sigma^d} \text{End}(MV)$, we have:

$$\text{End}(M) \subseteq \left( \bigcap_{V \in \Sigma^d} \text{End}(MV) \right) \cap \left( \bigcap_{V \in \Sigma \setminus \Sigma^d} \text{End}(MV) \right) \cap \text{End}(MR)$$

$$= \left( \bigcap_{V \in \Sigma^d} V_{p_V} \right) \cap \left( \bigcap_{V \in \Sigma \setminus \Sigma^d} V \right) \cap \text{End}(MR)$$

$$\subseteq \left( \bigcap_{V \in \Sigma} (V :_F MV) \right) \cap (R :_F MR) = M^{-1}.$$ 

Therefore, all inclusions are equalities, and hence:

$$\text{End}(M) = \left( \bigcap_{V \in \Sigma^d} V_{p_V} \right) \cap \left( \bigcap_{V \in \Sigma \setminus \Sigma^d} V \right) \cap \text{End}(MR).$$

Thus $E = \text{End}(M) \cap R = (\bigcap_{V \in \Sigma^d} V_{p_V}) \cap (\bigcap_{V \in \Sigma \setminus \Sigma^d} V) \cap R$. □
Proof. We apply Peskine’s version of Zariski’s Main Theorem to the integrally closed domain $A_N$. If $T$ is any overring of $A_N$ that is a finitely generated $A_N$-algebra with $A_N \neq T$, then for any prime ideal $Q$ of $T$ lying over $N$, $Q$ cannot be both maximal and minimal among prime ideals of $T$ that contract to $N$ [18]. Before making use of this fact, we observe first that for each maximal ideal $L \neq N$ of $A$, $\text{End}(N)A_L = A_L$, so that $CA_L = A_L$. Therefore, since $C \neq A$, it must be that $CA_L \neq A_N$. (We are using here the fact that two overrings of $A$ are equal if and only if they are equal at each localization of $A$ at a maximal ideal.) Choose $x \in C \setminus A_N$. Then $x \in \text{End}(N)$, so that $NA_N$ is an ideal of $A_N[x]$. Moreover, since by assumption $N$ is a prime ideal of $C$, $NA_N$ is a prime ideal of $A_N[x] \subseteq C_N$. Hence every prime ideal of $A_N[x]$ contained in $NA_N$ must also be a prime ideal of $A_N$, so that $NA_N$ is minimal among prime ideals of $A_N[x]$ lying over $NA_N$. As noted, Zariski’s Main Theorem asserts then that $NA_N$ is properly contained in some prime ideal $Q$ of $A_N[x]$. Thus $P := Q \cap A[x]$ is a prime ideal of $A[x]$ that properly contains $N$. (Indeed, if $P = N$, then since $A_N[x] = A[x]A_N$, it follows that $Q = Q \cap (A[x]A_N) = (Q \cap A[x])A_N = NA_N$, a contradiction.) In particular, $N$ is a nonmaximal prime ideal of $A[x]$. Let $B$ be the integral closure of $A[x]$ in its quotient field. By assumption $C$ is an integrally closed domain. Thus, since $A[x] \subseteq C$, it must be that $B \subseteq C$. Moreover, $B$ is integral over $A[x]$, $N$ is a nonmaximal prime ideal of $A[x]$ and $N$ is a prime ideal of $B$ (since it is a prime ideal of the larger ring $C$), so $N$ is a nonmaximal prime ideal of $B$ also.

Lemma 4.2. Let $B \subseteq C$ be integrally closed overrings of $H$. If $Q$ is a nonzero prime ideal of $C$ lying over $M$ such that $Q \cap B \neq B$, then there exists $V \subset \text{Zar}_C(H)$ such that $V_{|B} = Q$.

Proof. Let $P = Q \cap B$. By Proposition 2.3(4), $B_P$ is a DVR. Thus since $B_P \subseteq C_Q$, it follows that $B_P = C_Q$. Define $L = PB_P$, and for each $D$-submodule $X$ of the quotient field $F$ of $D$, set $X_* := (X + L)/L$. Then $H^* \subseteq B^* \subseteq (B_P)^*$. Observe that $H^*$ is a field, since $M = L \cap H$. On the other hand, since $P = L \cap B$ is a nonmaximal prime ideal of $B$, we have by Proposition 2.3(1), that $B^*$ has Krull dimension 1. Thus $(B_P)^*$ is not algebraic over $H^*$, and since the algebraic closure of $H^*$ in $(B_P)^*$ is the intersection of all valuation rings in $\text{Zar}(H^*) \cap (B_P)^*$, there exists a valuation ring $U$ in $\text{Zar}(H^*) \cap (B_P)^*$ such that $B^* \not\subseteq U$. Let $V$ be a subring of $F$ such that $V/L = U$. Since $V^* = V/L$ has quotient field $(B_P)^*$ and $B_P$ is a DVR with maximal ideal $L$, $V$ is a valuation ring with quotient field $F$; see for example [4, Proposition 1.1.8]. By the choice of $V$, we have $V \subseteq B_P = C_Q$ but $B \not\subseteq V$. Since $C_Q$ is a DVR, $V_{|B} = C_Q$. Finally, since $B \not\subseteq V$ and $B \subseteq C$, it must be that $C \not\subseteq V$.

Lemma 4.3. Suppose that $H \neq E$ and there are only finitely many prime ideals $Q_1, \ldots, Q_n$ of $E$ minimal over $M$. For each $i = 1, \ldots, n$, let $U_i = E_{Q_i}$, $N_i = Q_iE_{Q_i}$, $K_i = (H + N_i)/N_i$, $A_i = (E + N_i)/N_i$, and $F_i = U_i/N_i$. Then:

1. $\{P_{|B} : V \in \text{Zar}_E(H)\} = \{U_1, \ldots, U_n\}$.
2. For each $i$, $\text{Zar}(K_i|F_i) = \{V/N_i : V \in \text{Zar}(H), V \subseteq U_i\}$.
3. For each $i$, $F_i$ is a finitely generated field extension of $K_i$ of transcendence degree 1.
4. For each $i$, $A_i$ has quotient field $F_i$.

Proof. (1) Let $V \in \text{Zar}_E(H)$. Then $E \not\subseteq V$, and by Proposition 3.2.3, $E \subseteq V_{|B}$. Thus by Proposition 2.7(e), since $E$ is by Proposition 3.5 an integrally closed overring of $D$, we have $E_{p_V \cap E} = V_{|B}$. By (2.1), $V_{|B}$ is a DVR, so $p_V \cap E$ is a height 1 prime ideal of $E$. Moreover, by Proposition 3.2, $M \subseteq p_V \cap E$, so $p_V \cap E$ is a prime ideal of $E$ minimal over $M$. Hence $p_V \cap E = Q_i$ for some $i$, so that $V_{|B} = E_{p_V \cap E} = E_{Q_i} = U_i$. Therefore, $\{V_{|B} : V \in \text{Zar}_E(H)\} \subseteq \{U_1, \ldots, U_n\}$. To prove the reverse inclusion, we consider two cases.

Case 1: $n = 1$. In this case, since by Proposition 3.5, $M$ is a radical ideal of $E$, we have $M = Q_1$. Hence by Lemma 4.1 there exists an integrally closed overring $B$ of $H$ such that $B \subseteq E$ and $M$ is a nonmaximal prime ideal of $B$. Thus by Lemma 4.2 there exists $V \in \text{Zar}_E(H)$ such that $V_{|B} = E_1 = U_1$. Hence $U_1 \in \{V_{|B} : V \in \text{Zar}_E(H)\}$.

Case 2: $n > 1$. Since $E/M$ is an indecomposable ring and $M = Q_1 \cap \cdots \cap Q_n$, it must be that each $Q_i$ is a nonmaximal prime ideal of $E$. Thus by Lemma 4.2 there exists for each $i = 1, \ldots, n$, $V_i \in \text{Zar}_E(H)$ such that $(V_i)_{|p_{V_i}} = U_i$. Therefore, $\{U_1, \ldots, U_n\} \subseteq \{V_i : V \in \text{Zar}_E(H)\}$.

(2) Let $i \in \{1, \ldots, n\}$, and suppose that $V \in \text{Zar}(H)$ with $V \subseteq U_i$. Clearly, $V/N_i$ is a valuation ring containing $K_i$. Also, by (1), $V_{|B} = U_i$, so it follows that $V/N_i$ has quotient field $F_i$. Hence $V/N_i \in \text{Zar}(K_i|F_i)$. Conversely, suppose that $W \in \text{Zar}(K_i|F_i)$. Then $W = V/N_i$ for some subring $V$ of $U_i$. Moreover, since $U_i$ is a valuation ring and $V/N_i$ has quotient field $F_i$, it follows that $V$ is a valuation overring of $H$ contained in $U_i$; see [4, Proposition 1.1.18]. Statement (2) follows.
(3) Let \( i \in \{1, 2, \ldots, n\} \). By (1), \( U_i = V_{p_i} \) for some \( V \in \text{Zar}_E(H) \), so it is enough by (2.2) to show that \( N_i \cap D \) is a height 2 prime ideal of \( D \). Let \( \overline{D} \) denote the integral closure of \( D \) in its quotient field, and note that since \( D \) has Krull dimension 2, the Krull–Akizuki Theorem implies that \( \overline{D} \) is a Noetherian ring. Suppose \( N_i \cap D \) is a height 1 prime ideal of \( D \). Then \( p_i := N_i \cap \overline{D} \) is a height 1 prime ideal of \( \overline{D} \). Since \( M \cap \overline{D} \subseteq N_i \cap \overline{D} = p_i \), this forces \( M \cap \overline{D} = p_i \). Hence, \( \overline{D}_{p_i} \subseteq H_M \), and since \( \overline{D}_{p_i} \) is a DVR, we have \( \overline{D}_{p_i} = H_M \) and \( H_M \) is a DVR. However, since \( H \neq E \), we have by Proposition 3.5 that \( H_M \) is not a valuation domain. This contradiction implies that \( N_i \cap D \) is a height 2 prime ideal and hence \( F_i \) is a finitely generated field extension of \( K_i \) of transcendence degree 1.

(4) Since \( U_i = E_{Q_i} \), it follows easily that \( A_i \) has quotient field \( F_i \). \( \square \)

**Remark 4.4.** The hypothesis of Lemma 4.3 that there are only finitely many minimal prime ideals of \( M \) in \( E \) is satisfied if and only if \( H \) has a Noetherian \( E \)-representation. This is a consequence of Proposition 4.6.

**Lemma 4.5.** If \( H \neq E \) and \( H \) has a Noetherian \( E \)-representation, then \( \text{Zar}_E(H) \) is a Noetherian \( E \)-representation of \( H \) and it is the unique strongly irredundant \( E \)-representation of \( H \). Moreover, \( \{V_{p_i} : V \in \text{Zar}_E(H)\} \) is a finite set.

**Proof.** By Proposition 2.4, \( \Sigma : = \text{Rep}_E(H) \) is a Noetherian \( E \)-representation of \( H \) and it is the unique strongly irredundant \( E \)-representation of \( H \). We claim that \( \Sigma = \text{Zar}_E(H) \). By Proposition 3.2, for every member \( V \) of \( \text{Zar}_E(H) \), \( V \) has Krull dimension 2 and \( M \subseteq p_V \). Hence since \( \Sigma \) is a Noetherian space, \( \Sigma_1 \) is finite by Proposition 2.4(5) and Remark 2.5. Moreover, by Proposition 3.2, \( E \subseteq V_{p_i} \) for every \( V \in \text{Zar}_E(H) \). Hence the hypotheses of Proposition 2.6 are satisfied, so we deduce that \( \Sigma = \text{Zar}_E(H) \), and the lemma is proved. \( \square \)

**Proposition 4.6.** If \( H \neq E \), then the following statements are equivalent.

1. \( H \) has a Noetherian \( E \)-representation.
2. The set \( \{V_{p_i} : V \in \text{Zar}_E(H)\} \) is finite.
3. \( \text{Zar}_E(H) \) is a Noetherian subspace of \( \text{Zar}(H) \).
4. \( \text{Zar}_E(H) \) is a strongly irredundant Noetherian \( E \)-representation of \( H \).
5. \( E/M \) is a ring having finitely many minimal prime ideals.
6. \( E/M \) is a reduced indecomposable Noetherian ring.
7. \( E/M \) is a Noetherian ring.

**Proof.** By Proposition 3.5, \( H_M \) is a not a valuation domain and \( E \) is an integrally closed domain. Let \( \Sigma = \text{Zar}_E(H) \). Since \( E \neq H \) and \( H \) is an integrally closed ring, it must be that \( \Sigma \) is nonempty. Also, by Proposition 3.2, \( \Sigma = \Sigma^d \).

The equivalence of (1), (3) and (4) now follows from Lemma 4.5. Moreover, the equivalence of (2) and (3) follows from Proposition 2.6 and Lemma 4.5.

(2) \( \Rightarrow \) (5) By (2) we may write \( \Sigma_1 = \{U_1, \ldots, U_n\} \). Now by Proposition 3.2, each member of \( \Sigma \) has Krull dimension 2, each \( U_i \) is a DVR centered on \( M \) and \( E \subseteq U_1 \cap \cdots \cap U_n \). For each \( i = 1, \ldots, n \), define \( P_i = m_{U_i} \cap E \).

We claim that \( M = P_1 \cap \cdots \cap P_n \). For each \( i \), \( m_{U_i} \subseteq V \) for all \( V \in \Sigma \) such that \( V \subseteq U_i \). Thus for each \( i \), \( P_i \subseteq \bigcap_{V \in \Sigma, V \subseteq U_i} V \). Hence \( M \subseteq P_1 \cap \cdots \cap P_n \subseteq \bigcap_{V \in \Sigma} V \cap E = H \), and so \( M = P_1 \cap \cdots \cap P_n \) since \( M \) is a maximal ideal of \( H \). Thus there are finitely many prime ideals of \( E \) minimal over \( M \), and by Lemma 4.3(1), each of these prime ideals is a height 1 prime ideal of \( E \). Statement (5) follows.

(5) \( \Rightarrow \) (6) By Proposition 3.5, \( E \) is an integrally closed domain and \( E/M \) is a reduced indecomposable ring. Write \( P_1, \ldots, P_n \) for the prime ideals of \( E \) minimal over \( M \). For each \( i = 1, \ldots, n \) we have by Proposition 2.3(3) that \( E/P_i \) is a Noetherian domain. Since \( E/M \) is a subdirect sum of the Noetherian rings \( E/P_i \), \( E/M \) is a Noetherian ring [14, (3.16), p. 11].

(6) \( \Rightarrow \) (7) This is clear.

(7) \( \Rightarrow \) (1) Since \( E/M \) is a Noetherian ring, there are finitely many prime ideals of \( E \) that are minimal over \( M \). Hence by Lemma 4.3(1), \( \{V_{p_i} : V \in \text{Zar}_E(H)\} \) is finite, so by Proposition 2.6, \( \text{Zar}_E(H) \) is a Noetherian \( E \)-representation of \( H \). \( \square \)

Applying Corollary 4.8 of [16], we observe in the next corollary that when the equivalent conditions of Proposition 4.6 hold, then there is a one-to-one correspondence between subsets of \( \text{Zar}_E(H) \) and integrally closed domains \( B \) with \( H \subseteq B \subseteq E \).
Corollary 4.7. If $E/M$ is a Noetherian ring, then for every integrally closed domain $B$ with $H \subseteq B \subseteq E$, there exists a unique subset $\Sigma$ of $\text{Zar}_E(H)$ such that $B = (\bigcap_{V \in \Sigma} V) \cap E$.

Proof. By Proposition 3.2, $E$ is a subring of every valuation overring of $H$ of Krull dimension 1, and by Proposition 4.6, $\{V_{p_{\nu}} : V \in \text{Zar}_E(H)\}$ is a finite set. A consequence of [16, Corollary 4.8] is that these properties are sufficient for the uniqueness assertion of the corollary to hold. □

The next corollary provides a good source of examples of overrings of $D$ having Noetherian $R$-representations. It requires the following basic lemma.

Lemma 4.8. Suppose that $A \subseteq B$ are domains; $N$ is a maximal ideal of $A$; $B$ is integrally closed; and there exist prime ideals $P_1, \ldots, P_n$ of $B$ such that $N = P_1 \cap \cdots \cap P_n$. Then $B/N$ is an indecomposable ring if and only if $N$ is a maximal ideal of the integral closure $\widehat{A}$ of $A$ in its quotient field.

Proof. Suppose first that $B/N$ is an indecomposable ring. For each $i = 1, \ldots, n$, let $Q_i = P_i \cap \widehat{A}$. Then $N = Q_1 \cap \cdots \cap Q_n$. Suppose that the set $\{Q_1, \ldots, Q_n\}$ consists of more than one prime ideal. Without loss of generality we assume that $1 < k \leq n$ and $Q_1, Q_2, \ldots, Q_k$ are distinct prime ideals of $\widehat{A}$ with $\{Q_1, \ldots, Q_k\} = \{Q_1, \ldots, Q_n\}$. Since $N$ is a maximal ideal of $A$, $Q_1, \ldots, Q_k$ are maximal ideals of $\widehat{A}$. Set $I = Q_2 \cap \cdots \cap Q_k$. Then $Q_1 \cap I = N$ and $Q_1 + I = \widehat{A}$, so the ring $\widehat{A}/N$ contains nontrivial idempotents. Hence since $\widehat{A} \subseteq B$ the ring $B/N$ also contains nontrivial idempotents, a contradiction. Therefore, the set $\{Q_1, \ldots, Q_n\}$ consists of one ideal, and $N$ must be a maximal ideal of $\widehat{A}$. Conversely, if $N$ is a maximal ideal of $\widehat{A}$, then by Lemma 3.4, $\text{End}(N)/N$ is an indecomposable ring, and since $B \subseteq \text{End}(N)$, $B/N$ is an indecomposable ring. □

Corollary 4.9. Suppose that $J$ is an ideal of $R$ such that $D \cap J$ is a maximal ideal of $D$ and $R/J$ is a reduced indecomposable ring with finitely many minimal prime ideals. Then every integrally closed overring $C$ of $D$ such that $J \subseteq C \subseteq R$ has a Noetherian $R$-representation.

Proof. Since $J$ is a maximal ideal of $A := D + J$, $J$ is a maximal ideal of the integral closure $\widehat{A}$ of $A$ by Lemma 4.8 (where $R$ plays the role of “$B$” in the lemma). Moreover, since $J$ is an ideal of $R$, we have $\text{End}(J) \cap R = R$. Thus by Proposition 4.6 (where $A$ plays the role of “$H$” in the proposition), $\text{Zar}_R(\widehat{A})$ is a Noetherian subspace of $\text{Zar}(D)$. If $C$ is an integrally closed overring of $D$ with $J \subseteq C \subseteq R$, then $C$ has an $R$-representation $\Sigma \subseteq \text{Zar}_R(\widehat{A})$. Since every subspace of a Noetherian space is Noetherian, the corollary follows. □

5. Finite character $E$-representations

In this section we single out a special class of Noetherian representations, those that have finite character. To do so, we need a valuation-theoretic description of when a subalgebra of a function field in one variable is finitely generated. To obtain this we expand upon a characterization due to Alamelu [2] by combining it with the following lemma.

Lemma 5.1 ([16, Lemma 4.1]). Let $K$ be a field, and let $F$ be a finitely generated field extension of $K$ of transcendence degree 1. Let $A$ be a proper $K$-subalgebra of $F$ having quotient field $F$, and let $\Sigma \subseteq \text{Zar}(K|F)$. Suppose that there is a valuation ring $U \in \text{Zar}(K|F)$ such that $(\bigcap_{V \in \Sigma} V) \cap A \subseteq U$. Then $U \in \Sigma$ or $A \subseteq U$. □

Lemma 5.1 is an application of the Strong Approximation Theorem for valuations of function fields in one variable; see for example [7, Theorem 2.2.13].

Lemma 5.2. Let $K$ be a field, and let $F$ be a finitely generated field extension of $K$ of transcendence degree 1. Denote by $\overline{K}$ the algebraic closure of $K$ in $F$, and let $A$ be a $K$-subalgebra of $F$. Then the following statements are equivalent.

1. $A$ is a finitely generated $K$-algebra of Krull dimension 1.
2. $\text{Zar}_A(K|F)$ is a finite nonempty set.
3. There exist $V_1, \ldots, V_n \in \text{Zar}_A(K|F)$ such that $V_1 \cap \cdots \cap V_n \cap A \subseteq \overline{K}$. 

1806

Proof. The equivalence of (1) is (2) is proved by Alamelu in Corollary 1.3 of [2]. That (2) implies (3) is a consequence of the fact that since \( \overline{K} \) is the intersection of all valuation rings in \( \text{Zar}(K|F) \) and there are only finitely many valuation rings \( V_1, \ldots, V_n \) in \( \text{Zar}(K|F) \) not containing \( A \), then \( V_1 \cap \cdots \cap V_n \cap A \subseteq \overline{K} \). It remains to prove that (3) implies (2). Let \( F' \) denote the quotient field of \( A \), so that \( F' \subseteq F \). If \( F' \subseteq \overline{K} \), then every valuation ring in \( \text{Zar}(K|F) \) contains \( A \), a contradiction. Thus we may assume that \( F' \) is a finitely generated field extension of \( K \) of transcendence degree 1. Now \( (V_1 \cap F') \cap \cdots \cap (V_n \cap F') \cap A \subseteq \overline{K} \cap F' \), so if \( W \) is any member of \( \text{Zar}_A(K|F) \), then by Lemma 5.1, \( V_i \cap F' = W \cap F' \) for some \( i \). Since \( F \) is a finite extension of \( F' \), each \( V_i \cap F' \) extends to only finitely many valuation rings in \( \text{Zar}_A(K|F) \) [3, Theorem 3.2.9, p. 65]. It follows that \( \text{Zar}_A(K|F) \) is a finite nonempty set. □

Using Lemma 5.2 we obtain a version of Proposition 4.6 for finite character representations.

Proposition 5.3. If \( H \neq E \), then the following statements are equivalent.

1. \( H \) has a finite character \( E \)-representation.
2. \( H \) has a finite \( E \)-representation.
3. \( \text{Zar}_E(H) \) is a strongly irredundant finite \( E \)-representation of \( H \).
4. \( E/M \) is a reduced indecomposable finitely generated \( H/M \)-algebra of Krull dimension 1.
5. \( E/M \) is a finitely generated \( H/M \)-algebra.

Proof. (1) \( \Rightarrow \) (2) Let \( \Sigma \) be a finite character \( E \)-representation of \( H \). We may assume that \( \Sigma \subseteq \text{Zar}_E(H) \). Thus by Proposition 3.2, \( M \subseteq \mathfrak{m}_V \) for all \( V \in \Sigma \), so that \( \Sigma \) has finite character, \( \Sigma \) must be finite (Remark 2.5).

(2) \( \Rightarrow \) (3) Suppose that \( \Sigma \) is a finite \( E \)-representation of \( H \). Then by Proposition 4.6, \( \text{Zar}_E(H) \) is a strongly irredundant \( E \)-representation of \( H \). Thus \( \text{Zar}_E(H) = \{ V \in \Sigma : E \nsubseteq V \} \), so that \( \text{Zar}_E(H) \) is a finite set.

(3) \( \Rightarrow \) (4) By Proposition 4.6, \( E/M \) is a reduced indecomposable Noetherian ring. In particular, there are only finitely many prime ideals \( Q_1, \ldots, Q_n \) of \( E \) minimal over \( M \). Thus we may apply Lemma 4.3, and in doing so, we assume the same notation as the lemma. Let \( i \in \{1, 2, \ldots, n\} \). We have by Lemma 4.3(3) that \( F_i \) is a finitely generated field extension of \( K_i \) of transcendence degree 1. By assumption the set \( \{ V \in \text{Zar}_E(H) : V \subseteq U_i \} \) is finite, and by Lemma 4.3(1) this set is nonempty. Hence by Lemma 4.3(2), \( \text{Zar}_{A_i}(K_i|F_i) \) is finite. It is also nonempty since \( U_i \) is a DVR and by Proposition 3.2, \( \text{Zar}_E(H) \) consists of valuation rings of Krull dimension 2. Therefore, by Lemma 5.2, \( A_i \) is a finitely generated \( K_i \)-algebra of Krull dimension 1. Also, \( M = N_1 \cap \cdots \cap N_n \cap E \), so \( E/M \) is a subdirect product of the one-dimensional finitely generated \( H/M \)-algebras \( A_i \). Hence \( E/M \) is a finitely generated \( H/M \)-algebra of Krull dimension 1 [2, Lemma 2.2.].

(4) \( \Rightarrow \) (5) This is clear.

(5) \( \Rightarrow \) (1) Let \( Q_1, \ldots, Q_n \) denote the prime ideals of \( E \) that are minimal over \( M \). We use the notation of Lemma 4.3. By Proposition 4.6, \( \text{Zar}_E(H) \) is a Noetherian \( E \)-representation of \( H \), and by Lemma 4.3, \( \{ V_{p_i} : V \in \text{Zar}_E(H) \} = \{ U_1, \ldots, U_n \} \). For each \( i = 1, \ldots, n \), we set \( \Sigma_i = \{ V \in \text{Zar}_E(H) : V \subseteq U_i \} \), so that \( \text{Zar}_E(H) = \Sigma_1 \cup \cdots \cup \Sigma_n \). Now let \( i \in \{1, 2, \ldots, n\} \). By Lemma 4.3(2), \( \text{Zar}_{A_i}(K_i|F_i) = \{ V/N_i : V \in \Sigma_i \} \). Also, by Lemma 4.3(3) and (4), \( A_i \) is a finitely generated \( K_i \)-algebra of Krull dimension 1 with quotient field \( F_i \), so by Lemma 5.2, \( \{ V/N_i : V \in \Sigma_i \} \) is a finite set. Hence \( \Sigma_i \) is a finite set. Since this holds for each choice of \( i \), and since \( \text{Zar}_E(H) = \Sigma_1 \cup \cdots \cup \Sigma_n \), we have established statement (1). □

From Proposition 5.3 we derive the finite character version of Corollary 4.9:

Corollary 5.4. Suppose that there exists an ideal \( J \) of \( R \) such that \( D \cap J \) is a maximal ideal of \( D \) and \( R/J \) is a reduced indecomposable ring that is finitely generated as an algebra over \( D/(D \cap J) \). Then every integrally closed overring \( C \) of \( D \) such that \( J \subseteq C \subseteq R \) has a finite \( R \)-representation.

Proof. Since \( J \) is a maximal ideal of \( A := D + J \), \( J \) is a maximal ideal of \( \overline{A} \) by Lemma 4.8. Moreover, \( \text{End}(D) \cap R = R \), so by Proposition 5.3, \( \text{Zar}_R(\overline{A}) \) is a finite set. If \( C \) is an integrally closed overring of \( D \) with \( J \subseteq C \subseteq R \), then \( C \) has an \( R \)-representation \( \Sigma \subseteq \text{Zar}_R(\overline{A}) \), so that \( \Sigma \) is finite. □

It is easy to give examples that illustrate Corollary 5.4:

Example 5.5. Suppose that \( R \) is a finitely generated \( D \)-algebra. For a maximal ideal \( m \) of \( D \), let \( J \) be an intersection of prime ideals of \( R \) minimal over \( mR \) such that \( R/J \) is an indecomposable ring (e.g., choose \( J \) to be a prime ideal of \( R \) minimal over \( mR \)). Then by Corollary 5.4 the integral closure \( \overline{D} + J \) of \( D + J \) has a finite \( R \)-representation.
6. Noetherian $R$-representations of quasilocal overrings

In this section we use the results of Sections 4 and 5 to classify integrally closed quasilocal overrings of $D$ having Noetherian or finite character $R$-representations. Recall the ring $H_1 = (\bigcap_{P \in X_1} H_P) \cap R$ defined in Section 2. We see in Lemma 6.2 that when $H$ has a Noetherian $R$-representation, then $H_1$ has a finite character $R$-representation consisting of DVRs. This is a consequence of the fact that $X_1$ is a finite character set of prime ideals. (A collection $X$ of prime ideals of a ring $A$ has finite character if each nonzero element of $A$ is contained in at most finitely many members of $X$. Thus $X$ has finite character if and only if $\{A_P : P \in X\}$ is a finite character collection of rings.) First, to illustrate the usefulness of the set $X_1$ in our context, we make an observation concerning flat extensions of quasilocal integrally closed overrings of $D$ such that $\{P \in \text{Spec}(H) : H_P$ is a DVR$\}$ has finite character.

**Proposition 6.1.** Suppose that $H$ is quasilocal and $\{P \in \text{Spec}(H) : H_P$ is a DVR$\}$ has finite character. If $B$ is a flat overring of $H$, then $H = B$, $B$ is a Dedekind domain or $B$ is the quotient field of $H$.

**Proof.** Assume that $H \neq B$ and $B$ is not the quotient field of $H$. Since $B$ is a flat overring of $H$, every localization of $B$ at a prime ideal is equal to a localization of $H$ at one of its prime ideals [19, Theorem 2]. Hence $B_N = H_{N \cap H}$ for each maximal ideal $N$ of $B$. Thus since $H \neq B$ and $H$ is a quasilocal ring, it follows that for every maximal ideal $N$ of $B$, $N \cap H$ is a nonmaximal prime ideal of $H$, and by Proposition 2.3(4), $H_{N \cap H} = B_N$ is a DVR. By assumption $\{P \in \text{Spec}(H) : H_P$ is a DVR$\}$ is a finite character collection of height 1 prime ideals of $H$, so $I^\prime := \{B_N : N \in \text{Max}(B)\}$ is a finite character collection of DVRs. Thus $B$ is a finite character intersection of DVRs, so $B$ must be a Noetherian domain (Theorem 1.1). Therefore, $B$ is an integrally closed Noetherian domain of Krull dimension 1; that is, $B$ is a Dedekind domain. \(\square\)

In light of the next lemma, if $H$ is quasilocal and has a Noetherian representation, then $\{P \in \text{Spec}(H) : H_P$ is a DVR$\}$ has finite character and Proposition 6.1 applies.

**Lemma 6.2.** If there exists a Noetherian $R$-representation of $H$, then $X_1$ has finite character and the ring $\bigcap_{P \in X_1} H_P$ is an integrally closed Noetherian domain.

**Proof.** By Proposition 2.4(3), $\Sigma := \text{Rep}_R(H)$ is a Noetherian $R$-representation of $H$. First we observe that $\{H_P : P \in X_1\} \subseteq \Sigma_1$. For let $P \in X_1$. Then by Proposition 2.4(1), we have $H_P = (\bigcap_{V \in \Sigma} V_P) \cap R_P$. Since $H_P$ is a DVR and $R \not\subseteq H_P$, it must be that $H_P = V_P$ for some $V \in \Sigma$, proving that $H_P \in \Sigma_1$. By Proposition 2.4(5), $\Sigma_1$ has finite character, so $\{H_P : P \in X_1\}$, as a subset of $\Sigma_1$, also has finite character. Hence $X_1$ has finite character. Moreover, by Theorem 1.1, every Krull overring of a two-dimensional Noetherian domain is a Noetherian domain. Therefore, $\bigcap_{P \in X_1} H_P$ is an integrally closed Noetherian domain. \(\square\)

The next lemma gives a sufficient condition, one that occurs when considering $H_1$-representations, for an irredundant $R$-representation to be strongly irredundant.

**Lemma 6.3.** Suppose that $H$ has an irredundant $R$-representation $\Sigma$, and $R \subseteq H_P$ for all $P \in \text{Spec}(H)$ such that $H_P$ is a DVR and $H_P \in \Sigma_1$. Then the $R$-representation $\Sigma$ is strongly irredundant.

**Proof.** Let $W \in \Sigma$. If $W$ is a localization of $H$, then $W$ is strongly irredundant in $\Sigma$ by Proposition 2.7(b). Suppose that $W$ is not a localization of $H$, and define $P = p_W \cap H$. If $P$ is a maximal ideal of $H$, then $W$ is strongly irredundant in $\Sigma$ by Proposition 2.7(b). We claim that the remaining case, where $P$ is nonmaximal and $W$ is not a localization, cannot occur. To this end, suppose that $P$ is a nonmaximal prime ideal of $H$. Then by Proposition 2.3(4), $H_P$ is a DVR, so that $H_P = W_{p_W}$. Let $W_1 = \bigcap_{V \in \Sigma \setminus \{W\}} V$, so that $H = W \cap W_1 \cap R$. By assumption $W$ is not a localization of $H$, so by Proposition 2.7(b), $H = H_P \cap R_1 \cap R$. But then since $R \subseteq H_P$, we have $H = R_1 \cap R$, contrary to the irredundance of the $R$-representation $\Sigma$ of $H$. Therefore, $W$ is strongly irredundant in $\Sigma$. \(\square\)

Our strategy in this section is to trade a given Noetherian $R$-representation of the domain $H$ for a Noetherian $H_1$-representation $\Sigma$, and use this new representation to classify $H$. Note that by Lemma 6.2, $H_1$ is an intersection of an integrally closed Noetherian domain and $R$. Another advantage in trading representations, as noted in the next lemma, is that when $H$ is quasilocal and not a valuation ring, the members of the new representation $\Sigma$ are either irreducible or have Krull dimension 2. Moreover, they are all centered on the maximal ideal of $H$:
Lemma 6.4. Suppose that $H$ has a Noetherian $R$-representation. If $H$ is a quasilocal domain and $H$ is not a valuation ring, then:

1. If $V \in \text{Rep}_R(H) \setminus \text{Rep}_R^d(H)$, then $m_V \cap H \subseteq M$, so by Proposition 2.3(4), $V = H_{m_V} \cap H$ and $V$ is a DVR. Thus since $R \nsubseteq V$, it follows that $V \subseteq \{H_P : P \in X_1\}$. This shows that $\text{Rep}_R(H) \setminus \text{Rep}_R^d(H) \subseteq \{H_P : P \in X_1\}$. Now suppose that $P \in X_1$. Let $\Sigma = \text{Rep}_R(H)$. Then by Proposition 2.4(2), $H = (\bigcap_{V \in \Sigma} V) \cap R$, and by Proposition 2.4(1), $H_P = (\bigcap_{V \in \Sigma} V_P) \cap R$. Since $H_P$ is a DVR and $R \nsubseteq H_P$, it must be that $H_P = V_P$ for some $V \in \Sigma$, so that $m_V \cap H = P$. If $V \neq V_P$, then $V$ has Krull dimension 2 and $V \subseteq H_P$, so by Proposition 2.7(d), $V = H$, contrary to the assumption that $H$ is not a valuation domain. Thus $H_P = V_P = V \in \text{Rep}_R(H)$, which proves that $\{H_P : P \in X_1\} \subseteq \text{Rep}_R(H) \setminus \text{Rep}_R^d(H)$.

2. First we show that $\text{Rep}_R^d(H) \subseteq \text{Rep}_{H_1}(H)$.

3. Suppose that $V \in \text{Rep}_R^d(H)$. Then there exists an irredundantly closed domain $R_1 \subseteq R$ such that $H = V \cap R_1$, $M = m_V \cap H$ and $V$ is strongly irredundant in this intersection. If $m_V \cap H = M$, then $H$ has Krull dimension 2, so by Proposition 2.7(d), $H = V$, contrary to the assumption that $H$ is not a valuation ring. Thus $m_V \cap H = M$. Now for all $P \in X_1$, since $M = m_V \cap H \subseteq P$ (for otherwise $H = H_P$ and $H$ is a DVR), it follows that $V_P$ is the quotient field of $H$. If $V$ is not a field, then there exists a nonzero prime ideal $Q$ of $V$ such that $Q \cap V_P = V_P$, and hence $Q \cap H \subseteq P$. Necessarily, $v \in Q$, since $V$ is a DVR. This then implies that $R_1 \subseteq (\bigcap_{P \in X_1} H_P) \cap R = H_1$. Consequently, since $V$ is strongly irredundant in $H = V_1$, we have $V \in \text{Rep}_{H_1}(H)$, and hence $\text{Rep}_R^d(H) \subseteq \text{Rep}_{H_1}(H)$.

Next we prove that $\text{Rep}_{H_1}(H) \subseteq \text{Rep}_R^d(H)$.

4. Suppose that $\Sigma$ is an irredundant $H_1$-representation of $H$. We claim that $\Sigma$ is a strongly irredundant $H_1$-representation, for once this proved, it follows from (2) and Proposition 2.4(3) that $\Sigma = \text{Rep}_{H_1}(H) = \text{Rep}_R^d(H)$. In fact, it suffices by Lemma 6.3 to note that $H_1 \subseteq H_P$ for all $P \in \text{Spec}(H)$ such that $H_P$ is a DVR. This is indeed the case, since either $H_1 \subseteq R \subseteq H_P$, or $R \nsubseteq H_P$, in which case $P \in X_1$ and $H_1 = (\bigcap_{Q \in X_1} H_Q) \cap R \subseteq H_P$.

5. Observe that if $V \in \text{Rep}_R^d(H)$ and $V$ is rational, then by Proposition 2.7(a), $H = V$, contrary to the assumption that $H$ is not a valuation ring. Thus by (2.1) every member of $\text{Rep}_R^d(H)$ has Krull dimension 2 or is irrational. □

In light of Lemma 6.4(5), the next two theorems when taken together give a complete description of the quasi-local overrings of $D$ having a Noetherian or finite character $R$-representation.

Theorem 6.5. Suppose that $H$ is a quasilocal ring that is not a valuation domain. Then the following statements are equivalent.

1. $H$ has a Noetherian (finite character) $R$-representation and no member of $\text{Rep}_R^d(H)$ is irrational.

2. $H$ has a Noetherian (resp., finite character) $R$-representation and every member of $\text{Rep}_R^d(H)$ has Krull dimension 2.

3. $E/M$ is a Noetherian ring (resp., finitely generated $H/M$-algebra) and $E = B \cap R$ for some integrally closed Noetherian overring $B$ of $H$. 

\textbf{Proof.} (1) \iff (2) If $\text{Rep}_R^d(H)$ is empty, then clearly (1) is equivalent to (2). Otherwise, if $\text{Rep}_R^d(H)$ is nonempty, then the equivalence follows from Lemma 6.4(5).

(2) \implies (3) Since $H$ has a Noetherian $R$-representation, then by Proposition 2.4(2), $\text{Rep}_R(H)$ is an $R$-representation of $H$. Suppose first that $\text{Rep}_R^d(H)$ is empty. Then by Proposition 3.6, $H = E$, and hence $E/M$ is a field. Also, Lemma 6.4(1) shows that since $\text{Rep}_R^d(H)$ is empty, it must be that $\text{Rep}_R(H) = \{ H_P : P \in X_1 \}$. Thus $H = E = (\bigcap_{P \in X_1} H_P) \cap R$, and (3) follows from Lemma 6.2.

Now suppose that $\text{Rep}_R^d(H)$ is nonempty. By Lemma 6.4(3), $\text{Rep}_R^d(H)$ is a Noetherian $H_1$-representation of $H$, and by Proposition 3.2, $E \subseteq H_1$, so it follows that $\text{Rep}_R^d(H)$ is a Noetherian $E$-representation of $H$. Thus by Proposition 4.6, $E/M$ is a Noetherian ring. Moreover, if $H$ has a finite character $R$-representation, then since $E \subseteq R$, $H$ has a finite character $E$-representation. Therefore, if $H$ has a finite character $R$-representation, we may apply Proposition 5.3 to obtain that $E/M$ is a finitely generated $H/M$-algebra.

It remains to show in the case that $\text{Rep}_R^d(H)$ is nonempty that $E = B \cap R$ for an integrally closed Noetherian overring $B$ of $H$. Let $\Sigma = \text{Rep}_R^d(H)$, and let $B = (\bigcap_{U \in \Sigma_1} U) \cap (\bigcap_{P \in X_1} H_P)$. Since $H$ is not a valuation domain, we have by Proposition 2.7(b), that for all $V \in \text{Rep}_R^d(H)$, $M = p_V \cap H$. Hence for every $V \in \Sigma_1$, $MV \neq V$, and since by Proposition 2.4(5), $\Sigma_1$ has finite character, this implies that $\Sigma_1$ must be finite (Remark 2.5). Also, since by assumption every member of $\Sigma$ has Krull dimension 2, $\Sigma_1$ consists of DVRs by (2.1). Hence by Lemma 6.2, $\Sigma_1 \cup \{ H_P : P \in X_1 \}$ is a finite character set of DVRs, so that by Theorem 1.1, $B$ is an integrally closed Noetherian domain. Finally, by Lemma 6.4(1), $\text{Rep}_R(H) \setminus \text{Rep}_R^d(H) = \{ H_P : P \in X_1 \}$, so by Proposition 3.6, $E = B \cap R$. This verifies statement (3).

(3) \implies (2) Suppose first that $H = E$. Then since $B$ has a finite character $R$-representation consisting of DVRs, $H$ has a finite character $R$-representation consisting of DVRs. Thus by Proposition 2.4(4), $\text{Rep}_R(H)$ consists of DVRs. However, since $H$ is not a valuation domain and each member of $\text{Rep}_R(H)$ is a DVR, we have by Proposition 2.7(a) that $\text{Rep}_R^d(H)$ is empty, and so clearly (2) holds.

Now suppose $H \neq E$, and let $\Sigma = \text{Zar}_E(H)$. Owing to the equivalence of (1) and (2) in Proposition 4.6, $\Sigma$ is a strongly irredundant Noetherian $E$-representation of $H$. If in addition $E/M$ is a finitely generated $H/M$-algebra, then by Proposition 5.3, $\Sigma$ is finite. Now since $E = B \cap R$, where $B$ is Noetherian and integrally closed, $\Sigma \cup \{ B_P : P \in \text{Spec}(B) \}$ is a Noetherian $R$-representation of $H$. By Proposition 2.4(4), $\text{Rep}_R(H) \subseteq \Sigma \cup \Sigma_1 \cup \{ B_P : P \in \text{Spec}(B) \}$ is a DVR. Thus $\text{Rep}_R(H)$ consists only of valuation rings of Krull dimension 2 and DVRs. (Each $V \in \Sigma$ has Krull dimension 2 by Proposition 3.2.) If $V \in \text{Rep}_R^d(H)$, then since $H$ is not a valuation ring, $V$ is by Proposition 2.7(b) not a DVR. Hence every member of $\text{Rep}_R^d(H)$ has Krull dimension 2.

\hfill \Box

\textbf{Theorem 6.6.} Suppose that $H$ is a quasilocal ring that is not a valuation domain. Then the following statements are equivalent.

(1) $H$ has a Noetherian (finite character) $R$-representation and $\text{Rep}_R^d(H)$ contains at least one irrational valuation ring.

(2) $E/M$ is a Noetherian ring (resp., finitely generated $H/M$-algebra) and $E = A \cap B \cap R$, where $A$ is a finite intersection of irrational valuation overrings of $D$, $B$ is an integrally closed Noetherian overring and $A$ cannot be omitted from this intersection.

\textbf{Proof.} (1) \implies (2) Let $\Sigma = \text{Rep}_R(H)$. Then $\Sigma^d = \text{Rep}_R^d(H)$, and by Proposition 3.6 and Lemma 6.4(1),

\[ E = \left( \bigcap_{v \in \Sigma^d} V_{p_v} \right) \cap \left( \bigcap_{P \in X_1} H_P \right) \cap R = \left( \bigcap_{v \in \Sigma^d} V_{p_v} \right) \cap H_1. \]  

(1)

We define two sets:

$\bullet$ $I_1 = \{ V_{p_v} : V \in \Sigma^d \text{ and } V \text{ has Krull dimension 2} \}$, and

$\bullet$ $I_2 = \{ V \in \Sigma^d : V \text{ is an irrational valuation ring} \}$.

By (1), $I_2$ is nonempty. Also, by Lemma 6.4(5), $\{ V_{p_v} : V \in \Sigma^d \} = I_1 \cup I_2$. Now by Lemma 6.2, $X_1$ has finite character. Thus, since by Proposition 2.4(5), $I_1$ has finite character, it must be that $I_1 \cup \{ H_P : P \in X_1 \}$ has finite character. Therefore, by Theorem 1.1, $B := (\bigcap_{U \in I_1} U) \cap \left( \bigcap_{P \in X_1} H_P \right)$ is a Noetherian domain. Define $A = \bigcap_{U \in I_2} U$. Since $I_2$ has finite character and for all $V \in I_2$, we have $MV \neq V$, it follows from Remark 2.5 that $I_2$ is a finite set. Moreover, by Eq. (1), $E = A \cap B \cap R$. If $E$ is an intersection of an integrally closed Noetherian
overring of \( H \) and \( R \), then by Theorem 6.5, \( \text{Rep}^d_R(H) \) has no members that are irrational valuation rings, contrary to assumption. Hence \( A \) cannot be omitted from the intersection \( E = A \cap B \cap R \). Finally, if \( H = E \), then \( E/M \) is trivially a finitely generated \( H/M \)-algebra. On the other hand, if \( H \neq E \), then by Proposition 4.6, \( E/M \) is a Noetherian ring, and if also \( H \) has a finite character \( R \)-representation, then by Proposition 5.3, \( E/M \) is a finitely generated \( H/M \)-algebra.

(2) \( \Rightarrow \) (1) Define \( C = A \cap R \). Since \( A \) is an intersection of valuation rings of Krull dimension 1, \( A \) is a completely integrally closed domain, so by Remark 3.1, \( \text{End}(MA) = A \), which implies that \( \text{End}(M) \subseteq A \). Observe that \( \text{End}(M) \cap C = \text{End}(M) \cap A \cap R = \text{End}(M) \cap R = E \). By (2), \( E/M \) is a Noetherian ring and \( E = B \cap C \), where \( B \) is an integrally closed Noetherian domain. By Theorem 6.5 (with \( C \) playing the role of “\( R \)” in the theorem), \( H \) has a Noetherian \( C \)-representation and every element of \( \text{Rep}^d_C(H) \) has Krull dimension 2. Also, since \( C \) is a finite intersection of valuation rings with \( R \), \( H \) has a Noetherian \( R \)-representation. Moreover, by the theorem, if \( E/M \) is a finitely generated \( H/M \)-algebra, then \( H \) has a finite character \( R \)-representation.

It remains to prove that \( \text{Rep}^d_R(H) \) contains an irrational valuation ring. Suppose by the way of contradiction that no member of \( \text{Rep}^d_R(H) \) is an irrational valuation ring. Then by Theorem 6.5, \( E \) is an intersection of an integrally closed Noetherian domain and \( R \). Hence \( E \) has a finite character \( R \)-representation consisting of DVRs, and by Proposition 2.4(5), \( \text{Rep}_R(E) \) consists of DVRs. Yet \( A \) is a finite intersection of irrational valuation rings, and since \( A \) cannot be omitted from \( E = A \cap B \cap R \), it must be that one of these irrational valuation rings is an irredundant \( R \)-representative of \( E \). Since an irrational valuation ring has Krull dimension 1, it follows that \( E \) has a strongly irredundant \( R \)-representative that is an irrational valuation ring, a contradiction to the previous assertion that \( \text{Rep}_R(E) \) consists of DVRs. Therefore, we conclude that \( \text{Rep}^d_R(H) \) contains an irrational valuation ring. \( \square \)

Corollary 6.7. Suppose that \( H \) is a quasilocal domain that is not a valuation domain. Then the following statements are equivalent.

1. \( H \) has a Noetherian (finite character) \( R \)-representation.
2. \( E/M \) is a Noetherian ring (resp., finitely generated \( H/M \)-algebra) and \( E = A \cap B \cap R \), where \( B \) is a Noetherian integrally closed overring and \( A \) is either the quotient field of \( H \) or a finite intersection of irrational valuation overrings.

Proof. If \( H \) has a Noetherian (finite character) \( R \)-representation and \( H \) is not a valuation domain, then by Theorems 6.5 and 6.6, \( E/M \) is a Noetherian ring (resp., finitely generated \( H/M \)-algebra) and \( E = A \cap B \cap R \), where \( A \) is either the quotient field of \( H \) or a finite intersection of irrational valuation rings and \( B \) is a Noetherian integrally closed overring of \( H \). The converse also follows from Theorems 6.5 and 6.6. \( \square \)

7. Prime spectra and Noetherian \( R \)-representations

We turn now to an examination of \( \text{Spec}(H) \) when \( H \) has a Noetherian \( R \)-representation. In this case we pinpoint in Theorem 7.4 the prime ideals in \( \text{Spec}(H) \) that are contracted from prime ideals of valuation rings in \( \text{Rep}_R(H) \), and in Theorem 7.8 we account for all the remaining prime ideals of \( H \). In doing so we obtain the image of \( \text{Rep}_R(H) \) in \( \text{Spec}_R(H) \). This image turns out to be a special class of Zariski–Samuel associated prime ideals of principal ideals, a notion we review first.

Let \( A \) be a commutative ring, and let \( I \) be a proper ideal of \( A \). A prime ideal \( P \) of \( A \) is a weak Bourbaki associated prime ideal of \( I \) if there exists \( x \in A \) such that \( P \) is a minimal prime ideal of \( (I : A x) \). The prime ideal \( P \) is a Zariski–Samuel associated prime ideal of \( I \) if there exists \( x \in A \) such that \( P = \sqrt{I : A x} \). These are two among several possible choices for the notion of an associated prime for not-necessarily-Noetherian rings; see [5,11] and their references for background and applications of these classes of associated primes.

As usual, we want to relativize this notion for an extension ring \( B \) of \( A \). We say that \( P \) is a weak Bourbaki \( B \)-associated prime ideal of \( I \) if there exists \( x \in IB \cap A \) such that \( P \) is a minimal prime ideal of \( (I : A x) \). If in fact \( P = \sqrt{I : A x} \), then we say that \( P \) is a Zariski–Samuel \( B \)-associated prime ideal of \( I \). We are interested in the following special case: \( A \subseteq B \) are domains having common quotient field, and \( P \) is a weak Bourbaki or Zariski–Samuel \( B \)-associated prime ideal of some nonzero principal ideal of \( A \). This happens if and only if there exists \( 0 \neq b \in B \) such
that $P$ is a minimal prime ideal of $A \cap b^{-1}A$, or, in the Zariski–Samuel case, $P = \sqrt{A \cap b^{-1}A}$. It is these formulations that we use below.

For domains $A \subseteq B$, we define $wB_B(A)$ to be the set of all prime ideals $P$ of $A$ such that $P$ is a weak Bourbaki $B$-associated prime ideal of some nonzero principal ideal of $A$; that is,$$
 wB_B(A) = \{ P \in \text{Spec}(A) : P \text{ is a minimal prime ideal of } A \cap b^{-1}A \text{ for some } 0 \neq b \in B \}.
$$Similarly, we define:

$$
ZS_B(A) = \{ P \in \text{Spec}(A) : P = \sqrt{A \cap b^{-1}A} \text{ for some } 0 \neq b \in B \}.
$$

When $B$ is the quotient field of $A$, we omit the subscript and write $wB(A)$ and $ZS(A)$ instead of $wB_B(A)$ and $ZS_B(A)$. Turning now to our usual context where $H \subseteq R$ are integrally closed overrings of $D$, we single out in Lemma 7.1 a useful subcollection of $wB_R(H)$.

**Lemma 7.1.** There is a containment of sets:

$$
X_1 \cup \{ P \in \text{Spec}(H) : H \not\subseteq H_P \} \subseteq wB_R(H).
$$

If also $X_1$ has finite character (e.g., if $H$ has a Noetherian $R$-representation), then these two sets are equal.

**Proof.** Let $P \in X_1$. Then $R \not\subseteq H_P$, so there exists $0 \neq r \in R$ such that $H \cap r^{-1}H \subseteq P$. Since $P$ is a height 1 prime ideal of $H$, $P \in wB_R(H)$. Thus $X_1 \subseteq wB_R(H)$. Next suppose that $P$ is a prime ideal of $H$ such that $H \not\subseteq H_P$. Let $r \in H_1 \setminus H_P$. Then $I := H \cap r^{-1}H \subseteq P$ but $I \not\subseteq Q$ for any $Q \in X_1$. If $P$ is not a minimal prime ideal of $I$, then there exists a nonmaximal prime ideal $P_1$ of $H$ such that $H \cap r^{-1}H \subseteq P_1 \subseteq P$. But then $R \not\subseteq H_{P_1}$ (since $r \not\in H_{P_1}$), and by Proposition 2.3(4), $H_{P_1}$ is a DVR. This then implies that $P_1 \in X_1$, a contradiction to the fact that $I$ is contained in no member of $X_1$. Thus $P$ is a minimal prime ideal of $I$, and hence $P \in wB_R(H)$.

Now suppose that $X_1$ has finite character. Let $P \in wB_R(H)$, and suppose that $P \not\in X_1$. We claim that $H \not\subseteq H_P$. Suppose by the way of contradiction that $H \subseteq H_P$. Since $X_1$ has finite character, $\{ H_Q : Q \in X_1 \}$ is a finite character collection of DVRs. Hence by Proposition 2.4(1):

$$
\left( \bigcap_{Q \in X_1} (H_Q)_P \right) \cap R_P = (H_1)_P = H_P.
$$

Since each $Q \in X_1$ has height 1, $(H_Q)_P$ is the quotient field of $H$ when $Q \not\subseteq P$, or $(H_Q)_P = H_Q$ when $Q \subseteq P$. Thus

$$
\left( \bigcap_{Q \in X_1, Q \subseteq P} H_Q \right) \cap R_P \subseteq H_P.
$$

Since $P \in wB_R(H)$, there exists $0 \neq r \in R$ such that $P$ is a minimal prime ideal of $H \cap r^{-1}H$. Thus since $P \not\in X_1$, no member $Q$ of $X_1$ contained in $P$ contains $H \cap r^{-1}H$ (for otherwise $H \cap r^{-1}H \subseteq Q \subseteq P$ forces $P = Q \in X_1$, a contradiction). Therefore, $r \in (\bigcap_{Q \in X_1, Q \subseteq P} H_Q) \cap R$ but $r \not\in H_P$. This contradiction implies that $H \not\subseteq H_P$. \hfill $\Box$

**Lemma 7.2.** If $V \in \text{Rep}_R(H)$, then $m_V \cap H \in ZS_R(H)$.

**Proof.** Let $P = m_V \cap H$. By assumption there exists an integrally closed overruling $R_1$ of $H$ such that $R_1 \subseteq R$, $H = V \cap R_1$ and $V$ is strongly irredundant in this intersection. Let $Q$ be a prime ideal of $H$ with $P \not\subseteq Q$. Then $H_Q = V_Q \cap (R_1)_Q$, and since $m_V \cap H = P \not\subseteq Q$, it must be that $V_Q$ is a proper overruling of $V$. Let $V_1$ be the smallest proper overruling of $V$. Then we have $V_1 \cap R_1 \subseteq H_Q$ for every prime ideal $Q$ with $P \not\subseteq Q$. Since $V$ is strongly irredundant in $H = V \cap R_1$, we may choose $r \in (V_1 \cap R_1) \setminus V$. Then $r \in (\bigcap_{Q \subseteq P} H_Q) \setminus H_P$, so that if $Q$ is any prime ideal of $H$ with $H \cap r^{-1}H \subseteq Q$, then $P \subseteq Q$. Thus $P = \sqrt{H \cap r^{-1}H}$. \hfill $\Box$

In the proof of Theorem 7.4 we need a useful consequence of Proposition 2.4(1):
Lemma 7.3. Suppose that $H$ has a Noetherian $R$-representation. Let $S$ be a multiplicatively closed subset of $H$, and let $Y$ be a collection of multiplicatively closed subsets of $H$. Then

$$\left(\bigcap_{T \in Y} H_T\right) \cap R_S \subseteq \left(\bigcap_{T \in Y} (H_T)_S\right) \cap R_S.$$  

Proof. Let $\Sigma$ be a Noetherian $R$-representation of $H$, and for each $T \in Y$, define $\Sigma_T = \{V_T : V \in \Sigma\}$. Then for each $T \in Y$, Proposition 2.4(1) implies that $H_T = \left(\bigcap_{V \in \Sigma_T} V\right) \cap R_T$. Let $B = \bigcap_{T \in Y} H_T$. Then $B = \bigcap_{T \in Y} (V \in \Sigma_T) \cap R_T$. By Proposition 2.4(5), $\Sigma \cup \Sigma_1 \cup \{F\}$ is a Noetherian space (where as always $F$ is the quotient field of $H$), so necessarily $\bigcup_{T \in Y} \Sigma_T$, as a subspace of $\Sigma \cup \Sigma_1 \cup \{F\}$, is a Noetherian space. Then by Proposition 2.4(1),

$$B_S \cap R_S = \bigcap_{T \in Y} \left(\bigcap_{V \in \Sigma_T} V\right) \cap \left(\bigcap_{T \in Y} R_T\right) \cap R_S = \left(\bigcap_{T \in Y} (H_T)_S\right) \cap R_S.$$  

This proves the lemma. $\Box$

Theorem 7.4. Suppose that there exists a Noetherian $R$-representation of $H$, and let $\Sigma = \text{Rep}_R(H)$. Then:

$$ZS_R(H) = \text{wB}_R(H) = \{m_V \cap H : V \in \Sigma \cup \Sigma_1, R \not\subseteq V\}.$$  

Proof. We show that $ZS_R(H) \subseteq \text{wB}_R(H) \subseteq \{m_V \cap H : V \in \Sigma \cup \Sigma_1, R \not\subseteq V\} \subseteq ZS_R(H)$. The first inclusion is clear, so we verify the middle inclusion: $\text{wB}_R(H) \subseteq \{m_V \cap H : V \in \Sigma \cup \Sigma_1, R \not\subseteq V\}$. By Lemma 7.1 it suffices to prove that $X_1 \cup \{P \in \text{Spec}(H) : H_1 \not\subseteq H_P\} \subseteq \{m_V \cap H : V \in \Sigma \cup \Sigma_1, R \not\subseteq V\}$. By Lemma 7.1 it suffices to prove that $X_1 \cup \{P \in \text{Spec}(H) : H_1 \not\subseteq H_P\} \subseteq \{m_V \cap H : V \in \Sigma \cup \Sigma_1, R \not\subseteq V\}$. Let $P \in X_1$, then by Proposition 2.4(1), $H_P = \left(\bigcap_{V \in \Sigma} V\right) \cap R_P$, and since $R \not\subseteq H_P$ and $H_P$ is a DVR, $H_P = V_p$ for some $V \in \Sigma$. Hence there is $V \in \Sigma$ such that $H_P = V_p$, so that $P = p_V \cap H$. Since $R \not\subseteq H_P = V_p$, it follows that $X_1 \subseteq \{m_V \cap H : V \in \Sigma \cup \Sigma_1, R \not\subseteq V\}$.

Next we suppose that $P \in \text{Spec}(H)$ with $H_1 \not\subseteq H_P$, and we show that $P = m_V \cap H$ for some $V \in \Sigma \cup \Sigma_1$ with $R \not\subseteq V$. Suppose that no member $V$ of $\Sigma \cup \Sigma_1$ with $R \not\subseteq V$ is centered on $P$. Then for each $V \in \Sigma \cup \Sigma_1$ with $R \not\subseteq V$, we have either $V_P$ is the quotient field of $H$ or $V \not\subseteq V \subseteq P$. Since by assumption $P \neq m_V \cap H$, there are three cases to consider: (a) $V_P$ is the quotient field of $H$, (b) $m_V \cap H \not\subseteq P$, or (c) $P = p_V \cap H \not\subseteq m_V \cap H$. We observe that in all three cases, $H_1 \not\subseteq V_P$. For in case (b), by Proposition 2.3(4), $V = H_{m \cap H}$ and $V$ is a DVR, so that $m_V \cap H \not\subseteq V_P$, whence $H_1 \not\subseteq H_{m \cap H} \not\subseteq V_P$. In case (c), by Proposition 2.3(4), since $P$ is a nonmaximal prime ideal, $H_P$ is a DVR, and $H_P = V_{p_V}$. Also, since $P \not\subseteq m_V \cap H$, we conclude that $V_P = V_{p_V} = H_P$. Hence in either case (a), (b) or (c), it must be that $H_1 \not\subseteq V_P$. Therefore, if no $V \in \Sigma \cup \Sigma_1$ with $R \not\subseteq V$ is centered on $P$, it must be that $H_1 \not\subseteq \bigcap_{V \in \Sigma} V_P$, which, since by Proposition 2.4(1), $H_P = \left(\bigcap_{V \in \Sigma} V_P\right) \cap R_P$, implies that $H_1 \not\subseteq H_P$, a contradiction. Thus $P$ is the center of some $V \in \Sigma \cup \Sigma_1$ with $R \not\subseteq V$. This proves that $\text{wB}_R(H) \subseteq \{m_V \cap H : V \in \Sigma \cup \Sigma_1, R \not\subseteq V\}$.

Finally, we verify the last inclusion: $\{m_V \cap H : V \in \Sigma \cup \Sigma_1, R \not\subseteq V\} \subseteq ZS_R(H)$. Let $P = m_V \cap H$ for some $V \in \Sigma \cup \Sigma_1$ with $R \not\subseteq V$. If $P$ is a maximal ideal of $H$, then we may as well assume $V \in \Sigma$. Hence by Lemma 7.2, $P \in ZS_R(H)$. Otherwise, suppose that $P$ is a nonmaximal prime ideal of $H$. Then by Proposition 2.3(4), $H_P$ is a DVR, so that $H_P = V_{p_V}$; hence, $R \not\subseteq H_P$. Let $Y = \{Q \in \text{Spec}(H) : P \subseteq Q\}$. We claim that there exists some $r \in \left(\bigcap_{Q \in Y} H_Q\right) \cap R$, for then $P = \sqrt{H \cap r^{-1}H}$, so that $P \in ZS_R(H)$. Suppose that $\left(\bigcap_{Q \in Y} H_Q\right) \cap R \subseteq H_P$. Then by Lemma 7.3:

$$\left(\bigcap_{Q \in Y} (H_Q)_P\right) \cap R_P \subseteq H_P.$$  

For each $Q \in Y$, since $P \not\subseteq Q$ and $P$ is a height 1 ideal of $H$, it follows that $(H_Q)_P$ is the quotient field of $H$. Therefore, by (2), $R \subseteq H_P$, a contradiction. We conclude that $P \in ZS_R(H)$. $\Box$

Corollary 7.5. Suppose that $H$ has a Noetherian R-representation. If $P$ is a prime ideal of $H$ not in $ZS_R(H)$, then $H_P = C \cap R_P$ for some integrally closed Noetherian overring $C$ of $H$.  

By Theorem 7.4, $P \notin \text{wB}_R(H)$, so by Lemma 7.1, $H_P = (H_1)_P$. By Lemma 6.2, $H_1 = B \cap R$ for an integrally closed Noetherian overring $B$ of $H$. Thus $H_P = (H_1)_P = B_p \cap R_P$, and the corollary follows by setting $C = B_p$.

**Corollary 7.6.** Suppose $H$ has a Noetherian representation, and that $P$ is a height 2 prime ideal of $H$. Then $H_P$ is a Noetherian ring if and only if $P \notin \text{ZS}(H)$.

**Proof.** Suppose that $H_P$ is a Noetherian ring and there exists $0 \neq x$ in the quotient field of $H$ with $P = \sqrt{H \cap x^{-1}H}$. Since $H_P$ is a Krull domain, $H_P = \bigcap_{Q \subseteq P} H_Q$, where $Q$ ranges over the prime ideals of $H$ such that $Q \subseteq P$ and $H_Q$ is a DVR. So since $x \notin H_P$, we have $x \notin H_Q$ for some height 1 prime ideal $Q$ of $H$. But then $H \cap x^{-1}H \subseteq Q$, a contradiction to the assumption that $P = \sqrt{H \cap x^{-1}H}$ has height 2. Therefore, $P \notin \text{ZS}(H)$. The converse is a consequence of Corollary 7.5 (with $R = F$ in the corollary).

**Corollary 7.7.** If there exists a Noetherian $R$-representation of $H$, then $\text{ZS}_R(H)$ is a Noetherian subspace of $\text{Spec}(H)$. If also there exists a finite character $R$-representation of $H$, then $\text{ZS}_R(H)$ is a finite character collection of prime ideals.

**Proof.** By Proposition 2.4(5), $\text{Rep}_R(H) \cup \{V_p : V \in \text{Rep}_R(H)\}$ is a Noetherian subspace of $\text{Zar}(H)$, so since the mapping $\text{Zar}(H) \rightarrow \text{Spec}(H) : V \mapsto m_V \cap H$ is continuous [20, Lemmas 1 and 4, pp. 116-117], Theorem 7.4 implies that $\text{ZS}_R(H)$ is a Noetherian subspace of $\text{Spec}(H)$. Similarly, if $H$ has a finite character $R$-representation, then $\text{ZS}_R(H)$ has finite character.

Even when $R$ is the quotient field of $H$, it need not be the case that $\text{Spec}(H) = \{0\} \cup \text{ZS}_R(H)$. (For example, if $A$ is a local Noetherian domain of Krull dimension 2, then $\text{ZS}(A)$ consists of the nonmaximal prime ideals of $A$.) However, using the ring $H_1$, we can account for the other prime ideals of $H$:

**Theorem 7.8.** If there exists a Noetherian $R$-representation of $H$, then

$$\text{Spec}(H) = \text{ZS}_R(H) \cup \{Q \cap H : Q \in \text{Spec}(H_1)\}.$$ 

**Proof.** Let $P \in \text{Spec}(H) \setminus \text{ZS}_R(H)$, and let $\Sigma = \text{Rep}_R(H)$. We claim that $P$ is contracted from a prime ideal of $H_1$. Now $H = \bigcap_{V \in \Sigma} V \cap H_1$, so by Proposition 2.4(1),

$$H_P = \left(\bigcap_{V \in \Sigma} V_P\right) \cap (H_1)_P. \tag{3}$$

Suppose first that $P$ is a nonmaximal prime ideal of $H$. Then by Proposition 2.3(4), $H_P$ is a DVR, so by Eq. (3), $H_P = V_P$ for some $V \in \Sigma$ or $H_P = (H_1)_P$. In the latter case, $P$ is contracted from a prime ideal of $H_1$, as claimed, so suppose that $H_P \neq (H_1)_P$ (and in particular, $H_1 \not\subseteq H_P$). Then $H_P = V_P$ for some $V \in \Sigma$, and since $H_P$ is a DVR, we have $H_P \in \Sigma_1$. Also, $R \not\subseteq H_P$ since $H_1 \subseteq R$ and $H_1 \not\subseteq H_P$. But then $P$ is contracted from a prime ideal of a valuation ring $V \in \Sigma \cup \Sigma_1$ with $R \not\subseteq V$, which by Theorem 7.4 is contrary to the assumption that $P \notin \text{ZS}_R(H)$. We conclude that $P$ is contracted from a prime ideal of $H_1$.

Still assuming that $P \notin \text{ZS}_R(H)$, suppose next that $P$ is a maximal ideal of $H$. Since by Theorem 7.4, $P$ is a maximal ideal of $H$ not contracted from a prime ideal of a member of $\Sigma$, we have for each $V \in \Sigma$, $V_P$ is the quotient field of $H$ or $p_V \cap H$ is properly contained in $P$. In the latter case, we have by Proposition 2.3(4) that $V_P = V_p = H_P \cap H$ and $H_p \cap H$ is a DVR, so that $H_1 \subseteq V_P$. Thus for every $V \in \Sigma$, $H_1 \subseteq V_P$, and we conclude from Eq. (3) that $H_P = (H_1)_P$. Therefore, $P$ is contracted from a prime ideal of $H_1$.

If $H$ has a Noetherian $R$-representation, then by Proposition 2.4(1) and (5), $H_M$ has a Noetherian $R_M$-representation for each maximal ideal $M$ of $H$. Moreover, by Theorem 7.4 and Corollary 7.7, $\text{wB}_R(H)$ is a Noetherian space. In the next theorem we show that the converse holds when “Noetherian” is replaced by “finite character.” We do not whether the converse holds for the more general case of Noetherian representations.

**Theorem 7.9.** There exists a finite character $R$-representation of $H$ if and only if $\text{wB}_R(H)$ is a finite character set of prime ideals and for each maximal ideal $N$ of $H$ in $\text{wB}_R(H)$, $H_N$ has a finite character $R_N$-representation.
\textbf{Proof.} Let \( Y = \text{Max}(H) \cap wB_R(H) \), and suppose that there exists a finite character \( R \)-representation \( \Sigma \) of \( H \). By \textbf{Proposition 2.4}, \( \text{Rep}_R(H) \subseteq \Sigma \cup \Sigma_1 \) and \( \text{Rep}_R(H) \) is an \( R \)-representation of \( H \). Thus since \( \Sigma \cup \Sigma_1 \) has finite character, \( \text{Rep}_R(H) \) is a finite character \( R \)-representation of \( H \). Hence by \textbf{Proposition 2.4}(1), for each \( N \in Y \), \( \{V_N : V \in \text{Rep}_R(H)\} \) is a finite character \( R_N \)-representation of \( H_N \). Moreover, by \textbf{Theorem 7.4} and \textbf{Corollary 7.7}, \( wB_R(H) \) is a finite character set of prime ideals of \( H \).

Conversely, suppose that \( wB_R(H) \) has finite character, and for each \( N \in Y \), \( H_N \) has a finite character \( R_N \)-representation. We have by \textbf{Lemma 7.1} that \( Y_0 := \{N \in \text{Max}(H) : H_1 \not\subseteq H_N\} \subseteq wB_R(H) \). For each \( N \in \text{Max}(H) \), let \( X(N) = \{P \in X : P \subseteq N\} \). We claim that for each \( N \in Y_0 \):

\[
H_N = \left( \bigcap_{V \in \text{Rep}_R^d(H_N)} V \right) \cap \left( \bigcap_{P \in X(N)} H_P \right) \cap R_N. \tag{4}
\]

Let \( N \in Y_0 \). Since \( H_1 \subseteq R \) and \( H_1 \not\subseteq H_N \), it must be that \( R_N \not\subseteq H_N \). Thus it is clear that if \( H_N \) is a valuation ring, then \( H_N \in \text{Rep}_R^d(H_N) \), and therefore, when \( H_N \) is a valuation ring, \( H_N \) trivially satisfies Eq. (4). Otherwise, suppose that \( H_N \) is not a valuation ring. In case \( \text{Rep}_R(H_N) \) is empty, then by \textbf{Lemma 6.4}(1), \( \text{Rep}_R^d(H_N) = \{H_P : P \in X(N)\} \), so that \( H_N = \bigcap_{P \in X(N)} H_P \cap R_N \), in which case Eq. (4) holds. On the other hand, if \( \text{Rep}_R(H_N) \) is not empty, then we may apply \textbf{Lemma 6.4}(3) to the pair \( H_N \subseteq R_N \) to obtain the equality in (4). (Note that in this application of \textbf{Lemma 6.4}(3) to \( H_N \), the set “\( X_1 \)” in the statement of the lemma is \( \{P_N : P \in \text{Spec}(H) \subseteq N, H_P \text{ is a DVR and } R_N \not\subseteq H_P\} = \{P_N : P \in X(N)\} \).

Now for each maximal ideal \( N \), since \( X(N) \subseteq X_1 \), we have \( H_1 \subseteq \bigcap_{P \in X(N)} H_P \). Also, since \( H = \bigcap_{N \in \text{Max}(H)} H_N \), it follows that \( H = (\bigcap_{N \in Y_0} H_N) \cap H_1 \). Therefore, having established Eq. (4), we have that since \( H_1 \subseteq \bigcap_{P \in X(N)} H_P \) for all \( N \in \text{Max}(H) \), the set

\[
\Sigma := \bigcup_{N \in Y_0} \text{Rep}_R^d(H_N)
\]

is an \( H_1 \)-representation of \( H \). We claim in fact that \( \Sigma \) is a finite character \( H_1 \)-representation of \( H \). Let \( 0 \neq x \in H \). Then since \( Y_0 \) has finite character, \( x \) is a member of only finitely many members of \( Y_0 \). Moreover, if \( N \in Y_0 \) and \( V \in \Sigma \) such that \( NV \neq V \), then necessarily \( V \in \text{Rep}_R^d(H_N) \). Thus, since by \textbf{Remark 2.5}, \( \text{Rep}_R^d(H_N) \) is finite, it follows that there are at most finitely many members of \( \Sigma \) centered on \( N \). Consequently, \( x \) is a nonunit in at most finitely many members of \( \Sigma \). Therefore, \( \Sigma \) is a finite character collection of valuation overrings of \( H \).

Finally, we claim that \( H \) has a finite character \( R \)-representation. Indeed, since \( \Sigma \) is a finite character \( H_1 \)-representation of \( H \), it suffices to note that \( H_1 \) has a finite character \( R \)-representation \( \Gamma \), for this then implies that \( \Sigma \cup \Gamma \) is a finite character \( R \)-representation of \( H \). By \textbf{Lemma 7.1}, \( X_1 \subseteq wB_R(H) \), so \( X_1 \), as a subset of a finite character collection, has finite character. Hence \( \Gamma := \{H_P : P \in X_1\} \) is a finite character collection of DVR overrings of \( H \). Moreover, \( \Gamma \) is (by the definition of \( H_1 \)) an \( R \)-representation of \( H_1 \), so the claim is proved.

\section{Noetherian representations}

Recall that an overring \( H \) of \( D \) has a \textit{Noetherian representation} if there exists a Noetherian subspace \( \Sigma \) of \( \text{Zar}(H) \) such that \( H = \bigcap_{V \in \Sigma} V \). Thus a Noetherian representation of \( H \) is a Noetherian \( R \)-representation, where \( R \) is the quotient field of \( D \). In the previous section, taking \( R \) to be the quotient field of \( D \), we can deduce characterizations of overrings of \( D \) having Noetherian representations. In this section we consider further such rings. Applying the analysis of prime ideals of \( H \) from the last section, we obtain:

\textbf{Theorem 8.1.} If there exists a Noetherian representation of \( H \), then \( \text{Spec}(H) \) is a Noetherian space.

\textbf{Proof.} In the setting of \textbf{Corollary 7.7}, \( ZS_R(H) \) is a Noetherian subspace of \( \text{Spec}(H) \). Thus if \( \text{Spec}(H) \) is a Noetherian space, then since the contraction mapping \( \text{Spec}(H_1) \to \text{Spec}(H) \) is continuous and the union of two Noetherian subspaces is a Noetherian space, we conclude from \textbf{Theorem 7.8} that \( \text{Spec}(H) \) is a Noetherian space. In particular, if \( H \) has a Noetherian representation, then, taking \( R \) to be the quotient field of \( H \), we have by \textbf{Lemma 6.2} that \( H_1 \) is a Noetherian ring. This proves the theorem. \( \square \)
In contrast to Theorem 8.1, if there exists a finite character representation of $H$, it need not be the case that Spec$(H)$ has finite character. For example, consider the case where $D = H = K[X, Y]$, with $K$ a field and indeterminates $X$ and $Y$.

**Corollary 8.2.** If $H$ has a Noetherian representation and $J$ is a nonzero radical ideal of $H$, then $H/J$ is a Noetherian ring.

**Proof.** By Theorem 8.1, Spec$(H)$ is a Noetherian space. Thus $J$ has finitely many minimal prime ideals $P_1, \ldots, P_n$ [15]. Since $H/J$ is a subdirect sum of the ring $H/P_1 \times \cdots \times H/P_n$, and each factor in this product is a Noetherian ring (Proposition 2.3(3)), it follows that $H/J$ is a Noetherian ring [14, (3.16), p. 11].

**Corollary 8.3.** The ring $H$ has a Noetherian representation if and only if End$(M)$ has a Noetherian representation.

**Proof.** Suppose that $H$ has a Noetherian representation $\Sigma$. Then by Proposition 2.4(5), $\Sigma \cup \Sigma_1$ is a Noetherian representation of $H$, and thus from Proposition 3.6, it follows that End$(M)$ has a Noetherian representation. Conversely, suppose that End$(M)$ has a Noetherian representation $\Gamma$. By Lemma 3.4, $M$ is a radical ideal of End$(M)$, so by Corollary 8.2, End$(M)/M$ is a Noetherian ring. Thus by Proposition 4.6, $H$ has a Noetherian End$(M)$-representation $\Delta$. Since the union of two Noetherian subspaces is a Noetherian space, $\Gamma \cup \Delta$ is a Noetherian representation of $H$.

In the next theorem we classify the quasilocal overrings of $D$ having a Noetherian representation. Comparing this to Corollary 6.7, we see that the requirement in the corollary that $E/M$ is a Noetherian ring is redundant in our present context.

**Theorem 8.4.** Suppose that $H$ is quasilocal. Then $H$ has a Noetherian representation if and only if $H$ is a valuation domain, a Noetherian domain or End$(M) = A \cap B$, where $B$ is a Noetherian overring of $H$ and $A$ is either the quotient field of $H$ or a finite intersection of irrational valuation overrings.

**Proof.** If $H$ has a Noetherian representation, and $H$ is not a valuation domain or a Noetherian domain, then by Corollary 6.7, End$(M) = A \cap B$, where $B$ is a Noetherian overring of $H$ and $A$ is either the quotient field of $H$ or a finite intersection of irrational valuation overrings. Conversely, End$(M)$ has a Noetherian representation, so by Corollary 8.3, $H$ has a Noetherian representation.

In the next theorem, we show that it is possible to distinguish among quasilocal overrings of $D$ having a Noetherian representation the three possibilities for Rep$_d^d(H)$ arising in Theorems 6.5 and 6.6:

**Theorem 8.5.** Suppose that $H$ is quasilocal and $H$ has a Noetherian representation, but $H$ is neither a Noetherian domain nor a valuation domain. Then every valuation ring in Rep$_d^d(H)$ is irrational or has Krull dimension 2. Moreover:

1. Every member of Rep$_d^d(H)$ is an irrational valuation ring if and only if $H$ is completely integrally closed.
2. Every member of Rep$_d^d(H)$ is a valuation ring of Krull dimension 2 if and only if End$(M)$ is a Noetherian domain.
3. Rep$_d^d(H)$ contains both irrational valuation rings and valuation rings of Krull dimension 2 if and only if End$(M) \neq H$ and End$(M)$ is not a Noetherian domain.

**Proof.** We may assume that $R = F$ (where as always $F$ denotes the quotient field of $D$), so that $H_1 = \bigcap_P H_P$, where $P$ ranges over the prime ideals of $H$ such that $H_P$ is a DVR. By Lemma 6.4(3), Rep$_d^d(H)$ is an $H_1$-representation of $H$, and by Lemma 6.2, $H_1$ is an integrally closed Noetherian domain. Also, by Lemma 6.4(5), every member of Rep$_d^d(H)$ is irrational or has Krull dimension 2.

1. If Rep$_d^d(H)$ contains only irrational valuation rings, then since Rep$_d^d(H)$ is an $H_1$-representation of $H$ and $H_1$ is an integrally closed Noetherian domain, it is the case that $H$ is an intersection of completely integrally closed overrings, and hence is completely integrally closed. Conversely, if $H$ is a completely integrally closed domain, then by Remark 3.1, End$(M) = H$. By Lemma 6.4(1), Rep$(H) \setminus$ Rep$_d^d(H)$ consists of DVRs, and by Proposition 3.6, $H$ is the intersection of the valuation rings in $\{V_{P_v} : V \in$ Rep$(H)\}$. However, by Proposition 2.4(1), Rep$(H)$ is a strongly irredundant representation of $H$. This forces Rep$(H) = \{V_{P_v} : V \in$ Rep$(H)\}$, so that Rep$_d^d(H)$ consists of valuation overrings of $H$ of Krull dimension 1. Thus by Lemma 6.4(5), Rep$_d^d(H)$ contains only irrational valuation rings.
(2) This is a consequence of Theorem 6.5.
(3) Suppose that \( \text{Rep}^d(H) \) contains both irrational valuation rings and valuation rings of Krull dimension 2. By Proposition 2.4(3), \( \text{Rep}(H) \) is a strongly irredundant representation of \( H \), so since \( \text{Rep}^d(H) \) contains members of Krull dimension 2, we have by Proposition 3.6 that \( H \neq \text{End}(M) \). Moreover, by (2), \( \text{End}(M) \) is not a Noetherian domain. The converse of (3) follows from (1) and (2) and the fact that if \( H \) is a completely integrally closed domain, \( \text{End}(M) = H \) (Remark 3.1). \( \square \)

As a special case, we classify next the one-dimensional quasilocal overrings of \( D \) having a Noetherian representation.

**Theorem 8.6.** Suppose that \( H \) is not a valuation ring. Then the following statements are equivalent.

1. \( H \) is a quasilocal domain of Krull dimension 1 that has a Noetherian representation.
2. There exists a hidden prime divisor \( U \) of \( D \) such that \( H = \bigcap_{V \subseteq U} V \), where \( V \) ranges over the valuation overrings of \( D \) contained in \( U \).
3. There exists a hidden prime divisor \( U \) of \( D \) such that \( H \) is the integral closure of the ring \( D + m_U \) in its quotient field.

**Proof.** In the proof we use the following observation: (\( \dagger \)) If \( U \) is a valuation overring of a domain \( A \) and \( U \) has Krull dimension 1, then for every valuation overring \( V \) of \( A \) that is not a field, \( m_U \subseteq V \) if and only if \( V \subseteq U \). For it is clear that if \( V \subseteq U \), then \( m_U \subseteq V \). Conversely, suppose that \( V \) is a valuation overring of \( A \) that is not field such that \( m_U \subseteq V \). If \( m_U \subseteq V \), then \( U = \text{End}(m_U) \subseteq V \), but since \( U \) has Krull dimension 1, this forces \( U = V \), a contradiction to \( m_U \subseteq V \). Hence \( m_U \subseteq m_V \). Now if \( x \in V \setminus U \), then \( x^{-1} \in m_U \subseteq m_V \), so that \( m_V \) contains a unit of \( V \), a contradiction. Therefore, \( V \subseteq U \).

(1) \( \Rightarrow \) (2) Suppose that (1) holds. By Proposition 2.4, \( \Sigma := \text{Rep}(H) \) is the unique strongly irredundant Noetherian representation of \( H \) and \( \Sigma_1 \) has finite character. Since \( M \) is the only nonzero prime ideal of \( H \), we have that \( MV \neq V \) for all \( V \in \Sigma_1 \). Thus, since \( \Sigma_1 \) has finite character, \( \Sigma_1 \) is a finite set (Remark 2.5). Since \( H \) is a quasilocal ring of Krull dimension 1, we have by Proposition 3.6 that \( \text{End}(M) = U_1 \cap \cdots \cap U_n \), where \( \Sigma_1 = \{ U_1, \ldots, U_n \} \). A consequence of [9] is that if a quasilocal one-dimensional domain \( A \) has an irredundant representative \( U \), then \( A = U \). Thus if one of the rings \( U_i \) is in \( \Sigma \), then \( U_i \) is an irredundant representative of \( H \) of Krull dimension 1, so \( H \) is a quasilocal ring of Krull dimension 1, \( H = U_i \), contrary to the assumption that \( H \) is not a valuation ring. Thus each \( U_i \) is in \( \Sigma_1 \setminus \Sigma \) and hence is by (2.1) a prime divisor of \( D \). Hence \( \text{End}(M) \), as a finite intersection of DVRs, is a PID [14, (11.11), p. 38]. By Proposition 4.6, \( \text{End}(M)/M \) is a reduced indecomposable ring, so since \( \text{End}(M) \) is a PID, \( M \) must be a height 1 maximal ideal of \( \text{End}(M) \). Applying Lemma 4.3(1), we obtain \( n = 1 \), \( \text{End}(M) = U_1 \) and \( M = m_{U_1} \).

Let \( H' = \bigcap_{V \subseteq U_1} V \), where \( V \) ranges over the valuation rings in Zar\( \text{ar}(D) \) that are subrings of \( U_1 \). We show that \( H = H' \). If \( V \subseteq U_1 \) is a valuation overring of \( D \) and \( H \subseteq V \), then by Proposition 2.7(c), since \( H \) has Krull dimension 1, it must be that \( H = U_1 \), contrary to the assumption that \( H \) is not a valuation ring. Hence \( H \subseteq H' \). The reverse inclusion also holds since each valuation overring \( V \) of \( H \) contains \( M = m_{U_1} \), and hence if not a field is by (\( \dagger \)) a subring of \( U_1 \). This proves that \( H = H' \).

Finally, we claim that \( U_1 \) is a hidden prime divisor of \( D \). For otherwise, \( \overline{D_D} = U_1 \) for some height 1 prime ideal \( P \) of \( \overline{D} \), so that \( M = P \overline{D_D} \) and the one-dimensional Noetherian ring \( \overline{D + M}/M \) has quotient field \( U_1/M \). But then since \( H/M \) is a field, this forces \( H/M = U_1/M \), and hence \( H = U_1 \), contrary to the assumption that \( H \) is not a valuation ring.

Therefore, \( U_1 \) is a hidden prime divisor of \( D \), and statement (2) holds.

(2) \( \Rightarrow \) (3) Let \( A \) be the integral closure of \( D + m_U \) in its quotient field. Clearly, \( A \subseteq H \), since \( H \) is an integrally closed domain. If \( V \) is any valuation overring of \( A \), then \( m_U \subseteq V \), so if \( V \) is not a field, then \( V \subseteq U \) by (\( \dagger \)). Thus \( H \subseteq V \), and it follows that \( A = H \), since \( A \) is the intersection of all its valuation overrings.

(3) \( \Rightarrow \) (1) By (3), there exists a hidden prime divisor \( U \) of \( D \) such that \( H \) is the integral closure of the ring \( D + m_U \) in its quotient field. From (\( \dagger \)) it follows that every valuation overring of \( D \) containing \( H \) is contained in \( U \). Hence, since \( H \) is an integrally closed domain, is an intersection of its valuation overrings, \( \Sigma := \{ V \in \text{Zar}(D) : V \subseteq U \} \) is a representation of \( H \). From Proposition 2.6 we obtain that this representation is Noetherian. Next we show that \( H \) has Krull dimension 1. Since \( U \) is a hidden prime divisor of \( D \), \( m_U \) is a maximal ideal of \( D + m_U \). Indeed, \( (D + m_U)/m_U \cong D/(m_U \cap D) \), and by (2.2), \( m_U \cap D \) is necessarily a maximal ideal of \( D \). Thus \( (D + m_U)/m_U \) is a field, and \( m_U \) is a maximal ideal of \( D + m_U \). Therefore, by Lemma 4.8, since \( U/m_U \) is a field, we have that \( m_U \) is a maximal ideal of \( H \). Suppose that there exists a nonzero nonmaximal prime ideal \( P \) of \( H \). Then by Proposition 2.3(4),
Lemma 8.7. Suppose that there exists a strongly irredundant representative $V$ of $H$ such that $V$ is not a DVR and $M = m_V \cap H$. If some nonzero prime ideal of $H_M$ is a finitely generated ideal, then $M H_M$ is a finitely generated ideal of $H_M$ and $H_M = V$.

Proof. Write $H = V \cap R_1$, where $R_1$ is an integrally closed overring of $H$ and $V$ is strongly irredundant in this intersection. Since $V$ is irredundant in $H = V \cap R_1$, the ring $V$ is not a field. Thus by (2.1), $V$ has Krull dimension 1 or 2. Moreover, $H_M = V \cap (R_1)_M$, and by Proposition 2.7(b), $V$ is strongly irredundant in this intersection. Thus we assume without loss of generality that $H$ is quasilocal with maximal ideal $M$.

If $H$ has Krull dimension 1, then necessarily $M$ is a finitely generated ideal of $H$ and $H$ is an integrally closed Noetherian ring, hence a DVR. Thus we assume that $H$ has Krull dimension 2, and we claim that the maximal ideal $M$ of $H$ is a finitely generated ideal. By assumption some nonzero prime ideal $P$ of $H$ is finitely generated. If $M = P$, then obviously $M$ is a finitely generated ideal of $H$, so suppose that $P \neq M$. By Proposition 2.3(3), $H/P$ is a Noetherian domain. Consequently, $M/P$ is a finitely generated ideal of $H/P$, and since $P$ is a finitely generated ideal of $H$, it follows that $M$ is a finitely generated ideal of $H$.

Next, we show that $V = H_M$. In Theorem 1.1 of [10], Heinzer and Ohm prove that if $A$ is a domain, $U$ is a valuation overring of $A$ of Krull dimension 1 and $B$ is an overring of $A$ such that $A = U \cap B$ and $U$ is irredundant in this intersection, then either $U$ is a DVR or $A$ contains elements of arbitrarily small positive value with respect to the valuation corresponding to $U$. In other words, if $U$ is not a DVR, then $m_U = (m_U \cap A)U$. We apply this to the valuation ring $V$ in the following way. If $V$ has Krull dimension 1, then since $V$ is not a DVR, we have by the result of Heinzer and Ohm that $MV$ is the maximal ideal of $V$. As by assumption, $M$, whence $MV$, is finitely generated, it must be principal since $V$ is a valuation ring. But this is an impossibility since $V$ is not a DVR. On the other hand, if $V$ has Krull dimension 2, then by Proposition 2.7(d), $V = H_M$ or $\text{End}(MV) \neq V$. But the latter case is impossible since $M$ is finitely generated and $V$ is a valuation ring. Thus $V = H_M$. □

Lemma 8.8. Suppose that $H$ is a quasilocal ring, and there exists a Noetherian $R$-representation of $H$. If there exists a finitely generated height 1 prime ideal of $H$, then $H = H_1$.

Proof. If $H$ is a valuation ring, then $H$ is a DVR, since a nonzero finitely generated prime ideal of a valuation must be a maximal ideal. Suppose that $H$ is not a valuation ring and $H_1 \neq H$. Then $\text{Rep}_H(H)$ is nonempty, and by Lemma 6.4, $\text{Rep}_H^d(H)$ is a strongly irredundant Noetherian $H_1$-representation of $H$, and every member of $\text{Rep}_H^d(H)$ is irrational or has Krull dimension 2. But then Lemma 8.7 implies that $H$ is a valuation ring, contrary to the assumption. Thus $H = H_1$. □

Theorem 8.9. Suppose that $H$ is quasilocal, $H$ is not a field and $H$ has a Noetherian representation. Then $H$ is a Noetherian domain if and only if $H$ has a height 1 finitely generated prime ideal.

Proof. By Lemma 6.2, the ring $\bigcap_p H_p$, where $P$ ranges over the prime ideals of $H$ such that $H_P$ is a DVR, is a Noetherian domain. Thus the theorem follows from Lemma 8.8 (where we set $R$ in the lemma to be the quotient field $F$ of $H$). □

9. Countable Hilbert Noetherian domains

We wish now to use countable Hilbert Noetherian domains of Krull dimension 2 to contrast finite character representations in the quasilocal and global cases. In the quasilocal case an overring $H$ of $D$ that has a finite character $R$-representation can be decomposed into an intersection of a Noetherian domain, a Prüfer domain and $R$. (An integral
domain $A$ is a Prüfer domain if for each maximal ideal $M$ of $A$, $A_M$ is a valuation ring.) Of course when $R$ is the quotient field of $H$, this decomposition is particularly nice, since then $R$ is unnecessary. We state this now more formally:

**Proposition 9.1.** Suppose that $H$ has nonzero Jacobson radical. Then there exists a finite character $R$-representation of $H$ if and only if $H = A \cap B \cap R$, where $A$ is an integrally closed Noetherian overring and $B$ is a Prüfer overring having finitely many prime ideals.

**Proof.** Let $J$ be the Jacobson radical of $H$, and suppose that there exists a finite character $R$-representation $\Sigma$ of $H$. If $V \in \Sigma$ such that $J V = V$, then $m_V \cap H$ is a nonmaximal prime ideal of $H$, so that by Proposition 2.3(2), $V = H_{m_V \cap H}$ and $V$ is a DVR. If $R \subseteq V$, then $m_V \cap H \in X_1$, so that $H_1 \subseteq V$. Thus in all cases $H_1 \subseteq V$, and it follows that $I := \{ V \in \Sigma : J V \neq V \}$ is an $H_1$-representation of $H$.

Let $B = \bigcap_{V \in I} V$. Since $\Sigma$ has finite character, $I$ is by Remark 2.5 a finite set. Thus $B$, as a finite intersection of valuation rings of Krull dimension $\leq 2$, is a Prüfer domain having finitely many prime ideals [14, (11.11), p. 38]. Finally, by Lemma 6.2, $A := \bigcap_{P \in X_1} H_P$ is an integrally closed Noetherian domain. Since $H = A \cap B \cap R$, the claim is proved. The converse follows from the fact that $A$ and $B$, hence $A \cap B$, each have a finite character representation. □

This proposition raises the question of whether every overring $H$ of $D$ having a finite character $R$-representation can be written as an intersection of a Prüfer overring, a Noetherian overring and $R$. We show in Corollary 9.4 that this is not the case. First, we need a simple lemma. Recall that an integral domain is a Hilbert domain if every prime ideal is an intersection of maximal ideals.

**Lemma 9.2.** Let $A$ be a Hilbert domain, and let $p_1$, $p_2$, $p_3$, . . . be a sequence of incomparable nonmaximal prime ideals of $A$. Then for any maximal ideal $n$ of $A$, there exists a sequence of distinct maximal ideals $m_1$, $m_2$, $m_3$, . . . different from $n$ such that for every $i > 0$, $p_i \subseteq m_i$, but for every $j > i$, $p_j \not\subseteq m_j$.

**Proof.** For each radical ideal $j$ of $A$, let $V(j)$ denote the set of maximal ideals of $A$ that contain $j$. Since $A$ is a Hilbert domain, $j = \bigcap_{m \in V(j)} m$. Thus: ($\ast$) if $j_1$ and $j_2$ are radical ideals of $R$ such that $V(j_2) \subseteq V(j_1)$, then $j_1 \subseteq j_2$. Let $m_1$ be a maximal ideal distinct from $n$ containing $p_1$, and for each $i > 1$, let $m_i \in V(p_i) \setminus (V(p_1 p_2 \cdots p_{i-1}) \cup \{n\})$. (Such a maximal ideal $m_i$ exists by ($\ast$) and the assumption that $p_1$, . . . , $p_i$ are incomparable nonmaximal prime ideals.) Thus for each $i > 1$, the only maximal ideals in $\{ m_i : i > 0 \}$ possibly containing $p_i$ are $m_1$, $m_2$, . . . , $m_i$. □

If $A$ is a domain and $\Sigma$ is a representation of $A$, then we say $\Sigma$ is an essential representation of $A$ if for each $V \in \Sigma$, $V = A p$ for some prime ideal $P$ of $A$.

**Proposition 9.3.** Let $A$ be a countable integrally closed Noetherian Hilbert domain such that each maximal ideal of $A$ has height 2. Then there exists an overring $B$ of $A$ having a finite character essential representation consisting of valuation rings of Krull dimension 2, and such that $B$ is not a Prüfer domain and $B$ cannot be represented as an intersection of two overrings properly containing $B$, one of whose integral closure is completely integrally closed.

**Proof.** Let $n$ be a maximal ideal of $A$. Since $A$ is a countable Noetherian ring, the set of all ideals of $A$ is countable, and we may write the height one prime ideals of $A$ as a sequence $p_1$, $p_2$, $p_3$, . . . , where $p_i \subseteq n$ if and only if $i$ is an even integer. By Lemma 9.2 there exists a sequence $\{m_i\}_{i=1}^\infty$ of distinct maximal ideals of $A$ different from $n$ such that for every $i > 0$, $p_i \subseteq m_i$, but for every $j > i$, $p_j \not\subseteq m_j$. Since $A$ is Noetherian and each $p_i$ has height 1, each nonzero element $x$ of $A$ is a member of at most finitely many members of the sequence $\{p_i\}_{i=1}^\infty$. Thus since the minimal primes of $xA$ are all members of this sequence, it follows that the collection $\{ m_i : i > 0 \}$ of maximal ideals of $A$ has finite character.

For each positive even integer $i$, let $V_i$ be a valuation overring of $A$ of Krull dimension 2 such that $m_i = m_{V_i} \cap A$ and $p_i = p_{V_i} \cap A$. Define $\Sigma = \{ V_i : i$ is a positive even integer $\}$, and observe that $\Sigma$ has finite character since $\{ m_i : i > 0 \}$ has finite character. Define $B = \bigcap_{V \in \Sigma} V$.

We claim that for every positive even integer $j$, $V_j = B_{m_{V_j} \cap B}$. Fix for the moment a positive even integer $j$, and define $S = B \setminus (m_{V_j} \cap B)$. Then by Proposition 2.4(1), $B_S = V_j \setminus \bigcap_{i \neq j} (V_i)_S$. If $(V_j)_S$ is not a field, then $p_{V_j} \cap B \subseteq m_{V_j} \cap B$, so that $p_j = p_{V_j} \cap A \subseteq m_{V_j} \cap A = m_j$. Hence if $(V_j)_S$ is not a field, then $i \leq j$. Consequently,
(V_i)_S is a field for all even integers i > j. Hence the quasilocal ring B_S is a finite intersection of valuation rings, which forces B_S to be a valuation ring [14, (11.11), p. 38]. Thus B_S = V_j.

Suppose now that B = C ∩ C', where C’ is an overring of B and C is an overring of B having a completely integrally closed integral closure. Let i be a positive even integer, and define S_i = B \ (m_{V_i} ∩ B). Then V_i = B_{S_i} = C_{S_i} ∩ C_{S_i}. Since C_{S_i} has a completely integrally closed integral closure and V_i has Krull dimension 2, V_i ≠ C_{S_i}. Since V_i is a valuation domain, this forces C’ ⊆ V_i. Hence C’ ⊆ ∩_{i∈2N} V_i = B. Thus B cannot be represented as an intersection of two overrings properly containing B, one of whose integral closure is completely integrally closed.

Finally, we observe that B is not a Prüfer domain. For B ⊆ ∩_{i∈2N} (V_i)p_{V_i} = ∩_{i∈2N} A_{B} = A_{n}. (We have used here that by Proposition 2.3(4), A_{B} = (V_i)p_{V_i}.) Thus B has a Noetherian overring of Krull dimension 2. Since every overring of a Prüfer domain is Prufer, and a Prüfer Noetherian domain is Dedekind, it follows that B cannot be a Prüfer domain.

Since an integrally closed Noetherian domain is a completely integrally closed domain, we obtain the following corollary, which shows that the hypothesis of having nonzero Jacobson radical cannot be removed from Proposition 9.1.

**Corollary 9.4.** If A is a countable Noetherian Hilbert domain such that each maximal ideal has height 2, then there is an overring B of A such that B has a finite character representation and B cannot be written as an intersection of a Prüfer overring and a Noetherian overring.

We give finally another example to show how the global case differs from the local case. If B is an overring of D having nonzero Jacobson radical, and B has a finite character representation Σ consisting of valuation rings of Krull dimension 2, then it follows from Remark 2.5 that Σ is finite. Hence B is a Prüfer domain having finitely many maximal ideals [14, (11.11), p. 38]. This can fail in a striking way when the Jacobson radical of B is 0:

**Proposition 9.5.** Suppose A is a countable integrally closed Noetherian Hilbert domain such that each maximal ideal of A has height 2. Then there exists a finite character representation of A consisting of valuation rings of Krull dimension 2.

**Proof.** Let n be a maximal ideal of A. Since A is a countable Noetherian ring, the set of ideals of A is countable. Thus we may enumerate the height one prime ideals of A as p_1, p_2, p_3, . . . . By Lemma 9.2, there exists a sequence {m_i}_{i=1}^{∞} of distinct maximal ideals such that for each i > 0, p_i ⊆ m_i and the only maximal ideals in {p_i : i > 0} possibly containing p_i are m_1, m_2, . . . , m_i. Moreover, as in the proof of Proposition 9.3, the set {m_i : i > 0} has finite character. Now for each pair p_i ⊆ m_i, there is a valuation overring V_i of Krull dimension 2 centered on m_i and whose smallest nonzero prime ideal lies over p_i. By Proposition 2.3(4), V ⊆ A_p. Therefore, A = ∩_{i=1}^{∞} A_p = ∩_{i=1}^{∞} V_i.

**Acknowledgement**

I thank the referee for many helpful comments, and especially for the suggestions that improved the presentation.

**References**