The Number of Periodic Solutions of 2-Dimensional Periodic Systems

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Communicated by Jack K. Hale

Received December 4, 1981; revised April 26, 1982

Concerning the number of periodic solutions of 2-dimensional periodic systems, there are several interesting results. For example, it is well known that the Duffing equation with periodic forcing term

\[ \ddot{u} + cu + au +Bu^3 = B \cos t \quad \left( \frac{d}{dt} \right). \]  

where \( c \geq 0, \alpha \geq 0, \beta > 0, \) and \( B \) are constants, has one and only one \( 2\pi \)-periodic solution if the damping coefficient \( c \) is large enough, can have at least three \( 2\pi \)-periodic solutions in the case where \( c = 0 \) and \( \alpha = 0 \). For reference, see [2, pp. 195-202; 3; 4; 6]. The method used in [2] for the Duffing equation considered there can be adapted to a \( 2\pi \)-periodic equation like (1) by making a change of time scale; \( \omega t \to t. \) In fact, unless this is done, the basic averaging theorem [2, Theorem 3.2, p. 190], which deals with \( T \)-periodic systems, \( T \) independent of \( e \), cannot be used since in this equation \( \omega = \sqrt{1 + e^2} \) for some \( \beta > 0, \) i.e., \( T = 2\pi/\sqrt{1 + e^2} \).

The purpose of this paper is to give sufficient conditions on a 2-dimensional \( \omega \)-periodic system that it has a finite number of \( \omega \)-periodic solutions. Since an \( \omega \)-periodic system is also an \( n\omega \)-periodic system for any positive integer \( n \), our conditions will also show that under them, only a finite number of \( n\omega \)-periodic solutions are possible for each such positive integer \( n \). For stronger sufficient conditions that higher dimensional periodic systems have at most a finite number of periodic solutions, cf. [9].
We shall denote by $R^n$ $n$-dimensional real Euclidean space, set $R^1 = R$ and define its topology by means of the Euclidean norm. We shall consider a 2-dimensional periodic system

$$\begin{align*}
\dot{u} &= U(t, u, v), \\
\dot{v} &= V(t, u, v),
\end{align*}$$

(2)

where $U(t, u, v)$ and $V(t, u, v)$ are continuous in $(t, u, v) \in R^3$ and periodic in $t$ of period $\omega > 0$; that is,

$$\begin{align*}
U(t + \omega, u, v) &= U(t, u, v), \\
V(t + \omega, u, v) &= V(t, u, v)
\end{align*}$$

for all $(t, u, v) \in R^3$.

Furthermore, we assume that solutions of (2) are uniquely determined by initial conditions and denote by $(u(t, x, y), v(t, x, y))$ the solution of (2) through $(x, y) \in R^2$ at $t = 0$.

**Definition 1.** A point $(x, y) \in R^2$ is called an $\omega$-periodic point if the solution $(u(t, x, y), v(t, x, y))$ is periodic in $t$ of period $\omega$.

**Definition 2.** System (2) is said to be dissipative, if the following conditions are satisfied:

(i) the solution $(u(t, x, y), v(t, x, y))$ exists for all $t \geq 0$ and for $(x, y) \in R^2$,

(ii) there is a compact subset $D$ of $R^2$ such that for any $(x, y) \in R^2$, we have

$$(u(t, x, y), v(t, x, y)) \in D \quad \text{for all large} \quad t \geq 0.$$ 

When system (2) is dissipative, we can see that the periodic points are contained in $D$, and hence the set of periodic points is bounded.

**Remark.** System (2) is dissipative if and only if its solutions are ultimately bounded in the sense of the definition in [8, p. 36].

Our main result is the following:

**Theorem.** Assume that system (2) is dissipative, $U(t, u, v)$ and $V(t, u, v)$ of (2) are analytic in $(u, v) \in R^2$, and

$$\frac{\partial U}{\partial u} (t, u, v) + \frac{\partial V}{\partial v} (t, u, v) < 0 \quad \text{for} \quad (t, u, v) \in R^3.$$ 

Then the number of $\omega$-periodic points is finite.
For the proof we require Lemmas 1–5. Even though some of these appear in the standard literature, we have for the sake of completeness included proofs for all but the first. In Lemmas 1–3, we assume that solutions \((u(t, x, y), v(t, x, y))\) exist for \(0 < t < \omega\) and for \((x, y) \in \mathbb{R}^2\).

**Lemma 1.** If \(U(t, u, v)\) and \(V(t, u, v)\) are analytic in \((u, v) \in \mathbb{R}^2\), then \(u(t, x, y)\) and \(v(t, x, y)\) are analytic in \((x, y) \in \mathbb{R}^2\).

For the proof, see [1, p. 36].

Let \(M(t, x, y)\) be the Jacobian matrix of \(u(t, x, y)\) and \(v(t, x, y)\) with respect to \(x\) and \(y\), that is,

\[
M(t, x, y) = \begin{pmatrix}
\frac{\partial u}{\partial x}(t, x, y) & \frac{\partial u}{\partial y}(t, x, y) \\
\frac{\partial v}{\partial x}(t, x, y) & \frac{\partial v}{\partial y}(t, x, y)
\end{pmatrix}.
\]

**Lemma 2.** Suppose condition (3) holds for \(0 < t < \omega\) and for \((u, v) \in \mathbb{R}^2\). Then for every \((x, y)\), at least one eigenvalue of \(M(\omega, x, y)\) is not 1.

**Proof:** Matrix \(M(t, x, y)\) is a fundamental matrix of the variational equation of (2);

\[
\dot{z} = A(t)z,
\]

where \(A(t)\) is the \(2 \times 2\) matrix defined by

\[
A(t) = \begin{pmatrix}
\frac{\partial U}{\partial u}(t, u, v) & \frac{\partial U}{\partial v}(t, u, v) \\
\frac{\partial V}{\partial u}(t, u, v) & \frac{\partial V}{\partial v}(t, u, v)
\end{pmatrix}
\]

with \(u = u(t, x, y)\) and \(v = v(t, x, y)\). Since \(M(0, x, y)\) is the \(2 \times 2\) unit matrix, it follows from Abel's equality that

\[
\det M(\omega, x, y) = \exp \left\{ \int_0^\omega \left[ \frac{\partial U}{\partial u}(s, u, v) + \frac{\partial V}{\partial v}(s, u, v) \right] ds \right\},
\]

where \(u = u(s, x, y)\) and \(v = v(s, x, y)\) in the integrand, and hence condition (3) yields

\[0 < \det M(\omega, x, y) < 1,
\]

which implies our conclusion.
Here we shall consider an $n$-dimensional $\omega$-periodic system;

$$\dot{u} = U(t, u) \quad \left( \cdot = \frac{d}{dt} \right),$$  \hspace{1cm} (5)

where $u = (u_1, u_2, \ldots, u_n)$, $U(t, u) = (U_1(t, u), \ldots, U_n(t, u))$, $U(t, u)$ and its first partial derivatives with respect to the components of $u$ are continuous on $\mathbb{R} \times \mathbb{R}^n$, and $\omega$-periodic in $t$. Denoting by $u(t, x)$ the solution of (5) through $x \in \mathbb{R}^n$ at $t = 0$, we consider the $\omega$-period map $T$.

$$T x = u(\omega, x) \quad \text{for} \quad x \in \mathbb{R}^n.$$  

A subset $D$ of $\mathbb{R}^n$ is said to be invariant under $T$, if the image of $D$ by $T$. $TD$, satisfies $TD = D$.

Clearly, if $D$ is invariant under $T$, then $T^{-1}D = D$, and so $T^mD = D$ for all integers $m$.

**Lemma 3.** Assume that

$$\sum_{i=1}^{n} \frac{\partial U_i}{\partial u_i}(t, u) \neq 0 \quad \text{for all} \quad (t, u) \in \mathbb{R} \times \mathbb{R}^n.$$  \hspace{1cm} (6)

If $D \subset \mathbb{R}^n$ is bounded and invariant under $T$, then $D$ has no interior.

**Proof:** We can assume that (6) implies

$$\sum_{i=1}^{n} \frac{\partial U_i}{\partial u_i}(t, u) < 0 \quad \text{for} \quad (t, u) \in \mathbb{R} \times \mathbb{R}^n;$$

if not, make the change of $t$; $t \rightarrow -t$. Since $D$ is invariant under $T$, the set \( \{ u(t, x) \in \mathbb{R}^n; t \in \mathbb{R}, x \in D \} \) is bounded in $\mathbb{R}^n$, and hence, using the periodicity of $U$, there is a constant $a > 0$ such that

$$\sum_{i=1}^{n} \frac{\partial U_i}{\partial u_i}(t, u(t, x)) < -a \quad \text{for all} \quad t \in \mathbb{R} \quad \text{and} \quad x \in D.$$  \hspace{1cm} (7)

Now, supposing that $D$ has an interior, we can choose an $n$-dimensional closed ball $S$ such that $S \subset D$, and hence

$$T^mS \subset T^mD = D \quad \text{for all integers} \quad m.$$  \hspace{1cm} (8)

Denote the volume of $T^mS$ by $|T^mS|$; we have clearly

$$|T^mS| = \int_{T^mS} du.$$
Under the change of variable $x; x \to u$, defined by $u = u(m\omega, x)$, we have
\[ du = \left| \det \frac{\partial u(m\omega, x)}{\partial x} \right| \, dx \]
and
\[ |T^m S| = \iint_S \left| \det \frac{\partial u(m\omega, x)}{\partial x} \right| \, dx. \]
Since $w = \frac{\partial u(t, x)}{\partial x}$ satisfies
\[ \dot{w} = \frac{\partial U}{\partial u}(t, u(t, x)) \, w, \]
it follows from Abel's equality that
\[ \det \left( \frac{\partial u}{\partial x}(m\omega, x) \right) = \exp \int_0^{m\omega} \left\{ \sum_{i=1}^n \frac{\partial U_i}{\partial u_i}(s, u(s, x)) \right\} \, ds, \]
and hence by (7),
\[ \left| \det \frac{\partial u}{\partial x}(m\omega, x) \right| \geq e^{-m\omega a} \quad \text{for all integers } m \leq 0. \]
Therefore we have
\[ |T^m S| \geq \iint_S e^{-m\omega a} \, dx = e^{-m\omega a} |S| \quad \text{for } m \leq 0 \]
and
\[ \lim_{m \to -\infty} |T^m S| = +\infty, \]
because $|S| \neq 0$. This shows that $T^m S$ is unbounded as $m \to -\infty$, which contradicts (8).

**Lemma 4.** In system (2), suppose that condition (3) holds for all $(t, u, v) \in \mathbb{R}^3$. Then there exists no Jordan curve in $\mathbb{R}^2$ which consists of only $\omega$-periodic points.

**Proof.** Suppose that there exists such a Jordan curve. Clearly it is invariant under the $\omega$-period map $T$ for system (2). If $D$ is the open set interior of this curve, we can see from the uniqueness of the initial value problem that $D$ is invariant under $T$. For if not, some solution $(x(t), y(t))$ of
(2) would satisfy \((x(0), y(0)) \in D, (x(\omega), y(\omega)) \in D\), and so \((\tilde{x}(t_1), \tilde{y}(t_1)) = (x(t_1), y(t_1))\) for some \(t_1 \in [0, \omega]\) and solution \((\tilde{x}(t), \tilde{y}(t))\) of (2) such that \((\tilde{x}(0), \tilde{y}(0)) \in \partial D\), the Jordan curve. This contradicts Lemma 3 and proves the lemma.

Let \(F(x, y)\) be a function on \(\mathbb{R}^2\) to \(\mathbb{R}^n\) and the set of its zeros be \(0(F)\), that is

\[
0(F) = \{(x, y) \in \mathbb{R}^2 : F(x, y) = 0\}.
\]

**Lemma 5.** Assume that \(F(x, y)\) is analytic in \(x\) and \(y\) and that we have

\[
\frac{\partial F}{\partial x} (x, y) \neq 0 \quad \text{or} \quad \frac{\partial F}{\partial y} (x, y) \neq 0 \quad \text{for each} \quad (x, y) \in \mathbb{R}^2. \tag{9}
\]

If \(0(F)\) is bounded and contains infinitely many elements, then at least one component of \(0(F)\) is a smooth Jordan curve.

**Proof.** Since \(0(F)\) has infinitely many elements, there exists a cluster point in \(0(F)\), and let \(C\) be the component of \(0(F)\) containing it. Then any point of \(C\) is a cluster point in \(0(F)\).

First of all, we shall show that \(C\) is a one-dimensional smooth manifold. For any \(p_0 = (x_0, y_0) \in C\), let \(L(p_0)\) be a relative neighbourhood of \(p_0\), that is,

\[
L(p_0) = C \cap U
\]

for a neighbourhood \(U\) of \(p_0\). It is sufficient to show that there exists an analytic function of the form either \(y(x)\) or \(x(y)\) such that

\[
L(p_0) = \{(x, y(x)) ; x \in I\} \tag{10}
\]

or

\[
L(p_0) = \{(x(y), y) ; y \in J\}, \tag{11}
\]

where \(I\) and \(J\) are the domains of \(y(x)\) and \(x(y)\), respectively, and are open intervals.

We show that the condition in (9)

\[
\frac{\partial F}{\partial y} (x_0, y_0) \neq 0
\]

implies (10); a similar argument, which we omit, shows that the other case in (9) implies (11). Therefore, we may assume that

\[
F(x, y) = (f(x, y), g(x, y)),
\]
where \( f(x, y) \in \mathbb{R}^1 \), \( g(x, y) \in \mathbb{R}^{n-1} \) and

\[
\frac{\partial f}{\partial y}(x_0, y_0) \neq 0.
\]

Since \( f(x_0, y_0) = 0 \), it follows from the application of the implicit function theorem to \( f(x, y) = 0 \) that there exists an analytic function \( y = y(x) \), which is defined on an open interval \( I \) containing \( x_0 \), such that

\[
y_0 = y(x_0), \quad f(x, y(x)) = 0 \quad \text{on} \quad I.
\]  

(12)

and any point \((x, y)\) in \( L(p_0) \) can be represented as

\[
(x, y(x))
\]  

(13)

provided \( L(p_0) \) is assumed to be a sufficiently small neighbourhood of \( p_0 \). Therefore for such a \( L(p_0) \) we have

\[
L(p_0) \subset \{(x, y(x)); x \in I\}.
\]

It remains to show that

\[
L(p_0) \supset \{(x, y(x)); x \in I\},
\]

that is,

\[
(x, y(x)) \subset O(F) \quad \text{for} \quad x \in I,
\]

which is equivalent to

\[
g(x, y(x)) = 0, \quad x \in I,
\]

(14)

by (12). Since \( p_0 \) is a cluster point in \( O(F) \), there exist infinitely many points \( \{p_k\}_{k=1}^\infty \) of \( O(F) \) such that

\[
p_k \to p_0 \quad \text{as} \quad k \to \infty.
\]

Setting \( p_k = (x_k, y_k) \), we have

\[
y_k = y(x_k) \quad \text{by} \ (13)
\]

and

\[
g(x_k, y_k) = 0.
\]
Setting \( \phi(x) = g(x, y(x)) \) for \( x \in I \), we can see that \( \phi(x) \) is analytic on \( I \) and \( \phi(x_k) = g(x_k, y(x_k)) = g(x_k, y_k) = 0 \). Since \( \{x_k\}_{k=1}^{\infty} \) has a cluster point \( x_0 \), it follows from the theorem of unicity that
\[
\phi(x) = 0 \quad \text{on} \quad I,
\]
which is (14). Thus \( C \) is a one-dimensional analytic manifold, and moreover it is compact, because \( 0(F) \) is compact. Since a compact, smooth and one-dimensional manifold is a Jordan curve (cf. [5, pp. 55-57]), \( C \) is a Jordan curve. This proves Lemma 5.

We shall now complete the proof of our theorem. Consider \( F(x, y) = (f(x, y), g(x, y)) \) for \( (x, y) \in \mathbb{R}^2 \), where
\[
f(x, y) = u(\omega, x, y) - x, \quad g(x, y) = v(\omega, x, y) - y.
\]
Clearly, zeros of \( F(x, y) \) are identical with \( \omega \)-periodic points of system (2). Therefore our purpose is to prove that \( 0(F) \) consists of a finite number of elements.

We have that \( F(x, y) \) is analytic in \( (x, y) \in \mathbb{R}^2 \) by Lemma 1. Moreover we shall show that condition (9) in Lemma 5 holds for the function \( F(x, y) \). Since
\[
\frac{\partial F}{\partial x}(x, y) = \begin{pmatrix}
\frac{\partial f}{\partial x}(x, y) \\
\frac{\partial g}{\partial x}(x, y)
\end{pmatrix} = \begin{pmatrix}
\frac{\partial u}{\partial x}(\omega, x, y) - 1 \\
\frac{\partial v}{\partial x}(\omega, x, y)
\end{pmatrix}
\]
and
\[
\frac{\partial F}{\partial y}(x, y) = \begin{pmatrix}
\frac{\partial f}{\partial y}(x, y) \\
\frac{\partial g}{\partial y}(x, y)
\end{pmatrix} = \begin{pmatrix}
\frac{\partial u}{\partial y}(\omega, x, y) \\
\frac{\partial v}{\partial y}(\omega, x, y) - 1
\end{pmatrix}
\]
we have the \( 2 \times 2 \) matrix
\[
\begin{pmatrix}
\frac{\partial F}{\partial x}(x, y), \\
\frac{\partial F}{\partial y}(x, y)
\end{pmatrix} = M(\omega, x, y) = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\]
where \( M(\omega, x, y) \) is the matrix in Lemma 2. Therefore, if \( \frac{\partial F(x, y)}{\partial x} = 0 \) and \( \frac{\partial F(x, y)}{\partial y} = 0 \), then we have
\[
M(\omega, x, y) = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\]
which contradicts Lemma 2.
Since system (2) is dissipative, \( O(F) \) is bounded. Therefore, if \( O(F) \) has infinitely many elements, then it follows from Lemma 5 that there exists a Jordan curve consisting of only \( \omega \)-periodic points, which contradicts Lemma 4. The proof is complete.

We now apply our theorem to the Duffing equation (1) which is equivalent to the system

\[
\dot{u} = v, \quad \dot{v} = -cv - au - \beta u^3 + B \cos t. \tag{15}
\]

System (15) is dissipative (cf. [7]), the right-hand side is analytic in \( u \) and \( v \) and

\[
\frac{\partial U}{\partial u} + \frac{\partial V}{\partial v} = -c < 0.
\]

Therefore by our theorem and a previous remark, we can conclude that for any integer \( n > 0 \), (15), and hence also (1), can have at most a finite number of \( 2\pi n \)-periodic solutions.

The following example shows that there exist nonautonomous \( \omega \)-periodic systems like (2) which are dissipative, satisfy the divergence condition (3), are \( C^\infty \) (infinitely differentiable) in \( (u, v) \), and have an infinite set of \( \omega \)-periodic solutions. Define

\[
g(x) = x^2 \exp(-1/x^2) \sin(1/x), \quad x \neq 0,
\]

\[
= 0, \quad x = 0,
\]

and consider the system

\[
x' = y, \quad y' = -cy - g(x), \tag{16}
\]

where \( c > 0 \) is a fixed constant. It follows easily that (16) has constant solutions (critical points) \( (x_n(t), y_n(t)) = (1/n\pi, 0) \), \( n = 1, 2, \ldots \), which for \( n \) odd are asymptotically stable focii or nodes, and for \( n \) even are unstable saddles; the characteristic exponents associated with these critical points all have nonzero real parts.

By standard perturbation results (cf. [2]) there exists for each \( n = 1, 2, \ldots \), an \( \varepsilon_n > 0 \) such that

\[
x' = y, \quad y' = -cy - g(x) + \varepsilon_n \sin t \tag{17n}
\]

has a \( 2\pi \)-periodic solution \( (\bar{x}_n(t), \bar{y}_n(t)) \) near \( (1/n\pi, 0) \). Clearly, \( 1/(n+2) \pi < \bar{x}_{n+1}(t) < 1/n\pi \) for all \( t \) and \( n = 1, 2, \ldots \), and

\[
\bar{x}_1(t) > 1/2\pi \quad \text{for all} \quad t.
\]
Define the closed bounded interval $I_n$ to be the range of $\tilde{x}_n(t)$; clearly, $I_{n+1} \subset [1/(n+2)\pi, 1/n\pi]$, $n = 1, 2, \ldots$, and $I_1 \subset [1/2\pi, \infty)$. By choosing $\varepsilon_n$ appropriately small, $n = 1, 2, 3, \ldots$, we may assume $I_j \cap I_k$ empty for $j \neq k$.

Let $h(x)$ be any $C^{\infty}$ function on $\mathbb{R}$ such that $h(x) = h(-x)$, $h(x) = \varepsilon_n$ for $x \in I_n$, $n = 1, 2, \ldots$.

It follows that the $2\pi$-periodic system

\begin{align*}
 x' &= y, \\
 y' &= -cy - g(x) + h(x) \sin t
\end{align*}

has for each $n = 1, 2, \ldots$, the $2\pi$-periodic solution $(\tilde{x}_n(t), \tilde{y}_n(t))$.

The divergence condition (3) holds trivially for (17). To show (17) dissipative we may use [7, Theorem II].

Finally we give an example of a 3-dimensional $\omega$-periodic system which is analytic, dissipative, satisfies the divergence condition (6), and yet has an infinite number of distinct $\omega$-periodic solutions. The system is as follows:

\begin{align*}
 \dot{x} &= y, \\
 \dot{y} &= \varepsilon(1 - x^2)y - x, \\
 \dot{z} &= - [\varepsilon(1 - x^2) + M^2] z + A \sin \frac{2\pi}{\omega} t,
\end{align*}

where $\varepsilon$, $a$, $A$, and $M$ are positive constants with $\varepsilon < 1$.

Since the first two equations of (18) form a van der Pol equation, this 2-dimensional system is dissipative and has a nontrivial periodic solution of least period $T = T(\varepsilon) > 0$. We choose $M$ such that $|x(t)| \leq M$ and $|y(t)| \leq M$ for all $t$ sufficiently large and choose $\omega = 2\pi/T$.

It is straightforward to verify that (6) holds for $n = 3$; that (18) is dissipative also follows easily; we omit the detail.

If $(\tilde{x}(t), \tilde{y}(t))$ is a periodic solution of the first two equations of (18), clearly for any $\theta$, $0 < \theta < T$, $(x(t + \theta), y(t + \theta))$ is also a periodic solution. Putting $x = \tilde{x}(t)$ in the third equation of (18), and using the fact that the resulting coefficient of $z$ is less than $-\varepsilon$ for all $t$, we see easily that the third equation in (18) has a unique $\omega$-periodic solution $\tilde{z}(t, \theta)$. But the $\omega$-periodic solutions $(\tilde{x}(t + \theta), \tilde{y}(t + \theta), \tilde{z}(t, \theta))$ of (18) are distinct for distinct $\theta$ in $(0, T) = (0, 2\pi/\omega)$. Thus (18) has an infinite number of $\omega$-periodic solutions, one for each $\theta$ in $(0, T)$. This verifies that the example is as asserted, and shows that our basic theorem holds only for 2-dimensional systems.

For another example of a periodic system with an infinite number of stable periodic solutions in a bounded region, cf. [10].

Acknowledgments

The authors wish to thank Professors T. Yoshizawa and J. Kato at Tohoku University and R. K. Miller of Iowa State University for their invaluable discussions, comments, and suggestions.
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