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On Graphs Admitting Arc-Transitive Actions of Almost Simple Groups

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Let Γ be a finite connected regular graph with vertex set $V\Gamma$, and let G be a subgroup of its automorphism group $\mathrm{Aut}\,\Gamma$. Then Γ is said to be *G-locally primitive* if, for each vertex α, the stabilizer G_α is primitive on the set of vertices adjacent to α. In this paper we assume that G is an almost simple group with socle $\mathrm{soc}\,G = S$; that is, S is a nonabelian simple group and $S \trianglelefteq G \leq \mathrm{Aut}\,S$. We study nonbipartite graphs Γ which are G-locally primitive, such that S has trivial centralizer in $\mathrm{Aut}\,\Gamma$ and S is not semiregular on vertices. We prove that one of the following holds: (i) $S \trianglelefteq \mathrm{Aut}\,\Gamma \leq \mathrm{Aut}(S)$, (ii) $G < Y \leq \mathrm{Aut}\,\Gamma$ with Y almost simple and $\mathrm{soc}\,Y \neq S$, or (iii) S belongs to a very restricted family of Lie type simple groups of characteristic p, say, and $\mathrm{Aut}\,\Gamma$ contains the semidirect product $Z_p^d{:}G$, where Z_p^d is a known absolutely irreducible G-module. Moreover, in certain circumstances we can guarantee that $S \trianglelefteq \mathrm{Aut}\,\Gamma \leq \mathrm{Aut}(S)$. For example, if Γ is a connected $(G,2)$-arc transitive graph with $\mathrm{Sz}(q) \leq G \leq \mathrm{Aut}(\mathrm{Sz}(q))$ $(q = 2^{2n+1} \geq 8)$ or $G = \mathrm{Ree}(q)$ $(q = 3^{2n+1} \geq 27)$, then $G \leq \mathrm{Aut}\,\Gamma \leq \mathrm{Aut}(G)$.

1. INTRODUCTION

A fundamental problem in determining the structure of a graph Γ is the problem of finding its full automorphism group $\mathrm{Aut}\,\Gamma$. We are interested in certain families of finite vertex-transitive graphs for which membership is determined by the existence of a vertex-transitive subgroup of the automorphism group possessing a certain property. In this context the

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problem is to determine AutΓ, given a certain vertex-transitive subgroup of it. For some families of finite vertex-transitive graphs, for example, the class of finite 2-arc transitive graphs, the subfamily consisting of those graphs admitting an almost simple subgroup of automorphisms with the appropriate property has been identified as having special significance and being worthy of special study (see, for example, [18]). The purpose of this paper is to study the automorphism groups of finite graphs Γ which admit a vertex-transitive subgroup G of automorphisms, with G almost simple, that is, with the socle $\mathrm{soc}\, G$ a nonabelian simple group S. In certain circumstances we can guarantee that $S \trianglelefteq \mathrm{Aut}\,\Gamma \leq \mathrm{Aut}(S)$.

Let Γ be a finite simple undirected graph with vertex set $V\Gamma$ and edge set $E\Gamma$ (which we identify with a subset of unordered pairs of vertices from $V\Gamma$). For $\alpha \in V\Gamma$ let $\Gamma(\alpha)$ denote the set of all vertices adjacent to α, and let G be a subgroup of Aut Γ. Then Γ is said to be *G-locally primitive* if, for each $\alpha \in V\Gamma$, the stabilizer G_α is primitive on $\Gamma(\alpha)$.

It is not difficult to see that, for a connected G-locally primitive graph Γ, either the group G is transitive on vertices, or Γ is bipartite and the G-orbits on vertices are the two parts of the bipartition. We shall focus on the nonbipartite case: we study the automorphism groups of finite, connected, nonbipartite, G-locally primitive graphs Γ with G an almost simple group such that the socle of G is not semiregular on vertices. (A permutation group G on a set X is said to be *semiregular* if the only element of G which fixes a point of X is the identity element.) In this case both G and its simple socle are transitive on vertices. Our main result is the following.

THEOREM 1.1. *Let G be a finite almost simple group with socle S and let Γ be a connected nonbipartite G-locally primitive graph. Suppose that S is not semiregular on vertices and that $\mathrm{C}_{\mathrm{Aut}\,\Gamma}(S) = 1$. Then one of the following holds.*

(a) $S \trianglelefteq \mathrm{Aut}\,\Gamma \leq \mathrm{Aut}(S)$; *or*

(b) $G < Y \leq \mathrm{Aut}\,\Gamma$, *$Y$ is almost simple with* $\mathrm{soc}\, Y \neq S$, *and either* $\mathrm{N}_Y(G)$ *is maximal in Y, or* $\Gamma = K_8$, $G = \mathrm{PSL}(2,7)$, *and* $Y = A_8$ *or* S_8; *or*

(c) $\mathrm{Aut}\,\Gamma$ *contains a subgroup $N.G$, where $N = Z_p^d$ for some prime p and $d > 1$, and $S = S(q)$ is a Lie type simple group over a field of order $q = p^e$ for some e dividing d. Further, $S, d/e, N$ are as in Table* 1 *or* 2, *and N is intransitive and semiregular on $V\Gamma$.*

Remark 1.2. (a) In part (b) of Theorem 1.1 with $\Gamma \neq K_8$, more information about the socles of G and Y is available from [13] and [14]. Note that since $\mathrm{C}_{\mathrm{Aut}\,\Gamma}(S) = 1$, we have that $\mathrm{N}_Y(G) \leq \mathrm{Aut}\, S$, that is, $H = \mathrm{N}_Y(G)$ is almost simple with socle S. Thus we have H maximal in Y and $Y = HY_\alpha$ such that neither H nor Y_α contains $\mathrm{soc}\, Y$. Let K be maximal such that $Y_\alpha \leq K < Y$ and $K \not\supseteq \mathrm{soc}\, Y$. If K is maximal in Y then $Y = HK$ is a

TABLE I
Unbounded Lie Rank

S	d/e	Comments on N
$A_l^{\pm}(q)$	$l+1$	Natural module
$A_l(q)$	$l(l+1)/2$	q odd; see Remark 2.10
$B_l(q)$	$2l+1$	Natural module
$C_l(q)$	$2l$	Natural module
$D_l^{\pm}(q)$	$2l$	Natural module

maximal factorization of the almost simple group Y with the factor H also almost simple, and neither H nor K contains soc Y. All such factorizations are known explicitly and their classification is given in [13]. On the other hand, if K is not maximal in Y then the factorization $Y = HK$ was called a $\max^-$ factorization in [14] (see also Definition 3.1 in this paper) and in that paper all $\max^-$ factorizations of almost simple groups were classified explicitly. The special case of Theorem 1.1(b) in which H is primitive on the vertices of Γ has been investigated further in [15].

(b) In case (c) of Theorem 1.1, let Γ_N be the quotient graph obtained by taking the N-orbits as vertices and joining two N-orbits by an edge if there is at least one edge in Γ joining a point in the first to a point in the second N-orbit. Then Γ is a cover of Γ_N and Γ_N is also a connected nonbipartite G-locally primitive graph (see Remark 2.5 for more details). For the groups $A_l(q)$ with $d/e = l(l+1)/2$ and ${}^3D_4(q)$ with $d/e = 12$ in Tables I and II, respectively, more details are given in Remark 2.10. In particular, q is odd.

Theorem 1.1 will be proved in Section 2. In Section 3 we explore some applications of it. First we prove a somewhat technical result (Theorem 3.3) which illustrates in a general way that case (b) can often be avoided.

TABLE II
Bounded Lie Rank

S	d/e	Comments on N
$B_l(q)$, $l = 3$ or 4	8, 16	Spin module
$C_l(q)$, $l = 3$ or 4	8, 16	Spin module
$D_l^{\pm}(q)$, $l = 4$ or 5	8, 16	Spin module
${}^3D_4(q)$	12	q odd; see Remark 2.10
$E_6^{\pm}(q)$	27	
$E_7(q)$	56	
$G_2(q)$	6	q even, symplectic

Then we look at the two infinite families of almost simple groups G which motivated this investigation. We show that Theorem 1.1(a) holds for these families in the case of $(G, 2)$-arc transitive actions. (A 2-arc of a graph Γ is a triple (α, β, γ) of vertices such that $\{\alpha, \beta\}$ and $\{\beta, \gamma\}$ are edges of Γ, and Γ is said to be *$(G, 2)$-arc transitive* if G is a subgroup of $\mathrm{Aut}\,\Gamma$ such that G is transitive on the 2-arcs of Γ.)

THEOREM 1.3. *Let Γ be a connected $(G, 2)$-arc transitive graph, where either* $\mathrm{Sz}(q) \le G \le \mathrm{Aut}(\mathrm{Sz}(q))$ $(q = 2^{2n+1} \ge 8)$ *or* $G = \mathrm{Ree}(q)$ $(q = 3^{2n+1} \ge 27)$. *Then* $G \le \mathrm{Aut}\,\Gamma \le \mathrm{Aut}(G)$.

2. PROOF OF THEOREM 1.1

Let Γ be a connected, nonbipartite, G-locally primitive graph of valency d_Γ, and let H be the stabilizer in G of a vertex α. Then G is transitive on $V\Gamma$, H is transitive on $\Gamma(\alpha)$, and so G is transitive on the 1-arcs of Γ, that is, the ordered pairs of adjacent vertices of Γ. Thus (see [20]) there exists a 2-element $g \in G$ such that $\langle H, g\rangle = G$, $\beta := \alpha^g \in \Gamma(\alpha)$, and $\alpha = \beta^g$, and hence $H \cap H^g = G_{\alpha\beta}$ has index d_Γ in H. Moreover $\Gamma \cong \Gamma^* := \Gamma(G, H, HgH)$, where Γ^* is defined by

$$V\Gamma^* = \{Hx \mid x \in G\}, \qquad E\Gamma^* = \big\{\{Hx, Hy\} \mid x, y \in G,\ xy^{-1} \in HgH\big\}. \tag{1}$$

All connected regular G-locally primitive graphs considered in this paper will be defined in terms of a subgroup H and a 2-element g as in (1).

Now let Γ and G be as in Theorem 1.1 so that G is an almost simple subgroup of $\mathrm{Aut}\,\Gamma$ with socle S such that $C_{\mathrm{Aut}\,\Gamma}(S) = 1$ and S is not semiregular on vertices. First we record an upper bound on the order of the Sylow subgroups of S in terms of the minimal index $m(S)$ of proper subgroups of S.

LEMMA 2.1. *For each prime p, the p-part of $|S|$ is p^a, where $a \le (m(S) - 1)/(p - 1)$ and if $p = 2$ then $a \le m(S) - 2$.*

Proof. Set $r := m(S)$. Since S is isomorphic to a subgroup of S_r a Sylow p-subgroup of S has order p^a at most the power of p dividing $r!$. Thus $a \le (r - 1)/(p - 1)$. If $p = 2$ then $S \le A_r$ and so $a \le (r - 1) - 1$. ∎

Suppose now that Theorem 1.1(a) does not hold. Then clearly we may assume further that $G = \mathrm{Aut}(S) \cap \mathrm{Aut}\,\Gamma$. Let $G < Y \le \mathrm{Aut}\,\Gamma$ with G maximal in Y. Let N be a minimal normal subgroup of Y. Then $N = T^d$ for some simple group T and positive integer $d \ge 1$. The next proposition gives some information about Y and N. For a subgroup H of $\mathrm{Aut}\,\Gamma$ which fixes a subset X of $V\Gamma$ setwise, H^X will denote the permutation group on X induced by H. Note that the assertions of Proposition 2.2 and Lemmas 2.3–2.9 concerning the action and structure of Y remain true if the assumption $\mathrm{C}_{\mathrm{Aut}\,\Gamma}(S) = 1$ is replaced by the weaker condition $\mathrm{C}_Y(S) = 1$.

PROPOSITION 2.2. *Let Y and N be as given above. Then the following hold.*

(i) *S is transitive on the* 1*-arcs of* Γ, *and S is not a normal subgroup of Y.*

(ii) *Either N is semiregular on $V\Gamma$ or N is transitive on the* 1*-arcs of* Γ.

(iii) *Either Theorem* 1.1(b) *holds, or $Y = NG$ and $G \cap N = 1$.*

Proof. Suppose that S is normal in Y, and consider the natural map $\varphi\colon Y \to \mathrm{Aut}(S)$ determined by conjugation. If $\mathrm{Ker}\,\varphi = 1$ then Y is isomorphic to a subgroup of $\mathrm{Aut}(S)$, contradicting our assumption that $G = \mathrm{Aut}\,\Gamma \cap \mathrm{Aut}(S)$. Hence $\mathrm{Ker}\,\varphi \neq 1$. Further $\mathrm{Ker}\,\varphi \cap G = 1$ since $\mathrm{C}_G(S) = 1$. Then since both S and $\mathrm{Ker}\,\varphi$ are normal in Y, they centralize each other, contradicting $\mathrm{C}_{\mathrm{Aut}(\Gamma)}(S) = 1$. Thus S is not normal in Y.

Suppose next that N is not semiregular on $V\Gamma$. Then $N_\alpha \neq 1$ and by the connectivity of Γ it follows that $N_\alpha^{\Gamma(\alpha)}$ is a nontrivial normal subgroup of $Y_\alpha^{\Gamma(\alpha)}$. Since by assumption $G_\alpha^{\Gamma(\alpha)}$ is primitive, also $Y_\alpha^{\Gamma(\alpha)}$ is primitive. Hence $N_\alpha^{\Gamma(\alpha)}$ is transitive and so N has at most two orbits in $V\Gamma$. Since Γ is not bipartite it follows that N is transitive on $V\Gamma$; hence N is transitive on the 1-arcs of Γ. A similar argument shows that S is transitive on the 1-arcs of Γ (since S is not semiregular on $V\Gamma$). Thus parts (i) and (ii) are proved.

If $G \cap N = 1$ then $Y = NG$ since G is maximal in Y, and part (iii) holds. Thus we may assume that $G \cap N \neq 1$. Then $S \leq N \cap G$. Since S is transitive on $V\Gamma$, N is also. Thus $Y = GY_\alpha$ and $N = SN_\alpha$. From $\mathrm{C}_{\mathrm{Aut}\,\Gamma}(S) = 1$ it follows that $\mathrm{C}_Y(S) = 1$ and hence $\mathrm{C}_Y(N) = 1$. Recall that $N = T^d$ with T simple and $d \geq 1$. Suppose first that $d = 1$. Then $N = T \leq Y \leq \mathrm{Aut}(T)$, that is, Y is almost simple. Also $T \neq S$ by part (i), and so Theorem 1.1(b) holds. Thus we may assume that $d > 1$. If $G \leq N$ then, as G is almost simple, G is a proper subgroup of N. Then, as G is maximal in Y, $Y = N$, contradicting the fact that N is a minimal normal subgroup of Y. Thus G is not contained in N, and since G is maximal in Y, we have $Y = NG$.

Write $N = T_1 \times \cdots \times T_d$ with each $T_i \cong T$, and for each i, let $\pi_i\colon N \to T_i$ denote the natural projection map. If $\pi_i(G \cap N) = 1$ for some i then T_i centralizes $G \cap N$ and hence $T_i \subseteq \mathrm{C}_Y(S)$ which is a contradiction. Hence $\pi_i(G \cap N) \neq 1$, for each i. Let $M := \prod_{i=1}^{d} \pi_i(G \cap N)$. Then M is a nontrivial G-invariant subgroup of N which contains $G \cap N$ as a proper subgroup (since $G \cap N$ is almost simple). Since G is maximal in Y it follows that $Y = MG$ and hence that M is normal in Y. By the minimality of N it follows that $M = N$. Thus $\pi_i(G \cap N) = T_i$, for each i, and we deduce that $T_i \cong T \cong S = G \cap N$. Without loss of generality we may assume that $G \cap N = \{(s, \dots, s) \mid s \in S\}$.

Now N is the unique minimal normal subgroup of Y, for if U were a second minimal normal subgroup of Y then $U \leq \mathrm{C}_Y(N) \leq \mathrm{C}_Y(S)$, which is not the case. This means in particular, since N is transitive on $V\Gamma$, that Y is quasiprimitive on $V\Gamma$ (that is, each nontrivial normal subgroup of Y is transitive on $V\Gamma$). Moreover N is not regular on $V\Gamma$ since S is not regular. By the structure theorem in [18] classifying various types of finite quasiprimitive permutation groups (see the description in [1]) it follows immediately that Y is quasiprimitive of type SD, CD, or PA. In the first two cases, using the notation of [1, Section 2], N_α is a product of pairwise disjoint nontrivial strips of N, as is S, and since $N = N_\alpha S$, we have a contradiction to [1, Lemma 2.4]. Thus Y has type PA, which means that N_α is a subdirect subgroup of R^d for some proper subgroup R of S. Choose x, y to lie in different right cosets of R in S. Since $N = N_\alpha S$ the element $(x, y, \dots, y)$ of N may be expressed as a product $(r_1, r_2, \dots, r_d)(s, s, \dots, s)$ for some $r_1, \dots, r_d \in R$ and $s \in S$. Thus we have $s = r_1^{-1}x = r_2^{-1}y \in Rx \cap Ry$, which is a contradiction. This completes the proof. ∎

From now on we shall assume that neither (a) nor (b) of Theorem 1.1 holds and hence that $Y = NG$ and $G \cap N = 1$. Let $\varphi\colon Y \to \mathrm{Aut}(N)$ denote the natural map induced by conjugation. Since $\mathrm{C}_{\mathrm{Aut}\,\Gamma}(S) = 1$ it follows that $\mathrm{Ker}\,\varphi \cap G = 1$. Thus G is isomorphic to a subgroup of $\mathrm{Aut}(N)$. In the next lemma we show that N is elementary abelian.

LEMMA 2.3. *$N = Z_p^d$ for some prime p and integer $d > 1$.*

Proof. If $d = 1$ then from the remarks above it follows that $N = T$ is a nonabelian simple group and if $\mathrm{Ker}\,\varphi = 1$ then T has an insoluble outer automorphism group, contradicting the "Schreier conjecture." Thus $\mathrm{Ker}\,\varphi \neq 1$ and as G is maximal in Y, $Y = \mathrm{Ker}\,\varphi\, G$ so $\mathrm{Ker}\,\varphi \cong N = T$. Moreover, $\mathrm{Ker}\,\varphi \cap N = 1$, and it follows that $\mathrm{Ker}\,\varphi \cong S$ and $\mathrm{Ker}\,\varphi$ centralizes N. Thus, $|\mathrm{Ker}\,\varphi| = |N| = |S| > |V\Gamma|$ so by Proposition 2.2, both N and $\mathrm{Ker}\,\varphi$ are transitive non-regular subgroups which centralize each other. This is a contradiction. Thus $d > 1$. Suppose that T is a nonabelian simple group. Then $N = T_1 \times \cdots \times T_d$ with each $T_i \cong T$, and Y, and hence G, permutes the simple direct factors $T_1, \dots, T_d$ transitively.

Suppose that S permutes these simple direct factors nontrivially. Then S acts faithfully as a permutation group of degree d and hence $d \geq m(S)$. If N were semiregular on $V\Gamma$, then $|N| = |T|^d$ would divide $|V\Gamma|$, which in turn divides $|S|$, contradicting Lemma 2.1. Hence N is not semiregular on

$V\Gamma$, and so by Proposition 2.2(ii), N is transitive on the 1-arcs of Γ. If Y had a minimal normal subgroup M different from N, then M would centralize the transitive group N, and hence M would be semiregular on $V\Gamma$. In particular $|M| \leq |V\Gamma| < |S|$. However, as such a group M is isomorphic to a normal subgroup of $Y/N \cong G$, M must have a normal subgroup isomorphic to S, which is a contradiction. Hence N is the unique minimal normal subgroup of Y and, as N is transitive on $V\Gamma$, Y is quasiprimitive on $V\Gamma$. By [18], and using the notation of [1], Y is of type SD, CD, or PA. In each case it is easy to see that there is some prime p such that $p^{m(S)}$ divides $|V\Gamma|$ and hence divides $|S|$, contradicting Lemma 2.1

Thus S normalizes each of the T_i, so $\varphi(NS)$ is a subgroup of $\mathrm{Aut}(T)^d$, and hence of $\mathrm{Inn}(T)^d$. It follows that $\varphi(NS) \cong N$ and $K := \mathrm{Ker}\,\varphi \cap NS \cong S$. Thus K is a normal subgroup of Y which is not semiregular on $V\Gamma$ (since $|K| = |S| > |V\Gamma|$) and hence, by Proposition 2.2(ii), K is transitive on $V\Gamma$. Since now both N and K are transitive nonregular subgroups which centralize each other, we have a contradiction. Hence $T = Z_p$ for some prime p. ∎

Next we show that Y is a subgroup of affine transformations of N and that N is intransitive on $V\Gamma$. In the proof we identify the exceptional example $\Gamma = K_8$ of Theorem 1.1(b). We remind the reader that we are assuming that Y does not satisfy Theorem 1.1(b) and so the possibility $\Gamma = K_8$ does not occur in the statement.

LEMMA 2.4. *$N = Z_p^d$ is the unique minimal normal subgroup of Y so that G is ismorphic to an irreducible subgroup of* $\mathrm{GL}(d,p)$. *Moreover N is intransitive and semiregular on $V\Gamma$.*

Proof. Suppose that M is a minimal normal subgroup of Y and $M \neq N$. Then M is isomorphic to a subgroup of $Y/N \cong G$, and it follows that $M \cong S$. However, this means that $\mathrm{Ker}\,\varphi$ contains $M \times N$ and hence that $\varphi(Y)$ is isomorphic to a quotient of $Y/(M \times N)$ which is isomorphic to a subgroup of $\mathrm{Out}(S)$. This contradicts the fact that $\varphi(G) \cong G$. Hence N is the unique minimal normal subgroup of Y. In particular $\mathrm{Ker}\,\varphi = N$, so $G \cong \varphi(G) \leq \mathrm{GL}(d,p)$ and $\varphi(G)$ is irreducible on N.

Now suppose that N is transitive on $V\Gamma$. Then Y is quasiprimitive on $V\Gamma$ of affine type *HA*, and so Y is primitive on $V\Gamma$ (see [18]). The pair (G,Y) corresponds to an inclusion of a quasiprimitive subgroup in a primitive subgroup in $\mathrm{Sym}(V\Gamma)$. It follows from [1] that G is primitive on $V\Gamma$, and then it follows from [17, Table 1 of the Theorem] that $G = \mathrm{PSL}(2,7)$. However, in this case Y is 2-transitive on $V\Gamma$, so $\Gamma = K_8$ and Theorem 1.1(b) holds, contrary to our assumption. Thus N is intransitive and it follows from Proposition 2.2(ii) that N is semiregular on $V\Gamma$. ∎

Remark 2.5. Let $B := \alpha^N$, the N-orbit containing α. Then $|B| = p^d$, and the setwise stabilizer S_B is transitive on B and contains S_α as a subgroup of index p^d. By Lemma 2.4, $B \neq V\Gamma$, and by [16] the quotient graph Γ_N of Γ modulo the N-orbits is again a 1-arc transitive graph, and Γ is a cover of Γ_N. The group induced by Y on Γ_N is $Y/N \cong G$, and since $Y_B = NG_B$, $Y_B^{\Gamma_N(B)} = G_B^{\Gamma_N(B)}$. This latter group contains as a subgroup $G_\alpha^{\Gamma_N(B)}$ which is permutationally isomorphic to $G_\alpha^{\Gamma(\alpha)}$ and hence is primitive. Thus G (and Y) are locally primitive on Γ_N, and S is transitive on the 1-arcs of Γ_N.

We now consider the various possibilities for the simple group S.

LEMMA 2.6. *S is not an alternating group.*

Proof. Suppose that $S = A_n$ for some $n \geq 5$. Since $|N| = p^d$ divides $|S|$ it follows from Lemma 2.1 that $d \leq (n-1)/(p-1) < n$. Thus by [7], [22], and [23], N is the deleted permutation module for S over the field GF(p), and $d = n-2$ if p divides n and $d = n-1$ otherwise. It follows that $p = 2$ and (again by Lemma 2.1) that $d = n-2$ is even and 2^{n-2} divides $|A_n|$. This is only possible if $n = 2^b$ for some $b \geq 3$ and S_B contains a Sylow 2-subgroup of S. Thus $|S_\alpha|$ is odd, and, since $S_\alpha^{\Gamma(\alpha)}$ is transitive, $|\Gamma(\alpha)|$ is odd. Also the number of N-orbits, namely, $|S : S_B|$, is odd since S_B contains a Sylow 2-subgroup of S. The quotient graph Γ_N thus has an odd number $|S : S_B|$ of vertices, and odd valency $|\Gamma_N(B)| = |\Gamma(\alpha)|$. Hence Γ_N has an odd number of 1-arcs, which is impossible. ∎

LEMMA 2.7. *S is not a sporadic simple group.*

Proof. Suppose to the contrary that S is one of the sporadic simple groups. Then d is greater than or equal to the least degree of a faithful linear representation of S over the field GF(p), that is, $d \geq x(S)$, where $x(S)$ is given in the table in [13, 2.3.2]. Since we also require that p^d divides $|S|$, it follows that $p = 2$, that S is one of M_{12}, M_{22}, J_2, Suz, and that d is one of 6, 6 or 7, 6 or 7, 12 or 13, respectively. However, since we require a representation of S, rather than one of its proper covering groups, it follows from [8] that $S \neq M_{12}$ or M_{22}. Moreover, if $S = J_2$ then by [8], $d = 6$ and the Brauer character involves the irrationality $b5$ which (see [8, p. 284]) has a minimal polynomial of degree 2 over GF(2). It follows from [6, p. 155, Corollary 9.23] that a faithful representation of J_2 in characteristic 2 of degree 6 must be over an even degree extension field of GF(2), so $S \neq J_2$. Finally if $S =$ Suz then the argument of [13, p. 141] shows that there is no faithful representation of the simple group S of degree 12 or 13. (There is, however, such a representation of $3 \cdot$ Suz.) ∎

Thus S is a simple group of Lie type, say $S = S(q)$, over a field of order q. First we show that q is a power of p. (For a prime p and a group G,

$|G|_p$ denotes the p-part of $|G|$, and $O_p(G)$ denotes the largest normal p-subgroup of G.)

LEMMA 2.8. *$S = S(q)$ is a simple group of Lie type over the field* $\mathrm{GF}(q)$, *where $q = p^e$ for some $e \geq 1$.*

Proof. By Lemma 2.4, N is semiregular on $V\Gamma$ and hence $|N| = p^d$ divides $|S|$. Also since G is an irreducible subgroup of $\mathrm{GL}(d, p)$, $d \geq e(S)$, where $e(S)$ is as given in [10, Table 5.3.A] (or see [11]). Thus

$$p^{e(S)} \mid p^d \mid |S|.$$

Checking this condition carefully we find that only the following possibilities remain:

S	$e(S)$	p	d
$\mathrm{L}_2(5)$	2	2	2
$\mathrm{L}_2(7)$	2	2	3
$\mathrm{PSp}_4(3)$	4	2	$4 \leq d \leq 6$
$\mathrm{U}_4(2)$	4	3	4
$\mathrm{U}_4(3)$	6	2	$6 \leq d \leq 7$

We show that none of these cases arises. Since S is transitive on the set of 1-arcs of Γ, we have by (1) that $\Gamma \cong \Gamma(S, K, KgK)$, for some 2-element $g \in S$, with $K = S_\alpha$. By Remark 2.5, S is transitive on the 1-arcs of the quotient graph Γ_N, and Γ is a cover of Γ_N. So $\Gamma_N \cong \Gamma(S, H, HgH)$, for the same $g \in S$, with $H = S_B$. Note that $|H| = p^d|K|$.

Since $\mathrm{GL}(2, 2)$ is soluble certainly $S \neq \mathrm{L}_2(5)$. If $S = \mathrm{L}_2(7) \cong \mathrm{L}_3(2)$ then $G = S$ and $Y = NS = \mathrm{AGL}(3, 2)$. Since $|S : K| = 8|S : H| > 8$, $|G_\alpha|$ is a proper divisor of 21, and so $|G_\alpha| = 3$ or 7. If $|G_\alpha| = 3$ then $G_\alpha \cong Z_3$, $d_\Gamma = 3$, and $|V\Gamma| = 56$. However, by [16], Γ is an eight-fold cover of Γ_N and as the action induced by Y on $V\Gamma_N$ is 2-transitive it follows that $\Gamma_N = K_7$, whence Γ should have valency 6, which is a contradiction. If $|G_\alpha| = 7$ then $G_\alpha \cong Z_7$, $d_\Gamma = 7$, and $|V\Gamma| = 24$. Arguing as above we have $\Gamma_N \cong K_3$, whence Γ should have valency 2, which is impossible.

Now we look at $S = \mathrm{PSp}_4(3) \cong \mathrm{U}_4(2)$. We have $S \leq G \leq \mathrm{Aut}(S) = S.2$. Suppose first that $p = 2$. By [8, p. 60], d is either 4 or 6. For $d = 4$ the Brauer character involves the irrationalities $b27$ and $z3$ which (see [8, p. 284]) have minimal polynomials of degree 2 over $\mathrm{GF}(2)$. Arguing as in the proof of Lemma 2.7 we conclude that a faithful representation of $\mathrm{PSp}_4(3)$ in characteristic 2 of degree 4 must be over an even degree extension field of $\mathrm{GF}(2)$, so $d \neq 4$. Hence $d = 6$. Since $|N| = 2^6$ divides $|H|$, H contains a Sylow 2-subgroup P of S and $|K|$ is odd. This implies that $d_\Gamma = |\Gamma(\alpha)|$ must be odd. We claim that $K \cap K^g \neq 1$. Suppose to the contrary that $K \cap K^g = 1$. Then $H \cap H^g$ is a Sylow 2-subgroup of S. Since $H \cap H^g$ is normalized by g, it follows that $\langle H \cap H^g, g\rangle$ is a 2-subgroup and hence is equal to $H \cap H^g$. Thus $g \in H \cap H^g$ and so

$\langle H, g \rangle \neq S$ contradicting our remarks before (1) at the beginning of this section. Hence $K \cap K^g \neq 1$. In particular $|K|$ is odd and composite. By [2, p. 26], H is contained in either $2^4{:}A_5$ or $2 \cdot (A_4 \times A_4).2$. Since A_5 has no subgroup of order 15 it follows that $H = 2 \cdot (A_4 \times A_4).2$, $K = Z_3 \times Z_3$, and $d_\Gamma = 3$. Thus $Z(H) = \langle a \rangle \cong Z_2$ and also $Z(H \cap H^g) = \langle a \rangle$. Since $Z(H \cap H^g)$ is a characteristic subgroup of $H \cap H^g$ it follows that $g \in \mathrm{N}_S(\langle a \rangle)$. On the other hand $H = \mathrm{N}_S(\langle a \rangle)$, so that $g \in H$, which is impossible.

Thus if $S = \mathrm{PSp}_4(2) \cong \mathrm{U}_4(2)$ then $p = 3$ and $d = 4$. Here $|S|_3 = 3^4$, so H contains a Sylow 3-subgroup of S and $|K|$ is prime to 3. Now $\langle K, g \rangle = S$ and $g \in \mathrm{N}_S(K \cap K^g) \cap \mathrm{N}_S(H \cap H^g)$. By [2, p. 26], H is contained in either $3_+^{1+2}{:}2A_4$ or $3^3{:}S_4$. So $|K|$ divides 8 and hence d_Γ is 2^2 or 2^3, and $|K \cap K^g| \leq 2$. Thus $|H \cap H^g|$ is 3^4 or 2.3^4 and so $O_3(H \cap H^g)$ is a Sylow 3-subgroup of S which is normalized by g. However, in this case $\langle H, g \rangle$ is contained in the parabolic subgroup $3_+^{1+2}{:}2A_4$ or $3^3{:}S_4$ of S containing H, contradicting the fact that $\langle H, g \rangle = S$.

Finally we consider the case $S = \mathrm{U}_4(3)$. While $\mathrm{U}_4(3)$ certainly has a representation in characteristic 2 of dimension $d = 6$ or 7, since $|\mathrm{U}_4(3)|$ is divisible by 7 and $|\mathrm{GL}(7,2)|$ is not divisible by 7 it follows that $\mathrm{U}_4(3) \nleq \mathrm{GL}(7,2)$. So this representation is not realizable over GF(2). Thus $S \neq \mathrm{U}_4(3)$. ∎

Now we must consider the simple groups of Lie type $S = S(p^e)$. We shall denote by $R_p(S)$ the minimal dimension of a faithful, irreducible, projective KS-module, where K is an algebraically closed field of characteristic p. The values of $R_p(S)$ are given by [10, Table 5.4.C] for all simple groups of Lie type. In the proof of the next lemma we use Lie notation for the simple groups $S(p^e)$.

LEMMA 2.9. *The integer e divides d and $S = S(p^e)$, d/e, and N are as given in one of the lines of Table I or II.*

Proof. The group G is an irreducible subgroup of $\mathrm{GL}(d, p)$. Let f be the largest divisor of d such that G is conjugate to a subgroup of $\mathrm{GL}(d/f, p^f)$. Without loss of generality we may assume that $G \leq \mathrm{GL}(d/f, p^f)$. Then we may regard N as a (d/f)-dimensional faithful, absolutely irreducible projective $\mathrm{GF}(p^f)G$-module. If G were realizable over some proper subfield of $\mathrm{GF}(p^f)$ then we would have a proper divisor c of f with G conjugate to a subgroup of $\mathrm{GL}(d/f, p^c)$. However, in this case N would have a proper G-invariant subgroup of order $p^{cd/f}$, contradicting the fact that G is irreducible on N. Thus G is realizable over no proper subfield of $\mathrm{GF}(p^f)$, so the representation $G < \mathrm{GL}(d/f, p^f)$ satisfies the conditions of [10, Proposition 5.4.6]. Using this result and [10, Remark 5.4.7] we have one of the following:

(i) f divides e and $d/f \geq R_p(S)^{e/f}$; or

(ii) f divides $2e$ (but not e), S is of type ${}^2A_l$, ${}^2D_l$, or ${}^2E_6$ and $d/f \geq R_p(S)^{2e/f}$; or

(iii) f divides $3e$ (but not e), S is of type ${}^3D_4$, and $d/f \geq R_p(S)^{3e/f}$.

Note that, since $|N| = p^d$ divides $|S|_p = p^{et}$, say, we have $d \leq et$. Thus in all cases

$$et \geq d \geq fR_p(S)^{e/(e,f)}. \tag{2}$$

If $(e, f) \neq e$ then using [10, Table 5.4C] we see that (2) does not hold for any simple group of Lie type $S(p^e)$. For example, if $S = A_l(p^e)$ then $R_p(S) = l + 1$ and $t = l(l+1)/2$, and it is easy to see that $f(l+1)^{e/(e,f)} > el(l+1)/2$. Thus we find that $(e, f) = e$; that is, e divides f so we have $f = e$, $2e$, or $3e$ in case (i), (ii), or (iii), respectively, and $t \geq d/e \geq (f/e)R_p(S)$. Again using [10, Table 5.4.C] we see that one of the following holds.

S	f/e	Values for d/e	S	f/e	Values for d/e
$G_2(2^e)$	1	6	${}^2D_l(p^e)$	2	$[4l, l(l-1)]$
$E_7(p^e)$	1	$[56, 63]$	$C_l(p^e)$	1	$[2l, l^2]$
$E_6^{\pm}(p^e)$	1	$[27, 36]$	$B_l(p^e)$	1	$[2l+1, l^2]$
${}^3D_4(p^e)$	1	$[8, 12]$	$A_l^{\pm}(p^e)$	1	$[l+1, l(l+1)/2]$
$D_l^{\pm}$	1	$[2l, l(l-1)]$	${}^2A_l(p^e)$	2	$[2(l+1), l(l+1)/2]$

Now applying [10, Propositions 5.4.11, 5.4.12] it follows that either S is as in Table I or II, or $S = {}^3D_4(p^e) \leq \mathrm{GL}(d/e, p^e)$ and $8 \leq d/e \leq 12$, or $S = A_l^{\pm}(p^e) \leq \mathrm{GL}(d/e, p^e)$ with $d/e = l(l+1)/2$. The fact that p is odd in Table 1, line 2, and Table 2, line 4, will follow from Remark 2.10 below. In the case of $S = {}^3D_4(p^e)$, $|S|$ is divisible by $p^{8e} + p^{4e} + 1$ and as $S \leq \mathrm{GL}(d/e, p^e)$ we must have $d/e = 12$. In the latter case the twisted group $S = {}^2A_l(p^e)$ is realized only over the field $\mathrm{GF}(p^{2e})$ in this representation. ∎

Remark 2.10. In the case of $S = A_l^{\pm}(p^e)$, $d/e = l(l+1)/2$ in Table I, and $S = {}^3D_4(p^e)$, $d/e = 12$ in Table II, we have $|S|_p = p^d$, so S_B contains a Sylow p-subgroup of S, that is, S_B is a parabolic subgroup of S, and $|S_\alpha|$ and $|S : S_B|$ are both prime to p. Thus $|\Gamma(\alpha)|$ is prime to p (since $|\Gamma(\alpha)|$ divides $|S_\alpha|$). If $p = 2$ this implies that the number of 1-arcs of the quotient graph Γ_N, namely, $|S : S_B|\,|\Gamma(\alpha)|$, is odd, which is not the case. Hence p is odd. We have $S_B = UL$, where L is a Levi subgroup and $U = O_p(S_B)$, and $S_\alpha \cong S_\alpha U/U$ is a Hall p'-subgroup of $S_B/U \cong L$. The fact that L has a Hall p'-subgroup places strong restrictions on the possibilities for the parabolic subgroup S_B.

Theorem 1.1 follows from the results of this section.

3. APPLICATIONS OF THEOREM 1.1

In this section we illustrate the way in which Theorem 1.1 can be used. First we explore a fairly general situation in which we can prove that case (b) of Theorem 1.1 does not arise. We need to avoid the possibilities of certain factorizations of almost simple groups involving the given almost simple group G. To explain the restriction on G we need some notation concerning factorizations of almost simple groups.

DEFINITION 3.1. For an almost simple group H with socle T, a factorization $H = AB$ is called a *max^+ factorization* if A and B are maximal subgroups of H and neither A nor B contains T. A factorization $H = AB$ is called a *max^- factorization* if A and B are maximal among the subgroups of H which do not contain T, but at least one of A, B is not maximal in H.

DEFINITION 3.2. Suppose that G is almost simple with socle S and that G acts faithfully and transitively on a set X. We shall say that the permutation group (G, X) *has property $\mathscr{P}$* if the following holds (where $x \in X$).

If H is an almost simple group with socle $T \neq S$, H has a maximal subgroup (isomorphic to) G such that $G < H < \mathrm{Sym}(X)$, and there is a $\max^+$ or a $\max^-$ factorization $H = GK$ with $G \cap K \geq G_x$, then $H \cong A_n$ or S_n and $K \cong A_{n-1}$ or S_{n-1}, respectively, where $n = |H : K|$.

Imposing property $\mathscr{P}$ on the transitive group $(G, V\Gamma)$ together with a restriction on the valency of Γ helps us to avoid case (b) of Theorem 1.1. Recall that, for a nonabelian simple group S, $m(S)$ denotes the minimal index of a proper subgroup of S.

THEOREM 3.3. *Let G be an almost simple group with socle S and let Γ be a connected, nonbipartite, G-locally primitive graph of valency $d_\Gamma \leq m(S)/2$. Suppose that $(G, V\Gamma)$ has property $\mathscr{P}$, that S is not semiregular on vertices, and that $\mathrm{C}_{\mathrm{Aut}\,\Gamma}(S) = 1$. Then either $S \leq \mathrm{Aut}\,\Gamma \leq \mathrm{Aut}(S)$ or case* (c) *of Theorem* 1.1 *holds.*

Proof. Suppose that case (c) of Theorem 1.1 does not hold and that $\mathrm{Aut}\,\Gamma \not\leq \mathrm{Aut}(S)$. We may assume that $G = \mathrm{Aut}\,\Gamma \cap \mathrm{Aut}(S)$. Then $G \neq \mathrm{Aut}\,\Gamma$ and by Theorem 1.1 there is an almost simple subgroup Y of $\mathrm{Aut}\,\Gamma$ with socle T such that $T \neq S$ and G is maximal in Y. If K is maximal among the subgroups of Y which contain Y_α but do not contain T, then

$Y = GK$ and so, by property $\mathscr{P}$, $Y = A_n$ or S_n and $K = A_{n-1}$ or S_{n-1}, respectively, where $n = |Y : K|$, and n divides $|V\Gamma|$. Set $B := Y_{\alpha\beta}$. Since G is transitive on the 1-arcs of Γ, $Y = GB$. Let p be a prime dividing $|B|$ and let P be a Sylow p-subgroup of B. Then $P \neq 1$, and since Γ is connected it follows that $P^{\Gamma(\alpha)} \neq 1$. Hence $p \leq |\Gamma(\alpha) \setminus \{\beta\}| < m(S)/2$. Then from $|Y| = |G|.|B|/|G \cap B|$ it follows that each prime r dividing $|Y|$, such that $r \geq m(S)/2$, must divide $|G|$. By Bertrand's postulate (see [5, p. 68]) there is a prime r satisfying $n \geq r > n/2 \geq m(S)/2$, and this prime must therefore divide $|G|$. An element of G of order r must be a cycle of length r (in the natural representation of Y of degree n), and it follows that G is primitive of degree n. If $r \leq n - 3$ then by a result of Jordan (see [24, Theorem 13.9]), G contains A_n which is not the case. Thus there are no primes r satisfying $n/2 < r \leq n - 3$, and hence $n \leq 7$. Since S is a simple proper subgroup of T, $n \neq 5$. If $n = 7$ then 35 divides $|G|$ and again G contains A_n, which is a contradiction. Hence $n = 6$ and $S \cong A_5$. However, this means that Γ has valency at most $m(S)/2 < 3$; that is, Γ is a cycle, which is not the case since $S \leq \mathrm{Aut}\,\Gamma$. ∎

As another application of Theorem 1.1 we prove Theorem 1.3. Its proof will be given separately for the Suzuki and Ree groups in the following two lemmas. For a $(G, 2)$-arc transitive graph Γ, G_α is 2-transitive on $\Gamma(\alpha)$ and hence G_α is primitive on $\Gamma(\alpha)$. Thus if G is transitive on $V\Gamma$ and transitive on the 2-arcs of Γ, then Γ must be G-locally primitive. Because Theorem 1.3 is used in [3] and [4] to determine the automorphism groups of the graphs constructed in those papers, we need to clarify the ways in which some results from [3] and [4] are used in the proof of Lemmas 3.4 and 3.5 below. The results from [3] and [4] used here are those which prove that $C_{\mathrm{Aut}\,\Gamma}(S) = 1$ (where $S = \mathrm{Sz}(q)$ or $\mathrm{Ree}(q)$) for each of the possibilities for G_α. This condition then allows us to apply Theorem 1.1. We considered it appropriate to prove Lemmas 3.4 and 3.5 in this paper as an illustration of the use of our general result, Theorem 1.1.

LEMMA 3.4. *Let* $\mathrm{Sz}(q) \leq G \leq \mathrm{Aut}(\mathrm{Sz}(q))$, *where* $q = 2^m$ *for some odd integer* $m \geq 3$, *and let* Γ *be a connected* $(G, 2)$-*arc transitive graph. Then* $\mathrm{Sz}(q) \leq \mathrm{Aut}\,\Gamma \leq \mathrm{Aut}(\mathrm{Sz}(q))$.

Proof. As in (1), we may assume that $\Gamma = \Gamma(G, H, HgH)$ for some core-free subgroup H and 2-element g. It was shown in [3, Proposition 4.4] that Γ is not bipartite and that $S := \mathrm{Sz}(q)$ is transitive on the 1-arcs of Γ, so in particular S is not semiregular of $V\Gamma$. Moreover, all possibilities for the subgroup H have been determined in [3, Proposition 4.4]; by [3, Proposition 4.4 and Lemma 7.3] in all but one case $C_{\mathrm{Sym}(V\Gamma)}(S) = 1$ and in the remaining case $C_{\mathrm{Aut}\,\Gamma}(S) = 1$ (although S has nontrivial centralizer in

Sym($V\Gamma$)). Thus Theorem 1.1 applies. We need to show that Theorem 1.1(b) cannot hold.

Suppose to the contrary that there is an almost simple subgroup Y of $\mathrm{Aut}\,\Gamma$ such that $G = \mathrm{N}_Y(G)$ is a maximal subgroup of Y, and $\mathrm{soc}\,Y \neq S$. Since S is transitive on the 1-arcs of Γ, $Y = Y_{\alpha\beta}S$, where $Y_{\alpha\beta}$ is the stabilizer of a 1-arc (α, β) of Γ. Then by [13] and [14] either $\mathrm{soc}\,Y = \mathrm{Sp}_4(q) = O_4^+(q)S$, or Y has socle A_n and $Y = AS$, where A is a subgroup of Y not containing A_n such that either $A_{n-k} \lhd A \leq S_{n-k} \times S_k$, for some $k \leq 5$ with S acting k-homogeneously of degree n, or n is 6, 8, or 10. In the former case set $Y_1 := \mathrm{Sp}_4(q) \cap Y_{\alpha\beta}$. Then by [13, 5.1.7b] we may assume that $Y_1 \leq \mathrm{O}_4^+(q)$. Now $\mathrm{soc}\,Y = \mathrm{Sp}_4(q) = \mathrm{O}_4^+(q)S = Y_1 S$. Write $s = |S_{\alpha\beta}|$. Then we have $|\mathrm{O}_4^+(q) : Y_1| = |S \cap \mathrm{O}_4^+(q)|/s$. From $\mathrm{soc}\,Y = \mathrm{O}_4^+(q)S$ it follows that $|S \cap \mathrm{O}_4^+(q)| = 2(q-1)$, and hence

$$|\mathrm{O}_4^+(q) : Y_1| = 2(q-1)/s. \tag{3}$$

On the other hand $\mathrm{O}_4^+(q)$ has no subgroups of index $2(q-1)/s$, for $1 \leq s < q-1$. Hence s is $q-1$ or $2(q-1)$. Since $|S_\alpha| > 2(q-1)$ it follows that S_α is contained in a parabolic subgroup of S. It then follows from [3, Proposition 4.4] that $d = q$ and $S_\alpha \cong Z_2^m{:}Z_{2^m-1}$. Hence $s = q-1$ and $|\mathrm{O}_4^+(q) : Y_1| = 2$, so $Y_1 = \mathrm{L}_2(q) \times \mathrm{L}_2(q)$. Now $|(\mathrm{soc}\,Y)_\alpha| = q|Y_1|$ and $\mathrm{soc}\,Y = (\mathrm{soc}\,Y)_\alpha S = BS$, where $(\mathrm{soc}\,Y)_\alpha \leq B < \mathrm{soc}\,Y$ with B maximal in $\mathrm{soc}\,Y$. However, by [13] there are no such factorizations for $\mathrm{soc}\,Y \cong \mathrm{Sp}_4(q)$. Hence $\mathrm{soc}\,Y \neq \mathrm{Sp}_4(q)$. In the latter case, since $m(\mathrm{Sz}(q)) = q^2 + 1 > 10$, n is not one of 6, 8, or 10. Since $\mathrm{Sz}(q)$ has no k-homogeneous representation for $k \geq 3$ by [9], we must have $k \leq 2$. Also $n \geq q^2 + 1$. Let p_m denote the largest prime divisor of $|\mathrm{Aut}\,\Gamma|$. By [3, Proposition 4.4], the valency of Γ is less than $2q$, which implies that the largest prime divisor of $|(\mathrm{Aut}\,\Gamma)_\alpha|$ is less than $2q$. Also (see [21]) the largest prime divisor of $|\mathrm{Sz}(q)|$ is less than $2q$ and $\mathrm{Sz}(q)$ is transitive on $V\Gamma$. Thus $p_m \leq 2q$. However, there is a prime r satisfying $2q < r < q^2 \leq n$, and hence $|\mathrm{Aut}\,\Gamma|$ is divisible by r, contradicting $p_m \leq 2q$. Thus no such subgroup Y exists. ∎

LEMMA 3.5. *Let $q = 3^m$ for some odd integer $m \geq 3$, and let Γ be a connected $(\mathrm{Ree}(q), 2)$-arc transitive graph. Then $\mathrm{Ree}(q) \leq \mathrm{Aut}\,\Gamma \leq \mathrm{Aut}(\mathrm{Ree}(q))$.*

Proof. As in (1), we may assume that $\Gamma = \Gamma(G, H, HgH)$ for some core-free subgroup H and 2-element g. Since $G := \mathrm{Ree}(q)$ has no subgroups of index 2, Γ is not bipartite, and since Γ is $(G, 2)$-arc transitive G is certainly not semiregular on vertices. All possibilities for the subgroup H have been determined in [4, Proposition 3.5]; and by [4, Proposition 3.5 and Lemmas 5.3 and 6.3] in all cases $\mathrm{C}_{\mathrm{Aut}\,\Gamma}(G) = 1$. Thus Theorem 1.1 applies, and we need to show that case (b) cannot hold. Suppose to the contrary that there is an almost simple subgroup Y of $\mathrm{Aut}\,\Gamma$ with $\mathrm{soc}\,Y \neq G$

such that $N_Y(G)$ (a subgroup of $\mathrm{Aut}(G)$) is maximal in Y. Since G is transitive on the 1-arcs of Γ, $Y = GY_{\alpha\beta}$ for a 1-arc (α, β) of Γ. By [13] and [14] either $\mathrm{soc}\, Y = G_2(q) = \mathrm{SL}(3,q)G$, with $(\mathrm{soc}\, Y)_{\alpha\beta} \le \mathrm{SL}(3,q)$, or Y has socle A_n and $Y = AG$ with A a subgroup of Y not containing A_n such that either $A_{n-k} \lhd A \le S_{n-k} \times S_k$, for some $1 \le k \le 5$ and G is k-homogeneous of degree n, or n is 6, 8, or 10. In the first case set $Y_1 := G_2(q) \cap Y_{\alpha\beta}$. Now $G_2(q) = \mathrm{SL}(3,q)G = Y_1 G$ and $|G \cap \mathrm{SL}(3,q)| = q - 1$ (see [13, Proposition B of Section 8.3]). Set $s = |G_{\alpha\beta}|$. Then arguing as in the proof of Lemma 3.4,

$$|\mathrm{SL}(3,q) : Y_1| = (q-1)/s. \tag{4}$$

Suppose that $s = q - 1$. Then $G_{\alpha\beta} = G \cap \mathrm{SL}(3,q)$ has order $q - 1$ and by [4, Proposition 3.5], the only possibility for $H = G_\alpha$ is $H \cong Z_3^m : Z_{3^m - 1}$. This means that $(\mathrm{soc}\, Y)_\alpha$ has order $q|Y_1|$ and $\mathrm{soc}\, Y = G_2(q) = G(\mathrm{soc}\, Y)_\alpha = GB$, where $(\mathrm{soc}\, Y)_\alpha \le B < G_2(q)$ with B maximal in $G_2(q)$. However, by [13] and [14] there are no such factorizations of $G_2(q)$. Hence $1 < |\mathrm{SL}(3,q) : Y_1| \le q - 1$. However, the minimal index of a proper subgroup of $\mathrm{SL}(3,q)$ is $(q^3 - 1)/(q-1) = q^2 + q + 1$, and hence $\mathrm{SL}(3,q)$ has no proper subgroup Y_1 satisfying (4), which is a contradiction. The second possibility with $Y = A_n$ leads to a contradiction by an argument similar to that in the proof of Lemma 3.4. ∎

REFERENCES

1. R. W. Baddeley and C. E. Praeger, "On Primitive Overgroups of Quasiprimitive Permutation Groups," technical report, University of Leicester, 1996.
2. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, "Atlas of Finite Groups," Clarendon Press, Oxford, 1985.
3. X. G. Fang and C. E. Praeger, "Finite Two Arc Transitive Graphs Admitting a Suzuki Simple Group," *Comm. Algebra* (to appear).
4. X. G. Fang and C. E. Praeger, "Finite Two Arc Transitive Graphs Admitting a Ree Simple Group," *Comm. Algebra* (to appear).
5. M. Hall, "The Theory of Groups," Macmillan, New York, 1959.
6. M. Issacs, "Character Theory of Finite Groups," Academic Press, New York, 1976.
7. G. D. James, On the minimal dimensions of irreducible representations of symmetric groups, *Math. Proc. Cambridge Philos. Soc.* **94** (1983), 417–424.
8. C. Jansen, K. Lux, R. Parker, and R. Wilson, "An Atlas of Brauer Characters," Clarendon Press, Oxford, 1995.
9. W. M. Kantor, k-homogeneous groups, *Math. Z.* **124** (1972), 274–292.
10. P. Kleidman and M. Liebeck, "The Subgroup Structure of the Finite Classical Groups," Cambridge Univ. Press, Cambridge, UK, 1990.
11. V. Landazuri and G. M. Seitz, On the minimal degrees of projective representations of the finite Chevalley groups, *J. Algebra* **32** (1974), 418–473.
12. M. W. Liebeck, On the orders of maximal subgroups of the finite classic groups, *Proc. London Math. Soc. (3)* **50** (1985), 426–446.

13. M. W. Liebeck, C. E. Praeger, and J. Saxl, The maximal factorizations of the finite simple groups and their automorphism groups, *Mem. Amer. Math. Soc.* **86**(432) (1990).
14. M. W. Libeck, C. E. Praeger, and J. Saxl, On factorizations of almost simple groups, *J. Algebra* **185** (1996), 409–419.
15. M. W. Liebeck, C. E. Praeger, and J. Saxl, On the automorphism groups of finite vertex-primitive, edge-transitive graphs and directed graphs, in preparation.
16. C. E. Praeger, Imprimitive symmetric graphs, *Ars Combin.* **19A** (1985), 149–163.
17. C. E. Praeger, The inclusion problem for finite primitive permutation groups, *Proc. London Math. Soc. (3)* **60** (1990), 68–88.
18. C. E. Praeger, On the O'Nan–Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs, *J. London Math. Soc. (2)* **47** (1993), 227–239.
19. R. Ree, A family of simple groups associated with the simple Lie algebra of type (G_2), *Amer. J. Math* **83** (1961), 432–462.
20. G. O. Sabidussi, Vertex-transitive graphs, *Monatsh. Math* **68** (1964), 426–438.
21. M. Suzuki, On a class of doubly transitive groups, *Ann. of Math.* **75**(1) (1962), 105–145.
22. A. Wagner, The faithful linear representations of least degree of S_n and A_n over a field of characteristic 2, *Math. Z.* **151** (1976), 127–137.
23. A. Wagner, The faithful linear representations of least degree of S_n and A_n over a field of odd characteristic, *Math. Z.* **154** (1977), 103–114.
24. H. Wielandt, "Finite Permutation Groups," Academic Press, New York, 1964.