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ScienceDirect
Topology and its Applications 154 (2007) 144–154
Topology and its Applications
www.elsevier.com/locate/topol

Resolvability of spaces having small spread or extent ☆

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Received 28 December 2005; accepted 10 April 2006

Abstract

In a recent paper O. Pavlov proved the following two interesting resolvability results:

(1) If a T_1-space X satisfies $\Delta(X) > \mathrm{ps}(X)$ then X is maximally resolvable.
(2) If a T_3-space X satisfies $\Delta(X) > \mathrm{pe}(X)$ then X is ω-resolvable.

Here $\mathrm{ps}(X)$ ($\mathrm{pe}(X)$) denotes the smallest successor cardinal such that X has no discrete (closed discrete) subset of that size and $\Delta(X)$ is the smallest cardinality of a non-empty open set in X.

In this note we improve (1) by showing that $\Delta(X) > \mathrm{ps}(X)$ can be relaxed to $\Delta(X) \geqslant \mathrm{ps}(X)$, actually for an arbitrary topological space X. In particular, if X is any space of countable spread with $\Delta(X) > \omega$ then X is maximally resolvable.

The question if an analogous improvement of (2) is valid remains open, but we present a proof of (2) that is simpler than Pavlov's.

MSC: 54A25; 54B05

Keywords: κ-resolvable space; Maximally resolvable space; Dispersion character; Spread; Extent

1. Introduction

Given a cardinal $\kappa > 1$, a topological space is called *κ-resolvable* iff it contains κ many disjoint dense subsets. Denoting by $\tau^*(X)$ the family of non-empty open subsets of a topological space X, we say that the space X is *maximally resolvable* iff it is $\Delta(X)$-resolvable, where $\Delta(X) = \min\{|G|: G \in \tau^*(X)\}$ is the so-called *dispersion character* of X. A space is called $(<\kappa)$-resolvable iff it is μ-resolvable for all $\mu < \kappa$. In this introduction we shall give three lemmas that provide sufficient conditions for κ-resolvability. Finally, a space that is not κ-resolvable is also called *κ-irresolvable*.

☆ The preparation of this paper was supported by OTKA Grant no. 37758.
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0166-8641/$ – see front matter
doi:10.1016/j.topol.2006.04.004

El'kin proved in [3] that, for any cardinal κ, every space may be written as the disjoint union of a hereditarily κ-irresolvable open subset and a κ-resolvable closed subset. As Pavlov observed in the introduction of [10], this statement has the following reformulation.

Lemma 1.1. *A topological space X is κ-resolvable iff every non-empty open subspace of X includes a non-empty κ-resolvable subset, in other words: iff X has a π-network consisting of κ-resolvable subsets.*

For any topological space X we let $\mathrm{ls}(X)$ denote the minimum number of left-separated subspaces needed to cover X. The following lemma is implicit in the proof of [10, Theorem 2.8] and easily follows from the well-known fact that every space has a *dense* left-separated subspace, see e.g. [7, 2.9.c].

Lemma 1.2. *If for each $U \in \tau^*(X)$ we have $\mathrm{ls}(U) \geqslant \kappa$, that is no non-empty open set in X can be covered by fewer than κ many left separated sets, then X is κ-resolvable.*

Our next lemma generalizes Propositions 2.3 and 3.3 from [10]. We believe that our present approach is not only more general but also simpler than that in [10]. To formulate the lemma, we need to introduce a piece of notation.

Given a family of sets $\mathcal{A}$ and a cardinal κ, we denote by $S_\kappa(\mathcal{A})$ the collection of all *disjoint* subfamilies of $\mathcal{A}$ of size less than κ, i.e.

$$S_\kappa(\mathcal{A}) = \{\mathcal{A}' \in [\mathcal{A}]^{<\kappa}\colon\ \mathcal{A}' \text{ is disjoint}\}.$$

Lemma 1.3. *Let us be given a topological space X, a dense set $D \subset X$, an infinite cardinal $\kappa \geqslant |D|$, moreover a family $\mathcal{I} \subset \mathcal{P}(X)$ of subsets of X. If for each $x \in D$ and for any $\mathcal{Y} \in S_\kappa(\mathcal{I})$ there is a set $Z \in \mathcal{I}$ such that $\bigcup \mathcal{Y} \cap Z = \emptyset$ and $x \in \overline{Z}$ then X is κ-resolvable.*

Proof. Let $\{x_\alpha\colon\ \alpha < \kappa\} = D$ be a κ-abundant enumeration of D, that is for any point $x \in D$ we have $a_x = \{\alpha\colon\ x_\alpha = x\} \in [\kappa]^\kappa$. By a straightforward transfinite recursion on $\alpha < \kappa$ we may then choose sets $Z_\alpha \in \mathcal{I} \cap \mathcal{P}(X \setminus \bigcup_{\nu<\alpha} Z_\nu)$ with $x_\alpha \in \overline{Z_\alpha}$ for all $\alpha < \kappa$. (Note that we have $\{Z_\nu\colon\ \nu < \alpha\} \in S_\kappa(\mathcal{I})$ along the way.)

For any ordinal $i < \kappa$ and for any point $x \in D$ let α_i^x be the ith element of the set a_x and set

$$D_i = \bigcup \{Z_{\alpha_i^x}\colon\ x \in D\}.$$

Then clearly $D \subset \overline{D_i}$, hence $\{D_i\colon\ i < \kappa\}$ is a disjoint family of dense sets, witnessing that X is κ-resolvable. □

As an illustration, note that if $|X| = \Delta(X) = \kappa > \lambda$ and $t(x, X) \leqslant \lambda$ holds for all points $x \in D$ of a set D which is dense in the space X, then D, X, κ, and $\mathcal{I} = [X]^{\leqslant\lambda}$ satisfy the conditions of Lemma 1.3 and so X is κ-resolvable. Thus we obtain the following result as an immediate corollary of Lemma 1.3.

Corollary 1.4. *If $\Delta(X) > \sup\{t(x, X)\colon\ x \in D\}$ for some dense set $D \subset X$ then X is maximally resolvable. In particular, if $\Delta(X) > t(X)$ then X is maximally resolvable.*

The second statement is a theorem of Pytkeev from [11].

2. Improving Pavlov's result concerning spread

As was mentioned in the abstract, in [10] Pavlov defined $\mathrm{ps}(X)$ as the smallest successor cardinal such that X has no discrete subset of that size. We recall from [7, 1.22] the related definition of $\hat{s}(X)$ that is the smallest uncountable cardinal such that X has no discrete subset of that size. Clearly, one has $\hat{s}(X) \leqslant \mathrm{ps}(X)$ and $\hat{s}(X) = \mathrm{ps}(X)$ iff $\hat{s}(X)$ is a successor. Finally, let us define $\mathrm{rs}(X)$ as the smallest uncountable regular cardinal such that X has no discrete subset of that size. Then we have $\hat{s}(X) \leqslant \mathrm{rs}(X) \leqslant \mathrm{ps}(X)$ and $\hat{s}(X) = \mathrm{rs}(X)$ iff $\hat{s}(X)$ is regular.

In [10] it was shown that if a space X satisfies $\Delta(X) > \mathrm{ps}(X)$ then X is maximally (i.e. $\Delta(X)$) resolvable. The aim of this section is to improve this result by showing that the assumption $\Delta(X) > \mathrm{ps}(X)$ can be relaxed to $\Delta(X) \geqslant \mathrm{rs}(X)$.

Before doing that, however, we have to give an auxiliary result that involves the cardinal function $\mathrm{h}(X)$, or more precisely its "hatted" version $\hat{\mathrm{h}}(X)$. We recall that $\hat{\mathrm{h}}(X)$ is the smallest uncountable cardinal such that X has no right separated subset of that size, or equivalently, the smallest uncountable cardinal κ with the property that any family $\mathcal{U}$ of open sets in X has a subfamily $\mathcal{V}$ of size $< \kappa$ such that $\bigcup \mathcal{V} = \bigcup \mathcal{U}$, see e.g. [7, 2.9.b].

Lemma 2.1. *If κ is an uncountable regular cardinal and*

$$|X| \geqslant \kappa \geqslant \hat{\mathrm{h}}(X)$$

then X contains a κ-resolvable subspace X^*.

Proof. We can assume without loss of generality that $X = \langle \kappa, \tau \rangle$. Let us denote by $\mathrm{NS}(\kappa)$ the ideal of non-stationary subsets of κ and set $\mathcal{G} = \{U \in \tau \colon U \in \mathrm{NS}(\kappa)\}$. Since $\hat{\mathrm{h}}(X) \leqslant \kappa$ there is $\mathcal{G}' \in [\mathcal{G}]^{<\kappa}$ with $\bigcup \mathcal{G}' = \bigcup \mathcal{G} = G$. Then $G \in \mathrm{NS}(\kappa)$ because the ideal $\mathrm{NS}(\kappa)$ is κ-complete.

Let us now consider the set

$$T = \{x \in \kappa \colon \exists C_x \subset \kappa \text{ club } (\forall S \subset C_x \text{ if } S \in \mathrm{NS}(\kappa) \text{ then } x \notin \overline{S})\}.$$

Claim 2.1.1. $T \in \mathrm{NS}(\kappa)$.

Assume, on the contrary, that T is stationary in κ. Fix for each $x \in T$ a club C_x as above. Then the diagonal intersection

$$C = \triangle\{C_x \colon x \in T\}$$

is again club and so $C \cap T$ is stationary in κ as well. We may then choose a set $S \in [C \cap T]^\kappa$ that is non-stationary. But then for each $x \in S$ we have

$$S \setminus (x+1) \subset C \setminus (x+1) \subset C_x,$$

hence by the choice of C_x we have $x \notin \overline{S \setminus (x+1)}$. Consequently, S is right separated in its natural well-ordering, contradicting the assumption $\hat{\mathrm{h}}(X) \leqslant \kappa$, and so our claim has been verified.

Finally, put $X^* = X \setminus (G \cup T)$ and $\mathcal{I} = \mathrm{NS}(\kappa) \cap \mathcal{P}(X^*)$. Then Lemma 1.3 can be applied to the space X^*, with itself as a dense subspace, the cardinal κ, and the family $\mathcal{I}$. Indeed, for any point $x \in X^*$ and for any non-stationary set $Y \subset X^*$ there is a club set $C \subset X^* \setminus Y$, and then $x \notin T$ implies that $x \in \overline{Z}$ for some non-stationary set $Z \subset C$. (We have, of course, used here that $\mathcal{I}$ is κ-complete.) This shows that X^* is indeed κ-resolvable. □

We are now ready to formulate and prove the promised improvement of Pavlov's theorem.

Theorem 2.2. *Let X be a space and κ be a regular cardinal such that*

$$\hat{s}(X) \leqslant \kappa \leqslant \Delta(X),$$

then X is κ-resolvable. Consequently, if $\Delta(X) \geqslant \mathrm{rs}(X)$ holds for a space X then X is maximally resolvable. In particular, any space of countable spread and uncountable dispersion character is maximally resolvable.

Proof. In view of Lemma 1.1 it suffices to show that any non-empty open subset G of X includes a κ-resolvable subspace. To this end, note that, trivially, for each $G \in \tau^*(X)$ we have either

(i) $\mathrm{ls}(H) \geqslant \kappa$ for all $H \in \tau^*(G)$, or
(ii) $\mathrm{ls}(H) < \kappa$ for some $H \in \tau^*(G)$.

In case (i) G itself is κ-resolvable by Lemma 1.2. In case (ii) we claim that $\hat{\mathrm{h}}(H) \leqslant \kappa$ holds true and therefore H (and hence G) contains a κ-resolvable subset by Lemma 2.1. Assume, on the contrary, that $R \subset H$ is right-separated and has cardinality κ. Since $H = \bigcup\{L_\alpha \colon \alpha < \mathrm{ls}(H)\}$, where the sets L_α are all left-separated, there is an $\alpha < \mathrm{ls}(H) < \kappa$ such that $|R \cap L_\alpha| = \kappa$ because κ is regular. But then the subspace $R \cap L_\alpha$ is both right and left separated, hence (see e.g. [7, 2.12]) it contains a discrete subset of size $|R \cap L_\alpha| = \kappa$, contradicting our assumption that $\hat{s}(X) \leqslant \kappa$.

If $\Delta(X)$ is regular then this immediately yields that X is maximally resolvable, while if $\Delta(X)$ is singular then, as $\mathrm{rs}(X)$ is regular, we have

$$\Delta(X) > \mathrm{rs}(X)^+ \geqslant \mathrm{ps}(X),$$

hence Pavlov's result [10, 2.9] may be applied to get the second part, of which the third is a special case. (We should mention here that, formally, the reference to Pavlov's result applies only to T_1 spaces as in [10] all spaces are assumed to be T_1. However, a closer examination shows that the result we referred to is proved there without the use of any separation axiom. Below we shall present our version of this proof.) □

It is natural to raise the question if Theorem 2.2 could be further improved by replacing $\mathrm{rs}(X)$ with $\hat{s}(X)$ in it. Of course, this is really a problem only in the case when

$$\Delta(X) = \hat{s}(X) = \lambda$$

is a singular cardinal. Recall now that Hajnal and Juhász proved in [6] (see also [7, 4.2]) that $\hat{\mathrm{s}}(X)$ cannot be singular *strong limit* for a Hausdorff space X. Consequently, the above mentioned strengthening is valid for Hausdorff spaces provided that all singular cardinals are strong limit, in particular if GCH holds.

Corollary 2.3. *Assume that for every (infinite) cardinal κ the power 2^κ is a finite successor of κ (or equivalently, all singular cardinals are strong limit). Then every Hausdorff space X satisfying $\Delta(X) \geqslant \hat{\mathrm{s}}(X)$ is maximally resolvable.*

It is also known (see e.g. [7, 4.3]) that $\hat{\mathrm{s}}(X)$ cannot have countable cofinality for a strongly Hausdorff, in particular for a T_3 space X. Hence the first interesting ZFC question that is left open by Theorem 2.2 is the following.

Problem 2.4. Assume that X is a T_3 space satisfying

$$\hat{\mathrm{s}}(X) = \Delta(X) = \aleph_{\omega_1}.$$

Is X then (maximally) resolvable?

It is clear that if in Theorem 2.2 we have $\Delta(X) = \lambda > \mathrm{rs}(X)$ then the first part may be applied to any regular cardinal κ with $\mathrm{rs}(X) \leqslant \kappa \leqslant \lambda$, hence if λ is singular then we obtain that X is $(<\lambda)$-resolvable without any reference to Pavlov's result. This is of significance because the proof of Pavlov's theorem in the case when $\Delta(X)$ is singular is rather involved. However, if in addition λ has countable cofinality then no reference to Pavlov's proof is needed because of the following result of Bhaskara Rao.

Theorem. (Bhaskara Rao [1]) *If $\mathrm{cf}(\lambda) = \omega$ and the space X is $(<\lambda)$-resolvable then X is also λ-resolvable.*

The question if the analogous result can be proved for singular cardinals of uncountable cofinality is one of the outstanding open problems in the area of resolvability and was already formulated in [8]. We just repeat it here.

Problem 2.5. Assume that λ is a singular cardinal with $\mathrm{cf}(\lambda) > \omega$ and the space X is $(<\lambda)$-resolvable. Is it true then that X is also λ-resolvable?

We close this section by giving a partial affirmative answer to Problem 2.5. At the same time we shall also show how the first part of Theorem 2.2 implies the second in case $\Delta(X)$ is singular, thus making our proof of 2.2 self-contained. To do this, we shall first fix some notation.

Definition 2.6. For any space X we let $\mathcal{D}(X)$ denote the family of all dense subsets of X. Next, we set

$$\mathcal{F}(X) = \bigcup \{\mathcal{D}(U):\ U \in \tau^*(X)\};$$

we call the members of $\mathcal{F}(X)$, i.e. dense subsets of (non-empty) open sets, *fat* sets in X.

For a subspace $Y \subset X$ and a cardinal ν we let

$$\mathcal{H}(Y, \nu) = \mathcal{F}(X) \cap [Y]^{\leqslant \nu},$$

in other words, $\mathcal{H}(Y,\nu)$ is the family of all fat (in X!) subsets of Y of size at most ν. It is easy to see that if $c(X)\leqslant\nu$ and $\mathcal{H}(Y,\nu)$ is non-empty then there is a member $H(Y,\nu)\in\mathcal{H}(Y,\nu)$ of maximal closure, i.e. such that

$$\overline{H(Y,\nu)}=\overline{\bigcup\mathcal{H}(Y,\nu)}.$$

(If $\mathcal{H}(Y,\nu)$ is empty then we set $H(Y,\nu)=\emptyset$.) Clearly, if $Y\subset Z\subset X$ and $c(X)\leqslant\nu$ then we have

$$\overline{H(Y,\nu)}\subset\overline{H(Z,\nu)}.$$

Finally, we define the *local density* $d_0(X)$ of the space X by

$$d_0(X)=\min\{d(U)\colon U\in\tau^*(X)\}.$$

Clearly, we have

$$d_0(X)=\min\{|A|\colon A\in\mathcal{F}(X)\}=\min\{\Delta(D)\colon D\in\mathcal{D}(X)\}.$$

The following result is obvious but very useful.

Lemma 2.7. *Let X be a space and λ a singular cardinal such that every $D\in\mathcal{D}(X)$ is $({<}\lambda)$-resolvable. Then X is λ-resolvable.*

As an immediate consequence of Lemma 2.7 and of the first part of Theorem 2.2 we obtain that if λ is singular and $s(X)<\lambda\leqslant d_0(X)$ then X is λ-resolvable. (Of course, here $s(X)<\lambda$ is equivalent with $\hat{s}(X)<\lambda$ or with $\mathrm{ps}(X)<\lambda$.)

The following lemma shows that, under certain simple and natural conditions, if a space X is not μ-resolvable for some cardinal μ then some open set $V\in\tau^*(X)$ satisfies a condition just slightly weaker than $\mu\leqslant d_0(V)$.

Lemma 2.8. *Let X and μ be such that $c(X)<\mu\leqslant\Delta(X)$. Then either X is μ-resolvable or*

(∗) *there is $V\in\tau^*(X)$ such that for each $\kappa<\mu$ there is $T\in[V]^{<\mu}$ with $d_0(V\setminus T)>\kappa$.*

If μ is regular then $V\in\tau^*(X)$ and $T\in[V]^{<\mu}$ may even be chosen so that $d_0(V\setminus T)\geqslant\mu$.

Proof. Let us first consider the case when μ is regular and assume that for all $V\in\tau^*(X)$ and $T\in[V]^{<\mu}$ we have $d_0(V\setminus T)<\mu$. We define pairwise disjoint dense sets $D_\alpha\in\mathcal{D}(X)\cap[X]^{<\mu}$ for $\alpha<\mu$ by transfinite recursion as follows.

Assume that $\{D_\beta\colon \beta\in\alpha\}\subset\mathcal{D}(X)\cap[X]^{<\mu}$ have already been defined and set $T=\bigcup\{D_\beta\colon \beta\in\alpha\}$, then $|T|<\mu$ as μ is regular. Let $\mathcal{W}$ be a maximal disjoint collection of open sets $W\in\tau^*(X)$ such that $d(W\setminus T)<\mu$. By our assumption, then $\bigcup\mathcal{W}$ is dense in X and hence so is $\bigcup\{W\setminus T\colon W\in\mathcal{W}\}$. So if for each $W\in\mathcal{W}$ we fix $D_W\in\mathcal{D}(W\setminus T)$ with $|D_W|<\mu$ then $D_\alpha=\bigcup\{D_W\colon W\in\mathcal{W}\}$ is dense in X as well and clearly $|D_\alpha|<\mu$. The family $\{D_\alpha\colon \alpha<\mu\}$ witnesses that X is μ-resolvable.

So let us assume now that μ is singular and fix a strictly increasing sequence $\langle\mu_\alpha\colon \alpha<\mathrm{cf}(\mu)\rangle$ of regular cardinals converging to μ with $c(X)\cdot\mathrm{cf}(\mu)<\mu_0$.

We then define a $\mathrm{cf}(\mu)\times\mu$ type matrix $\{A^\alpha_\xi\colon \alpha<\mathrm{cf}(\mu),\ \xi<\mu\}$ of pairwise disjoint subsets of X, column by column in $\mathrm{cf}(\mu)$ steps, as follows:

$$X_\alpha=X\setminus\bigcup\{A^\beta_\xi\colon \beta<\alpha,\ \xi<\mu\},$$
$$A^\alpha_\xi=\mathrm{H}\Big(X_\alpha\setminus\bigcup\{A^\alpha_\zeta\colon \zeta<\xi\},\mu_\alpha\Big).$$

Observe that we have $|A^\alpha_\xi|\leqslant\mu_\alpha$, moreover

$$\overline{A^\alpha_\xi}\supseteq\overline{A^\alpha_\eta}\quad\text{whenever } \alpha<\mathrm{cf}(\mu) \text{ and } \xi\leqslant\eta<\mu. \tag{†}$$

Let us put $A_\xi=\bigcup\{A^\alpha_\xi\colon \alpha<\mathrm{cf}(\mu)\}$ for $\xi<\mu$. The sets A_ξ are pairwise disjoint, so if they are all dense in X then X is μ-resolvable. Thus we can assume that at least one of them is not dense in X, hence there is a non-empty open set $V\subset X$ and an ordinal $\xi^\star<\mu$ such that $V\cap A_{\xi^\star}=\emptyset$. Then we also have

$$V\cap A_\eta=\emptyset\quad\text{for each } \eta\geqslant\xi^\star \tag{‡}$$

because of (†).

For $\kappa < \mu$ pick $\beta < \mathrm{cf}(\mu)$ with $\kappa \leqslant \mu_\beta$ and put

$$T = \bigcup \{A^\alpha_\xi\colon \alpha \leqslant \beta,\ \xi < \xi^*\}.$$

Then $|T| \leqslant \mu_\beta \cdot |\xi^*| < \mu$ and it is immediate from our definitions that then we have

$$d_0(V \setminus T) > \mu_\beta \geqslant \kappa. \qquad \square$$

Before giving our next result we introduce a refined version of the family of fat sets $\mathcal{H}(Y,\nu)$ defined above and of the associated operator $H(Y,\nu)$. If a cardinal $\varrho < \nu$ is also given, then we let

$$\mathcal{H}(Y,\varrho,\nu) = \{A \in \mathcal{H}(Y,\nu)\colon \Delta(A) \geqslant \varrho\}.$$

Again, if $c(X) \leqslant \nu$ and $\mathcal{H}(Y,\varrho,\nu)$ is non-empty then $\mathcal{H}(Y,\varrho,\nu)$ has a member $H(Y,\rho,\nu)$ of maximal closure. (If $\mathcal{H}(Y,\varrho,\nu)$ is empty then we set $H(Y,\varrho,\nu) = \emptyset$.)

Lemma 2.9. *Assume that X is a topological space and μ is a singular cardinal with $c(X) < \mu \leqslant \Delta(X)$, moreover X satisfies condition (∗) from Lemma 2.8, i.e. for every $\kappa < \mu$ there is a set $T \in [X]^{<\mu}$ such that $d_0(X \setminus T) > \kappa$. Then we have either* (i) *or* (ii) *below.*

(i) *There is a disjoint family $\{D_\alpha\colon \alpha < \mathrm{cf}(\mu)\} \subset \mathcal{F}(X) \cap [X]^{<\mu}$ such that $\Delta(D_\alpha)$ converges to μ, moreover*

$$\bigcup \{D_\gamma\colon \gamma \geqslant \alpha\} \in \mathcal{D}(X)$$

for all $\alpha < \mathrm{cf}(\mu)$.

(ii) *There are an open set $W \in \tau^*(X)$ and a set $T \in [X]^{<\mu}$ with $d_0(W \setminus T) \geqslant \mu$.*

Proof. Fix the same strictly increasing sequence $\langle \mu_\alpha\colon \alpha < \mathrm{cf}(\mu)\rangle$ of regular cardinals converging to μ with $c(X) \cdot \mathrm{cf}(\mu) < \mu_0$ as in the above proof. Note that then for each $\alpha < \mathrm{cf}(\mu)$ we have

$$\mu_\alpha^- = \sup\{\mu_\beta\colon \beta < \alpha\} < \mu_\alpha.$$

Then by a straightforward transfinite recursion on $\alpha < \mathrm{cf}(\mu)$ we define disjoint sets $D_\alpha \in [X]^{<\mu}$ as follows.

If D_β has been defined for each $\beta < \alpha$ then set

$$D_\alpha = H\big(X \setminus \cup \{D_\beta\colon \beta < \alpha\}, \mu_\alpha^-, \mu_\alpha\big).$$

(Note that D_α may be empty but it is a member of $\mathcal{F}(X)$ if it is not.) Next, for each $\alpha < \mathrm{cf}(\mu)$ we let

$$E_\alpha = \bigcup \{D_\gamma\colon \gamma \geqslant \alpha\}.$$

Assume first that $E_\alpha \in \mathcal{D}(X)$ for all $\alpha < \mathrm{cf}(\mu)$. In particular, then $D_\alpha \neq \emptyset$ for cofinally many $\alpha < \mathrm{cf}(\mu)$, hence by re-indexing we may actually assume that $D_\alpha \neq \emptyset$ for all $\alpha < \mathrm{cf}(\mu)$. Now, $\Delta(D_\alpha) > \mu_\alpha^-$ immediately implies that $\Delta(D_\alpha)$ converges to μ, hence (i) is satisfied.

Next, assume that some E_α is not dense, hence there is a $W \in \tau^*(X)$ with $W \cap E_\alpha = \emptyset$. Since X satisfies (∗) there is a set $S \in [X]^{<\mu}$ such that $d_0(X \setminus S) > \mu_\alpha$. Let us set

$$T = \bigcup \{D_\beta\colon \beta < \alpha\} \cup S,$$

then $|T| < \mu$ as well, moreover we claim that $d_0(W \setminus T) = \kappa \geqslant \mu$.

Assume, indirectly, that $U \in \tau^*(W)$ and $d(U \setminus T) = \kappa < \mu$. Since $U \setminus T \subset X \setminus S$ we have $\kappa > \mu_\alpha$, hence if $\delta < \mathrm{cf}(\mu)$ is chosen so that

$$\mu_\delta^- \leqslant \kappa < \mu_\delta$$

then $\alpha < \delta$. Let A be any dense subset of $U \setminus T$ of size κ, then clearly $\Delta(A) = \kappa$ as well, moreover $A \subset X \setminus \cup \{D_\beta\colon \beta < \delta\}$ holds because $W \cap E_\alpha = \emptyset$. But then, by our definition, we have

$$A \in \mathcal{H}\big(X \setminus \cup \{D_\beta\colon \beta < \delta\}, \mu_\delta^-, \mu_\delta\big),$$

hence $A \subset \overline{D_\delta}$, contradicting that $W \cap \overline{D_\delta} = \emptyset$. □

We now give one more easy result that, for a limit cardinal λ, may be used to conclude λ-resolvability.

Lemma 2.10. *Let X be a space and λ a limit cardinal and assume that $\{D_\alpha\colon \alpha < \mathrm{cf}(\lambda)\}$ are disjoint subsets of X such that*

$$\bigcup\{D_\alpha\colon \beta \leqslant \alpha < \mathrm{cf}(\lambda)\} \in \mathcal{D}(X)$$

for every $\beta < \mathrm{cf}(\lambda)$. Assume also that D_α is κ_α-resolvable for each $\alpha < \mathrm{cf}(\lambda)$ and the sequence $\langle \kappa_\alpha\colon \alpha < \mathrm{cf}(\lambda)\rangle$ converges to λ. Then X is λ-resolvable.

Proof. For each $\alpha < \mathrm{cf}(\lambda)$ fix a disjoint family

$$\{E_\xi^\alpha\colon \xi < \kappa_\alpha\} \subset \mathcal{D}(D_\alpha),$$

then for any $\xi < \lambda$ set

$$E_\xi = \bigcup\{E_\xi^\alpha\colon \xi < \kappa_\alpha\}.$$

Since the κ_α converge to λ, for any fixed $\xi < \lambda$ we eventually have $\xi < \kappa_\alpha$ and so E_ξ is dense in X. Consequently the disjoint family $\{E_\xi\colon \xi < \lambda\}$ witnesses that X is λ-resolvable. □

From the above results and the first part of Theorem 2.2 we may now easily obtain the "missing" second part. Indeed, assume that λ is singular and $s(X) < \Delta(X) = \lambda$. Reasoning inductively, we may assume that if $s(Y) < \Delta(Y) < \lambda$ then Y is maximally, that is $\Delta(Y)$-resolvable.

Now, by Lemma 1.1, to prove that X is λ-resolvable it suffices to show that some subspace of X is. Since $c(X) \leqslant s(X)$, from Lemmas 2.8 and 2.9 it follows that, if X itself is not λ-resolvable, then either there are a $W \in \tau^*(X)$ and a $T \in [W]^{<\lambda}$ such that $d_0(W\setminus T) \geqslant \lambda$ or there is a $V \in \tau^*(X)$ with disjoint sets $\{D_\alpha\colon \alpha < \mathrm{cf}(\lambda)\} \subset \mathcal{F}(V)$ such that $\Delta(D_\alpha)$ converges to λ and

$$\bigcup\{D_\gamma\colon \alpha \leqslant \gamma < \mathrm{cf}(\lambda)\} \in \mathcal{D}(X)$$

for all $\alpha < \mathrm{cf}(\lambda)$. But we have seen that in the first case $W\setminus T$ (and hence W), while in the second V is λ-resolvable.

We are now ready to present our result that, under certain conditions, enables us to deduce λ-resolvability from $({<}\lambda)$-resolvability for a singular cardinal λ. We first recall that $\hat{c}(X)$ is defined as the smallest (uncountable) cardinal such that X has no disjoint family of open sets of that size. As was shown in [4] (see also [7, 4.1]), $\hat{c}(X)$ is always a regular cardinal. We also note that if λ is a limit cardinal then every $({<}\lambda)$-resolvable space S has dispersion character $\Delta(S) \geqslant \lambda$.

Theorem 2.11. *Assume that X is a topological space, λ is a singular cardinal, and $\hat{c}(X) \leqslant \mathrm{cf}(\lambda) < \lambda \leqslant \Delta(X)$. If every dense subspace $S \subset X$ satisfying $\Delta(S) \geqslant \lambda$ is $({<}\lambda)$-resolvable then X is actually λ-resolvable.*

Proof. Let us start by pointing out that if A is fat in X then $S = A \cup (X\setminus\overline{A}) \in \mathcal{D}(X)$, moreover $\Delta(A) \geqslant \lambda$ implies $\Delta(S) \geqslant \lambda$. So, every fat set $A \in \mathcal{F}(X)$ that satisfies $\Delta(A) \geqslant \lambda$ is $({<}\lambda)$-resolvable. It immediately follows from this that the conditions on our space X are inherited by all non-empty open subspaces, hence by Lemma 1.1 it is again sufficient to prove that X has some λ-resolvable subspace.

Now, if some $A \in \mathcal{F}(X)$ satisfies $d_0(A) \geqslant \lambda$ then $\Delta(B) \geqslant \lambda$ holds for every $B \in \mathcal{D}(A)$, hence all dense subsets of A are $({<}\lambda)$-resolvable. But then, by Lemma 2.7, A is λ-resolvable.

Therefore, from here on we may assume that $d_0(A) < \lambda$ for all $A \in \mathcal{F}(X)$. Actually, we claim that then even $d(A) < \lambda$ holds whenever $A \in \mathcal{F}(X)$. Indeed, if $A \in \mathcal{D}(U)$ for some $U \in \tau^*(X)$ then let $\mathcal{W}$ be a maximal disjoint family of open sets $W \subset U$ such that $d(A \cap W) < \lambda$. Then $\hat{c}(X) \leqslant \mathrm{cf}(\lambda) = \kappa$ implies $|\mathcal{W}| < \kappa$, moreover $\bigcup\mathcal{W}$ is clearly dense in U by our assumption. But then $\bigcup\mathcal{W} \cap A$ is dense in A and so

$$d(A) \leqslant d\Bigl(\bigcup\mathcal{W} \cap A\Bigr) = \sum\{d(W \cap A)\colon W \in \mathcal{W}\} < \lambda.$$

(We note that this is the only part of the proof where $\hat{c}(X) \leqslant \mathrm{cf}(\lambda)$ is used rather than the weaker assumption $c(X) < \lambda$.)

By Lemma 2.8, if X itself is not λ-resolvable then there is a $V \in \tau^*(X)$ that satisfies condition (∗). We shall show that then V is λ-resolvable.

To see this, first fix a strictly increasing sequence $\langle \lambda_\alpha \colon \alpha < \kappa \rangle$ of cardinals converging to λ and then, using (∗), fix for each $\alpha < \kappa$ a set $T_\alpha \in [V]^{<\lambda}$ with $d_0(V \setminus T_\alpha) > \lambda_\alpha$. Having done this, we define disjoint sets $D_\alpha \in \mathcal{D}(V) \cap [V]^{<\lambda}$ by transfinite induction on $\alpha < \kappa$ as follows.

Assume that $\alpha < \kappa$ and $D_\beta \in \mathcal{D}(V) \cap [V]^{<\lambda}$ has been defined for each $\beta < \alpha$. Set

$$Z_\alpha = X \setminus \Big(\bigcup \{D_\beta \colon \beta < \alpha\} \cup T_\alpha \Big),$$

then Z_α is dense in V because $\Delta(X) \geqslant \lambda$. But then $d(Z_\alpha) < \lambda$, hence we may pick $D_\alpha \in \mathcal{D}(Z_\alpha) \subset \mathcal{D}(V)$ with $|D_\alpha| < \lambda$. Note that as $D_\alpha \subset V \setminus T_\alpha$ we also have $\Delta(D_\alpha) > \lambda_\alpha$.

Now consider any partition $\{J_\xi \colon \xi < \kappa\}$ of κ into κ many sets of size κ and for each $\xi < \kappa$ put

$$E_\xi = \bigcup \{D_\alpha \colon \alpha \in J_\xi\}.$$

Then each E_ξ is dense in V and clearly $\Delta(E_\xi) = \lambda$, hence it is $(<\lambda)$-resolvable. But the E_ξ's are pairwise disjoint, hence obviously V is λ-resolvable. □

We do not know if the assumption $\hat{c}(X) \leqslant \mathrm{cf}(\lambda)$ can be relaxed to $c(X) < \lambda$ in Theorem 2.11, or even if it can be dropped altogether.

3. A simpler proof of Pavlov's theorem concerning extent

The *extent* $\mathrm{e}(X)$ of a space X is defined as the supremum of sizes of all closed discrete subspaces of X. (This is Archangelskiĭ's notation, in [10] $\mathrm{ext}(X)$ and in [7] $\mathrm{p}(X)$ is used to denote the same cardinal function.) Similarly as in the previous section for the spread $\mathrm{s}(X)$, we may define $\hat{\mathrm{e}}(X)$ as the smallest infinite (but not necessarily uncountable) cardinal such that X has no closed discrete subset of that size. Note that a space X is countably compact iff $\hat{\mathrm{e}}(X) = \omega$. Clearly, one has $\hat{\mathrm{e}}(X) \leqslant \mathrm{pe}(X)$ (the latter was defined in the abstract).

In [10] it was proved that $\Delta(X) > \mathrm{pe}(X)$ implies the ω-resolvability of X for any T_3 space X. In this section we shall present our proof of the slightly stronger result in which only $\Delta(X) > \hat{\mathrm{e}}(X)$ is used. We believe that this proof is significantly simpler than the one given in [10], although it follows the same steps.

We start with giving our simplified proof of the following result of Pavlov concerning spaces that are finite unions of left separated subspaces.

Theorem 3.1. (Pavlov [10, Lemma 3.1]) *Assume that* $\mathrm{ls}(X) < \omega$ *and* $\kappa \leqslant |X|$ *is an uncountable regular cardinal. Then there is a strictly increasing and continuous sequence* $\langle F_\alpha \colon \alpha < \kappa \rangle$ *of closed subsets of X with* $|F_\alpha| < \kappa$ *for all* $\alpha < \kappa$.

Proof. We prove the theorem by induction on $\mathrm{ls}(X)$. So assume that it is true for $\mathrm{ls}(X) = k$ and consider $X = \bigcup_{0 \leqslant i \leqslant k} L_i$ where the L_i are disjoint and left separated, moreover $\omega < \kappa \leqslant |X|$. We may clearly assume that the left separating order type of each L_i is $\leqslant \kappa$.

Assume that S is an initial segment of some L_i with $\mathrm{tp}(S) < \kappa$ and $|\overline{S}| \geqslant \kappa$ (closures are always taken in X). Since $\overline{S} \cap L_i = S$ we may apply the inductive hypothesis to $\overline{S} \setminus S$ and find an increasing and continuous κ-sequence $\langle F_\alpha \colon \alpha < \kappa \rangle$ of its closed subsets of size $< \kappa$. But then the traces $\overline{F_\alpha} \cap S$ will stabilize and $|\overline{F_\alpha}| \leqslant |F_\alpha| + |S| < \kappa$, hence a suitable final segment of $\langle \overline{F_\alpha} \colon \alpha < \kappa \rangle$ is as required. Almost the same argument shows that the inductive step can also be completed if $|L_i| < \kappa$ for some i. So we may assume that $\mathrm{tp}\, L_i = \kappa$ for each i and that $|\overline{A}| < \kappa$ whenever $A \in [X]^{<\kappa}$.

Let y_α denote the αth member of L_0 and use the inductive assumption to find an increasing and continuous κ-sequence $\langle F_\alpha \colon \alpha < \kappa \rangle$ of closed subsets of $\bigcup_{1 \leqslant i \leqslant k} L_i$ of size $< \kappa$, and then consider the set

$$I = \{\alpha < \kappa \colon y_\alpha \in \overline{F_\alpha}\}.$$

Assume first that $|I| < \kappa$ and hence $\sigma = \sup I < \kappa$. We claim that then the set

$$J = \left\{\beta > \sigma\colon\ \overline{F_\beta} \neq \bigcup_{\gamma<\beta} \overline{F_\gamma}\right\}$$

is non-stationary in κ. Indeed, for each $\beta \in J$ there must be some $g(\beta) < \kappa$ with $y_{g(\beta)} \in \overline{F_\beta} \setminus \bigcup_{\gamma<\beta} \overline{F_\gamma}$. Since $g(\beta) \geqslant \beta > \sigma$ would imply $g(\beta) \notin I$ and hence

$$y_{g(\beta)} \notin \overline{F_{g(\beta)}} \supset \overline{F_\beta},$$

we must have $g(\beta) < \beta$. But the regressive function g is clearly one-to-one on J, hence by Fodor's (or Neumer's) pressing down theorem J is non-stationary. So there is a club set C in κ with $C \cap J = \emptyset$, and then the sequence $\langle \overline{F_\alpha}\colon\ \alpha \in C\setminus\sigma\rangle$ clearly satisfies our requirements.

So we may assume that $|I| = \kappa$. For each $\alpha < \kappa$ let us put $H_\alpha = \overline{\{y_\gamma\colon\ \gamma \in I \cap \alpha\}}$. Note that we have $H_\alpha \subset \overline{F_\alpha}$ by the definition of I. Next, consider the set

$$J = \left\{\alpha < \kappa\colon\ \alpha \text{ is limit and } H_\alpha \neq \bigcup_{\gamma<\alpha} H_\gamma\right\}.$$

We claim that this set J is again non-stationary. Indeed, for every $\alpha \in J$ we may pick a "witness" $z_\alpha \in H_\alpha \setminus \bigcup_{\gamma<\alpha} H_\gamma$. Now, if $z_\alpha \in L_0$ then $z_\alpha = y_{g(\alpha)}$ for some $g(\alpha) < \alpha$ because L_0 is left separated. If, on the other hand, $z_\alpha \notin L_0$ then $z_\alpha \in H_\alpha \subset \overline{F_\alpha}$ implies $z_\alpha \in F_\alpha$ because F_α is closed in $X\setminus L_0$. But the sequence $\langle F_\alpha\colon\ \alpha \in \kappa\rangle$ is continuous, hence in this case we can choose an ordinal $g(\alpha) < \alpha$ such that $z_\alpha \in F_{g(\alpha)}$.

In other words, this means that if $g(\alpha) = \beta$ then $z_\alpha \in \{y_\beta\} \cup F_\beta$. Now, the sequence $\langle z_\alpha\colon\ \alpha \in J\rangle$ is obviously one-to-one, hence for each $\beta < \kappa$ we have $|g^{-1}\{\beta\}| \leqslant |F_\beta| + 1 < \kappa$, consequently, again by Fodor, J is not stationary. So there is a club $C \subset \kappa \setminus J$ and then $\langle H_\alpha\colon\ \alpha \in C\rangle$ is increasing and continuous, however maybe it is not strictly increasing. But $|I| = \kappa$ clearly implies that the union of the H_α's is of size κ and so an appropriate subsequence of $\langle H_\alpha\colon\ \alpha \in C\rangle$ will be both continuous and strictly increasing. □

Before proceeding further, we need a simple definition.

Definition 3.2. Let X be a space and μ an infinite cardinal number. We say that $x \in X$ is a T_μ point of X if for every set $A \in [X]^{<\mu}$ there is some $B \in [X\setminus A]^{<\mu}$ such that $x \in \overline{B}$. We shall use $T_\mu(X)$ to denote the set of all T_μ points of X.

For the reader familiar with Pavlov's paper [10] we note that his $\mathrm{tr}_{\nu^+,\nu}(X)$ is identical with our $T_{\nu^+}(X)$. Note also that if $Y \subset X$ then trivially any T_μ point in Y is a T_μ point in X, that is, we have $T_\mu(Y) \subset T_\mu(X)$. Finally, if μ is regular then the set $T_\mu(X)$ is clearly $(<\mu)$-closed in X, i.e. for every set $A \in [T_\mu(X)]^{<\mu}$ we have $\overline{A} \subset T_\mu(X)$.

Lemma 3.3. *Assume that the space X may be written as the union of a strictly increasing continuous chain $\langle F_\alpha\colon\ \alpha < \kappa\rangle$ of closed subsets of X with $|F_\alpha| < \kappa$ for all $\alpha < \kappa$, where κ is an uncountable regular cardinal. Then $T_\kappa(X) = \emptyset$ implies that there exists a set $D \subset X$ with $|D| = \kappa$ such that every subset $Y \in [D]^{<\kappa}$ is closed discrete in X. In particular, we have $\hat{\mathrm{e}}(X) \geqslant \kappa$.*

Proof. The assumption $T_\kappa(X) = \emptyset$ implies that for every point $x \in X$ we may fix a set $A_x \in [X]^{<\kappa}$ such that $x \notin \overline{B}$ whenever $B \in [X\setminus A_x]^{<\kappa}$. By the regularity of κ, the set

$$C = \{\alpha < \kappa\colon\ \forall x \in F_\alpha\ (A_x \subset F_\alpha)\}$$

is club in κ. For each $\alpha \in C$ let us pick a point $x_\alpha \in F_{\alpha+1}\setminus F_\alpha$ and then set $D = \{x_\alpha\colon\ \alpha \in C\}$.

To see that this D is as required, it remains to show that all "small" subsets of D are closed discrete. This in turn will follow if we show that all proper initial segments of D are. So let $\gamma < \kappa$ and consider the set $S = \{x_\alpha\colon\ \alpha \in C \cap \gamma\}$. For every point $y \in X$ there is a $\beta < \kappa$ such that $y \in F_{\beta+1}\setminus F_\beta$. Let δ be the largest element of C with $\delta \leqslant \beta$ and ε the smallest element of C above β, hence we have $\delta \leqslant \beta < \varepsilon$.

Then, on one hand, $\{x_\alpha\colon \alpha < \delta\} \subset F_\delta \subset F_\beta$, while on the other hand, $A_y \subset F_\varepsilon$ and $\{x_\alpha\colon \varepsilon \leqslant \alpha < \gamma\} \subset X \backslash F_\varepsilon$, which together imply that y has a neighbourhood U such that $U \cap S \subset \{x_\delta\}$. □

We need one more result making use of the operator T_μ.

Lemma 3.4. *If a space X satisfies $T_\mu(X) = X$ for a regular cardinal μ then X is μ-resolvable.*

Proof. Clearly, $T_\mu(X) = X$ implies $T_\mu(U) = U$ for all open subsets $U \subset X$, hence by Lemma 1.1 it suffices to show that X includes a μ-resolvable subspace Y.

Since every point of X is a T_μ point, for any set $A \in [X]^{<\mu}$ we may fix a *disjoint* family $\mathcal{B}(A) \subset [X\backslash A]^{<\mu}$ with $|\mathcal{B}(A)| = |A| < \mu$ such that

$$A \subset \bigcup\bigl\{\overline{B}\colon\ B \in \mathcal{B}(A)\bigr\}.$$

We now define sets A_α in $[X]^{<\mu}$ by induction on $\alpha < \mu$ as follows. Let $x \in X$ be any point and start with $A_0 = \{x\}$. Assume next that $0 < \alpha < \mu$ and the sets $A_\beta \in [X]^{<\mu}$ have been defined for all $\beta < \alpha$. Then we set

$$B_\alpha = \bigcup \mathcal{B}\Bigl(\bigcup\{A_\beta\colon\ \beta < \alpha\}\Bigr) \text{ and } A_\alpha = \bigcup\{A_\beta\colon\ \beta < \alpha\} \cup B_\alpha.$$

After the induction is completed we let

$$Y = \bigcup\{A_\alpha\colon\ \alpha < \mu\}.$$

It is clear from the construction that the B_α's are pairwise disjoint, moreover for every set $s \in [\mu]^\mu$ the union $\bigcup_{\alpha\in s} B_\alpha$ is dense in Y. But then Y is obviously μ-resolvable. □

We are now ready to state and prove our promised result.

Theorem 3.5. *Assume that the regular closed subsets of the space X form a π-network in X and $T_\mu(X)$ is dense in X for some regular cardinal $\mu > \hat{e}(X)$. Then X is ω-resolvable. In particular, any T_3 space X satisfying $\Delta(X) > \hat{e}(X)$ is ω-resolvable.*

Proof. Assume, indirectly, that X is ω-irresolvable. By Lemmas 1.1 and 1.2 then there is a regular closed subset K of X that is both *hereditarily* ω-irresolvable and satisfies $\operatorname{ls}(K) < \omega$.

Let us now define the sequence of sets $\{K_n\colon\ n < \omega\}$ by the following recursion: $K_0 = K$ and $K_{n+1} = T_\mu(K_n)$. Since $T_\mu(Y)$ is $(<\mu)$-closed in Y for any space Y, we may conclude by a simple induction that K_i is $(<\mu)$-closed in K and hence $\hat{e}(K_i) \leqslant \hat{e}(K) < \mu$ for all $i < \omega$.

We next claim that, for each $n < \omega$, $K_{n+1} = T_\mu(K_n)$ is dense in K_n and hence in K. For $n = 0$ this follows immediately from our assumption that $T_\mu(X) \in \mathcal{D}(X)$.

Clearly, any neighborhood of a T_μ point in any space must have size at least μ. Hence if our claim holds up to (and including) n then we also have $\Delta(K_n) \geqslant \mu$ and since $K_n \in \mathcal{D}(K)$ the regular closed subsets of K_n form a π-network in K_n. (The latter holds because the regular closed subsets of a dense subspace are exactly the traces of the regular closed sets in the original space.)

Now, let U be any non-empty open subset of K_n. We show first that then $|U \cap K_{n+1}| \geqslant \mu$, hence $\Delta(K_{n+1}) \geqslant \mu$. (In other words, K_{n+1} is not only dense but even μ-dense in K_n.) To see this, let $\emptyset \neq F \subset U$ be regular closed in K_n, then $|F| \geqslant \mu$ and $\operatorname{ls}(F) < \omega$ imply, in view of Theorem 3.1, the existence of a strictly increasing continuous sequence $\langle F_\alpha\colon\ \alpha < \mu\rangle$ of closed subsets of F (and hence of X) with $|F_\alpha| < \mu$. Then we may apply Lemma 3.3 to any final segment of the sequence $\langle F_\alpha\colon\ \alpha < \mu\rangle$ to conclude that $F_\alpha \cap T_\mu(K_n) = F_\alpha \cap K_{n+1} \neq \emptyset$ for cofinally many $\alpha < \mu$, hence $|U \cap K_{n+1}| \geqslant |F \cap K_{n+1}| \geqslant \mu$.

But $\Delta(K_{n+1}) \geqslant \mu$ implies that for any non-empty regular closed set H in K_{n+1} we have $|H| \geqslant \mu$, and so, using again $\operatorname{ls}(H) < \omega$ and $\hat{e}(K_n) < \mu$, we obtain from Theorem 3.1 and Lemma 3.3 that $T_\mu(H)$ is non-empty, i.e. K_{n+2} is indeed dense in K_{n+1}.

Now suppose that there is an $n < \omega$ such that $K_n \backslash K_{n+1}$ is not dense in K_n. This would imply that for some $U \in \tau^*(K_n)$ we have $U \subset K_{n+1}$ and hence $T_\mu(U) = U$. But that would imply by Lemma 3.4 that U is μ-resolvable, a contradiction. Therefore, we must have that $K_n \backslash K_{n+1}$ is dense in K_n and hence in K for all $n < \omega$. But then K would be ω-resolvable, which is again absurd. This contradiction then completes the proof of the first part of our theorem.

To see the second part note that, by Lemma 1.2 and by considering regular closed subsets of X, it suffices to prove the ω-resolvability of X under the additional condition $\mathrm{ls}(X) < \omega$. But then $T_\mu(X) \in \mathcal{D}(X)$ follows immediately from Theorem 3.1 and Lemma 3.3 with the choice $\mu = \hat{\mathrm{e}}(X)^+$. □

Since for any crowded (i.e. dense-in-itself) countably compact T_3 space X one has $\Delta(X) \geqslant \mathfrak{c} \geqslant \omega_1$, Theorem 3.5 immediately implies the following result of Comfort and Garcia-Ferreira.

Theorem. (Comfort and Garcia-Ferreira [2, Theorem 6.9]) *Every crowded and countably compact T_3 space is ω-resolvable.*

Note that the assumption of regularity in this theorem is essential because of the following two results.

Theorem. (Malykhin [9, Example 14]) *There is a countably compact, irresolvable T_2 space.*

Theorem. (Pavlov [10, Example 3.9]) *There is a countably compact, irresolvable Uryshon space.*

Pytkeev has recently announced in [12] that a crowded and countably compact T_3 space is even ω_1-resolvable. We have not seen his paper but would like to point out that this stronger result is an immediate consequence of an old (and deep) result of Tkačenko and of Lemma 1.2.

Tkačenko proved in [13] that *if X is a countably compact T_3 space with* $\mathrm{ls}(X) \leqslant \omega$ *then X is compact and scattered.* In [5] it was shown that this statement remains valid if T_3 is weakened to T_2, hence we get the following result.

Theorem 3.6. *If X is a crowded and countably compact T_2 space in which the regular closed subsets form a π-network then X is ω_1-resolvable.*

Proof. By the above result from [5], every non-empty regular closed subset $F \subset X$ must satisfy $\mathrm{ls}(F) \geqslant \omega_1$. But then X is ω_1-resolvable by Lemma 1.1. □

Any crowded and countably compact T_3 space has dispersion character $\geqslant \mathfrak{c}$. Hence the following interesting, and apparently difficult, problem is left open by Theorem 3.6.

Problem 3.7. Is a crowded and countably compact T_3 space $\mathfrak{c}$-resolvable or even maximally resolvable?

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