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Hamiltonian formulation of systems with linear velocities within Riemann–Liouville fractional derivatives

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Abstract

The link between the treatments of constrained systems with fractional derivatives by using both Hamiltonian and Lagrangian formulations is studied. It is shown that both treatments for systems with linear velocities are equivalent.

Keywords: Fractional derivative; Hamiltonian system; Nonconservative systems

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1. Introduction

The generalization of the concept of derivative and integral to a noninteger order α has been subjected to several approaches and some various alternative definitions of fractional derivatives appeared [1–6]. In the last few years fractional calculus was applied successfully in various areas, e.g., chemistry, biology, modelling and identification, electronics and wave propagation. Fractional calculus, has played an important role in engineering, science, and pure and applied mathematics [7–9]. Fractional derivatives were applied in recent studies of scaling phenomena [10–12]. Classical mechanics is one of the fields when fractional calculus found many applications [13–19]. Riewe has used the fractional calculus to obtain a formalism which can be applied for both conservative and nonconservative systems [13,14]. Although many laws of nature can be obtained using certain functionals and the theory of calculus of variations, not all laws can be obtained by this manner. As it is known, almost all systems contain internal damping, yet traditional energy based approach cannot be used to obtain equations describing the behavior of a nonconservative system [13,14]. Using the fractional calculus one can obtain the Lagrangian and the Hamiltonian equations of motion for the nonconservative systems.

The understanding of constrained dynamics [20], both at the classical and quantum level, has been a subject of long standing theoretical interest, which has seen important contributions ever since Dirac's quantization of the electromagnetic field. The path integral approach and the canonical one are two main approaches of quantization.

Recently, an extension of the simplest fractional problem and the fractional variational problem of Lagrange was obtained [17,18]. Even more recently, this approach was extended to Lagrangians with linear in velocities [21,22], which represents a typical example of second-class constrained systems in Dirac's classification [20]. These Lagrangians are important because their Euler–Lagrangian equations become systems of first order differential equations in contrast with second order corresponding to the regular ones. In addition, these systems may possess gauge symmetries and gauge ambiguities.

From these reasons it is interesting to study the fractional Hamiltonian formulation of constrained systems.

The aim of this paper is to obtain the fractional Hamiltonian equations of motion for Lagrangians with linear velocities.

The plan of our paper is as follows. In Section 2 some basic tools of fractional derivatives as well as Riewe's approach of the fractional Lagrangian and Hamiltonians are presented. In Section 3 the Euler–Lagrange equations were obtained using the Agrawal's approach and the fractional formulation of systems with constraints is introduced. In Section 4 the fractional Hamiltonian analysis of the systems possessing linear velocities is analyzed. Section 5 is dedicated to our conclusion.

2. Fractional Lagrangian and Hamiltonian formulations

In this section we briefly present the definition of the left and right derivatives together with Riewe's formulation of Lagrangian and Hamiltonian dynamics. The left Riemann–Liouville fractional derivative is defined as

$$ {}_a\mathbf{D}_t^{\alpha} f(t) \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, \tag{1} $$

and the right Riemann–Liouville fractional derivative has the form

$$ {}_t\mathbf{D}_b^{\alpha} f(t) \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)\, d\tau, \tag{2} $$

where the order α fulfills $n-1 \leqslant \alpha < n$ and Γ represents the Euler's gamma function. If α is an integer, these derivatives are defined in the usual sense, i.e.,

$$ {}_a\mathbf{D}_t^{\alpha} f(t) = \left(\frac{d}{dt}\right)^{\alpha}, \quad {}_t\mathbf{D}_b^{\alpha} f(t) = \left(-\frac{d}{dt}\right)^{\alpha}, \quad \alpha = 1, 2, \ldots. \tag{3} $$

Now we shall briefly review Riewe's formulation of fractional generalization of Lagrangian and Hamiltonian equations of motion [13,14]. The starting point is the action function of the form

$$ S = \int_a^b L\left(\left\{q_n^r, Q_{n'}^r\right\}, t\right) dt. \tag{4} $$

Here the generalized coordinates are defined as

$$ q_n^r := \left({}_a\mathbf{D}_t^{\alpha}\right)^n x_r(t), \qquad Q_{n'}^r := \left({}_t\mathbf{D}_b^{\alpha}\right)^{n'} x_r(t), \tag{5} $$

and $r = 1, 2, \ldots, R$ represents the number of fundamental coordinates, $n = 0, \ldots, N$, the sequential order of the derivatives defining the generalized coordinates q, and $n' = 1, \ldots, N'$ the sequential order of the derivatives in definition of the coordinates Q. A necessary condition for S to posses an extremum for given functions $x_r(t)$ is that $x_r(t)$ fulfill the Euler–Lagrange equations [13,14]

$$ \frac{\partial L}{\partial q_0^r} + \sum_{n=1}^{N} \left({}_t\mathbf{D}_b^{\alpha}\right)^n \frac{\partial L}{\partial q_n^r} + \sum_{n'=1}^{N'} \left({}_a\mathbf{D}_t^{\alpha}\right)^{n'} \frac{\partial L}{\partial Q_{n'}^r} = 0. \tag{6} $$

Using the references [13,14], the generalized momenta have the following form:

$$ \begin{aligned} p_n^r &= \sum_{k=n+1}^{N} \left({}_t\mathbf{D}_b^{\alpha}\right)^{k-n-1} \frac{\partial L}{\partial q_k^r}, \\ \pi_{n'}^r &= \sum_{k=n'+1}^{N'} \left({}_a\mathbf{D}_t^{\alpha}\right)^{k-n'-1} \frac{\partial L}{\partial Q_k^r}. \end{aligned} \tag{7} $$

Thus, the canonical Hamiltonian is given by

$$ H = \sum_{r=1}^{R} \sum_{n=0}^{N-1} p_n^r q_{n+1}^r + \sum_{r=1}^{R} \sum_{n'=0}^{N'-1} \pi_{n'}^r Q_{n'+1}^r - L. \tag{8} $$

The Hamilton's equations of motion are as follows:

$$\frac{\partial H}{\partial q^r_N} = 0, \qquad \frac{\partial H}{\partial Q^r_{N'}} = 0. \tag{9}$$

For $n = 1, \ldots, N$, $n' = 1, \ldots, N'$ we have the following equations of motion:

$$\frac{\partial H}{\partial q^r_n} = {}_t\mathbf{D}^\alpha_b p^r_n, \qquad \frac{\partial H}{\partial Q^r_{n'}} = {}_a\mathbf{D}^\alpha_t \pi^r_{n'}, \tag{10}$$

$$\frac{\partial H}{\partial q^r_0} = -\frac{\partial L}{\partial q^r_0} = {}_t\mathbf{D}^\alpha_b p^r_0 + {}_a\mathbf{D}^\alpha_t \pi^r_0. \tag{11}$$

The remaining equations are given by

$$\frac{\partial H}{\partial p^r_n} = q^r_{n+1} = {}_a\mathbf{D}^\alpha_t q^r_n, \qquad \frac{\partial H}{\partial \pi^r_{n'}} = Q^r_{n+1} = {}_t\mathbf{D}^\alpha_b Q^r_{n'}, \tag{12}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \tag{13}$$

where $n = 0, \ldots, N$, $n' = 1, \ldots, N'$.

3. Fractional Euler–Lagrange equations

Recently Agrawal has obtained the Euler–Lagrange equations for fractional variational problems [17]. In the following we like to present briefly his approach.

Consider the action function

$$S\left[q^1_0, \ldots, q^R_0\right] = \int_a^b L\left(\left\{q^r_n, Q^r_{n'}\right\}, t\right) dt, \tag{14}$$

subject to the independent constraints

$$\Phi_m\left(t, q^1_0, \ldots, q^R_0, q^r_n, Q^r_{n'}\right) = 0, \quad m < R, \tag{15}$$

where the generalized coordinates are defined as

$$q^r_n := \left({}_a\mathbf{D}^\alpha_t\right)^n x_r(t), \qquad Q^r_{n'} := \left({}_t\mathbf{D}^\beta_b\right)^{n'} x_r(t). \tag{16}$$

Then, the necessary condition for the curves $q^1_0, \ldots, q^R_0$ with the boundary conditions $q^r_0(a) = q^{ra}_0$, $q^r_0(b) = q^{rb}_0$, $r = 1, 2, \ldots, R$, to be an extremal of the functional given by Eq. (14) is that the functions q^r_0 satisfy the following Euler–Lagrange equations [17]:

$$\frac{\partial \bar{L}}{\partial q^r_0} + \sum_{n=1}^{N} \left({}_t\mathbf{D}^\alpha_b\right)^n \frac{\partial \bar{L}}{\partial q^r_n} + \sum_{n'=1}^{N'} \left({}_a\mathbf{D}^\alpha_t\right)^{n'} \frac{\partial \bar{L}}{\partial Q^r_{n'}} = 0, \tag{17}$$

where $\bar{L}$ has the form [17]

$$\bar{L}\left(\left\{q^r_n, Q^r_{n'}\right\}, t, \lambda_m(t)\right) = L\left(\left\{q^r_n, Q^r_{n'}\right\}, t\right) + \lambda_m(t)\Phi_m\left(t, q^1_0, \ldots, q^R_0, q^r_n, Q^r_{n'}\right). \tag{18}$$

Here the multiple $\lambda_m(t) \in R^m$ are continuous on $[a, b]$.

3.1. Fractional Hamiltonian formulation of Agrawal's approach

In order to obtain the Hamilton's equations for the fractional variational problems, we redefine the left and the right canonical momenta as

$$p_n^r = \sum_{k=n+1}^{N} \left({}_t\mathbf{D}_b^\alpha\right)^{k-n-1} \frac{\partial \bar{L}}{\partial q_k^r},$$

$$\pi_{n'}^r = \sum_{k=n'+1}^{N'} \left({}_a\mathbf{D}_t^\alpha\right)^{k-n'-1} \frac{\partial \bar{L}}{\partial Q_k^r}. \tag{19}$$

Using (19), the canonical Hamiltonian becomes

$$\bar{H} = \sum_{r=1}^{R} \sum_{n=0}^{N-1} p_n^r q_{n+1}^r + \sum_{r=1}^{R} \sum_{n'=0}^{N'-1} \pi_{n'}^r Q_{n'+1}^r - \bar{L}. \tag{20}$$

Then, the modified canonical equations of motion are obtained as

$$\left\{q_n^r, \bar{H}\right\} = {}_t\mathbf{D}_b^\alpha p_n^r, \qquad \left\{Q_{n'}^r, \bar{H}\right\} = {}_a\mathbf{D}_t^\alpha \pi_{n'}^r, \tag{21}$$

$$\left\{q_0^r, \bar{H}\right\} = {}_t\mathbf{D}_b^\alpha p_0^r + {}_a\mathbf{D}_t^\alpha \pi_0^r, \tag{22}$$

where $n = 1, \ldots, N$, $n' = 1, \ldots, N'$.

The other set of equations of motion are given by

$$\left\{p_n^r, \bar{H}\right\} = q_{n+1}^r = {}_a\mathbf{D}_t^\alpha q_n^r, \qquad \left\{\pi_{n'}^r, \bar{H}\right\} = Q_{n+1}^r = {}_t\mathbf{D}_b^\alpha Q_{n'}^r, \tag{23}$$

$$\frac{\partial \bar{H}}{\partial t} = -\frac{\partial \bar{L}}{\partial t}. \tag{24}$$

Here, $n = 0, \ldots, N$, $n' = 1, \ldots, N'$ and the commutator $\{\cdot\,,\cdot\}$ is the Poisson's bracket defined as

$$\{A, B\}_{q_n^r, p_n^r, Q_{n'}^r, \pi_{n'}^r} = \frac{\partial A}{\partial q_n^r}\frac{\partial B}{\partial p_n^r} - \frac{\partial B}{\partial q_n^r}\frac{\partial A}{\partial p_n^r} + \frac{\partial A}{\partial Q_{n'}^r}\frac{\partial B}{\partial \pi_{n'}^r} - \frac{\partial B}{\partial Q_{n'}^r}\frac{\partial A}{\partial \pi_{n'}^r}, \tag{25}$$

where $n = 0, \ldots, N$, $n' = 1, \ldots, N'$.

4. Equivalence of fractional Hamiltonian and Lagrangian formulations for systems with linear velocities

Recently, for $0 < \alpha \leqslant 1$, the Lagrangians with linear velocities were investigated in [21]. For example, the Euler–Lagrange equations of the following Lagrangian:

$$L' = a_j(q^i)\,{}_a\mathbf{D}_t^\alpha q^j - V(q^i), \tag{26}$$

were obtained as [21]

$$\frac{\partial a_j(q^i)}{\partial q^k}\,{}_a\mathbf{D}_t^\alpha q^j + {}_t\mathbf{D}_b^\alpha a_k(q^i) - \frac{\partial V(q^i)}{\partial q^k} = 0. \tag{27}$$

Now we would like to obtain the Hamiltonian equations of motion for the same model. Let us define

$$x_j^n = \left({}_a\mathbf{D}_t^\alpha\right)^n q_j, \quad n = 0, 1, \ldots, N-1, \ j = 1, 2, \ldots, R. \tag{28}$$

The generalized momenta are given by

$$p_j^0 = \frac{\partial L}{\partial x_j^0} = a_j\left(x_i^0\right), \qquad p_j^1 = \frac{\partial L}{\partial x_j^1} = 0. \tag{29}$$

The canonical Hamiltonian reads as

$$H = \left(p_j^0 - a_j\left(x_i^0\right)\right)x_j^1 + V\left(x_i^0\right). \tag{30}$$

The Hamiltonian equations of motion are calculated as

$$\frac{\partial H}{\partial x_j^1} = p_j^0 - a_j\left(x_i^0\right) = \left({}_t\mathbf{D}_b^\alpha\right)p_j^1 = 0. \tag{31}$$

In fact, this equation is the primary constraint in the Dirac's formalism. The other equations of motion are calculated as

$$\frac{\partial H}{\partial x_k^0} = -\frac{\partial a_j(x_i^0)}{\partial x_k^0}x_j^1 + \frac{\partial V(x_i^0)}{\partial q^k} = \left({}_t\mathbf{D}_b^\alpha\right)p_k^0, \tag{32}$$

$$\frac{\partial H}{\partial p_k^0} = x_k^1 = \left({}_a\mathbf{D}_t^\alpha\right)q_k. \tag{33}$$

Making use of Eq. (31) we obtain

$$\frac{\partial a_j(q^i)}{\partial q^k}\,{}_a\mathbf{D}_t^\alpha q^j + {}_t\mathbf{D}_b^\alpha a_k(q^i) - \frac{\partial V(q^i)}{\partial q^k} = 0. \tag{34}$$

It is obvious the equivalence between Eqs. (34) and (27). An interesting point to be specified here is that, primary constraints in the Dirac's formalism are present as equations of motion in our treatment, while the other Hamiltonian equations of motion are equivalent to the Lagrangian equations of motion as given in [21].

5. Conclusion

One of the main problems encountered in applying the fractional calculus to a given singular Lagrangian and Hamiltonian is the existence of multiple choices of the possible fractional generalizations. In addition, the solutions of the fractional Euler–Lagrange equations contain more information than the classical ones. In this paper, Hamiltonian equations have been obtained for systems with linear velocities, in the same manner as those obtained by using the formulation of Euler–Lagrange equations for variational problems introduced by one of us [21]. We study the general model for systems with linear velocities and it was observed that the Hamiltonian and Lagrangian equations which are obtained by the two methods are in exact agreement.

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