On Medial Semigroups*

J. L. Chrislock

University of California, Santa Cruz, California 95060

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1. Introduction

Tamura and Kimura [11] proved that any commutative semigroup is a semilattice of Archimedean semigroups. Subsequently, Tamura ([7], [10]), Petrich [4], and Hewitt and Zuckerman [2] have investigated commutative Archimedean semigroups. The purpose of this paper is to generalize some of their results to medial semigroups (\( xab y = xby \)).

A congruence \( \tau \) on a semigroup \( S \) will be called a semilattice congruence if \( S/\tau \) is a semilattice (\( x^2 = x, xy = yx \)). There exists a semilattive congruence on a semigroup \( S \) which is contained in all others. This congruence is unique, and we will denote it by \( \sigma_s \). Tamura ([6], [9]) and later Petrich [5] proved that a congruence class \( \tau' \) of \( S \) modulo \( \sigma_s \) has no nontrivial semilattice congruence, that is, \( T \) as a semigroup has \( a\sigma_s b \) for all \( a, b \in T \).

A semigroup \( S \) will be called Archimedean if for all \( a, b \in S \) there is an \( n \in \mathbb{N} = \{1, 2, 3, \ldots \} \) such that \( a^n \in \mathbb{S}S \) and \( b^n \in S\mathbb{S} \). In Section 2, we give \( \sigma_M \) explicitly for a medial semigroup \( M \) and prove that the congruence classes of \( \sigma_M \) are Archimedean semigroups. In Sections 3 and 4 we investigate Archimedean semigroups which contain idempotents and extend a result in [11]. Finally in Section 5, medial separative semigroups (\( x^2 = xy = y^2 \) implies \( x = y \)) are studied, and it is proved that any such semigroup can be embedded in a semigroup which is a union of groups.

Clifford and Preston [1] contains all undefined terms. In addition, it contains in Section 4.3 those results generalized here.

2. Medial Semigroups

A semigroup \( M \) is medial if \( xab y = xby \) for all \( x, a, b, y \in M \). Such a semigroup \( M \) satisfies \( (xy)^n = x^n y^n \) and \( (MxM)^n = M^n x^n M^n \) for all \( x, y \in M \) and \( n \in \mathbb{N} \).

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Theorem 2.1. For a medial semigroup $M$, $aa_Mb$ ($a, b \in M$) if and only if $a^n \in MbM$ and $b^n \in MaM$ for some $n \in \mathbb{N}$. Furthermore, the congruence classes of $\sigma_M$ are medial Archimedean semigroups.

Proof. Define a relation $\rho$ on $M$ by $a \rho b$ if $a^n \in MbM$ and $b^n \in MaM$ for some $n \in \mathbb{N}$. Reflexivity and symmetry of $\rho$ are obvious. To show its transitivity and compatibility, let $a \rho b$, $b \rho c$, and $d \in M$. Then $a^n \in MbM$, $b^n \in MaM$, $c^n \in McM$, and $e^n \in MbM$ for some $m, n \in \mathbb{N}$. Thus

$$a^m e (MbM)^m = M^m b^m M^m \subseteq M^m McM M^m \subseteq McM.$$

Similarly, $c^n e \in MaM$. Thus $a \rho c$. The proof that $\rho$ is compatible involves four statements like the following:

$$(ad)^{n+1} = a^{n+1} d^{n+1} \in MbMd^{n+1} \subseteq MbdMd^n \subseteq MbdM.$$

It now follows easily that $\rho$ is a semilattice congruence, for $x^y \in Mx^y M$, $(xy)^z \in Mxy M$, and $(yx)^z \in Myx M$ for every $x, y \in M$. Hence $\sigma_M \subseteq \rho$. Conversely, if $a \rho b$ then $a^n = ubv$ and $b^n = waz$ for some $n \in \mathbb{N}$ and $u, v, w, z \in M$. Thus

$$a_\sigma_M a_\sigma_M ubv_\sigma_M ubv_\sigma_M a_\sigma_M b_\sigma_M w_\sigma_M a_\sigma_M z_\sigma_M b_\sigma_M w_\sigma_M b_\sigma_M.$$

Therefore $\rho \subseteq \sigma_M$. Hence $\rho = \sigma_M$.

It remains to show that a congruence class $L$ of $\sigma_M$ is an Archimedean semigroup. By what was mentioned in the introduction, $aa_Lb$ for every $a, b \in L$. But $L$ is a medial semigroup, and we therefore can apply the explicit formulation of $\sigma_L$ above. Thus $L$ is Archimedean.

For a medial semigroup $M$, we will call the congruence classes of $M$ modulo $\sigma_M$ the Archimedean components of $M$ and will say that $M$ is a semilattice of Archimedean semigroups and write $M = U [M_a : a \in Y]$, where $Y = \{a_\sigma M : a \in M\}$.

3. Archimedean Semigroups

In this section we investigate Archimedean semigroups containing idempotents. We do not require mediality.

A subset $A$ of a semigroup $S$ will be called a root of $S$ if for each $a \in S$ there is some $n \in \mathbb{N}$ such that $a^n \in A$. A semigroup $S$ will be called rooted if it contains an idempotent and if the union of all its subgroups is a root for $S$.

The proof of the following is a direct application of the definition of an Archimedean semigroup.

Lemma 3.1. A semigroup $S$ is Archimedean if and only if each of its ideals is a root of $S$. 
**THEOREM 3.2.** A semigroup $S$ is Archimedean and contains an idempotent if and only if it possesses an ideal $K$ which is both a simple semigroup with an idempotent and a root of $S$.

*Proof.* Let $e$ be an idempotent of an Archimedean semigroup $S$. Since every ideal of $S$ is a root of $S$, $e$ belongs to all ideals of $S$. Thus $S$ contains a minimal ideal $K$. This ideal $K$ is obviously a root of $S$, and by Corollary 2.30 of [1], it is a simple semigroup.

Conversely, if $S$ possesses an ideal $K$ which is a simple semigroup, then $K$ must be the minimal ideal of $S$. Thus if it is a root of $S$, every ideal of $S$ is a root of $S$. By Lemma 3.1, $S$ is Archimedean.

**COROLLARY 3.3.** A semigroup $S$ is Archimedean and rooted if and only if it contains an ideal $K$ which is both a completely simple semigroup and a root of $S$.

*Proof.* If $S$ is Archimedean and rooted, it has a simple ideal $K$ as a root, by Theorem 3.2. But $K$ itself must be rooted if $S$ is. Thus by Theorem 2.55 of [1], $K$ is completely simple.

If a semigroup $S$ contains a completely simple ideal $K$ which is a root of $S$, then $S$ is Archimedean by Theorem 3.2. In addition, since $K$ is the union of all subgroups of $S$, $S$ is rooted.

**COROLLARY 3.4.** Let $S$ be Archimedean and rooted. For all idempotents $e, f$ of $S$, $eS$ is a right group, $Sf$ is a left group, and $eSf$ is a group.

*Proof.* The semigroup $S$ has an ideal $K$ which is completely simple. Since $K$ contains all idempotents of $S$, $eS = e(eS) \subseteq eK \subseteq eS$ for every idempotent $e$ of $S$. Hence $eS = eK$. Similarly, $Se = Ke$ and $eSf = eKf$ for all idempotents $e$ and $f$ of $S$. Using a representation of $K$ as triples ([1], Theorem 3.5), we can easily show, for instance, that $eK = eS$ is a right group.

We end our discussion of Archimedean semigroups by merely stating two results. A semigroup $T$ is right Archimedean if, for all $a, b \in T$, $a^n \in bT$, and $b^n \in aT$ for some $n \in \mathbb{N}$.

**THEOREM 3.5.** A semigroup $T$ is right Archimedean and contains an idempotent if and only if it possesses an ideal which is both a right group and a root of $T$.

**COROLLARY 3.6.** If $T$ is a right Archimedean semigroup containing idempotents $e$ and $f$, then $eT$ is a right group and $Te = fTe$ is a group. Furthermore, the set of all idempotents $E$ of $T$ is a right zero semigroup and $eT \cong Te$ for all $e \in E$. 

It is interesting to notice that the presence of an idempotent in a right Archimedean semigroup $T$ guarantees that $T$ will be rooted.

4. Medial Archimedean Semigroups

Theorem 2.1 certainly motivates us to study medial semigroups which are Archimedean. We begin by studying such semigroups which contain idempotents.

Let $L$ and $R$ be non-empty sets, and make both into semigroups by defining $l \cdot m = l(l, m \in L)$ and $r \cdot s = s(r, s \in R)$. The direct product $L \times R$ is called a rectangular band. If a semigroup is isomorphic to the direct product of a group and a rectangular band, it will be called a rectangular group. Ivan [3] has proved that a semigroup is a rectangular group if and only if it is a completely simple semigroup in which the idempotents form a subsemigroup.

**Theorem 4.1.** Let $M$ be a medial semigroup. $M$ is an Archimedean semigroup possessing an idempotent if and only if $M$ contains an ideal $J$ which is both a rectangular group and a root of $M$.

**Proof.** Let $M$ be Archimedean with an idempotent. By Theorem 3.2, $M$ contains a simple ideal $J$ which is a root of $M$. Let $e, f$, be idempotents of $J$ such that $ef = fe = e$. Since $J$ is simple, $f \cdot xey$ for some $x, y \in J$. Thus

$$f = f^2 = xef = xef = xeyef = fef = ef = e.$$ 

So every idempotent of $J$ is primitive ([1], p. 76), and hence $J$ is a completely simple semigroup ([1], p. 76). But the idempotents of a medial semigroup form a subsemigroup. Therefore, by Ivan’s result, $J$ is a rectangular group.

Since a rectangular group is simple with an idempotent, the converse is a trivial consequence of Theorem 3.2.

**Corollary 4.2.** Let $M$ and $J$ be as in Theorem 4.1. If $E$ is the set of idempotents of $M$, then $E$ is a rectangular band; $eMf$ is an abelian group for all $e, f, g \in E$, and $J \cong eMf \times E$ for all $e, f \in E$.

**Proof.** All idempotents of $M$ lie in $J$. But if $J \cong G \times \Gamma$, $G$ a group and $\Gamma$ a rectangular band, then $\Gamma$ is isomorphic to the set of all idempotents of $J$. Hence $E$ is isomorphic to the rectangular band $\Gamma$. Furthermore, since $M$ is rooted, $eMf$, for all $e, f, g \in E$, is a group by Corollary 3.4, and since $M$ is medial, $eMf$ is abelian. Finally, we saw in the proof of Corollary 3.4 that $eMf \cong G$. Thus $J \cong eMf \times E$ for all $e, f \in E$.

**Corollary 4.3.** Let $M, J$, and $E$ be as in Corollary 4.2. There exists a congruence relation $\rho$ on $M$ such that $M/\rho$ is isomorphic to the rectangular
band $E$ and such that each congruence class $M_e(e \in E)$ of $\rho$ contains an ideal $G_e$ which is both an abelian group and a root of the semigroup $M_e$.

Proof. Since $J$ is isomorphic to the direct product $G \times E$ of a group $G$ and the rectangular band of idempotents $E$, we can think of $J$ as a disjoint union of groups $G_e$, where $G_e = \{(g, e) : g \in G\}$. Now let

$$M_e = \{a \in M : a^n \in G_e \text{ for some } n \in N\}.$$

Since the $G_e$'s are disjoint subsemigroups and since $J = \bigcup_{e \in E} G_e$ is a root of $M$, the $M_e$'s induce an equivalence relation $\rho$ on $M$. To verify the compatibility of $\rho$, let $a^n, b^m \in G_e$ and $d \in M$. Clearly $d^p \in G_f$ for some $f \in E$ and $p \in N$. Therefore

$$(ad)^{np} = a^{np}d^m \in G_e G_f \subseteq G_{ef}.$$  

Similarly, $(bd)^{np} \in G_{ef}$. Thus $ad \rho bd$. Likewise, $da \rho db$.

The fact that $M/\rho$ is isomorphic to $E$ is easily seen since $E$ is a representative system for the congruence $\rho$ and at the same time a semigroup. Finally, let $M_e$ be a congruence class of $\rho$, $M_e \cap J = G_e$, and since $J$ is an ideal of $M$, $G_e$ is an ideal of $M_e$. The definition of $M_e$ clearly implies that $G_e$ is a root of $M_e$. The proof is complete.

It is obvious from Theorem 4.1 that a medial Archimedean semigroup is rooted if and only if it contains an idempotent. Consequently, every Archimedean component of a medial semigroup $N$ satisfies Theorem 4.1 if and only if $N$ is rooted.

5. Separative Semigroups

A semigroup is called separative if $x^2 = xy = y^2$ implies $x = y$. We will call a semigroup left [right] separative if $x^2 = xy$ and $y^2 = yx$ imply $x = y$. If a medial semigroup is left [right] separative, then it is separative. A rectangular band $L \times R$ with $|L| \geq 2$ and $|R| \geq 2$ shows that the converse is false.

**Theorem 5.1.** If $M$ is a medial semigroup with Archimedean components $M_a(x \in Y)$, then

1. $M$ is separative if and only if in each $M_a$ $ax = ay$ and $xb = yb$ implies $x = y$,

2. $M$ is left [right] separative if and only if each $M_a$ is left [right] cancellative, and

3. $M$ is left and right separative if and only if each $M_a$ is cancellative.
Proof. Let $M$ be a separative medial semigroup. If, for $n \geq 2$, $x^{n+1} = x^n y (x, y \in M)$, then

$$\begin{align*}
(x^n)^2 &= x^{n-1} x^{n+1} = x^{n-1} x^n y = x^n (x^{n-1} y) \\
&= x^{n-1} x^n y = x^n (x^{n-1} y) = (x^n y)^2,
\end{align*}$$

where $x^n y = y$ if $n = 2$. Thus $x^n = x^{n-1} y$, by the separativity of $M$. Repeating this process $n - 1$ times, we get $x^2 = xy$. Similarly, $x^{n+1} = y x^n$, $n \in N$, implies $x^2 = y x$.

Now suppose $a, x, y \in M_\alpha (\alpha \in Y)$ such that $ax = ay$. Then for some $u, v, \in M$ and $n \in N$, $x^n = u a v$.

Thus $x^{n-1} = u a v x = u a v y = u a v = x^n y$.

By the above, $x^2 = xy$. A similar argument proves $xb = yb(x, y, b \in M_\alpha)$ implies $yb = y$, since $M$ is separative, $ax = ay$ and $xb = yb(a, b, x, y \in M_\alpha)$ implies $x = y$. This proves half of (1).

Conversely, let each $M_\alpha (\alpha \in Y)$ satisfy this condition, and suppose $x^2 = xy = y^2$ for $x, y \in M$. Recalling the definition of $\sigma_M$,

$$x \sigma_M x^2 \sigma_M y^2 \sigma_M y.$$

Therefore, $x$ and $y$ are in the same Archimedean component. Since $xx = xy$, $xy = yx$, $x = y$. The proof of (1) is complete. The proof of (2) is similar, and (3) follows immediately from (2).

If $\rho$ is a congruence on a semigroup $S$, the elements of $S/\rho$ will be denoted by $\overline{x} (x \in S)$.

LEMMA 5.2. Let $L$ be a medial semigroup in which $ax \equiv ay$ and $xb \equiv yb$ imply $x \equiv y$. Define a relation $\eta$ on the semigroup $L^* := L \times L \times L$ by $(a, b, c) \eta (a', b', c')$ if $cab'c' = c' a' b c$. This relation is a congruence relation, and $L^*/\eta$ is a rectangular group. Moreover, the mapping $\varphi : L \to L^*/\eta$ defined by $(a) \varphi = (\overline{a}, a^2, a)$ is an isomorphism.

Proof. Let $L$, $\eta$, $L^*$, and $\varphi$ be as stated above. Reflexivity and symmetry of $\eta$ follow immediately. To prove transitivity, let $(a_1, b_1, c_1) \eta (a_2, b_2, c_2)$ and $(a_2, b_2, c_2) \eta (a_3, b_3, c_3)$. By the definition of $\eta$, $c_2 a_2 b_2 c_2 = c_3 a_3 b_2 c_3$ and $c_2 a_2 b_2 c_2 = c_2 a_2 b_2 c_2$. We get

$$c_2 a_2 b_2 c_2 c_2 a_2 b_2 c_2 = c_3 a_3 b_3 c_3.$$

$$c_2 a_2 b_2 c_2 c_2 a_2 b_2 c_2 = c_3 a_3 b_2 c_3.$$

Similarly, $(c_3 a_3 b_2 c_3) b_2 c_2 c_2 = (c_3 a_3 b_1 c_1) b_2 c_2 c_2$. 

Therefore, $x$ and $y$ are in the same Archimedean component. Since $xx = xy$, $xy = yx$, $x = y$. The proof of (1) is complete. The proof of (2) is similar, and (3) follows immediately from (2).

If $\rho$ is a congruence on a semigroup $S$, the elements of $S/\rho$ will be denoted by $\overline{x} (x \in S)$.
By our assumption on $L$, $c_1 a_1 b_1 c_2 = c_2 a_2 b_2 c_1$, in other words, $(a_1, b_1, c_1) (a_2, b_2, c_2)$. The proof that $\eta$ is compatible involves a routine application of mediality.

Since $L/\eta^*$ is obviously medial, the proof that it is a rectangular group will be complete once we show that it is simple and contains an idempotent (Theorem 4.1). The existence of an idempotent is easy. Any element of the form $(a, a, a)$ is idempotent. Now let $(a, b, c), (e, f, g) \in L^*/\eta$. Since $(b, a, g)(a, b, c)(e, f, g) \eta(e, f, g), L^*/\eta$ is simple.

To show that $\varphi$ is an isomorphism, let $\varphi(a) = \varphi(b) (a, b \in L)$. By the definition of $\varphi$, $(a, a^2, a) \eta(b, b^2, b)$. Thus

$$a^2 b^3 = a b b^2 = b b a^2 = b^2 a^3.$$ Consequently,

$$(b^2 a^2) a = b (b^2 a^2) = b (a^2 b^3) = (b^3 a^2) b,$$

and

$$a (a^2 b^3) - (a^2 b^3) (a b^2) - (b^2 a^3) (a b^2) - (b b a^3) ab$$

$$= b (a^2 b^3) ab = b (b^2 a^2) b^2 = b (a^2 b^3) b^2 = b (a b^3).$$

By our assumption on $L$, $a = b$. Finally, $\varphi$ is a homomorphism since $\varphi(a) \varphi(b) = (ab, a^2 b^2, ab) = (ab, (ab)^2, ab) = \varphi(ab)$.

**Theorem 5.3.** A medial semigroup can be embedded in a semigroup which is a union of groups if and only if it is separative.

**Proof.** Let $S$ be a semigroup which is a union of groups, and let $M$ be a subsemigroup of $S$. To prove separativity, suppose $x^2 = xy = y^2$ for $x, y \in M$. By ([1], p. 23), $S$ is a disjoint union of groups. Therefore $x$ and $y$ must be elements of a common subgroup of $S$. But in this group $x = y$. Thus $x = y$.

Conversely, Let $M$ be a separative medial semigroup with Archimedean components $M_\alpha (\alpha \in Y)$. Combining Theorem 5.1 and Lemma 5.2, we can embed each $M_\alpha$, using a congruence relation $\eta_\alpha$, into a rectangular group $R_\alpha$. Now if $(a, b, c) \in R_\beta$ and $(x, y, z) \in R_\gamma (\beta, \gamma \in Y)$, define a product on $R = U\{R_\alpha : \alpha \in Y\}$ by

$$(a, b, c) \cdot (x, y, z) = (ax, by, cz) \in R_\beta \gamma.$$ To show that the product is well defined, let $(a, b, c) \eta_\beta (a', b', c')$ and $(x, y, z) \eta_\gamma (x', y', z')$. Then in $R_\beta \gamma$,

$$(ax, by, cz) = (a' x', b' y', c' z')$$

since

$$(cz)(ax)(b'y')(c' z') = (cab' c')(zxy' z') = (c'a' bc)(z' x' yz)$$

$$= (c' z')(a' x')(by)(cz).$$
The operation is obviously associative. Finally, define a mapping \( \Psi : M \to R \) by 
\[
(a, a^2, a) \in R \] if \( a \in M \). Since the restriction of \( \Psi \) to each \( M \) is one-to-one (Lemma 5.2) and since \( (M \alpha) \Psi \cap (M \beta) \Psi = \emptyset \) if \( \alpha \neq \beta \), \( \Psi \) is 1 – 1. It is also a homomorphism since
\[
(ab) \Psi = (ab, (ab)^2, ab) = (ab, ab^2ab) = (a) \Psi \cdot (b) \Psi.
\]
Thus \( \Psi \) is embedded in the union of groups \( R \).

A congruence \( \xi \) on a semigroup \( S \) will be called a [left, right] separative congruence if \( S/\xi \) is [left, right] separative. The following theorem determines explicitly the least [left, right, left and right] separative congruence on a medial semigroup.

**Theorem 5.4.** Let \( M \) be a medial semigroup. The relation \( \xi \) defined by
\[
\xi = \{(a, b) \mid a^n = a^n b \text{ and } b^n = b^n a \text{ for some } n \in \mathbb{N}\}
\]
is the smallest left separative congruence on \( M \). The smallest right separative congruence \( \eta \) on \( M \) is defined dually. The relation \( \xi_0 = \xi_1 \cap \xi_2 \) is the smallest separative congruence on \( M \), and the relation \( \xi_3 \) defined by \( \xi_3 = \{(a, b) \mid a^n = a^n b a \text{ and } b^n = b^n a b \text{ for some } n \in \mathbb{N}\} \) is the smallest left and right separative congruence on \( M \).

**Proof.** Since all four proofs are similar, we will prove the theorem just for \( \xi_0 \). Both \( \xi_2 \) and \( \xi_3 \) are easily shown to be congruences. Hence \( \xi_0 = \xi_1 \cap \xi_2 \) is also. To show that \( \xi_0 \) is separative, let \( x \xi_0 y, x \xi_0 y' \) for some \( x, y \in M \). By the definition of \( \xi_0 \), we have \( x^2 \xi_1 xy, xy \xi_1 y^2 \), and \( x^2 \xi_1 y^2 \). Thus for some \( n \in \mathbb{N} \),

1. \( (x^2)^{n+1} = (x^2)^n \cdot xy \)
2. \( (y^2)^{n+1} = (y^2)^n \cdot xy \)
3. \( (y^2)^{n+1} = (y^2)^n \cdot x^2 \)

Multiplying (1) by \( x^2 \), we get
\[
(4) \quad x^{2n+4} = x^{2n+3}y,
\]
Multiplying (3) by \( y^2 \) and using (2), we get
\[
(5) \quad y^{2n+4} = y^{2n+2}x^2 = y(y^{2n}xy) = y^2y^{2n-2} = y^{2n+4}x.
\]
Statements (4) and (5) imply \( x \xi_1 y \). Similarly, \( x \xi_2 y \).

Hence \( x \xi_0 y \).

It remains to prove that \( \xi_0 \) is the smallest such congruence. Let \( \xi \) be another separative congruence on \( M \), and let \( a \xi b(a, b \in M) \). Since \( a \xi b \) and \( a \xi b \),
\[
a^{n+1} = a^n b \text{ and } b^{n+1} = ab^n \text{ for some } n \in \mathbb{N}.
\]
Letting \( M/\xi = P \), we get
\[
\bar{a} = \bar{a} \xi \bar{b} = \bar{a} \xi \bar{b}.
\]
Hence \( \bar{a} \) and \( \bar{b} \) are in the same Archimedean component of \( P \). But \( P \) is separative. Thus by (1) of Theorem 5.1, \( \bar{a} = \bar{b} \) since \( \bar{a}^{n+1} = \bar{a}^n \bar{b} \) and \( \bar{b}^{n+1} = \bar{a}^n \bar{b} \). In other words, \( a \xi b \). This completes the proof.
Note: Results have been avoided for Archimedean semigroups without an idempotent. In fact, the only result known in the commutative case requires cancellation. See [7] or ([1], p. 136). Related results appear in [4] and [10]. A similar characterization is possible for medial Archimedean semigroups without an idempotent if we require left cancellation (we will call these $N^\ast$-semigroups) and goes as follows. Let $H$ be a medial right group and $I$ be a function from $H \times H$ to $Z$ (the set of non-negative integers) which satisfies

1. $I(\alpha, \beta y) + I(\beta, y) = I(\alpha, \beta) + I(\alpha \beta, y) - I(\beta, \gamma) + I(\beta, \alpha),$

for each $\alpha \in H$ there is an $n \in N$ such that $I(\alpha^n, \alpha) > 0$, and

2. there is a left identity $e$ of $H$ such that $I(e, e) = 1$. Define a product on $Z \times H$ by

$$(n, \alpha) \cdot (m, \beta) = (n + m + I(\alpha, \beta), \alpha \beta).$$

Then $\{Z \times H, \cdot\}$ is an $N^\ast$-semigroup. Conversely, for an $N^\ast$-semigroup $S$ there is a right group $H$ and a function $I : H \times H \rightarrow Z$ satisfying (1), (2), and (3) such that $\{Z \times H, \cdot\}$ is isomorphic to $S$. We have also proved the following generalization of [4]. Let $S$ be an $N^\ast$-semigroup. $S$ is finitely generated if and only if each of its associated right groups is finite. Furthermore, each associated right group of $S$ is periodic if and only if for each $x, y \in S$ there exist $p, q \in N$ such that $x^p = y^q$.

References

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