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On the Mean-Square of the Riemann Zeta-Function on the Critical Line

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The function $E(T)$ is used to denote the error term in the mean-square estimate for the Riemann zeta-function on the half-line. In this paper we will prove a variety of new results concerning this function. The general aim is to extend the analogy of this function with the error term in Dirichlet's divisor problem. There are three main themes that we stress. The first theme is representations for the integral $E_1(T)=\int_0^T E(t)\,dt$. The forms these take are similar to (but more complicated than) the analogous formulas due to Voronoi in the divisor problem. The proof proceeds somewhat as the proof Atkinson used to get his representation for $E(T)$ itself, and is about as difficult. The extra averaging does not seem to aid the method significantly, as it did for Voronoi. The second theme is upper and lower bound results. Essentially we show that the best bounds known in the divisor problem also hold for this function as well. These include both omega-plus and omega-minus results for $E(T)$ and for $E_1(T)$. The results for $E_1(T)$ completely determine its order. The methods used here are again similar to ones used in the divisor problem. However, some recent innovations are needed to account for the lack of arithmetical structure and the complicated natures of our representation for $E_1(T)$ and Atkinson's for $E(T)$. Finally, we prove a mean-square estimate for $E_1(T)$. This estimate indicates that this function frequently achieves its maximal order.

1. Introduction

Mean-square estimates have always played a central role in the theory of the Riemann zeta-function $\zeta(s)$. Let, as usual, for $T \geqslant 2$,

$$E(T)=\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt - T\log(T/2\pi)-(2\gamma-1)\,T$$

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denote the error term in the mean-square formula for $\zeta(s)$ on the "critical line" $\Re e\, s = \frac{1}{2}$ (γ is Euler's constant). This function has a rich but difficult history, and we refer the reader to [18, Chap. 15] for a comprehensive account. An important explicit formula for $E(T)$ was established long ago by F. V. Atkinson [1]. If $AT < N < A'T$ for any two fixed constants $0 < A < A'$, then Atkinson's result may be written as

$$E(T) = \Sigma_1(T) + \Sigma_2(T) + O(\log^2 T), \tag{1.1}$$

where

$$\Sigma_1(T) = \left(\frac{2T}{\pi}\right)^{1/4} \sum_{n \leqslant N} (-1)^n \frac{d(n)}{n^{3/4}} e_1(T, n) \cos(f(T, n)), \tag{1.2}$$

$$\Sigma_2(T) = -2 \sum_{n \leqslant N'} \frac{d(n)}{n^{1/2}} (\log T/2\pi n)^{-1} \cos(g(T, n)), \tag{1.3}$$

and

$$e_j(T, n) = \left(1 + \frac{\pi n}{2T}\right)^{-1/4} \left\{\left(\frac{2T}{\pi n}\right)^{1/2} \operatorname{ar\,sinh}\left(\frac{\pi n}{2T}\right)^{1/2}\right\}^{-j} \qquad (j = 1, 2), \tag{1.4}$$

$$f(T, n) = 2T \operatorname{ar\,sinh}\left(\frac{\pi n}{2T}\right)^{1/2} + (2\pi n T + \pi^2 n^2)^{1/2} - \frac{\pi}{4}, \tag{1.5}$$

$$g(T, n) = T \log\left(\frac{T}{2\pi n}\right) - T + \frac{\pi}{4}, \tag{1.6}$$

$$N' = N'(T, N) = \frac{T}{2\pi} + \frac{N}{2} - \left(\frac{N^2}{4} + \frac{NT}{2\pi}\right)^{1/2}, \tag{1.7}$$

and as usual $d(n) = \sum_{\delta \mid n} 1$ is the number of divisors of n and $\operatorname{ar\,sinh} x = \log(x + \sqrt{x^2 + 1})$. (The function $e_2(T, n)$ will appear in the mean-value formula (2.4).) One of the motivations for Atkinson's work was a certain analogy between $E(T)$ and $2\pi\Delta(T/2\pi)$, where

$$\Delta(x) = \sum_{n \leqslant x} d(n) - x \log x - (2\gamma - 1)\, x$$

is the error term in the asymptotic formula for the summatory function of $d(n)$ (Dirichlet's divisor problem). Indeed, the first $o(T^{1/3})$ terms in $\Sigma_1(T)$ are asymptotically equal (apart from the $(-1)^n$ factor) to the corresponding terms in the famous Voronoi formula for $2\pi\Delta(T/2\pi)$ (see [18, Chap. 3]). This analogy is also present in the mean-square formulas

$$\int_2^T \Delta^2(t)\, dt = \left\{(6\pi^2)^{-1} \sum_{n=1}^{\infty} d^2(n)\, n^{-3/2}\right\} T^{3/2} + O(T^{5/4+\varepsilon}) \tag{1.8}$$

and

$$\int_2^T E^2(t)\,dt = \left\{\tfrac{2}{3}(2\pi)^{-1/2} \sum_{n=1}^{\infty} d^2(n)\,n^{-3/2}\right\} T^{3/2} + O(T^{5/4}\log^2 T). \tag{1.9}$$

The asymptotic formula (1.8) is a classical result of H. Cramér [6] (see also [18, Chap. 13] for a proof), and the error term has been improved to $O(T\log^5 T)$ by K.-C. Tong [27]. The mean-square result (1.9) was proved by D. R. Heath-Brown [14] who used Atkinson's formula and Cramér's method. The error term here has been improved only recently to $O(T\log^5 T)$ by T. Meurman [25]. Estimates for the higher power moments of both $\Delta(T)$ and $E(T)$ were derived by A. Ivić [17], and all these results show the analogy between the two functions.

The asymptotic formulas (1.8) and (1.9) provide at once the weak omega results $\Delta(T)=\Omega(T^{1/4})$ and $E(T)=\Omega(T^{1/4})$, where $f(x)=\Omega(g(x))$ as $x\to\infty$ means that $f(x)=o(g(x))$ does not hold. The latter result was proved, independently of (1.9), by A. Good [8]. Here the situation is markedly different, since there is a long history of improvements in both Ω_+ and Ω_- results for $\Delta(T)$, while nothing beyond $E(T)=\Omega(T^{1/4})$ seems to have been published heretofore. We recall that $f(x)=\Omega_+(g(x))$ means that there exists a positive constant C and a sequence x_n tending to infinity such that $f(x_n)>Cg(x_n)$ for all n. Analogously, $f(x)=\Omega_-(g(x))$ means that the inequality $f(y_n)<-Cg(y_n)$ holds for another sequence y_n. The best known omega results for $\Delta(T)$ are at present

$$\Delta(T)=\Omega_+\{(T\log T)^{1/4}(\log\log T)^{(3+\log 4)/4}\exp\{-C\sqrt{\log\log\log T}\}\} \tag{1.10}$$

and

$$\Delta(T)=\Omega_-\left\{T^{1/4}\exp\left(D\frac{(\log\log T)^{1/4}}{(\log\log\log T)^{3/4}}\right)\right\} \tag{1.11}$$

for some suitable positive constants C and D. Of these, (1.10) is due to J. L. Hafner [10], while (1.11) was proved by K. Corrádi and I. Kátai [5]. These papers contain references to previous work on the same subject. By analogy between $E(T)$ and $2\pi\Delta(T/2\pi)$, one expects that sharper omega results than $E(T)=\Omega(T^{1/4})$ should hold, as was hinted in [18, p. 482].

2. Statement of Results

Our main aim lies in obtaining omega results for $E(T)$ and establishing a sharp asymptotic formula for $\int_2^T E(t)\,dt$. In trying to establish the analogues of (1.10) and (1.11) for $E(T)$, one encounters the following difficulties. First, the Voronoi formula (weak version)

$$\Delta(T)=\frac{T^{1/4}}{\pi\sqrt{2}}\sum_{n\leqslant T}\frac{d(n)}{n^{3/4}}\cos(4\pi\sqrt{nT}-\pi/4)+O(T^{\varepsilon})$$

is considerably simpler than Atkinson's formula for $E(T)$, which contains two fairly complicated sums whose length is $O(T)$. Moreover, the sum $\Sigma_1(T)$ in Atkinson's formula contains the oscillating factor $(-1)^n$. It was expected by many (including the authors) that this oscillating factor would hinder an effective Ω_+ result for $E(T)$. It will be shown, however, that the method of [10] can be applied in this situation. The result is the following.

THEOREM 1. *There exists a positive constant C such that*

$$E(T)=\Omega_+\{(T\log T)^{1/4}(\log\log T)^{(3+\log 4)/4}\exp\{-C\sqrt{\log\log\log T}\}\}. \tag{2.1}$$

To deal with the analogue of (1.11) for $E(T)$ we shall use the ideas developed in [11], where sharp omega results for $\Delta(x; a, b)$ (the error term in the asymptotic formula for $\sum_{m^a n^b \leqslant x} 1$, $1\leqslant a<b$ fixed integers) are established. Therein some fundamental ideas of Corrádi and Kátai [5] are used, which lead to the proof of (1.11). Here we shall prove the following.

THEOREM 2. *There exists a positive constant D such that*

$$E(T)=\Omega_-\left\{T^{1/4}\exp\left(D\frac{(\log\log T)^{1/4}}{(\log\log\log T)^{3/4}}\right)\right\}. \tag{2.2}$$

The omega results of Theorems 1 and 2 are best possible in the sense that they correspond to currently best known omega results in Dirichlet's divisor problem. Since problems involving $E(T)$ are, in general, more difficult than the corresponding problems involving $\Delta(T)$, it is hard to imagine improvements over (2.1) and (2.2) which would not entail corresponding improvements in (1.10) and (1.11). In analogy with the classical conjecture $\Delta(T)\ll T^{1/4+\varepsilon}$, one expects also $E(T)\ll T^{1/4+\varepsilon}$ to hold, so actually both (2.1) and (2.2) should be fairly close to the truth.

In connection with the proof of Theorem 2, we should point out that the method used by Corrádi and Kátai [5] requires a functional equation for the generating functions of the arithmetical functions involved. In the case of $E(T)$ we are not dealing with a situation of this type. The new idea introduced in [11] is that it is really the Voronoi-type series representation for the error term that provides the essential tool and not the functional equation from which it is usually derived. But Atkinson's formula (1.1) is not an infinite series representation for $E(T)$, so that it cannot be used directly in this context. However, besides the explicit formula for $\Delta(T)$, G. F. Voronoi [28] also proved

$$\int_2^T \Delta(t)\,dt = \frac{T}{4} + \frac{T^{3/4}}{2\sqrt{2}\,\pi^2} \sum_{n=1}^{\infty} \frac{d(n)}{n^{5/4}} \sin(4\pi\sqrt{nT} - \pi/4) + O(1). \quad (2.3)$$

(This also follows from [18, Eqs. (3.26) and (3.31)].) Thus before proving Theorem 2 we shall establish the analogue of (2.3) for $E(T)$, which will then be used in proving Theorem 2. This is an interesting problem in itself and the result is an asymptotic formula with a sharp error term.

THEOREM 3. *For $T \geqslant 2$,*

$$\int_2^T E(t)\,dt = \pi T + \frac{1}{2}\left(\frac{2T}{\pi}\right)^{3/4} \sum_{n \leqslant T} (-1)^n \frac{d(n)}{n^{5/4}} e_2(T, n) \sin(f(T, n)),$$

$$-2 \sum_{n \leqslant c_0 T} \frac{d(n)}{n^{1/2}} (\log T/2\pi n)^{-2} \sin(g(T, n)) + O(T^{1/4}), \quad (2.4)$$

where $e_j(T, n)$, $f(T, n)$, and $g(T, n)$ are given by (1.4), (1.5), *and* (1.6), *respectively, and $c_0 = 1/2\pi + 1/2 - \sqrt{1/4 + 1/2\pi}$.*

This formula may be proved in a more general form (see (4.19)), with the ranges of summation $n \leqslant N$ and $n \leqslant N'$, respectively, as in Atkinson's formula. A rather simple consequence of (2.4) is the asymptotic formula

$$\int_2^T E(t)\,dt = \pi T + \frac{1}{2}\left(\frac{2T}{\pi}\right)^{3/4} \sum_{n=1}^{\infty} (-1)^n \frac{d(n)}{n^{5/4}} \sin\left(4\pi\sqrt{\frac{nT}{2\pi}} - \frac{\pi}{4}\right)$$

$$+ O(T^{2/3} \log T). \quad (2.5)$$

This will be the necessary tool in the proof of Theorem 2. The formulas (2.3) and (2.5) also support the analogy between $E(T)$ and $2\pi\Delta(T/2\pi)$, since apart from the factor $(-1)^n$ the series in (2.5) is the same as the one in (2.3) with $T/2\pi$ in place of T. To obtain (2.5) from (2.4) note that, by standard complex integration methods,

$$\sum_{n \leqslant c_0 T} \frac{d(n)}{n^{1/2}} (\log T/2\pi n)^{-2} \sin(g(T, n)) \ll T^{2\mu(1/2) + \varepsilon}, \quad (2.6)$$

where as usual $\mu(\sigma) = \limsup_{t \to \infty} \log |\zeta(\sigma + it)|/\log t$. Hence the classical bound $\mu(\frac{1}{2}) \leqslant \frac{1}{6}$ is already more than sufficient for the error term in (2.5). The first sum in (2.4) is then split at $T^{1/3}$. The terms for which $n \geqslant T^{1/3}$ are estimated trivially, and the remaining ones are simplified by using Taylor's formula. This leads to (2.5). We are aided here by the fact that the series in (2.5) is absolutely convergent.

We now define, for $T \geqslant 2$,

$$G(T) = \int_2^T E(t)\, dt - \pi T. \tag{2.7}$$

Then from (2.4) or (2.5) it follows easily that $G(T) = O(T^{3/4})$. It seemed interesting to investigate this function $G(T)$ more closely. Our results for this function are summarized in the following theorem.

THEOREM 4. *If $G(T)$ is defined as above, then*

$$\int_2^T G^2(t)\, dt = BT^{5/2} + O(T^2), \qquad B = \frac{\zeta^4(5/2)}{5\pi\sqrt{2\pi}\,\zeta(5)} = 0.079320\cdots \tag{2.8}$$

and moreover

$$G(T) = \Omega_{\pm}(T^{3/4}). \tag{2.9}$$

From (2.8) it follows immediately that $G(T) = \Omega(T^{3/4})$ but (2.9) is sharper, since it means that both $G(T) = \Omega_+(T^{3/4})$ and $G(T) = \Omega_-(T^{3/4})$ are true. Thus apart from the value of the numerical constants involved, the order of magnitude of $G(T)$ is precisely determined. Using (2.3), one can obtain the analogue of Theorem 4 for $\Delta(T)$, but the proof in this case would be simpler and we omit it.

Besides being needed for the proof of Theorem 2, the result of Theorem 3 may be used in various other problems involving $E(T)$. One is determining upper bounds for $E(T)$. Without any use of exponential sums we can prove very simply that

$$E(T) \ll T^{1/3} \log T. \tag{2.10}$$

This result was first obtained relatively recently by R. Balasubramanian [2], who integrated the classical Riemann–Siegel formula to derive an expression for $E(T)$ different from Atkinson's formula. Our proof of (2.10) uses (2.5), (7.1), and (7.2) with $x = T^{1/3}$. The details are analogous to those in the corresponding result for $\Delta(T)$, which can be found in [4, Chap. VIII]. For this reason we omit them, besides a better result is given below.

Naturally, deep exponential sum techniques should lead to sharper results than (2.10). Using the method of G. Kolesnik [23] it was indicated in [18, Chap. 15] that one can obtain the bound $E(T) \ll T^{35/108+\varepsilon}$. This result depended on an estimate of M. Jutila [20] [18, Eq. (15.84)], which is the analogue of the truncated Voronoi formula for $\Delta(x)$, for which Kolesnik's method yields the estimate $\Delta(x) \ll x^{35/108+\varepsilon}$. Meanwhile, Kolesnik [24] developed his techniques still a little further, so that now it is possible in both problems to replace the exponent $35/108 = 0.324074 \cdots$ by $139/429 = 0.324009 \cdots$. This is close to the theoretical limit of the method in question, which is $0.3239247 \cdots$, as discussed by S. W. Graham [9]. In both cases the "ε" can be replaced by a suitable logarithm factor, which in view of Jutila's estimate will be poorer in the case of $E(T)$. However, using Theorem 3, we obtain a result with a much better log-factor:

THEOREM 5. *We have*

$$E(T) \ll T^{139/429} (\log T)^{1467/429}. \tag{2.11}$$

Naturally, the above estimate holds also for $\Delta(T)$. But after the first version of this paper was written, we have been informed by H. Iwaniec and C. J. Mozzochi that they have just completed their important work [19], where they proved $\Delta(x) \ll x^{7/22+\varepsilon}$ (and the analogous bound for the circle problem). They started from the elementary formula

$$\Delta(x) = -2 \sum_{n \leqslant \sqrt{x}} \psi\left(\frac{x}{n}\right) + O(1) \qquad (\psi(t) = t - [t] - 1/2), \tag{2.12}$$

and used a method similar to the one developed recently by E. Bombieri and H. Iwaniec [3] in proving $\zeta(\frac{1}{2} + it) \ll t^{9/56+\varepsilon}$. The exponent $\frac{7}{22} = 0.3181818 \cdots$ is a considerable improvement over the exponent $\frac{139}{429} = 0.324009 \cdots$, which follows from Kolesnik's method. However, note that no analogue of (2.12) is known to hold for $E(T)$. Hence it does not seem possible at present to use this new technique to improve on (2.11), which is thus the sharpest known bound for $E(T)$. We thank Iwaniec and Mozzochi for kindly making available to us the preprint of their work [19].

3. PROOF OF THE Ω_+ RESULT

To avoid the square roots which appear in Atkinson's formula we pass from $E(T)$ to the more convenient function

$$E_0(t) = (2t)^{-1/2} E(2\pi t^2). \tag{3.1}$$

We use (1.1)–(1.7) and choose $N = t^2$ so that $N' = \alpha t^2$, with $\alpha = (3 - \sqrt{5})/2$. This gives

$$E_0(t) = E_A(t) + E_B(t) + O(1),$$

where

$$E_A(t) = \sum_{n \leq N} (-1)^n \frac{d(n)}{n^{3/4}} e_A(t, n) \cos(f_A(t, n)),$$

$$E_B(t) = -(2t)^{-1/2} \sum_{n \leq N'} \frac{d(n)}{n^{1/2}} (\log t/\sqrt{n})^{-1} \cos(g_B(t, n)),$$

and

$$e_A(t, n) = \left(1 + \frac{n}{4t^2}\right)^{-1/4} \left\{\frac{2t}{\sqrt{n}} \operatorname{ar\,sinh} \frac{\sqrt{n}}{2t}\right\}^{-1},$$

$$f_A(t, n) = 2\pi t \sqrt{n} \left\{\frac{2t}{\sqrt{n}} \operatorname{ar\,sinh} \frac{\sqrt{n}}{2t} + \left(1 + \frac{n}{4t^2}\right)^{1/2}\right\} - \frac{\pi}{4},$$

$$g_B(t, n) = 4\pi t^2 \log \frac{t}{\sqrt{ne}} + \frac{\pi}{4}.$$

Next for positive integers n we let $\lambda_n = 4\pi \sqrt{n}$ and

$$k_n(u) = K_{\lambda_n/2}(u) = \frac{\lambda_n}{2\pi} \left(\frac{\sin(\lambda_n u/2)}{\lambda_n u/2}\right)^2,$$

the Fejér kernel of index $\lambda_n/2$. Finally for a large positive integer M, let

$$E^*(t) = \int_{-1}^{1} E_0(t + u)\, k_M(u)\, du. \tag{3.2}$$

This averaging is essential to be able to eliminate the contribution from Σ_2 in Atkinson's formula as well as isolate the relevant terms of Σ_1. Because $k_M(u) > 0$ and $0 < \int_{-1}^{1} k_M(u)\, du < 1$, Theorem 1 will be an immediate consequence of the following statement: *There exist absolute positive constants A and C such that*

$$E^*(t) > A(t \log t)^{1/4} (\log\log t)^{(3 + \log 4)/4} \exp\{-C \sqrt{\log\log\log t}\} \tag{3.3}$$

holds for some arbitrarily large values of t. To obtain (3.3) we need a suitable expression for $E^*(t)$. This is contained in the following lemma.

LEMMA 1. *If* $1 \leq M \leq t^{1/2}$ *then*

$$E^*(t) = \sum_{n \leq M} (-1)^n \frac{d(n)}{n^{3/4}} \left(1 - \sqrt{\frac{n}{M}}\right) \cos(4\pi t \sqrt{n} - \pi/4) + O(1). \tag{3.4}$$

The sum in (3.4) closely resembles the corresponding expression for the Ω_+ result in the divisor problem (see [10]), only now the alternating factor $(-1)^n$ is present. This forces us to be more careful than in the divisor problem. The basic idea, however, is the same. By Dirichlet's approximation theorem (see, e.g., [18, Lemma 9.1]) we shall optimally select the cosines (of even index) to be positive on a thin set of integers. These integers will have about twice the normal number of distinct prime divisors. Such a set has been constructed in [10]. We formulate the result for our convenience:

LEMMA 2. *For each positive constant C and positive integer $K \geqslant 2$, there is a set $P_C \subseteq \{1, 2, ..., K\}$ such that uniformly*

$$\sum_{\substack{n \notin P_C \\ n \leqslant K}} \frac{d(n)}{n^{3/4}} \ll C^{-2} K^{1/4} \log K,$$

and if $|P_C|$ denotes the cardinality of P_C, then

$$|P_C| \ll K(\log K)^{1-\log 4} \exp\{C \sqrt{\log \log K}\}.$$

Postponing the proof of Lemma 1, we proceed to deduce (3.3) from (3.4). Let $K = [M/2]$ and P_C be as in Lemma 2 for this K and some C to be chosen later. By Dirichlet's approximation theorem there exists t satisfying

$$M^2 \leqslant t \leqslant M^2 (64)^{|P_C|} \tag{3.5}$$

and such that for each m in P_C, and $n = 2m$ we have $|t\sqrt{n} - x_n| \leqslant \frac{1}{64}$ for some integers x_n. For these n and this t, it follows that

$$\cos\left(4\pi t \sqrt{n} - \frac{\pi}{4}\right) \geqslant \cos\left(\frac{\pi}{16} + \frac{\pi}{4}\right) > \frac{1}{2}.$$

Note that each pair M, t constructed in this way satisfies the hypotheses of Lemma 1. Now (3.4) allows us to deduce that for this pair

$$E^*(t) \geqslant \left\{\frac{1}{2} \sum_{\substack{n \leqslant M \\ n = 2m \\ m \in P_C}} - \sum_{\substack{n \leqslant M \\ n = 2m \\ m \notin P_C}} - \sum_{\substack{n \leqslant M \\ n = 2m+1}}\right\} \frac{d(n)}{n^{3/4}} \left(1 - \sqrt{\frac{n}{M}}\right) + O(1)$$

$$= \left\{\frac{1}{2} \sum_{n \leqslant M} - \frac{3}{2} \sum_{\substack{n \leqslant M \\ n = 2m \\ m \notin P_C}} - \frac{3}{2} \sum_{\substack{n \leqslant M \\ n = 2m+1}}\right\} \frac{d(n)}{n^{3/4}} \left(1 - \sqrt{\frac{n}{M}}\right) + O(1). \tag{3.6}$$

But we have

$$\sum_{\substack{n\leqslant x\\ n=2m}} d(n) = \sum_{\substack{kl\leqslant x\\ l=2m}} 1 + \sum_{\substack{kl\leqslant x\\ k=2m}} 1 - \sum_{\substack{kl\leqslant x\\ k=2m_1,\, l=2m_2}} 1$$

$$= \frac{x}{2}\log x + \frac{x}{2}\log x - \frac{x}{4}\log x + O(x)$$

$$= \frac{3}{4}x\log x + O(x),$$

and thus also

$$\sum_{\substack{n\leqslant x\\ n=2m+1}} d(n) = \frac{1}{4}x\log x + O(x).$$

Partial summation gives easily the following:

$$\sum_{n\leqslant M} \frac{d(n)}{n^{3/4}}\left(1-\sqrt{\frac{n}{M}}\right) = \frac{8}{3}M^{1/4}\log M + O(M^{1/4}),$$

$$\sum_{n=2m+1\leqslant M} \frac{d(n)}{n^{3/4}}\left(1-\sqrt{\frac{n}{M}}\right) = \frac{2}{3}M^{1/4}\log M + O(M^{1/4}).$$

Using Lemma 2 and these estimates we infer from (3.6) for this pair t, M and C sufficiently large that

$$E^*(t) \geqslant (\tfrac{1}{3} + O(C^{-2}))\, M^{1/4}\log M + O(M^{1/4}) \geqslant \tfrac{1}{4}M^{1/4}\log M. \tag{3.7}$$

Now the right-hand side of the inequality (3.5) and the second part of Lemma 2 imply

$$M \gg \log t(\log\log t)^{\log 4-1}\exp\{-C\sqrt{\log\log\log t}\} \tag{3.8}$$

for some (perhaps other) $C>0$. From (3.7) and (3.8) we obtain (3.3) and so Theorem 1. It should be remarked that we may take t to be arbitrarily large by taking M large. This is guaranteed by the left-hand side of the inequality in (3.5).

It remains to prove Lemma 1. This will be achieved in two steps. The first step is to show that the sum $E_B(t)$ contributes at most $O(1)$ to (3.4), that is, if

$$E_B^*(t) = \frac{-1}{\sqrt{2}}\int_{-1}^{1}\frac{1}{\sqrt{t+u}}\sum_{n\leqslant \alpha(t+u)^2}\frac{d(n)}{n^{1/2}}$$
$$\times\{\log((t+u)/\sqrt{n})\}^{-1}\cos(g_B(t+u,n))\,k_M(u)\,du, \tag{3.9}$$

then

$$E_B^*(t) = O(1). \tag{3.10}$$

Observe first that in the sum in (3.9) we may replace the range of summation $n \leqslant \alpha(t+u)^2$ by $n \leqslant \alpha t^2$ $(\alpha = (3-\sqrt{5})/2)$ with an error of $O(t^{-1/2}\log t)$. Furthermore, by Taylor's formula we have

$$\frac{1}{\sqrt{t+u}}\{\log((t+u)/\sqrt{n})\}^{-1} = \frac{1}{\sqrt{t}}\{\log(t/\sqrt{n})\}^{-1} + O(t^{-3/2}).$$

Thus this error can be absorbed as well. This leaves the following expression to estimate:

$$\frac{-1}{\sqrt{2t}}\sum_{n \leqslant \alpha t^2}\frac{d(n)}{n^{1/2}}\{\log(t/\sqrt{n})\}^{-1}\left\{\Re e\int_{-1}^{1} e^{ig_B(t+u,n)}k_M(u)\,du\right\}. \quad (3.11)$$

The integral will be estimated by replacing the exponent by its Taylor polynomial of degree two. We need this many terms to make the error small enough. Thus

$$g_B(t+u,n) = g_B(t,n) + 8\pi t\log(t/\sqrt{n})\,u + 4\pi\{\log(t/\sqrt{n})+1\}\,u^2 + O(t^{-1}).$$

Again the error term here can be ignored. The remaining relevant integral is

$$\int_{-1}^{1} e^{i\phi(u)}k_M(u)\,du,$$

where for t fixed

$$\phi(u) = 8\pi t\log(t/\sqrt{n})\,u + 4\pi\{\log(t/\sqrt{n})+1\}\,u^2 = Xu + Yu^2,$$

say. We integrate by parts to obtain

$$\int_{-1}^{1} e^{i\phi(u)}k_M(u)\,du = \left.\frac{e^{i\phi(u)}k_M(u)}{i\phi'(u)}\right|_{-1}^{1} + i\int_{-1}^{1} e^{i\phi(u)}\frac{d}{du}\left(\frac{k_M(u)}{\phi'(u)}\right)du. \quad (3.12)$$

If we use

$$k_M(u) \ll \lambda_M\min(1,(\lambda_M u)^{-2}), \qquad k'_M(u) \ll \lambda_M^2\min(\lambda_M u,(\lambda_M u)^{-2}),$$
$$X \ll |\phi'(u)| \ll X, \qquad \phi'' = 2Y, \qquad X \gg t \gg \log t \gg Y,$$

then the left-hand side of (3.12) is seen to have order at most $M^{1/2}t^{-1}$. In view of the inequality $M \leqslant t^{1/2}$, the expression in (3.11) is then

$$\ll t^{-1/2}\sum_{n \leqslant \alpha t^2}\frac{d(n)}{n^{1/2}}\cdot M^{1/2}t^{-1} \ll 1,$$

and so (3.10) follows.

To complete the proof of Lemma 1 we must show

$$\begin{aligned} E_A^*(t) &= \int_{-1}^{1} E_A(t+u)\, k_M(u)\, du \\ &= \int_{-1}^{1} \sum_{n \leqslant (t+u)^2} (-1)^n \frac{d(n)}{n^{3/4}}\, e_A(t+u, n) \cos(f_A(t+u, n))\, k_M(u)\, du \\ &= \sum_{n \leqslant M} (-1)^n \frac{d(n)}{n^{3/4}} \left(1 - \sqrt{\frac{n}{M}}\right) \cos(4\pi t \sqrt{n} - \pi/4) + O(1), \qquad (3.13) \end{aligned}$$

where e_A and f_A are given after (3.1). We proceed as in the proof of (3.10), first to change the range of summation to $n \leqslant t^2$ and then using Taylor's formula to simplify the second integral in (3.13). Writing

$$f_A(t+u, n) = f_A(t, n) + \rho(u) + O(t^{-1}),$$

where for t fixed

$$\begin{aligned} \rho(u) &= \left\{ 8\pi t \operatorname{ar\,sinh} \left(\frac{\sqrt{n}}{2t} \right) \right\} u + \left\{ 4\pi \operatorname{ar\,sinh} \left(\frac{\sqrt{n}}{2t} \right) - \frac{2\pi \sqrt{n}}{\sqrt{t^2 + n/4}} \right\} u^2 \\ &= Vu + Wu^2, \end{aligned}$$

say, we obtain

$$E_A^*(t) = \sum_{n \leqslant t^2} (-1)^n \frac{d(n)}{n^{3/4}}\, e_A(t, n) \left\{ \Re e\, e^{if_A(t, n)} \int_{-1}^{1} e^{i\rho(u)} k_M(u)\, du \right\} + O(1). \qquad (3.14)$$

We shall evaluate the integral in (3.14) by using

$$\begin{aligned} I &= \int_{-1}^{1} e^{i\rho(u)} k_M(u)\, du \\ &= \begin{cases} O\left(\sqrt{\dfrac{M}{n}} \right), & \text{if } 1 \leqslant n \leqslant t^2, \\ O\left(\dfrac{1}{\sqrt{n}} + \dfrac{\sqrt{n}}{t} \right), & \text{if } n \geqslant M, \\ 1 - \sqrt{\dfrac{n}{M}} + O\left(\dfrac{1}{\sqrt{n}} + \dfrac{\sqrt{n}}{t} \right), & \text{if } 1 \leqslant n \leqslant M. \end{cases} \qquad (3.15) \end{aligned}$$

To see that (3.15) holds integrate first by parts as in (3.12) and note that for $1 \leqslant n \leqslant t^2$, we have $\sqrt{n} \ll V \leqslant \lambda_n$. This gives easily the first bound in

(3.15). For the second and third parts of (3.15) observe that $W \ll \sqrt{n}/t$, which yields the error term $O(\sqrt{n}/t)$. Next we recall the estimate

$$\int_{-1}^{1} e^{iVu} k_M(u)\, du = \begin{cases} 1 - \dfrac{V}{\lambda_M} + O\left(\dfrac{1}{V}\right), & \text{if } V \leqslant \lambda_M, \\ O\left(\dfrac{1}{V}\right), & \text{if } V \geqslant \lambda_M. \end{cases}$$

This can be found, for example, in [10]. Hence for $V \geqslant \lambda_M$ we have $n \geqslant M$ so that

$$I = O\left(\frac{1}{\sqrt{n}}\right) + O\left(\frac{\sqrt{n}}{t}\right).$$

This proves part of the second case. To complete the proof we only have to add the following claim: For $V \leqslant \lambda_M$

$$1 - \frac{V}{\lambda_M} = \max\left(0,\, 1 - \sqrt{\frac{n}{M}}\right) + O\left(\frac{1}{\sqrt{n}}\right).$$

Since $V \leqslant \lambda_n$ we easily deduce

$$1 - \frac{V}{\lambda_M} \geqslant \max\left(0,\, 1 - \sqrt{\frac{n}{M}}\right).$$

Also, since $V \leqslant \lambda_M$ implies $n \ll M \ll t$, there is a positive constant c such that

$$V \geqslant \lambda_n - cn^{3/2}t^{-2} \geqslant \lambda_n - c/\sqrt{n}.$$

Thus

$$\begin{aligned} 1 - \frac{V}{\lambda_M} &\leqslant 1 - \sqrt{\frac{n}{M}} + \frac{c}{\sqrt{n}} \\ &\leqslant \max\left(0,\, 1 - \sqrt{\frac{n}{M}}\right) + \frac{c}{\sqrt{n}}, \end{aligned}$$

as claimed. Combining the relevant formulas we complete the proof of (3.15).

We may now easily finish the proof of (3.12). The terms in (3.14) for which $M^{5/2} < n \leqslant t^2$ contribute, by the first part of (3.15), no more than

$$O\left(M^{1/2} \sum_{n > M^{5/2}} \frac{d(n)}{n^{5/4}}\right) = O(M^{-1/8} \log M) = O(1).$$

The contribution of the terms with $M < n \leqslant M^{5/2}$ is, by the second part of (3.15), bounded by

$$\sum_{n \leqslant M^{5/2}} \frac{d(n)}{n^{5/4}} + \frac{1}{t} \sum_{n \leqslant M^{5/2}} \frac{d(n)}{n^{1/4}} \ll 1 + t^{-1}(M^{5/2})^{3/4} \log M \ll 1$$

since $M \leqslant t^{1/2}$. Finally, the terms with $1 \leqslant n \leqslant M$ give

$$\sum_{n \leqslant M} (-1)^n \frac{d(n)}{n^{3/4}} e_A(t, n)\left(1 - \sqrt{\frac{n}{M}}\right) \cos(f_A(t, n)) + O(1).$$

This is still not exactly in the form given by (3.13). To obtain this note that for $1 \leqslant n \leqslant M \leqslant t^{1/2}$ we have

$$e_A(t, n) = 1 + O(nt^{-2}), \qquad f_A(t, n) = 4\pi t\sqrt{n} - \frac{\pi}{4} + O(n^{3/2}t^{-1}),$$

and that the contribution of the error terms to the above sum will be $O(1)$. This establishes (3.13), completing the proof of Lemma 1 and thus of Theorem 1.

We remark that there are two other approaches to the proof of Theorem 1. One alternative is to use the mean value of $E(T)$ given by (2.5) to provide a shorter proof of Lemma 1. (See [11], where the details are worked out in a similar problem.) Let

$$f(t) = 2^{-1/2} t^{-3/2} \int_2^t (E(2\pi y^2) - \pi)\, y\, dy,$$

so that by (2.5)

$$f(t) = \frac{1}{4\pi} \sum_{n=1}^{\infty} (-1)^n \frac{d(n)}{n^{5/4}} \sin(\lambda_n t - \pi/4) + O(1), \tag{3.16}$$

and by direct computation,

$$E_0(t) = \frac{d}{dt} f(t) + O(t^{-1/2}). \tag{3.17}$$

Using (3.17) in (3.2), integrating by parts, and using (3.16), we get for $t \geqslant M^2$,

$$\begin{aligned} E^*(t) &= -\int_{-1}^{1} f(t+u)\, k'_M(u)\, du + O(1) \\ &= -\frac{1}{4\pi} \sum_{n=1}^{\infty} (-1)^n \frac{d(n)}{n^{5/4}} \mathfrak{Im} \left\{ e^{i(\lambda_n t - \pi/4)} \int_{-1}^{1} e^{i\lambda_n u} k'_M(u)\, du \right\} + O(1). \end{aligned}$$

The last integral can be evaluated easily as

$$\int_{-1}^{1} e^{i\lambda_n u} k'_M(u)\, du = \begin{cases} -i\lambda_n\left(1-\dfrac{\lambda_n}{\lambda_M}\right)+O(1), & \text{if } n \leqslant M, \\ O(1), & \text{if } n > M. \end{cases}$$

The result is then exactly Lemma 1. We have given the longer proof here, first because the proof of (2.5) is even longer and so little is actually gained, and second, because of the historical point: the techniques used in the proof of Lemma 1 have been available certainly since Atkinson proved his formula. Thus, there has really been no obstable to an Ω_+ result at least as good as Hardy's classical theorem [12] for Dirichlet's divisor problem:

$$\Delta(x) = \Omega_+ \{(x \log x)^{1/4} \log\log x\}.$$

The second alternative to the proof of Theorem 1, which we found out about during the preparation of this paper, was found independently by T. Meurman. His proof is also based on the ideas of the first author's paper [10] (see Lemma 2), but there are differences between his proof and ours. While we work with the Fejér kernel, he employs an averaging technique similar to [18, Eq. (15.71)]. We thank T. Meurman for letting us know of his work.

4. The mean Value of $E(T)$

We shall now prove Theorem 3, which will be used in Section 5 in the proof of the Ω_- result for $E(T)$. In proving this theorem we would naturally wish to use Atkinson's formula, but unfortunately the error term $O(\log^2 T)$ in (1.1) is much too large for this purpose. However, his method of proof may be used, in the sense that we shall integrate the integrals which appear in the derivation of Atkinson's formula. We start from (all the notation and references are to [18, Chap. 15])

$$2i\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\, dt$$
$$= \log\frac{\Gamma(\frac{1}{2}+iT)}{\Gamma(\frac{1}{2}-iT)} + 2iT(2\gamma - \log 2\pi) + 2\int_{1/2-iT}^{1/2+iT} g(u, 1-u)\, du,$$

where g is the analytic continuation of the function defined, for $\Re u < 0$, by the series expansion

$$g(u, 1-u) = 2\sum_{n=1}^{\infty} d(n) \int_0^{\infty} y^{-u}(1+y)^{u-1} \cos(2\pi n y)\, dy.$$

Using Stirling's formula for the gamma-function it follows that

$$\int_T^{2T} E(t)\,dt = \int_T^{2T} \int_{-t}^{t} g(\tfrac{1}{2}+i\sigma, \tfrac{1}{2}-i\sigma)\,d\sigma\,dt + O(1)$$

$$= \int_T^{2T} \{I_1(t) - I_2(t) + I_3(t) - I_4(t)\}\,dt + O(1), \qquad (4.1)$$

where $I_n = I_n(t)$ is as in [18, Eq. (15.30)], only with T replaced by t and $N=[T]$. To prove Theorem 3, it will suffice to prove

$$\int_T^{2T} E(t)\,dt = \pi T + H(2T) - H(T) + K(2T) - K(T) + O(T^{1/4}), \qquad (4.2)$$

where (using the notation of (1.5) and (1.6))

$$H(x) = 2^{-3/2} \sum_{n \leqslant x} (-1)^n \frac{d(n)}{n^{1/2}} \left(\frac{x}{2\pi n} + \frac{1}{4}\right)^{-1/4} \times \left(\operatorname{ar\,sinh} \sqrt{\frac{\pi n}{2x}}\right)^{-2} \sin(f(x,n)), \qquad (4.3)$$

$$K(x) = -2 \sum_{n \leqslant c_0 x} \frac{d(n)}{n^{1/2}} (\log x/2\pi n)^{-2} \sin(g(x,n)), \qquad (4.4)$$

and then to replace T by $T2^{-j}$ and sum or $j=1,2,\ldots$. Here $c_0 = 1/2\pi + 1/2 - \sqrt{1/4 + 1/2\pi}$. The main term πT in (4.2) comes from $I_3(t)$, while the sums defined by H will appear in the evaluation of $\int_T^{2T} I_1(t)\,dt$, where, for $T \leqslant t \leqslant 2T$,

$$I_1(t) = 4 \sum_{n \leqslant T} d(n) \int_0^{\infty} \frac{\sin(t \log(1+1/y)) \cos(2\pi n y)}{y^{1/2}(1+y)^{1/2} \log(1+1/y)}\,dy. \qquad (4.5)$$

This exponential integral was evaluated directly in (15.39) of [18] but this contains the error term $O(T^{-1/4})$ which, when integrated, is too large for our purposes. To avoid this difficulty we take advantage of the extra averaging over t via the following lemma.

LEMMA 3. *Let α, β, γ, a, b, k, T be real numbers such that α, β, γ are positive and bounded, $\alpha \neq 1$, $0 < a < \frac{1}{2}$, $a < T/8\pi k$, $b \geqslant T$, $k \geqslant 1$, $T \geqslant 1$,*

$$U(t) = \left(\frac{t}{2\pi k} + \frac{1}{4}\right)^{1/2}, \qquad V(t) = 2\operatorname{ar\,sinh}\sqrt{\frac{\pi k}{2t}},$$

$$L(t) = \frac{1}{i}(2k\sqrt{\pi})^{-1} t^{1/2} V^{-\gamma-1}(t)\, U^{-1/2}(t) \left(U(t) - \frac{1}{2}\right)^{-\alpha} \left(U(t) + \frac{1}{2}\right)^{-\beta} \times \exp\left\{itV(t) + 2\pi i k U(t) - \pi i k + \frac{\pi i}{4}\right\},$$

and

$$J(T)=\int_T^{2T}\int_a^b y^{-\alpha}(1+y)^{-\beta}\left(\log\frac{1+y}{y}\right)^{-\gamma}$$
$$\times\exp\{it\log(1+1/y)+2\pi iky\}\,dy\,dt. \tag{4.6}$$

Then uniformly for $|\alpha-1|\geqslant\varepsilon$, $1\leqslant k\leqslant T+1$, *we have*

$$J(T)=L(2T)-L(T)+O(a^{1-\alpha})+O(Tk^{-1}b^{\gamma-\alpha-\beta})$$
$$+O((T/k)^{(\gamma+1-\alpha-\beta)/2}\,T^{-1/4}k^{-5/4}). \tag{4.7}$$

A similar result, without $L(2T)-L(T)$, *holds for the corresponding integral with* $-k$ *in place of* k.

This result is a modified version of [18, Lemma 15.1], originally due to F. V. Atkinson [1]. The proof follows from a theorem of Atkinson [1] (this is [18, Theorem 2.2]) on exponential integrals, similar to the proof of [18, Lemma 15.1], so we outline the key steps. One takes (in the proof of the aforementioned Theorem 2.2)

$$f(z)=\frac{t}{2\pi}\log\frac{1+z}{z},\ \Phi(x)=x^{-\alpha}(1+x)^{-\beta},\ F(x)=\frac{t}{1+x},\ \mu(x)=\frac{x}{2}.$$

The contribution of the integrals $I_{31}+I_{33}$, present on [18, p. 63], is contained in the O-terms in (4.7), since in our case we find that

$$I_{31}+I_{33}\ll\begin{cases}e^{-ck}, & \text{if } k\geqslant\log^2 T,\\ e^{-(tk)^{1/2}}, & \text{if } k\leqslant\log^2 T.\end{cases}$$

Likewise, the error terms

$$\Phi_a(|f'_a+k|+f''^{1/2}_a)^{-1},\qquad \Phi_b(|f'_b+k|+f''^{1/2}_b)^{-1}$$

give after integration $O(a^{1-\alpha})+O(Tk^{-1}b^{\gamma-\alpha-\beta})$, which are present in (4.7). The main contribution comes from I_{32}, only now one has to integrate over t. This leads to the same type of integral (the factor $1/i$ is unimportant) at $2T$ and T, respectively. The only change is that $\gamma+1$ appears instead of γ, because of the extra factor $\log(1+1/y)$ in the denominator. Hence the main terms will be $L(2T)-L(T)$, and as in [18, Theorem 2.2], the error term is $\Phi_0\mu_0F_0^{-3/2}$ with again $\gamma+1$ replacing γ. This gives the last O-term in (4.7) (see the analogous computation in [18, p. 453]), and completes the proof of Lemma 3.

We return now to the proof of Theorem 3. We first write

$$\int_T^{2T} I_1(t)\,dt$$

$$= 4 \sum_{n\leqslant T} d(n) \lim_{\alpha\to 1/2+0} \lim_{b\to\infty} \int_T^{2T} \int_0^b \frac{\sin(t\log(1+1/y))\cos(2\pi n y)}{y^{\alpha}(1+y)^{1/2}\log(1+1/y)}\,dy\,dt$$

$$= 2 \sum_{n\leqslant T} d(n) \lim_{\alpha\to 1/2+0} \lim_{b\to\infty} \mathfrak{Im} \left\{ \int_T^{2T} \int_0^b \frac{\exp(it\log(1+1/y)+2\pi i n y)}{y^{\alpha}(1+y)^{1/2}\log(1+1/y)}\,dy\,dt \right\}$$

$$+ O(T^{1/4}). \tag{4.8}$$

The first equality is justified because the integral defining $I_1(t)$ converges uniformly at ∞ and 0 for $\frac{1}{2}\leqslant\alpha\leqslant 1-\varepsilon$. Also, the second equality follows because the other two integrals coming from $\sin(\cdots)\cos(\cdots)$ in (4.5) are estimated by Lemma 3 as $O(T^{1/4})$, using the estimate for the case "$-k$ in place of k." We evaluate the double integral above by applying Lemma 3 with $\beta=\frac{1}{2}$, $\gamma=1$, $a\to 0$. Then we let $b\to\infty$ and $\alpha\to\frac{1}{2}+0$. We obtain

$$\int_T^{2T} I_1(t)\,dt = H(2T) - H(T)$$

$$- 2^{-3/2} \sum_{T\leqslant n\leqslant 2T} (-1)^n \frac{d(n)}{n^{1/2}} \left(\frac{2T}{2\pi n}+\frac{1}{4}\right)^{-1/4}$$

$$\times \left(\operatorname{ar\,sinh}\sqrt{\frac{\pi n}{4T}}\right)^{-2} \sin(f(2T,n)) + O(T^{1/4}), \tag{4.9}$$

where $H(x)$ is given by (4.3).

Henceforth we set for brevity $X=[T]+\frac{1}{2}$. The contribution of the integral

$$I_2(t) = 4\Delta(X) \int_0^{\infty} \frac{\sin(t\log(1+1/y))\cos(2\pi X y)}{y^{1/2}(1+y)^{1/2}\log(1+1/y)}\,dy$$

in (4.1) is estimated again by Lemma 3. Using $\Delta(X)\ll X^{1/3}$, it follows at once that

$$\int_T^{2T} I_2(t)\,dt \ll T^{-1/6}. \tag{4.10}$$

We now turn to

$$I_3(t) = -\frac{2}{\pi}(\log X + 2\gamma)\int_0^\infty \frac{\sin(t\log(1+1/y))\sin(2\pi Xy)}{y^{3/2}(1+y)^{1/2}\log(1+1/y)}\,dy$$
$$+\frac{1}{\pi i}\int_0^\infty \frac{\sin(2\pi Xy)}{y}\,dy\int_{1/2-it}^{1/2+it}(1+1/y)^u u^{-1}\,du$$
$$= -\frac{2}{\pi}(\log X + 2\gamma)\, I_{31}(t) + \frac{1}{\pi i} I_{32}(t),$$

say. It follows from [18, p. 457] that

$$I_3(t) = \pi + O(T^{-1/2}\log T) \qquad (T \leqslant t \leqslant 2T),$$

hence

$$\int_T^{2T} I_3(t)\,dt = \pi T + O(T^{1/2}\log T), \tag{4.11}$$

but the error term here is too large. Thus to prove (2.4) we must sharpen (4.11). We have first

$$\int_T^{2T} I_{31}(t)\,dt = \int_T^{2T}\int_0^{3T}\cdots + \int_T^{2T}\int_{3T}^\infty\cdots$$
$$= \int_0^{3T} \frac{\{\cos(T\log(1+1/y)) - \cos(2T\log(1+1/y))\}\sin(2\pi Xy)}{y^{3/2}(1+y)^{1/2}\log^2(1+1/y)}\,dy$$
$$+ O(T^{-1})$$

on estimating $\int_{3T}^\infty\cdots$ as $O(T^{-2})$ by writing the sine terms in $I_{31}(t)$ as exponentials, and using the standard elementary estimate for exponential integrals [18, Lemma 2.1]. The remaining integral above is split as

$$\int_0^{3T} = \int_0^{(2X)^{-1}} + \int_{(2X)^{-1}}^{3T} = I' + I'',$$

say. Using the second mean value theorem for integrals we have $I' \ll T^{-1/2}$ by the argument at the top of [18, p. 456], while $I'' \ll T^{-1/2}$ follows on applying [18, Lemma 15.1]. Hence

$$\int_T^{2T} I_{31}(t)\,dt \ll T^{-1/2}.$$

Next take $I_{32}(t)$ and write

$$I_{32}(t)=\int_0^1 \frac{\sin(2\pi Xy)}{y}\,dy\int_{1/2-it}^{1/2+it}(1+1/y)^u u^{-1}\,du$$

$$+\int_1^\infty \frac{\sin(2\pi Xy)}{y}\,dy\int_{1/2-it}^{1/2+it}(1+1/y)^u u^{-1}\,du$$

$$=I'_{32}(t)+I''_{32}(t),$$

say. The integral $I''_{32}(t)$ is $O(T^{-1}\log T)$, as is shown (by an integration by parts) in [18, p. 457]. This gives

$$\int_T^{2T} I''_{32}(t)\,dt \ll \log T.$$

In $I'_{32}(t)$ we have $0<y\leqslant 1$, hence by the residue theorem

$$\int_{1/2-it}^{1/2+it}(1+1/y)^u u^{-1}\,du=2\pi i-\left(\int_{1/2+it}^{-\infty+it}+\int_{-\infty-it}^{1/2-it}\right)(1+1/y)^u u^{-1}\,du,$$

and an integration yields

$$\int_T^{2T} I'_{32}(t)\,dt=\pi^2 iT-\int_T^{2T}\int_0^1 \frac{\sin(2\pi Xy)}{y}\int_{1/2+it}^{-\infty+it}(1+1/y)^u u^{-1}\,du\,dy\,dt$$

$$-\int_T^{2T}\int_0^1 \frac{\sin(2\pi Xy)}{y}\int_{-\infty-it}^{1/2-it}(1+1/y)^u u^{-1}\,du\,dy\,dt+O(1).$$

(We have used the elementary estimate

$$\int_0^1 \frac{\sin(2\pi Xy)}{y}\,dy=\frac{\pi}{2}+O(T^{-1}).)$$

The triple integrals are estimated similarly, each being $\ll T^{-1/2}$. Namely, changing the order of integration, and integrating by parts we have

$$\int_T^{2T}\int_{-\infty+it}^{1/2+it}(1+1/y)^u u^{-1}\,du\,dt$$

$$=\int_{-\infty}^{1/2}(1+1/y)^\sigma\left\{\int_T^{2T}(1+1/y)^{it}(\sigma+it)^{-1}\,dt\right\}d\sigma$$

$$=\int_{-\infty}^{1/2}(1+1/y)^\sigma\left\{\frac{(1+1/y)^{2iT}}{i(\sigma+2iT)\log(1+1/y)}-\frac{(1+1/y)^{iT}}{i(\sigma+iT)\log(1+1/y)}\right.$$

$$\left.+\int_T^{2T}\frac{(1+1/y)^{it}}{(\sigma+it)^2\log(1+1/y)}\,dt\right\}d\sigma$$

$$\ll T^{-1}\int_{-\infty}^{1/2}\frac{(1+1/y)^\sigma}{\log(1+1/y)}\,d\sigma \ll T^{-1}y^{-1/2}$$

for $0 < y \leqslant 1$, and

$$T^{-1} \int_0^1 |\sin(2\pi Xy)|\, y^{-3/2}\, dy$$

$$\ll T^{-1} \int_0^{X^{-1}} Xy^{-1/2}\, dy + T^{-1} \int_{X^{-1}}^{\infty} y^{-3/2}\, dy \ll T^{-1/2}.$$

Combining the preceding estimates we obtain

$$\int_T^{2T} I_3(t)\, dt = \pi T + O(\log T). \tag{4.12}$$

Finally, it remains to deal with the integral of $I_4(t)$ in (4.1), where

$$I_4(t) = -i \int_X^{\infty} \Delta(x) \left(\int_{1/2-it}^{1/2+it} \frac{\partial h(u, x)}{\partial x}\, du \right) dx,$$

and

$$h(u, x) = 2 \int_0^{\infty} y^{-u}(1+y)^{u-1} \cos(2\pi\, xy)\, dy$$

$$= 2 \int_0^{\infty} w^{-u}(x+w)^{u-1} \cos(2\pi w)\, dw.$$

It is easy to see that this integral for h is uniformly convergent and so we can differentiate under the integral sign to get (after changing variables again)

$$\frac{\partial}{\partial x} h(u, x) = \frac{2}{x}(u-1) \int_0^{\infty} y^{-u}(1+y)^{u-2} \cos(2\pi xy)\, dy.$$

This integral is absolutely convergent at both endpoints so we insert it in the definition for $I_4(t)$ to get

$$-\int_T^{2T} I_4(t)\, dt = 2i \int_X^{\infty} \frac{\Delta(x)}{x}\, dx \int_0^{\infty} \frac{\cos(2\pi xy)}{y^{1/2}(1+y)^{3/2}}\, dy$$

$$\times \int_T^{2T} \int_{1/2-it}^{1/2+it} (u-1) \left(\frac{1+y}{y} \right)^{u-1/2} du\, dt.$$

We can now evaluate explicitly the integrals with respect to u and t. We will see from subsequent estimates that what remains provides absolute convergence for the integral in x, so that this procedure is justified. We obtain

$$-\int_T^{2T} I_4(t)\, dt = 4 \int_X^{\infty} \frac{\Delta(x)}{x} \{ I(x, 2T) - I(x, T) + r(x, T) \}\, dx, \tag{4.13}$$

where

$$I(x, z) = \int_0^\infty \frac{-z \sin(z \log(1+1/y)) \cos(2\pi xy)}{y^{1/2}(1+y)^{3/2} \log^2(1+1/y)}\, dy,$$

$$r(x, T) = \int_0^\infty \frac{\{\cos(T \log(1+1/y)) - \cos(2T \log(1+1/y))\} \cos(2\pi xy)}{y^{1/2}(1+y)^{3/2} \log^2(1+1/y)}$$

$$\times \left\{\frac{1}{2} + \frac{2}{\log(1+1/y)}\right\} dy.$$

Write now

$$r(x, T) = \int_0^\infty = \int_0^{3T} + \int_{3T}^\infty = I' + I'',$$

say. In I'' we use

$$\cos(T \log(1+1/y)) - \cos(2T \log(1+1/y))$$
$$= 2 \sin\left(\frac{3T}{2} \log(1+1/y)\right) \sin\left(\frac{T}{2} \log(1+1/y)\right)$$

and the second mean-value theorem for integrals. We have, for some $c > 3T$,

$$I'' = \frac{3T^2}{2} \left\{\frac{\sin(3T/2 \log(1+1/3T))}{3T/2 \log(1+1/3T)} \cdot \frac{\sin(T/2 \log(1+1/3T))}{T/2 \log(1+1/3T)}\right\}$$
$$\times \int_{3T}^{c} \frac{\cos(2\pi xy)}{y^{1/2}(1+y)^{3/2}} \left\{\frac{1}{2} + \frac{2}{\log(1+1/y)}\right\} dy \ll Tx^{-1} \tag{4.14}$$

since the first expression in curly brackets is $O(1)$, and the integral in (4.14) is seen to be $O((Tx)^{-1})$ on applying [18, Lemma 2.1]. Hence by (1.8) and the Cauchy-Schwarz inequality we obtain

$$4 \int_X^\infty \Delta(x)\, x^{-1} I''\, dx$$
$$\ll T \left(\int_X^\infty \Delta^2(x)\, x^{-2}\, dx\right)^{1/2} \left(\int_X^\infty x^{-2}\, dx\right)^{1/2} \ll T^{1/4}. \tag{4.15}$$

In I' we use [18, Lemma 15.1] (treating the main term as an error term) to get the analogue of (4.15) for I'. The integral $I(x, 2T) - I(x, T)$ is also evaluated by [18, Lemma 15.1] with $\alpha \to \frac{1}{2} + 0$, $\beta = \frac{3}{2}$, $\gamma = 2$. The error terms will be $\ll T(T^{-1/4} x^{-5/4} + T^{-1} x^{-1/2})$, and their total contribution will be $\ll T^{1/4}$ as in (4.15). The main terms will be

$$\left\{-z(4x)^{-1}\left(\frac{z}{\pi}\right)^{1/2} V^{-2}U^{-1/2}\left(U-\frac{1}{2}\right)^{-1/2}\left(U+\frac{1}{2}\right)^{-3/2}\right.$$
$$\left.\times\sin\left(zV+2\pi xU-\pi x+\frac{\pi}{4}\right)\right\}\Bigg|_T^{2T},$$

where

$$U=\left(\frac{z}{2\pi x}+\frac{1}{4}\right)^{1/2},\qquad V=2\operatorname{ar\,sinh}\sqrt{\frac{\pi x}{2z}}.$$

Thus (4.13) becomes

$$\begin{aligned}-\int_T^{2T} I_4(t)\,dt&\\ &=-\int_X^{\infty}\Delta(x)\,x^{-3/2}\\ &\quad\times\left\{\sqrt{2}\,zV^{-2}U^{-1/2}\left(U+\frac{1}{2}\right)^{-1}\sin\left(zV+2\pi xU-\pi x+\frac{\pi}{4}\right)\Bigg|_T^{2T}\right\}dx\\ &\quad+O(T^{1/4}).\end{aligned}\tag{4.16}$$

The last integral bears close resemblance to the integral for I_4 itself at the top of [18, p. 458]. The difference is that instead of V^{-1} we have V^{-2} and sine (at T and $2T$) instead of cosine in (4.16). This difference is not essential, and after using the Voronoi series expansion for $\Delta(x)$ and changing the variable x to x^2 the above integral may be estimated by [18, Lemma 15.2]. The modification is that as on p. 454 we have $V=2\operatorname{ar\,sinh}(x_0\sqrt{\pi/2T})=\log(T/2\pi n)$; hence if we replace $\operatorname{ar\,sinh}(x\sqrt{\pi/2T})$ by its square in [18, (15.37)], we obtain in the main term an additional factor $2(\log(T/2\pi n))^{-1}$, and the error terms remain unchanged. With this remark one can proceed exactly as was done in the evaluation of I_4 in the proof of Atkinson's formula. For this reason we shall omit the details of the evaluation of the integral on the right-hand side of (4.16). We obtain

$$\begin{aligned}-\int_T^{2T} I_4(t)\,dt&=-2\sum_{n<Z}\frac{d(n)}{n^{1/2}}(\log z/2\pi n)^{-2}\sin(g(z,n))\Bigg|_T^{2T}+O(T^{1/4})\\ &=K(2T)-K(T)\\ &\qquad-2\sum_{N_2'\leqslant n\leqslant N_1'}\frac{d(n)}{n^{1/2}}(\log 2T/2\pi n)^{-2}\sin(g(2T,n))+O(T^{1/4}),\end{aligned}\tag{4.17}$$

where (as in [18, Eq. (15.43)]) in the notation of (1.7)

$$Z = N'(z, X) = \frac{z}{2\pi} + \frac{X}{2} - \left(\frac{X^2}{4} + \frac{Xz}{2\pi}\right)^{1/2},$$

$K(x)$ is given by (4.4), $g(T, n)$ by (1.6), and $N'_j = N'(2T, jT)$ by (1.7).

Thus except for the extra sums in (4.9) and (4.17) we are near the completion of the proof of Theorem 3. But the sums in question may be transformed into one another (plus a small error) by the method of M. Jutila [21]. Indeed, using [21] we obtain (analogously to [18, Eq. (15.45)])

$$-2 \sum_{N'_2 \leqslant n \leqslant N'_1} \frac{d(n)}{n^{1/2}} (\log 2T/2\pi n)^{-2} \sin(g(2T, n))$$

$$= 2^{-3/2} \sum_{T \leqslant n \leqslant 2T} (-1)^n \frac{d(n)}{n^{1/2}} \left(\frac{2T}{2\pi n} + \frac{1}{4}\right)^{-1/4}$$

$$\times \left(\operatorname{ar\,sinh} \sqrt{\frac{\pi n}{4T}}\right)^{-2} \sin(f(2T, n)) + O(\log^2 T),$$

the difference from (15.45) being in $(\log \cdots)^{-2}$ and $(\operatorname{ar\,sinh} \cdots)^{-2}$, and in $2T$ instead of T.

Hence from (4.1), (4.9), (4.10), (4.12), and (4.17) we obtain (4.2). This proves Theorem 3. We remark that, using the above procedure (i.e., [18, Eq. (15.45)]), we can reformulate Theorem 3 where the first sum in (2.4) will be $\sum_{n \leqslant N}$, and the second sum $\sum_{n \leqslant N'}$, for $T \ll N \ll T$, and N' given by (1.7). Written in this way, Theorem 3 becomes more similar to Atkinson's formula itself. Namely, for any two constants $0 < A < A'$, $AT < N < A'T$ and N' given by (1.7), Atkinson's formula is

$$E(T) = 2^{-1/2} \sum_{n \leqslant N} (-1)^n \frac{d(n)}{n^{1/2}} \left(\operatorname{ar\,sinh} \sqrt{\frac{\pi n}{2T}}\right)^{-1} \left(\frac{T}{2\pi n} + \frac{1}{4}\right)^{-1/4} \cos(f(T, n))$$

$$-2 \sum_{n \leqslant N'} \frac{d(n)}{n^{1/2}} \left(\log \frac{T}{2\pi n}\right)^{-1} \cos(g(T, n)) + O(\log^2 T). \tag{4.18}$$

Under the same hypotheses the above discussion shows that we have actually proved

$$\int_2^T E(t)\,dt = \pi T + 2^{-3/2} \sum_{n \leqslant N} (-1)^n \frac{d(n)}{n^{1/2}} \left(\operatorname{ar\,sinh} \sqrt{\frac{\pi n}{2T}}\right)^{-2}$$
$$\times \left(\frac{T}{2\pi n} + \frac{1}{4}\right)^{-1/4} \sin(f(T, n))$$
$$-2 \sum_{n \leqslant N'} \frac{d(n)}{n^{1/2}} \left(\log \frac{T}{2\pi n}\right)^{-2} \sin(g(T, n)) + O(T^{1/4}). \quad (4.19)$$

If we note that

$$\frac{\partial f(T, n)}{\partial T} = 2 \operatorname{ar\,sinh} \sqrt{\frac{\pi n}{2T}}, \qquad \frac{\partial g(T, n)}{\partial T} = \log \frac{T}{2\pi n},$$

then (disregarding the error terms), one may formally obtain (4.18) from (4.19) by differentiating the sine terms. The same phenomenon is present in the Voronoi formulas for $\Delta(T)$ and $\int_2^T \Delta(t)\,dt$. Moreover, the condition $AT < N < A'T$ could be relaxed to $T^{\delta} \ll N \ll T^2$ in both (4.18) and (4.19) at the cost of some extra error terms. This follows from the methods of M. Jutila [21] (see also [18, Theorem 15.2]) who treated $E(T)$, and similar analysis may be made for (4.19).

5. Proof of the Ω-Result

We are now going to prove Theorem 2, using (2.5), which is a consequence of the mean value formula given by Theorem 3. First we are going to prove a weak Ω-result, namely,

$$\liminf_{T \to \infty} \frac{E(T)}{T^{1/4}} = -\infty. \quad (5.1)$$

This will then be used in deriving the strong Ω_- result of Theorem 2.

To prove (5.1) it suffices to show that $\liminf_{T \to \infty} E^*(T) = -\infty$ where $E^*(T)$ is defined in (3.2). Now Lemma 1 implies that for $1 \leqslant M \leqslant T^{1/2}$ we have

$$E^*(T) = \sum_{n \leqslant M} (-1)^n \frac{d(n)}{n^{3/4}} \left(1 - \sqrt{\frac{n}{M}}\right) \cos(4\pi T \sqrt{n} - \pi/4) + O(1).$$

In this formula, write each $n \leqslant M$ in the form $n = v^2 q$, where q is the largest square-free divisor of n. By Kronecker's approximation theorem (see, e.g., [18, Lemma 9.3] or Hardy and Wright [13, Chap. XXIII]; for the linear independence of the numbers $\sqrt{q}$ over the integers see

K. Chandrasekharan [4, Chap. VIII]), there exists arbitrarily large T such that

$$T\sqrt{q}=\begin{cases} m_q+\delta_q, & \text{if } q \text{ is odd,} \\ \frac{1}{4}+m_q+\delta_q, & \text{if } q \text{ is even,} \end{cases}$$

with some integers m_q and $|\delta_q|<\delta$ for any fixed δ. With these T we conclude that

$$(-1)^n\cos(4\pi T\sqrt{n}-\pi/4)=-\varepsilon_n\cos\left(\frac{\pi}{4}\right)+O(\sqrt{n}\,\delta),$$

where

$$\varepsilon_n=\begin{cases} -1, & \text{if } n\equiv 0 \pmod 4, \\ 1, & \text{if } n\not\equiv 0 \pmod 4). \end{cases}$$

We deduce that

$$\liminf_{T\to\infty} E^*(T)\leqslant -\cos\left(\frac{\pi}{4}\right)\sum_{n\leqslant M}\frac{\varepsilon_n d(n)}{n^{3/4}}\left(1-\sqrt{\frac{n}{M}}\right)+O(\delta M^{3/4}\log M).$$

Now we let δ go to zero. Finally, we need to show that the sum can be made arbitrarily large. This is achieved by using an elementary technique similar to the one following (3.6), or by appealing directly to Theorem D3 of A. E. Ingham [16]. This concludes the proof of (5.1).

We pass now to our main result, namely the Ω_- result of Theorem 2. Let P_x be the set of odd primes less than or equal to x, and Q_x all square-free numbers composed only of primes from P_x (that is, Q_x is the set of odd square-free numbers all of whose prime factors are less than x). We let $|P_x|$ be the cardinality of P_x and $M=2^{|P_x|}$ be the cardinalty of Q_x. We then have

$$|P_x|\asymp\frac{x}{\log x} \qquad \text{and} \qquad M\ll\exp\{cx/\log x\} \tag{5.2}$$

for some positive constant c. We also have that the largest integer in Q_x is bounded by e^{2x}.

Next let S_x be the set of numbers defined by

$$S_x=\left\{\mu=\sum_{q\in Q_x} r_q\sqrt{q} : r_q\in\{\pm 1, 0\}; \sum r_q^2\geqslant 2\right\}.$$

Finally let

$$\tilde{\eta}(x)=\inf\{|\sqrt{m}+\mu| : m=1, 2, \ldots; \mu\in S_x\}.$$

The following lemma can be found in Gangadharan [7].

LEMMA 4. *Let* $q(x) = -\log \tilde{\eta}(x)$. *Then*

$$x \ll q(x) \ll \exp(cx/\log x)$$

for some positive constant c.

Let now

$$\tilde{E}(t) = \sqrt{2\pi}\,\{E(t^2/8\pi) - \pi\} \qquad \text{and} \qquad E_1(T) = \int_0^T \tilde{E}(t)\, t\, dt. \tag{5.3}$$

For our purpose, we need a representation for $E_1(T)$ which is ideally an infinite series of the Voronoi type. A result of this type, with a not-too-large error term, may be obtained from the asymptotic formula (2.5). With a change of variable, for $T \geqslant 10$, we get

$$E_1(T) = T^{3/2} \sum_{n=1}^{\infty} (-1)^n \frac{d(n)}{n^{5/4}} \sin(T\sqrt{n} - \pi/4) + O(T^{4/3} \log T). \tag{5.4}$$

If we could differentiate this series (and the O-term) we could deal with $E(T)$ directly. This is not possible, but when we refer to the "series" for $E(T)$ we mean what one would get formally by differentiating this series.

We require more notation. First we let

$$P(x) = \exp\{\alpha x/\log x\}$$

be such that

$$P(x) \geqslant \max\{q(x), M^2\}. \tag{5.5}$$

Next we let for each fixed x

$$\gamma_x = \sup_{u>0} \left\{ \frac{-\sqrt{2\pi}\, E(u^2/8\pi)}{u^{1/2 + 1/P(x)}} \right\}.$$

Now for $T \to 0+$, $E(T) \sim -T \log T$ so that the expression in brackets in the definition of γ_x is bounded for small u. If this expression is not bounded for all u then more than Theorem 2 would be true. Also, by our earlier Ω_- result there exists a $u > 0$ for which this expression is positive. Hence we can conclude that $0 < \gamma_x < \infty$, or, in other words,

$$\gamma_x u^{1/2 + 1/P(x)} + A + \tilde{E}(u) \geqslant 0$$

for all $u > 0$, where $A = \sqrt{2}\,\pi^{3/2}$.

Our next step is to describe the part of the kernel function we use to isolate certain terms of the "series" for $E(u)$, and to point them in an appropriate direction. Let

$$V(z)=2\cos^2\frac{z}{2}=\frac{e^{iz}+e^{-iz}}{2}+1$$

and set

$$T_x(u)=\prod_{q\in Q_x} V\left(u\sqrt{q}-\frac{5\pi}{4}\right).$$

Note that $T_x(u)\geqslant 0$ for all u. Finally, put $\sigma_x=\exp\{-2P(x)\}$ and

$$J_x=\sigma_x^{5/2}\int_0^\infty \{\gamma_x u^{1/2+1/P(x)}+A+\tilde{E}(u)\}\, ue^{-\sigma_x u}T_x(u)\,du.$$

From the remarks above we see immediately that $J_x\geqslant 0$. In the next two lemmas we provide the tools for an asymptotic expansion for J_x. In the first we cover the first two terms of J_x.

LEMMA 5. *For $\frac{1}{2}<\theta<2$, as $x\to\infty$ we have*

$$\int_0^\infty u^\theta e^{-\sigma_x u}T_x(u)\,du=\sigma_x^{-1-\theta}\Gamma(1+\theta)+o(\sigma_x^{-5/2}).$$

Proof. Expand the trigonometric polynomial $T_x(u)$ into exponential polynomials as

$$T_x(u)=T_0+T_1+\overline{T_1}+T_2,$$

where

$$T_0=1,\ T_1=\frac{e^{5\pi i/4}}{2}\sum_{q\in Q_x} e^{-iu\sqrt{q}},\ T_2=\sum_{\mu\in S_x} h_\mu e^{-iu\mu},$$

$\overline{T_1}$ is the complex conjugate of T_1, and h_μ are constants bounded by $\frac{1}{4}$ in absolute value.

Now T_0 contributes to the integral exactly the first term. So we concentrate on the other parts of T_x. The part T_1 contributes exactly

$$\frac{e^{5\pi i/4}\Gamma(1+\theta)}{2}\sum_{q\in Q_x}(\sigma_x+i\sqrt{q})^{-1-\theta}\ll\sum_{q\in Q_x} q^{-(1+\theta)/2}\ll M=o(\sigma_x^{-5/2})$$

since $\theta+1>0$ and (5.5) holds. The contribution of $\overline{T_1}$ is obviously no more than this. Finally, T_2 provides the term

$$\begin{aligned}&\Gamma(1+\theta)\sum_{\mu} h_\mu(\sigma_x+i\mu)^{-1-\theta}\\&\ll 3^M(\inf_{\mu\in S_x}|\mu|)^{-1-\theta}\ll 3^M\tilde{\eta}(x)^{-1-\theta}\\&\ll \exp\{c\sqrt{P(x)}+P(x)(1+\theta)\}=o(\sigma_x^{-5/2}),\end{aligned}$$

again by (5.5) and the fact that $1+\theta<3$.

In the next lemma we cover the contribution to J_x from $\tilde{E}(u)$. It is here that we appeal to the identity (5.4) for $E_1(T)$.

LEMMA 6. *For x tending to infinity,*

$$\int_0^\infty \tilde{E}(u)\,ue^{-\sigma_x u}T_x(u)\,du=-\frac{1}{2}\Gamma\left(\frac{5}{2}\right)\sigma_x^{-5/2}\sum_{q\in Q_x}\frac{d(q)}{q^{3/4}}+o(\sigma_x^{-5/2}).$$

Proof. Our first step is to integrate by parts to introduce $E_1(u)$ in the integral so that we can appeal to (5.4). Thus our integral can be written as

$$E_1(u)\,e^{-\sigma_x u}T_x(u)\Big|_0^\infty-\int_0^\infty E_1(u)\frac{d}{du}\{e^{-\sigma_x u}T_x(u)\}\,du.$$

Now since

$$E_1(u)=\begin{cases}O(u^2), & \text{if } 0\leqslant u\leqslant 10,\\ O(u^{3/2}), & \text{if } u\geqslant 10,\end{cases}$$

the integrated terms vanish. For the remaining integral, we wish to replace $E_1(u)$ by (5.4). However, we must be very careful how we deal with the error term. Write the integral in question as

$$\begin{aligned}&-\int_0^\infty h(u)\,u^{3/2}\frac{d}{du}\{e^{-\sigma_x u}T_x(u)\}\,du+O\left(\int_{10}^\infty u^{4/3}\log u\left|\frac{d}{du}\{\cdots\}\right|du\right)\\&\quad+\int_0^{10}h(u)\,u^{3/2}\frac{d}{du}\{\cdots\}\,du+O\left(\int_0^{10}u^2\left|\frac{d}{du}\{\cdots\}\right|du\right)\\&=I_1+O(I_2)+I_3+O(I_4),\end{aligned}$$

say, where $h(u)$ is defined by

$$h(u)=\sum_{n=1}^\infty(-1)^n\frac{d(n)}{n^{5/4}}\sin(u\sqrt{n}-\pi/4). \tag{5.6}$$

The integral I_3 is bounded by

$$I_3' = \int_0^{10} u^{3/2} \left| \frac{d}{du} \{ \cdots \} \right| du$$

and this dominates the last integral I_4. Hence, there are three things we need to do: estimate I_3' and I_2 and calculate I_1.

For the two integral estimates, we need a bound on the expression in absolute values. For this we note that from the definition and from the decomposition used in the proof of Lemma 5, we have

$$T_x(u) \ll 2^M,$$
$$T_x'(u) \ll 3^M e^{cx} M$$

so that

$$\left| \frac{d}{du} \{ e^{-\sigma_x u} T_x(u) \} \right| \ll e^{-\sigma_x u} 4^M.$$

In I_3' this contributes at most

$$4^M \int_0^{10} u^{-1/2}\, du \ll e^{c\sqrt{P(x)}} = o(\sigma_x^{-5/2}).$$

In I_2 the estimate becomes

$$4^M \int_{10}^{\infty} u^{4/3} e^{-\sigma_x u} \log u\, du \ll e^{c\sqrt{P(x)}} \sigma_x^{-7/3-\varepsilon} = o(\sigma_x^{-5/2}).$$

For I_1 we expand the expression $d/du \{ \cdots \}$ as

$$u^{-3/2} \frac{d}{du} \{ u^{3/2} e^{-\sigma_x u} T_x(u) \} - \frac{3}{2} u^{-1} e^{-\sigma_x u} T_x(u).$$

The last term contributes to I_1 at most (because $h(u)$ is bounded)

$$2^M \int_0^{\infty} u^{1/2} e^{-\sigma_x u}\, du \ll 2^M \sigma_x^{-3/2} = o(\sigma_x^{-5/2}).$$

Finally, we are left to deal with the following:

$$-\int_0^{\infty} h(u) \frac{d}{du} \{ u^{3/2} e^{-\sigma_x u} T_x(u) \}\, du.$$

We replace $h(u)$ by its series definition and integrate term by term. This is legitimate because everything converges absolutely and uniformly. We get the expression

$$-\sum_{n=1}^{\infty} (-1)^n \frac{d(n)}{n^{5/4}} \Im m \{e^{-\pi i/4} I(n)\}, \tag{5.7}$$

where

$$I(n) = \int_0^{\infty} e^{iu\sqrt{n}} \frac{d}{du} \{u^{3/2} e^{-\sigma_x u} T_x(u)\} \, du.$$

In this integral we can reintegrate by parts and expand $T_x(u)$ as we did in the proof of Lemma 5 to get

$$\begin{aligned} I(n) &= i\sqrt{n} \int_0^{\infty} e^{iu\sqrt{n}} u^{3/2} e^{-\sigma_x u} \{T_0(u) + T_1(u) + \overline{T_1(u)} + T_2(u)\} \, du \\ &= I_0(n) + I_1(n) + I_1^*(n) + I_2(n), \end{aligned}$$

say. Only $I_1(n)$ will contribute to our cause, as we will now see. First,

$$I_0(n) \ll \sqrt{n} \, |\sigma_x - i\sqrt{n}|^{-5/2} \ll n^{-3/4}.$$

Second,

$$I_1^*(n) \ll \sqrt{n} \sum_{q \in Q_x} |\sigma_x - i(\sqrt{n} + \sqrt{q})|^{-5/2} \ll n^{-3/4} M.$$

Third,

$$\begin{aligned} I_2(n) &\ll \sqrt{n} \sum_{\mu \in S_x} |\sigma_x - i(\sqrt{n} - \mu)|^{-5/2} \\ &\ll \begin{cases} 3^M n^{-3/4}, & \text{if } \; n > 2 \max\{|\mu| : \mu \in S_x\}, \\ 3^M \tilde{\eta}(x)^{-5/2} \sqrt{n}, & \text{if } \; n \leqslant 2 \max\{|\mu| : \mu \in S_x\}. \end{cases} \end{aligned}$$

This $\max\{|\mu|\}$ is bounded by Me^{cx}. Hence all of these contribute to our series (5.7) no more than

$$3^M \tilde{\eta}(x)^{-5/2} (Me^{cx})^{1/4+\varepsilon} = o(\sigma_x^{-5/2}),$$

as required. There remains only the contribution of $I_1(n)$. We need to distinguish two cases. If $n \neq q$ for all $q \in Q_x$, then we get a bound exactly as above for $I_2(n)$ but with M replacing the factor 3^M which comes from the number of terms in the sum. Now suppose $n = q$ for some q in Q_x. The term in the sum defining $T_1(u)$ corresponding to this q alone contributes exactly

$$ie^{5\pi i/4} \tfrac{1}{2} \Gamma(\tfrac{5}{2}) \sqrt{q} \, \sigma_x^{-5/2}.$$

The other terms contribute as in the case $n \neq q$. Combining all these contributions to (5.7) we see that the lemma is proved. It should be noted that each q in Q_x is odd so that the factor $(-1)^q$ in (5.7) is always negative for the significant terms.

We can now complete the proof of Theorem 2. We first have that $J_x \geqslant 0$. We also have by Lemmas 5 and 6 that as $x \to \infty$

$$J_x = \gamma_x \sigma_x^{-1/P(x)} \Gamma\left(\frac{5}{2} + \frac{1}{P(x)}\right) - \frac{1}{2} \Gamma\left(\frac{5}{2}\right) \sum_{q \in Q_x} \frac{d(q)}{q^{3/4}} + o(\gamma_x) + o(1).$$

Hence if x is sufficiently large we deduce that

$$\gamma_x \gg \sum_{q \in Q_x} \frac{d(q)}{q^{3/4}} \gg \prod_{2 < p \leqslant x} \{1 + 2p^{-3/4}\}$$
$$= \exp\left\{\sum_{2 < p \leqslant x} \log(1 + 2p^{-3/4})\right\} \gg \exp\left\{\frac{cx^{1/4}}{\log x}\right\}.$$

In other words, for each sufficiently large x there exists a u_x such that for some absolute constant $A > 0$

$$\frac{-E(u_x^2)}{u_x^{1/2}} \geqslant A \exp\left\{\frac{\log u_x}{P(x)} + \frac{cx^{1/4}}{\log x}\right\}.$$

This implies first that u_x tends to infinity with x. If the second term in the exponential dominates, then it is easy to see on taking logarithms and recalling the definition of $P(x)$ that

$$\log \log u_x \ll \frac{x}{\log x},$$

from which the theorem follows. If the opposite occurs then, without loss of generality, we may assume

$$\frac{(\log \log u_x)^{1/4}}{(\log \log \log u_x)^{3/4}} \gg \frac{\log u_x}{P(x)}$$

since otherwise the theorem holds again. But under this condition we again deduce that

$$\log \log u_x \ll \frac{x}{\log x},$$

so that the theorem holds in this last case as well.

This completes the proof of Theorem 2.

6. The Mean Square of $G(T)$

In this section we shall prove the mean square formula (2.8) for $G(T)$ (defined in 2.7)) and the $\Omega_\pm$ result (2.9). The method of proof of (2.8) is similar to the one used in proving (1.8) or (1.9). This is because (1.9) is based on Cramér's trick for dealing with (1.8). We use (2.4) to write, for $T \leqslant t \leqslant 2T$,

$$G(t) = 2^{-3/2} \sum_{n \leqslant t} (-1)^n \frac{d(n)}{n^{1/2}} \left(\frac{t}{2\pi n} + \frac{1}{4}\right)^{-1/4} \left(\operatorname{ar\,sinh} \sqrt{\frac{\pi n}{2t}}\right)^{-2} \sin(f(t, n))$$

$$+ \left\{ -2 \sum_{n \leqslant c_0 t} \frac{d(n)}{n^{1/2}} (\log t/2\pi n)^{-2} \sin(g(t, n)) + O(T^{1/4}) \right\}$$

$$= \Sigma_1(t) + \Sigma_2(t), \tag{6.1}$$

say. If follows that

$$\int_T^{2T} G^2(t)\, dt = \int_T^{2T} \Sigma_1^2(t)\, dt + \int_T^{2T} \Sigma_2^2(t)\, dt + 2 \int_T^{2T} \Sigma_2(t)\, \Sigma_2(t)\, dt.$$

The last integral is estimated by the Cauchy-Schwarz inequality as $\ll T^2$, since we have

$$\int_T^{2T} \Sigma_2^2(t)\, dt \ll T^{3/2}$$

and (as we will see)

$$\int_T^{2T} \Sigma_1^2(t)\, dt = B(2T)^{5/2} - BT^{5/2} + O(T^2), \tag{6.2}$$

where B is defined by (2.8). Namely we have

$$\int_T^{2T} \Sigma_2^2(t)\, dt \ll \int_T^{2T} \left| \sum_{n \leqslant c_0 t} \frac{d(n)}{n^{1/2}} (\log t/2\pi n)^{-2} n^{-it} \right|^2 dt$$

$$+ \int_T^{2T} T^{1/2}\, dt \ll T^{1+\varepsilon} + T^{3/2} \ll T^{3/2},$$

since the sum $\sum_{n \leqslant c_0 t}$ is essentially a Dirichlet polynomial which is $\ll T^\varepsilon$ in mean square. Now to prove (6.2) note that the left-hand side is

$$\frac{1}{8}\int_T^{2T} \sum_{n\leqslant t} \frac{d^2(n)}{n}\left(\frac{t}{2\pi n}+\frac{1}{4}\right)^{-1/2}\left(\text{ar sinh}\sqrt{\frac{\pi n}{2t}}\right)^{-4} \sin^2(f(t,n))\,dt$$

$$+\frac{1}{8}\int_T^{2T}\left\{\sum_{m\neq n\leqslant t} \frac{d(m)\,d(n)}{(mn)^{1/2}}\left(\frac{t}{2\pi m}+\frac{1}{4}\right)^{-1/4}\left(\frac{t}{2\pi n}+\frac{1}{4}\right)^{-1/4}\right.$$

$$\times\left(\text{ar sinh}\sqrt{\frac{\pi m}{2t}}\right)^{-2}\left(\text{ar sinh}\sqrt{\frac{\pi n}{2t}}\right)^{-2}$$

$$\left.\times \sin(f(t,m))\sin(f(t,n))\right\}dt$$

$$=\Sigma'+\Sigma'', \tag{6.3}$$

say, and the main terms in (6.2) will come from Σ'. We have

$$\Sigma'=\frac{1}{8}\sum_{n\leqslant\sqrt{T}}\frac{d^2(n)}{n}\int_T^{2T}\left(\frac{t}{2\pi n}+\frac{1}{4}\right)^{-1/2}$$

$$\times\left(\text{ar sinh}\sqrt{\frac{\pi n}{2t}}\right)^{-4}\sin^2(f(t,n))\,dt+O(T^2)$$

$$=\frac{1}{16}\sum_{n\leqslant\sqrt{T}}\frac{d^2(n)}{n}\int_T^{2T}\left(\frac{t}{2\pi n}\right)^{-1/2}\left(\frac{2t}{\pi n}\right)^{2}$$

$$\times\{1-\cos 2f(t,n)\}\,dt+O(T^2)$$

$$=(2\pi)^{-3/2}\sum_{n\leqslant\sqrt{T}}\frac{d^2(n)}{n^{5/2}}\int_T^{2T} t^{3/2}\{1-\cos 2f(t,n)\}\,dt+O(T^2)$$

$$=B(2T)^{5/2}-BT^{5/2}+O(T^2).$$

Here we used the fact that the contribution of the cosine terms is $O(T^2)$ after an application of [18, Lemma 2.1] on exponential integrals, and

$$\sum_{n\leqslant\sqrt{T}}\frac{d^2(n)}{n^{5/2}}=\frac{\zeta^4(5/2)}{\zeta(5)}+O\left(\frac{\log^3 T}{T^{3/4}}\right).$$

It remains to consider Σ''. We have by symmetry

$$\Sigma''\ll\sum_{n<m\leqslant 2T}\left|\int_{\max(m,T)}^{2T}(mn)^{\varepsilon-1/2}\left(\frac{t}{2\pi m}+\frac{1}{4}\right)^{-1/4}\left(\frac{t}{2\pi n}+\frac{1}{4}\right)^{-1/4}\right.$$

$$\times\left(\text{ar sinh}\sqrt{\frac{\pi m}{2t}}\right)^{-2}\left(\text{ar sinh}\sqrt{\frac{\pi n}{2t}}\right)^{-2}$$

$$\left.\times\sin f(t,m)\sin f(t,n)\,dt\right|,$$

and the sine terms will give exponentials of the form $\exp\{if(t, m) \pm if(t, n)\}$. The contribution of the terms with the plus sign is easily seen to be $\ll T^2$ by [18, Lemma 2.1]. For the remaining terms with the minus sign put $F(t) = f(t, m) - f(t, n)$ for any fixed $n < m \leqslant 2T$, so that

$$F'(t) = 2 \operatorname{ar\,sinh} \sqrt{\frac{\pi m}{2t}} - 2 \operatorname{ar\,sinh} \sqrt{\frac{\pi n}{2t}} \asymp T^{-1/2}(m^{1/2} - n^{1/2})$$

by the mean-value theorem. We have, again by [18, Lemma 2.1],

$$\Sigma'' \ll T^2 \sum_{n < m \leqslant 2T} (mn)^{\varepsilon - 5/4} (m^{1/2} - n^{1/2})^{-1}$$

$$= T^2 \left(\sum_{n \leqslant m/2} + \sum_{n > m/2} \right) = T^2(S_1 + S_2),$$

say. Trivially, $S_1 \ll 1$, and

$$S_2 \ll \sum_{m \leqslant 2T} m^{\varepsilon - 5/4} m^{1/2} \sum_{m/2 < n < m} \frac{m^{\varepsilon - 5/4}}{m - n} \ll 1.$$

Hence $\Sigma'' \ll T^2$, and putting together the preceding estimates it follows that

$$\int_T^{2T} G^2(t)\, dt = B(2T)^{5/2} - BT^{5/2} + O(T^2).$$

Replacing T by $T2^{-j}$ and summing over $j = 1, 2, \ldots$, we obtain (2.8).

To prove the second part of Theorem 4, namely (2.9), we proceed similarly as in the proof of (5.1). If $h(u)$ denotes the series defined in (5.6), it suffices to show that

$$\limsup_{u \to \infty} h(u) > 0 \qquad \text{and} \qquad \liminf_{u \to \infty} h(u) < 0. \tag{6.4}$$

To prove the first part of (6.4) let M be a large positive integer and $\delta > 0$ be given. For each $n \leqslant M$, write $n = v^2 q$ where q is square-free. Then by Kronecker's approximation theorem there exist arbitrarily large u satisfying

$$u\sqrt{q} = 2\pi \begin{cases} m_q + \delta_q, & \text{if } q \text{ is odd,} \\ \frac{1}{2} + m_q + \delta_q, & \text{if } q \text{ is even,} \end{cases}$$

for integers m_q and $|\delta_q| < \delta$. We deduce that for $n \leqslant M$,

$$(-1)^n \sin(u\sqrt{n} - \pi/4) = \varepsilon_n \sin\left(\frac{\pi}{4}\right) + O(\sqrt{n}\,\delta),$$

with $\varepsilon_n = -1$ if $n \equiv 0 \pmod 4$ and $\varepsilon_n = 1$ otherwise.

Now, since the series for $h(u)$ converges absolutely, we obtain

$$\limsup_{u\to\infty} h(u) \geqslant \sin\left(\frac{\pi}{4}\right) \sum_{n\leqslant M} \varepsilon_n \frac{d(n)}{n^{5/4}} + O(\delta M^{1/4}\log M) + O(M^{-1/4}\log M).$$

First let δ tend to zero and then M to infinity. To finish, we need only show that the resulting infinite series is positive. But for $s>1$

$$\sum_{n=1}^{\infty} \varepsilon_n \frac{d(n)}{n^s} = (1+2^{2-3s}-3\cdot 2^{1-2s})\,\zeta^2(s),$$

which is positive at $s=\frac{5}{4}$.

The second part of (6.4) is a bit easier. We proceed as above but apply Kronecker's theorem to find u such that

$$u\sqrt{q} = 2\pi \begin{cases} \frac{1}{2}+m_q+\delta_q, & \text{if } q \text{ is odd,} \\ m_q+\delta_q, & \text{if } q \text{ is even.} \end{cases}$$

In this case

$$(-1)^n \sin(u\sqrt{n}-\pi/4) = -\sin\left(\frac{\pi}{4}\right) + O(\sqrt{n}\,\delta),$$

so that we conclude

$$\liminf_{u\to\infty} h(u) \leqslant -\sin\left(\frac{\pi}{4}\right)\zeta^2\left(\frac{5}{4}\right) < 0,$$

as claimed. This completes the proof of Theorem 4.

7. The Upper Bound for $E(T)$

We are finally going to sketch the proof of the upper bound for $E(T)$ given by Theorem 5. This will be based on Theorem 3, and G. Kolesnik's estimates [23, 24] for exponential sums. His techniques were already used by W. G. Nowak [26] in the similar situation of the circle problem, namely the estimation of

$$P(T) = \sum_{n\leqslant T} r(n) - \pi T = \sum_{a^2+b^2\leqslant T} 1 - \pi T.$$

We start with an elementary estimate. From the definition of $E(T)$ we find that, for $0\leqslant x\leqslant T$,

$$E(T+x) - E(T) \geqslant -2Cx\log T$$

with some absolute positive constant C. Hence

$$\int_T^{T+x} E(t)\,dt = xE(T) + \int_0^x (E(T+u) - E(T))\,du$$

$$\geqslant xE(T) - 2C\int_0^x u \log T\,du = xE(T) - Cx^2 \log T.$$

Therefore we obtain

$$E(T) \leqslant x^{-1}\int_T^{T+x} E(t)\,dt + Cx\log T \qquad (0 < x \leqslant T), \tag{7.1}$$

and analogously

$$E(T) \geqslant x^{-1}\int_{T-x}^{T} E(t)\,dt - Cx\log T \qquad (0 < x \leqslant T). \tag{7.2}$$

One easily sees that (7.1) and (7.2) remain valid if $E(T)$ is replaced by $\Delta(T)$ and one obtains analogously also

$$P(T) \leqslant x^{-1}\int_T^{T+x} P(t)\,dt + Cx, \qquad P(T) \geqslant x^{-1}\int_{T-x}^{T} P(t)\,dt - Cx$$

for some $C > 0$ and $0 < x \leqslant T$. Note that in the bounds for $P(T)$ there is no log-factor present as in (7.1) and (7.2). Hence it is not a surprise that Nowak [26] obtained

$$P(T) \ll T^{139/429}(\log T)^{1384/429}$$

(now superseded by $T^{7/22+\varepsilon}$ of [19]), which has a better log-factor than (2.11).

We now evaluate $x^{-1}\int_T^{T+x} E(t)\,dt$ by Theorem 3, supposing that $T^{5/16} \leqslant x \leqslant T^{1/3}$ and truncating the first sum in (2.4) at $R = T^3 x^{-8}$. We have trivially

$$\sum_{R < n \leqslant T} (-1)^n \frac{d(n)}{n^{1/2}}\left(\frac{T}{2\pi n} + \frac{1}{4}\right)^{-1/4}\left(\operatorname{ar\,sinh}\sqrt{\frac{\pi n}{2t}}\right)^{-2} \sin(f(t, n))$$

$$\ll T^{3/4} \sum_{n > R} \frac{d(n)}{n^{5/4}} \ll T^{3/4} R^{-1/4} \log R \ll x^2 \log T.$$

Simplifying the terms for $1 \leqslant n \leqslant R$ by Taylor's formula we obtain

$$x^{-1} \int_T^{T+x} E(t)\,dt$$

$$= 2^{-1/4}\pi^{-3/4}x^{-1} \sum_{n \leqslant R} (-1)^n \frac{d(n)}{n^{5/4}} \{t^{3/4} \sin(f(t, n))\}\big|_T^{T+x} + O(x \log T).$$

But for $T \leqslant t \leqslant T+x$, $n \leqslant Tx^{-2}$, $T^{5/16} \leqslant x \leqslant T^{1/3}$, we have

$$2 \int_T^{T+x} t^{3/4} \operatorname{ar\,sinh} \sqrt{\frac{\pi n}{2t}} \cos(f(t, n))\,dt$$

$$= \int_T^{T+x} t^{3/4} \frac{d}{dt} \{\sin(f(t, n))\}$$

$$= \{t^{3/4} \sin(f(t, n))\}\big|_T^{T+x} + O(xT^{-1/4}),$$

and

$$\operatorname{ar\,sinh} \sqrt{\frac{\pi n}{2t}} = \sqrt{\frac{\pi n}{2T}} + O\left(\frac{n}{T}\right).$$

Hence we obtain

$$x^{-1} \int_T^{T+x} E(t)\,dt$$

$$= (2/\pi)^{1/4} x^{-1} T^{1/4} \sum_{n \leqslant Tx^{-2}} (-1)^n \frac{d(n)}{n^{3/4}} \int_T^{T+x} \cos(f(t, n))\,dt$$

$$+ 2^{-1/4}\pi^{-3/4}x^{-1}T^{3/4} \sum_{Tx^{-2} < n \leqslant T^3x^{-8}} (-1)^n \frac{d(n)}{n^{5/4}} \sin(f(t, n))\big|_T^{T+x}$$

$$+ O(x \log T), \tag{7.3}$$

and a similar formula holding also for $x^{-1} \int_{T-x}^{T} E(t)\,dt$. Therefore (7.1)–(7.3) yield, after some simplifying,

$$E(T) \ll x \log T$$

$$+ T^{1/4} \max_{T-x \leqslant t \leqslant T+x} \left| \sum_{n \leqslant Tx^{-2}} (-1)^n \frac{d(n)}{n^{3/4}} \exp(iF(t, n)) \right|$$

$$+ T^{3/4}x^{-1} \max_{T-x \leqslant t \leqslant T+x} \left| \sum_{Tx^{-2} < n \leqslant T^3x^{-8}} (-1)^n \frac{d(n)}{n^{5/4}} \exp(iF(t, n)) \right| \tag{7.4}$$

in the range $T^{5/16} \leqslant x \leqslant T^{1/3}$. Here we have replaced $f(t, u)$ with the simpler function

$$F(t, u) = b_1 t^{1/2} u^{1/2} + b_2 t^{-1/2} u^{3/2},$$

where b_1 and b_2 are real constants and the total error is absorbed in the term $x \log T$. This idea was already used by the second author in [18, pp. 192–193]. Moreover, as also shown in [18], the $(-1)^n$ in (7.4) is harmless and may be disregarded. The same bound holds also for $\Delta(T)$, without $(-1)^n$ and with $4\pi\sqrt{nt}$ in place of $F(t, n)$.

Having at our disposal (7.4), we proceed analogously as was done by Nowak [26] in the case of the circle problem, where detailed calculations are given. We estimate the term $T^{1/4} \max |\sum_{n \leqslant Tx^{-2}}|$ in (7.4) by using the estimate of Kolesnik [24, p. 118]. This produces a bound which is a sum of five terms, the largest of which is readily found to be

$$T^{195/692}(Tx^{-2})^{83/692} \log^4 T = x \log T \tag{7.5}$$

for $x = T^{139/429}(\log T)^{1038/429}$, so that in fact x is determined by (7.5). The remaining sum in (7.4) is estimated by splitting it as

$$\sum_{Tx^{-2} < n \leqslant T^3 x^{-8}} = \sum_{Tx^{-2} < n \leqslant V} + \sum_{V < n \leqslant T^3 x^{-8}} = \Sigma_1 + \Sigma_2,$$

say, where $V = T^{0.3531}$. With x given above, one has

$$T^{3/4} \max |\Sigma_1| \ll x^2 \log T \tag{7.6}$$

by the method used in estimating the previous sum. To estimate Σ_2 we use a subsidiary argument, furnished by Kolesnik [23, p. 116], where the relatively large size of V becomes prominent, and we find that (7.6) holds with Σ_1 replaced by Σ_2. Thus finally (7.4) gives

$$E(T) \ll x \log T = T^{139/429}(\log T)^{1467/429},$$

and Theorem 5 is proved.

Our method gives essentially the best possible log-factor obtainable from Kolesnik's method, since the first sum in (7.4) (which is the crucial one) is of length Tx^{-2}. Note that the above estimate for $E(T)$ may be used for bounding $\zeta(\frac{1}{2} + iT)$ itself. Namely, using [18, Lemma 7.1] we obtain

$$\begin{aligned} |\zeta(\tfrac{1}{2} + iT)|^2 &\ll \log T\left(1 + \int_{T-\log^2 T}^{T+\log^2 T} |\zeta(\tfrac{1}{2} + iu)|^2 \, du\right) \\ &\ll \log^4 T + \log T(E(T + \log^2 T) - E(T - \log^2 T)) \\ &\ll T^{139/429}(\log T)^{1896/429}, \end{aligned}$$

hence

$$\zeta(\tfrac{1}{2}+iT) \ll T^{139/858}(\log T)^{948/429}.$$

This result is superseded by $\zeta(\frac{1}{2}+iT) \ll T^{9/56+\varepsilon}$, proved recently by E. Bombieri and H. Iwaniec [3], and M. Huxley and N. Watt [15] improved their estimate by replacing T^{ε} with $\log^2 T$.

As was pointed out in [18, Chaps. 13 and 15], there are great similarities in the problem of the estimation of $\Delta(T)$, $P(T)$, $\zeta(\frac{1}{2}+iT)$, and $E(T)$. At the time of writing [18], one had for the exponents in these problems (disregarding the log-factors) the sequence $(A, A, A/2, A)$ with $A = \frac{35}{108}$ (due to G. Kolesnik [22, 23]), and then $A = \frac{139}{429}$ (also due to G. Kolesnik [24]). However, the new methods of Bombieri and Iwaniec [3] and Iwaniec and Mozzochi [19] have considerably changed the situation, and now the best known sequence of exponents is $(\frac{7}{22}, \frac{7}{22}, \frac{9}{56}, \frac{139}{429})$ which is no longer of the form $(A, A, A/2, A)$ for some $A > 0$. Whether this is a temporary phenomenon or a lasting one is too early to tell.

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