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Convex Invertible Cones and the Lyapunov Equation

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ABSTRACT

Convex cones of matrices which are closed under matrix inversion are defined, and their structure is studied. Various connections with the algebraic Lyapunov equation of general inertia are explored. © *Elsevier Science Inc., 1997*

I. INTRODUCTION

Convex cones play a central role in matrix theory: the sets of Hermitian matrices or real matrices with a prescribed sign pattern (e.g. entrywise positive or $\mathscr{Z}$ matrices) are all convex cones. Here, we shall focus on a special family of convex cones, namely those which are closed under matrix inver-

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sion, e.g. the sets of stable upper triangular matrices or dissipative matrices. Such a set will be referred to as a *convex invertible cone*, or cic for short.

Matrix cics have any interesting properties. In particular, typically they are divided into "basins of influence," each basin governed by a single involution (i.e. a matrix S with $S^2 = I$). The number of basins may be finite, countable, or a continuum.

A second property is generating sets. The set of extreme rays generates a cone by convex combinations; but if also inversion is allowed, one can choose small proper subsets of extreme rays as generators. For example, the cic generated by a, a complex scalar with a nonzero real part, is identical to the convex cone generated by a and its complex conjugate a^*.

There are also important connections with the algebraic Lyapunov equation,

$$HA + A^*H = Q, \qquad H \in \mathscr{H}, \quad Q \in \mathscr{P},$$

where $\mathscr{H} \subset \mathbb{C}^{n \times n}$ denotes the set of nonsingular Hermitian matrices, and $\mathscr{P} \subset \mathscr{H}$ is the subset of positive definite matrices. More generally, we study sets of matrices $\mathbf{X} \subset \mathbb{C}^{n \times n}$ satisfying the algebraic Lyapunov *inclusion*:

$$(HA + A^*H) \in \mathscr{P} \qquad \forall A \in \mathbf{X},$$

for some $H \in \mathscr{H}$. The matrix H is then called a *common Lyapunov solution* for $\mathbf{X}$.

As a first basic observation, it is shown that the set of all matrices sharing the same Lyapunov solution is a maximal open nonsingular cic. Hence, if a set $\mathbf{X}$ has a common Lyapunov solution, then the cic generated by $\mathbf{X}$ contains only nonsingular matrices. The converse however is not necessarily true. Nevertheless, two special cases where the nonsingularity of a cic does imply the existence of a common Lyapunov solution are presented: a *finitely* generated cic of upper triangular matrices and a cic generated by a pair of Hermitian matrices.

For a given matrix A one can define the set $\mathscr{H}_A$ of all possible Lyapunov solutions for this A, sometimes referred to as the image of the inverse Lyapunov transformation. For a set $\mathbf{X}$, this is generalized to be $\mathscr{H}_{\mathbf{X}} := \bigcap_{A \in \mathbf{X}} \mathscr{H}_A$. It is shown that if $\mathbf{X}$ and $\mathbf{Y}$ are two sets of generators for the same nonsingular cic, then $\mathscr{H}_{\mathbf{X}} = \mathscr{H}_{\mathbf{Y}}$. The converse is not true for complex matrices in general; it does hold over the reals when both sets are singletons. This nontrivial result goes back to [15].

In [8, 9] we looked at various applications of the cic structure to the study of systems and control theory. In particular, we interpret a well-known iterative procedure for solving the Lyapunov equation (the matrix sign function algorithm) in terms of the basin of attraction of the involution in a cic.

Convex invertible cones can be defined abstractly over any algebra with a unit element. Indeed, the discussion of the cic structure in [9] is extended to algebras beyond that of constant matrices. In particular, it is shown there that the set of positive real odd functions is a cic over the algebra of scalar real rational functions. Moreover, if a matrix B belongs to a cic generated by A, then there is always a scalar positive real odd function f such that $B = f(A)$.

The outline of this paper is as follows. In Section II basic properties of cics are investigated, including relations with involutions within a cic. Connections with the Lyapunov equation are explored in Section III. Nonsingular cics of Hermitian and upper triangular matrices are examined in Sections IV and V, respectively.

II. CONVEX INVERTIBLE CONES AND INVOLUTIONS

Our main goal in this paper is to study the Lyapunov equation in association with matrix cics. In this section we define these mathematical objects in general, and study their relevant properties: generating sets, nonsingularity, inertia, and involutions. We start with definitions, followed by several remarks.

DEFINITION.

(1) A set $\mathbf{X} \subset \mathbb{C}^{n \times n}$ is said to be invertible if along with any nonsingular matrix A in it, it contains also its inverse A^{-1}.

(2) A cic is a convex invertible cone of matrices, i.e., a set of matrices closed under addition, matrix inversion, and positive scaling.

(3) For $\mathbf{X} \subset \mathbb{C}^{n \times n}$ we denote by $\mathscr{C}(\mathbf{X})$ the cic generated by $\mathbf{X}$, namely, the smallest cic containing $\mathbf{X}$.

(4) Given a cic $\mathscr{C}$, every set $\mathbf{X} \subset \mathscr{C}$ such that $\mathscr{C} = \mathscr{C}(\mathbf{X})$ is called a generating set for $\mathscr{C}$. A generating set $\mathbf{X}$ is minimal if it does not strictly contain any other generating set.

(5) A cic (and in general, any set of matrices) will be called stable (nonsingular) if it contains exclusively stable (respectively, nonsingular) matrices.

Note that under this terminology invertibility and nonsingularity are unrelated properties. For example, the set $\mathbf{X} \subset \mathbb{R}^{2 \times 2}$ of matrices whose sign pattern is

$$\begin{pmatrix} + & - \\ + & + \end{pmatrix}$$

is a (nonsingular) convex cone, but not invertible [in fact $(\mathbf{X})^{-1} = (\mathbf{X})^T$]. On the other hand, the set $\bar{\mathscr{P}}$ of all Hermitian positive semidefinite matrices is a cic, although it contains singular matrices as well. Stability of the convex cone generated by $\mathbf{X}$ was characterized in [6] by means of a local Lyapunov condition.

Both nonsingularity and stability are preserved under positive scaling and matrix inversion, but in general not by taking convex combinations. It is therefore not enough to check a generating set $\mathbf{X}$ in order to establish these properties for $\mathscr{C}(\mathbf{X})$.

To get $\mathscr{C}(\mathbf{X})$ from a generating set $\mathbf{X}$, we may proceed by induction. We set $\mathbf{X}_0 = \mathbf{X}$, and $\mathbf{X}_{k+1}$ is obtained from $\mathbf{X}_k$ by taking positive combinations of members in $\mathbf{X}_k$ and their inverses. $\mathscr{C}(\mathbf{X})$ is the union of the increasing sequence $\mathbf{X}_k$.

Every cic has generating sets, but not necessarily *minimal* generating sets. The elements of a minimal generating set, if it does indeed exist, must belong to extreme rays of the cone. A minimal generating set is generically nonunique. We illustrate the concept of generation with a cic of sign pattern matrices, of the type studied in [9].

EXAMPLE 2.1. The set $\mathscr{K} \subset \mathbb{R}^{2\times 2}$ of matrices whose sign pattern is

$$\begin{pmatrix} + & + \\ + & - \end{pmatrix}$$

is a nonsingular cic. It can be considered as an orthant in $\mathbb{R}^{2\times 2}$. If we allow also zero entries, we get the closure $\bar{\mathscr{K}}$, which is a singular cic.

Recall that E_{jk} denotes the matrix with 1 at the j, k entry and zeros otherwise. The set $\mathbf{X} = \{E_{11}, E_{12}, E_{21}, -E_{22}\}$ is a minimal generating set for $\bar{\mathscr{K}}$. In fact, since $\bar{\mathscr{K}}$ is a convex cone with only four extreme rays, this generating set is essentially unique. However, the nonsingular cic $\mathscr{K}$ (the interior of $\bar{\mathscr{K}}$) has no minimal generating sets. In Observation 5.5 we show that the set of stable upper triangular matrices forms a cic with a countable generating set.

Complicated cics can be constructed from simpler ones using simple rules such as the following three. The proof is simple and has been omitted:

PROPOSITION 2.2.

(i) *The intersection of cics is a cic.*
(ii) *Let* $\mathbf{X} \subset \mathbb{C}^{n\times n}$, $\mathbf{Y} \subset \mathbb{C}^{m\times m}$ *be two matrix cics. Then the set of all*

$(n+m)\times(n+m)$ matrices of the form

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where $A \in \mathbf{X}$, $B \in \mathbf{Y}$, and $C \in \mathbb{C}^{n\times m}$ are arbitrary, is a cic as well.

(iii) *Each of the operations similarity, transposition, and complex conjugation defines a bijection between two cics, and in particular transforms a generating set to a generating set.*

These rules also preserve nonsingularity and stability of a cic.

Matrix inertia plays a major role in characterizing nonsingular cics. Recall that an $n\times n$ matrix has inertia (ν, δ, π) if among its eigenvalues ν have negative real part, π have positive real part, and δ are imaginary. In this context, the case $\delta = 0$ turns to be of a particular significance and will be referred to as *regular* inertia. We call a matrix unstable whenever $\nu \leqslant n-1$ and antistable when $\pi = n$.

The three operations mentioned in Proposition 2.2(iii), especially similarity, will be needed in the sequel. Note that these operations are inertia preserving. The implication of this observation for the cic structure will be reinforced in Proposition 2.6(a) and in the beginning of Section IV.

A matrix S is called an involution if $S^2 = I$, or equivalently if $S^{-1} = S$. The following are alternative definitions:

(1) $\Pi := \frac{1}{2}(S+I)$ is an idempotent, i.e. $\Pi^2 = \Pi$;
(2) S is similar to $\operatorname{diag}\{-I_\nu, I_{n-\nu}\}$ for some $0 \leqslant \nu \leqslant n$;
(3) S is unitarily similar to $\begin{pmatrix} -I_\nu & M \\ 0 & I_{n-\nu} \end{pmatrix}$ for some $M \in \mathbb{C}^{\nu\times(n-\nu)}$.

The scalar involutions are ± 1. For every $a \in \mathbb{C}$ with $\operatorname{Re} a \neq 0$, the function $\operatorname{Sign}(a) := (\operatorname{Re} a)/|\operatorname{Re} a|$ assigns an involution in a continuous way. For matrices, a rigorous definition of the analogous matrix sign function (see [1]) is given by

$$\operatorname{Sign}(A) = 2\Pi_A - I, \qquad \Pi_A := \frac{1}{2\pi i}\int_\gamma (tI - A)^{-1}\,dt, \qquad (2.1)$$

where γ is an arbitrary closed contour encircling all the right half plane eigenvalues of A. The idempotent matrix Π_A is the spectral projection associated with A and the right half plane. The continuity of Π_A as a function of A guarantees the continuity of Sign(A). Note that Sign is continuous even when the Jordan form is not. For example, consider

$$A = I_2 \quad \text{and} \quad B = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}.$$

We get Sign(A) = Sign(B) = I_2.

Alternatively, Sign(A) can be defined in terms of the Jordan representation. Namely, if VJV^{-1} and J is in Jordan form, then Sign(A) = $V\,\mathrm{diag}\{\mathrm{Sign}(J_{1,1}) \cdots \mathrm{Sign}(J_{n,n})\}V^{-1}$, irrespective of the number of Jordan blocks in J (see e.g. [8]). The equivalence of the two definitions is an immediate consequence of the following properties of the sign function: (1) Sign(A) $= -I$ if and only if A is stable; (2) $\mathrm{Sign}(\mathrm{diag}\{A, B\}) = \mathrm{diag}\{\mathrm{Sign}(A), \mathrm{Sign}(B)\}$; (3) $\mathrm{Sign}(VAV^{-1}) = V\,\mathrm{Sign}(A)V^{-1}$. These properties follow easily from (2.1). The uniqueness of the spectral projection Π_A shows that Sign(A) is independent of the Jordan representation, which is not unique. The matrix Sign function is studied in [Chapter 22].

The set of involutions is an unbounded algebraic variety in the space of $n \times n$ matrices. It has $n + 1$ connected components, determined by their (regular) inertia. The following elementary exercise provides us with a simple test for convexity of a set of involutions.

LEMMA 2.3. *For any two distinct involutions S_1, S_2, the following are equivalent*:

(i) *Any affine combination $aS_1 + (1 - a)S_2$ is an involution.*
(ii) *One affine combination other than S_1, S_2 is an involution.*
(iii) $S_1S_2 + S_2S_1 = 2I$.

A cic $\mathscr{C}$ contains involutions whenever it contains matrices with regular inertia; see Propositions 2.5, 2.6 below. The set $\{\mathrm{Sign}(\mathscr{C})\}$ of involutions in $\mathscr{C}$ is a relatively closed variety in $\mathscr{C}$. It is not necessarily bounded, convex, or even connected; see Proposition 2.6.

The cic of upper triangular $n \times n$ matrices has a rich variety of involutions (see Section V). The following is an additional example.

EXAMPLE 2.4. Consider $\bar{\mathscr{K}}$, the closure of the sign pattern cic introduced in Example 2.1. A matrix

$$\begin{pmatrix} a & b \\ c & -d \end{pmatrix} \qquad \text{where} \quad a, b, c, d \geqslant 0$$

is an involution in $\bar{\mathscr{K}}$ if and only if $a = d$ and $a^2 + bc = 1$. These equations describe a nonconvex, unbounded variety $\{\mathrm{Sign}(\bar{\mathscr{K}})\}$ in $\mathbb{R}^{2\times 2}$. The relative interior of the set $\{\mathrm{Sign}(\bar{\mathscr{K}})\}$ consists of the involutions of the open cic $\mathscr{K}$.

The set $\bar{\mathscr{K}}$ contains upper and lower triangular involutions of the form

$$S_U = \begin{pmatrix} 1 & b' \\ 0 & -1 \end{pmatrix}, \qquad S_L = \begin{pmatrix} 1 & 0 \\ c' & -1 \end{pmatrix}.$$

Each of these two sets is affine (see Example 5.1). In fact, every involution S in $\mathscr{K}$ splits as a positive combination of two such lower and upper triangular involutions,

$$S = \theta a \begin{pmatrix} 1 & \dfrac{b}{\theta a} \\ 0 & -1 \end{pmatrix} + (1-\theta) a \begin{pmatrix} 1 & 0 \\ \dfrac{c}{(1-\theta)a} & -1 \end{pmatrix}, \qquad 0 < \theta < 1.$$

This splitting is not unique.

We now state the property of a constant inertia of an arbitrary nonsingular cic. First we cover cics with a single generator:

PROPOSITION 2.5. *If $A \in \mathbb{C}^{n\times n}$ has regular inertia, then $\mathscr{C}(A)$ is nonsingular, and* Sign(A) *is the only involution it contains. If A has irregular inertia, then $\mathscr{C}(A)$ is singular and contains no involutions.*

Proof. Let A be given in its Jordan canonical form, namely $A = VJV^{-1}$; then $\mathscr{C}(A) = V\mathscr{C}(J)V^{-1}$. Note that the upper triangular matrix J may always be chosen so that it can be partitioned according to the inertia of A, namely, $J = \mathrm{diag}\{J_\nu, J_\delta, J_\pi\}$: The submatrix J_ν comprises all Jordan blocks corresponding to eigenvalues within the open left half plane. Similarly, all blocks corresponding to eigenvalues on the imaginary axis, and those within the open right half plane, are grouped in the submatrices J_δ and J_π respectively. Clearly, for a nonsingular matrix A one has that $A^{-1} = V\,\mathrm{diag}\{J_\nu^{-1}, J_\delta^{-1}, J_\pi^{-1}\}V^{-1}$, where each of the diagonal blocks is by itself upper triangular, so in particular $[J^{-1}]_{kk} = 1/[J]_{kk}$.

Assume first that inertia(A) is regular, i.e., $\delta = 0$. Then $\mathrm{Sign}(A) = \mathrm{Sign}(A^{-1}) = V\,\mathrm{diag}\{-I_\nu, I_\pi\}V^{-1}$. Obviously, $\mathrm{Sign}(\alpha A) = \mathrm{Sign}(A)$ for an arbitrary scalar $\alpha > 0$. Hence $\mathrm{Sign}(A) = \mathrm{Sign}(B)$ whenever $B \in \mathscr{C}(A)$. So nonsingularity of the cic and uniqueness of its involution are established.

Consider now the case where inertia(A) is irregular. If A is singular, we are done. So assume that the cic contains a nonsingular matrix B with an imaginary eigenvalue ia, $a \in \mathbb{R}$; then $(B + a^2 B^{-1})$ is singular. Moreover, in this case Sign(B) is not properly defined. So the claim is established. ■

As an illustration of the last result take A to be an arbitrary matrix with regular inertia which is not an involution. Let us denote $A(\theta) := \theta A + (1 - \theta)\,\mathrm{Sign}(A)$, where $0 \leqslant \theta \leqslant 1$. Then $\theta_1 > \theta_2$ implies that $\mathscr{C}(A(\theta_2))$ is a strict subcic of $\mathscr{C}(A(\theta_1))$. In particular $\mathscr{C}(A(\theta))$ is a nonsingular cic with a single involution. For general cics we obtain the following:

PROPOSITION 2.6.

(a) *For a cic $\mathscr{C}$ of complex matrices, the following properties are equivalent:*

(i) *$\mathscr{C}$ is nonsingular.*

(ii) *No matrices in $\mathscr{C}$ has imaginary eigenvalues.*

(iii) *All the matrices in $\mathscr{C}$ have the same regular inertia.*

(b) *A nonsingular cic has a nonempty connected set of involutions.*

Proof.

(a): Obviously, (iii) implies (i). Next, in order to show that (i) implies (ii), note that if $\mathscr{C}$ is regular and A belongs to it, then $\mathscr{C}(A) \subseteq \mathscr{C}$ is regular; hence A has regular inertia by Proposition 2.5. Finally, we show that (ii) implies (iii). If there exist two matrices $A, B \in \mathscr{C}$ with different inertias, consider the matrix $\theta A + (1 - \theta)B$. At some point $\theta \in [0, 1]$ the matrix changes inertia, so it must have an imaginary eigenvalue.

(b): If $\mathscr{C}$ is regular, we know that the Sign function is a continuous mapping from $\mathscr{C}$ into itself, whose image is the set of involutions in $\mathscr{C}$. By convexity, $\mathscr{C}$ is connected. Hence, by continuity, its set of involutions is also connected. By Proposition 2.5, this set is not empty. ■

In Proposition 2.3(iii) it was indicated that similarity, transposition, and complex conjugation preserve the cic structure. Recall that the inertia is invariant under these operations; hence nonsingularity of a cic is preserved as well.

The converse of Proposition 2.6(b) falls short of being true. As a simple counterexample, the set $[0, \infty)$ is a singular cic in $\mathbb{R}$, and it contains a single involution, $\{1\}$. Therefore, the nonsingularity of a cic cannot quite be determined by the connectedness of its set of involutions. In fact, a cic $\mathscr{C}$ with a connected set of involutions $\mathscr{S}$ can be decomposed into the union of two disjoint sets $\hat{\mathscr{C}}$ and $\mathbf{Y}$, where the nonsingular cic $\hat{\mathscr{C}}$ is such that $\mathscr{S} = \{\mathrm{Sign}(\hat{\mathscr{C}})\}$ and the set $\mathbf{Y}$ either is empty or consists only of matrices with irregular inertia.

The following useful observation is an immediate consequence of the last two propositions.

COROLLARY 2.7. *A cic containing a stable matrix is nonsingular if and only if it is stable. Moreover, a stable cic contains the matrix* $-I$ *as its unique involution.*

We conclude this section by pointing out a simple fact concerning commutativity and the Sign matrix. It puts into perspective the conditions in Lemmas 3.4 and 3.6 below.

LEMMA 2.8. *Let A, B be two matrices with regular inertia. Consider the following statements:*

(i) *The matrices A and B commute.*
(ii) *The matrices* $\mathrm{Sign}(A)$ *and B commute.*
(iii) *The matrix* $\mathrm{Sign}(A)\,\mathrm{Sign}(B)$ *is an involution.*

Then (i) *implies* (ii), *and* (ii) *implies* (iii). *The converse implications do not hold in general.*

Proof. (i) $\Rightarrow$ (ii): Using results in functional calculus, it follows that every matrix B which commutes with A commutes with the spectral projection Π_A, (2.1), and thus with $\mathrm{Sign}(A)$ as well.

(ii) $\Rightarrow$ (iii): From the previous part we have that $\mathrm{Sign}(A)\,\mathrm{Sign}(B) = \mathrm{Sign}(B)\,\mathrm{Sign}(A)$. Multiplying the relation on the right by $\mathrm{Sign}(A)\,\mathrm{Sign}(B)$ reveals that $[\mathrm{sign}(A)\,\mathrm{Sign}(B)]^2 = I$, so this direction is established.

(iii) $\nRightarrow$ (ii): Let $A = \mathrm{diag}\{-1, 1\}$, and let B be an arbitrary nondiagonal antistable 2×2 matrix. Then $\mathrm{Sign}(A)\,\mathrm{Sign}(B) = A = \mathrm{Sign}(A)$, but A and B do not commute.

(ii) $\nRightarrow$ (i): In the previous example interchange the roles of A and B. The proof is complete. ∎

III. SETS SHARING A COMMON LYAPUNOV SOLUTION

In this section we start exploring the connection between the notion of a matrix cic and the Lyapunov equation for not necessarily stable matrices (known as the inertia theorem); see Theorem 3.1 below. A major role is played by $\mathscr{H}$, the set of *nonsingular* Hermitian matrices. Note that $\mathscr{C}(\mathscr{H})$ comprises all Hermitian matrices of any rank (including zero). However, in the next section, we study nonsingular cics of Hermitian matrices.

Consider the equation

$$HA + A^*H = Q, \quad H \in \mathscr{H}, \quad Q \in \mathscr{P}. \tag{3.1}$$

The well-known inertia theorem states the following:

THEOREM 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be given. Then*:

(i) [13, *Theorem* 2.4.10] *There exists a pair of matrices $H \in \mathscr{H}$ and $Q \in \mathscr{P}$ such that Equation* (3.1) *is satisfied if and only if A has regular inertia. In this case* inertia(A) = inertia(H).

(ii) [13, *Corollary* 4.4.7] *For each $Q \in \mathscr{P}$ Equation* (3.1) *has a unique solution $H \in \mathscr{H}$ if and only if*

$$\lambda_j(A) + \lambda_k^*(A) \neq 0, \qquad 1 \leqslant j \leqslant k \leqslant n. \tag{3.2}$$

The original statement in [13, Corollary 4.4.7] is stronger than our Theorem 3.1(ii).

The existence condition in Theorem 3.1(i) coincides with the nonsingularity condition for singly generated cics (Proposition 2.5). This condition is weaker than the uniqueness condition (3.2).

We can define now $\mathscr{A}_H$, the set of matrices sharing a common solution $H \in \mathscr{H}$, to the Lyapunov equation,

$$\mathscr{A}_H := \{A \mid (HA + A^*H) \in \mathscr{P}\}.$$

For an arbitrary $H \in \mathscr{H}$ this set is not empty, since it always contains the matrix $A = H$.

Note that the set $\mathscr{A}_H$ is closed under inversion. Namely, if $(HA + A^*H) \in \mathscr{P}$, then due to Theorem 3.1(i) (the inertia theorem), the matrix A is nonsingular, and $(A^*)^{-1}(HA + A^*H)A^{-1} \in \mathscr{P}$. Testing for summation and positive scaling is trivial; hence the set $\mathscr{A}_H$ is a nonsingular cic. One can now use this fact in order to provide an alternative proof for the inertia part of Theorem 3.1(i).

OBSERVATION 3.2. *Let $H \in \mathscr{H}$ be arbitrary. Every matrix A in the set $\mathscr{A}_H$ satisfies* inertia(A) = inertia(H).

Proof. For a given $H \in \mathscr{H}$, the set $\mathscr{A}_H$ is a nonsingular cic. Hence, by Proposition 2.6(a) the cic $\mathscr{A}_H$ has a fixed inertia. Since $H \in \mathscr{A}_H$ as well, this inertia must be the same as that of H. ■

In Proposition 3.7 below we show that in fact $\mathscr{A}_H$ is a maximal open nonsingular cic. The proof there does not rely on the inertia theorem.

From the original Lyapunov theorem (or its generalization in Theorem 3.1) it follows that the set $\mathscr{A}_H$ consists of stable matrices if and only if the matrix H is negative definite.

As an important special case the subset $\mathscr{A}_I$ of antistable matrices coincides with the family of *dissipative* matrices, namely the matrices with positive definite Hermitian part:

$$\mathscr{A}_I = \{ A \mid (A + A^*) \in \mathscr{P} \}. \tag{3.3}$$

See e.g. [10, 17] for relevant material. Note that the antistable normal matrices form a (highly nonminimal) generating set for $\mathscr{A}_I$.

Explicit description of $\mathscr{A}_H$ is in general quite involved. First, let us assume that an $n \times n$ matrix H has a special structure $H = \operatorname{diag}\{-I_\nu, I_{n-\nu}\}$ for some $0 \leqslant \nu \leqslant n$:

OBSERVATION 3.3. *Given n and ν, where $0 \leqslant \nu \leqslant n$, every $A \in \mathscr{A}_{\operatorname{diag}\{-I_\nu, I_{n-\nu}\}}$ is of the form*

$$A = \begin{pmatrix} -P_\nu + Z_\nu & F + X \\ F^* & P_{n-\nu} + Z_{n-\nu} \end{pmatrix},$$

where the matrix P_ν is positive definite and Z_ν, $Z_{n-\nu}$ are skew Hermitian, all of the appropriate dimensions; $X, F \in \mathbb{C}^{(n-\nu)\times\nu}$ are arbitrary; and the $(n-\nu)\times(n-\nu)$ matrix $(P_{n-\nu} - \frac{1}{4}X^* P_\nu^{-1} X)$ is positive definite.

The proof is left to the reader. The special case $\nu = 0$ leads to Equation (3.3).

Now in order to extend the above characterization of the set $\mathscr{A}_H$ to the case where the matrix H has more general structure, recall that this matrix can always be written as $H = M^* \operatorname{diag}\{-I_\nu, I_{n-\nu}\} M$ for some nonsingular matrix M. Then use the following:

LEMMA 3.4. *Let $H \in \mathscr{H}$ and M be arbitrary nonsingular matrices. Then:*

(i) $M^{-1}\mathscr{A}_H M = \mathscr{A}_{M^*HM}$.
(ii) *If H and M commute, then* $M^*\mathscr{A}_H M = \mathscr{A}_H$.

Proof. Consider the equation $HA + A^*H = Q$ for an arbitrary $H \in \mathscr{H}$. Multiplying by a nonsingular M from the right and M^* from the left yields

$$(M^*HM)(M^{-1}AM) + (M^{-1}AM)^*(M^*HM) = M^*QM \qquad (3.4)$$

(i): Now, by definition $A \in \mathscr{A}_H$ is equivalent to requiring that Q be positive definite. The claim follows from Sylvester's law of inertia; see e.g. [12, Theorem 4.5.8].

(ii): The matrices H and M commute if and only if H and M^* do. Therefore, the claim follows from (3.4). ■

We need now to resort to the following decomposition of an arbitrary Hermitian matrix H with inertia $(\nu, 0, n - \nu)$, where $0 \leqslant \nu \leqslant n$,

$$H = L\,\mathrm{diag}\{-I_\nu, I_{n-\nu}\}\,L^*, \qquad L = (H^2)^{1/4}\,\mathrm{diag}\{U_\nu, U_{n-\nu}\}, \quad (3.5)$$

and the unitary matrices $U_\nu, U_{n-\nu}$ are *arbitrary* of the appropriate dimensions. Recall that an arbitrary $H \in \mathscr{H}$ can be written as $H = V\Lambda V^*$, where V is unitary and Λ is a real, nonsingular diagonal matrix. Hence, in (3.5) we in fact have $(H^2)^{1/4} = V|\Lambda|^{1/2}V^*$. This observation implies that due to Lemma 3.4(i), if the matrix H is of a given inertia $(\nu, 0, n - \nu)$, the cic $\mathscr{A}_H$ is similar to the set $\mathscr{A}_{\mathrm{diag}\{-I_\nu, I_{n-\nu}\}}$ described in Observation 3.3. Moreover, without loss of generality, the similarity matrix M in Lemma 3.4(i) can be taken to be positive definite.

The special case where in Lemma 3.4(i) we have $H = P \in \mathscr{P}$ is of a particular significance; it was already used in the proof of [18, Theorem 3]. Then, one can take the similarity matrix to be $M = P^{-1/2}U$, where U is an arbitrary unitary matrix, and thus obtain the cic of dissipative matrices (3.3).

As another special case of Lemma 3.4(i), following the decomposition in (3.5), we have that $M^{-1}\mathscr{A}_H M = \mathscr{A}_H$ whenever $H = P\,\mathrm{diag}\{-I_\nu, I_{n-\nu}\}\,P$, $P \in \mathscr{P}$, and $M = P^{-1}\,\mathrm{diag}\{U_\nu, U_{n-\nu}\}\,P$, where the unitary matrices $U_\nu, U_{n-\nu}$ are arbitrary.

If one takes H in 3.4(ii) to be a *scalar* matrix, i.e. $H = rI$ where $r \in \mathbb{R}$, $r \neq 0$, then it trivially commutes with an arbitrary matrix M, which results in the simple observation that the sets $\mathscr{A}_{\pm I}$ [see (3.3)] are invariant under a nonsingular congruence.

We can now state in the stable case the first fundamental connection between Lyapunov the equation and the novel notion of cic.

LEMMA 3.5. *The set $\mathscr{A}_{-P}$, where $P \in \mathscr{P}$, is a maximal open stable cic in $\mathbb{C}^{n \times n}$.*

Proof. Using the remark after Lemma 3.4, it suffices to show that $\mathscr{A}_{-I}$ is a maximal open stable cic. It is straightforward to show that $\mathscr{A}_{-I}$ is a stable cic and an open set. If it is not a maximal stable open cic, there must be a matrix $B \notin \overline{\mathscr{A}}_{-I}$ (where $\overline{\mathscr{A}}_{-I}$ is the closure of $\mathscr{A}_{-I}$) for which the cic $(\mathscr{A}_{-I}, B)$ is stable. We shall show that this is not possible.

By Proposition 2.6(a)(iii) the above B must be stable. If the eigenvalues of $B + B^*$ are $\lambda_1 \geqslant \cdots \geqslant \lambda_n$, then we must have $\lambda_1 > 0$, since otherwise $B \in \overline{\mathscr{A}}_{-I}$. Now, for an arbitrary $0 < \theta < 1$ the matrix $A = \frac{1}{2}(-\theta\lambda_1 I + B^* - B)$ is in $\mathscr{A}_{-I}$, but $A + B$ is unstable. ■

Consider the set $\overline{\mathscr{A}}_{-P}$, the closure of the cic $\mathscr{A}_{-P}$ in the usual matrix topology. This is not stable, as there are singulr matrices on its boundary. By similarity [see Lemma 3.4(i)] it is enough to verify this statement for $P = I$, and indeed $\partial\mathscr{A}_{-I}$ contains the set of singular negative semidefinite matrices.

The following simple lemma, closely related to Lemma 2.8, will turn out to be very useful in the sequel. Recall (e.g. [12, Theorem 2.5.5]) that two normal matrices (and Hermitian matrices in particular) commute if and only if they are simultaneously codiagonizable by the same unitary similarity transformation.

LEMMA 3.6. *Let $H, S \in \mathscr{H}$ be commuting matrices, where S is an involution. Then*

$$S\mathscr{A}_H = \mathscr{A}_H S = \mathscr{A}_{SH} = \mathscr{A}_{HS}.$$

Proof. Under the given premises,

$$\begin{aligned} HA + A^*H &= HSSA + A^*SSH = HS(SA) + (SA)^* SH \\ &= SH(SA) + (SA)^* SH. \end{aligned}$$

Thus $A \in \mathscr{A}_H$ is equivalent to $(SA) \in \mathscr{A}_{SH}$ $(= \mathscr{A}_{HS})$. Similarly,

$$S^*(HA + A^*H)S = SH(AS) + (AS)^* HS = SH(AS) + (AS)^* SH;$$

hence $A \in \mathscr{A}_H$ is equivalent to $(AS) \in \mathscr{A}_{SH}$ as well, so the proof is complete. ■

In particular this lemma shows that the set $S\mathscr{A}_H$ is invariant under the similarity transformation by S. Note that for an arbitrary $H \in \mathscr{H}$ we have that H Sign$(H) \in \mathscr{P}$ (this is obvious if H is already diagonalized). Hence, choosing in the above lemma $S = -\mathrm{Sign}(H)$ and using Lemma 3.5, one can obtain the main result of this section.

PROPOSITION 3.7. *For an arbitrary $H \in \mathscr{H}$ the set $\mathscr{A}_H$ is isomorphic to the stable set $\mathscr{A}_{-H\,\mathrm{Sign}(H)}$. In particular, $\mathscr{A}_H$ is a maximal open nonsingular cic in $\mathbb{C}^{n\times n}$.*

This result in particular implies that if $A \in \mathscr{A}_H$ for a given $H \in \mathscr{H}$, then the matrix $-A\mathrm{Sign}(H)$ is stable.

Define now the sets

$$\mathscr{H}_A := \{H \in \mathscr{H} | (HA + A^*H) \in \mathscr{P}\},$$

$$\mathscr{H}_{\mathbf{X}} := \bigcap_{A \in \mathbf{X}} \mathscr{H}_A.$$

In the case that A is stable, it follows from Theorem 3.1(i) that $\mathscr{H}_{-A} \subseteq \mathscr{P}$. Sets of the form $\mathscr{H}_A$ were previously studied e.g. in [19], where they were referred to as the image of the inverse Lyapunov transformation.

The Lyapunov equation puts into a duality relation sets of type $\mathscr{H}_A$ and sets of type $\mathscr{A}_H$. However, unlike $\mathscr{A}_H$, the nonsingular convex cone $\mathscr{H}_{\mathbf{X}}$ is not necessarily invertible. It obeys the modified inversion rule

$$\mathscr{H}_{\mathbf{X}^*} = (\mathscr{H}_{\mathbf{X}})^{-1}. \tag{3.6}$$

This follows from multiplying Equation (3.1) by H^{-1} on both sides. In a similar way, it is easy to verify that for arbitrary nonsingular A we have that $\mathscr{H}_A = \mathscr{H}_{A^{-1}}$. Note also that under the new notation Lemma 3.4(i) can be generalized to a set $\mathbf{X}$:

$$M^*\mathscr{H}_{\mathbf{X}} M = \mathscr{H}_{M^{-1}\mathbf{X}M}. \tag{3.7}$$

Explicit characterization of the geometry of sets of the form $\mathscr{H}_{\mathbf{X}}$ is known to be difficult even when $\mathbf{X}$ is a singleton (see e.g. [19]). In this case, the following uniqueness result is known.

THEOREM 3.8 [15].

(i) *Let* $A, B \in \mathbb{C}^{n\times n}$ *have regular inertia. Then* $\mathscr{H}_A = \mathscr{H}_B$ *if and only if* $B = (aI + ibA)(cA + idI)^{-1}$ *for some scalars* $a, b, c, d \in \mathbb{R}$ *satisfying* $ac + bd = 1$.

(ii) *Let* $A, B \in \mathbb{R}^{n\times n}$ *have regular inertia. Then* $\mathscr{H}_A \cap \mathbb{R}^{n\times n} = \mathscr{H}_B \cap \mathbb{R}^{n\times n}$ *if and only if* $B = \alpha A^{\pm 1}$ *for some scalar* $\alpha > 0$.

Theorem 3.8(ii) can be reformulated as saying that over the reals $\mathscr{H}_A = \mathscr{H}_B$ is equivalent to $\mathscr{C}(A) = \mathscr{C}(B)$. From Theorem 3.8(i) we know that this is no longer true over the complex field.

The rest of the section is devoted to further exploration of the relation between the existence of a common Lyapunov solution and the nonsingularity of the associated cic.

PROPOSITION 3.9. *Let* $\mathbf{X}, \mathbf{Y} \subset \mathbb{C}^{n\times n}$ *be sets of matrices. Then:*

(i) $\mathscr{H}_{\mathscr{C}(\mathbf{X})} = \mathscr{H}_{\mathbf{X}}$.
(ii) $\mathscr{C}(\mathbf{X}) \subseteq \mathbf{C}(\mathbf{Y})$ *implies* $\mathscr{H}_{\mathbf{Y}} \subseteq \mathscr{H}_{\mathbf{X}}$.

Proof.

(i): By definition $H \in \mathscr{H}_{\mathbf{X}}$ for some $H \in \mathscr{H}$ is equivalent to $\mathbf{X} \subseteq \mathscr{A}_H$. Now the claim follows from the maximality of the set $\mathscr{A}_H$; see Proposition 3.7.

(ii): If $H \in \mathscr{H}_{\mathbf{Y}}$ then due to (i) we have $H \in \mathscr{H}_{\mathscr{C}(\mathbf{Y})}$, and since $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{Y})$, necessarily $H \in \mathscr{H}_{\mathscr{C}(\mathbf{X})}$. Using (i) again, it follows that $H \in \mathscr{H}_{\mathbf{X}}$, so the proof is complete. ■

Proposition 3.9(i) implies that $\mathbf{X}$ and $\mathbf{Y}$ are two sets of generators to the same nonsingular cic then $\mathscr{H}_{\mathbf{X}} = \mathscr{H}_{\mathbf{Y}}$. The converse of 3.9(ii) does not hold in general. As was already remarked, over the complex field, one can use Theorem 3.8(i) to easily produce a counterexample even of a singly generated cic. Over the reals, we have the following example.

EXAMPLE 3.10. Let $\mathbf{X}, \mathbf{Y}$ be the sets of all matrices orthogonally similar to A and B, respectively, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 49 \end{pmatrix}, \qquad B = \begin{pmatrix} 7 & 24 \\ -24 & 7 \end{pmatrix}.$$

We are to show now that $\mathscr{C}(\mathbf{X})$ neither contains nor is contained in $\mathscr{C}(\mathbf{Y})$, although $\mathscr{H}_{\mathbf{X}} \cap \mathbb{R}^{2\times 2} = \mathscr{H}_{\mathbf{Y}} \cap \mathbb{R}^{2\times 2}$.

Clearly, the last equality is between two sets of matrices within $\mathscr{P} \cap \mathbb{R}^{2\times 2}$. Let us now introduce a parametrization of the set of all 2×2 real symmetric positive definite matrices.

Every 2×2 symmetric positive definite matrix with trace 2 can be parametrized as

$$\begin{pmatrix} 1+a & b \\ b & 1-a \end{pmatrix}, \qquad a, b \in \mathbb{R}, \quad a^2 + b^2 < 1.$$

Namely, the set of trace 2 matrices within $\mathscr{P} \cap \mathbb{R}^{2\times 2}$ is homeomorphic to the open unit disc in the $\{a, b\}$ plane. So up to a positive scaling, it covers the set of all 2×2 real positive definite matrices.

For an arbitrary set $\mathbf{Z}$ we shall denote by $\hat{\mathscr{H}}_{\mathbf{Z}}$, where $\hat{\mathscr{H}}_{\mathbf{Z}} = \hat{\mathscr{H}}_{\mathbf{Z}}(a, b)$, the restriction of the set $\mathscr{H}_{\mathbf{Z}} \cap \mathbb{R}^{n\times n}$ to those matrices with trace 2. Hence, if A is a real antistable matrix, then under the above parametrization, $\hat{\mathscr{H}}_A$ is an ellipse within this disc.

From (3.7) it now follows that $\mathscr{H}_{\mathbf{X}} \cap \mathbb{R}^{n\times n}$ is the intersection of the sets $U^*(\mathscr{H}_A \cap \mathbb{R}^{n\times n})U$ over all orthogonal matrices U. Using the above parametrization, this in turn amounts to rotating the set $\hat{\mathscr{H}}_A$ about the origin in the $\{a, b\}$ plane and then intersecting all the images. Namely, if

$$P = \begin{pmatrix} 1+a & b \\ b & 1-a \end{pmatrix}$$

is within $\hat{\mathscr{H}}_A$ and we use the usual parametrization for orthogonal matrices

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \qquad 0 \leqslant \theta < 2\pi,$$

then $\hat{\mathscr{H}}_{\mathbf{X}}$ contains the intersection over all θ, $0 \leqslant \theta < 2\pi$, of all matrices of the form

$$I + \sqrt{a^2 + b^2}\begin{pmatrix} \sin(2\theta + \psi) & \cos(2\theta + \psi) \\ \cos(2\theta + \psi) & -\sin(2\theta + \psi) \end{pmatrix},$$

where $\cos\psi = b/\sqrt{a^2 + b^2}$ and $\sin\psi = a/\sqrt{a^2 + b^2}$.

Note now that for all $|b| < 7/25$, the matrix

$$\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$$

is within $\hat{\mathscr{H}}_A$. Using the above analysis, it follows that $\hat{\mathscr{H}}_{\mathbf{X}}$ contains all matrices of the form

$$I + b\begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}, \qquad \text{where } 0 \leqslant \theta < \pi \text{ and } |b| < 7/25.$$

Actually, this is exactly the required set $\hat{\mathscr{H}}_{\mathbf{X}}$.

In principle, a similar analysis can be carried out for $\hat{\mathscr{H}}_B$ and $\hat{\mathscr{H}}_{\mathbf{Y}}$, respectively. In fact, it can be simplified by exploiting the fact that $\frac{1}{25}B$ is an orthogonal matrix. First note that $\mathbf{Y} = \{B, B^T\}$. Then, using (3.6), it directly follows that $\mathscr{H}_B = \mathscr{H}_{\mathbf{Y}}$ and in turn $\mathscr{H}_{\mathbf{X}} = \mathscr{H}_{\mathbf{Y}}$. See also Figure 1.

However, we shall show now that $\mathscr{C}(\mathbf{X})$ neither contains nor is contained in $\mathscr{C}(\mathbf{Y})$.

$\mathscr{C}(\mathbf{Y}) \not\subset (\mathscr{C}(\mathbf{X})$: It is easy to verify that $\mathscr{C}(\mathbf{Y}) = \{\alpha B + \beta B^T \mid \alpha, \beta \geqslant 0, \alpha + \beta > 0\}$. This in particular implies that the elements on the main diagonal of every matrix in $\mathscr{C}(\mathbf{Y})$ are identical and hence $A \notin \mathscr{C}(\mathbf{Y})$.

$\mathscr{C}(\mathbf{Y}) \not\subset (\mathscr{C}(\mathbf{X})$: Clearly, $\mathscr{C}(\mathbf{X}) \subset \mathscr{P}$, but since the matrix B has a nonreal spectrum, it cannot be in $\mathscr{C}(\mathbf{X})$.

We conclude this section by essentially reformulating Proposition 3.9(i).

COROLLARY 3.11. *If for a set of matrices* $\mathbf{X}$ *we have that* $\mathscr{H}_{\mathbf{X}} \neq \emptyset$, *then* $\mathscr{C}(\mathbf{X})$ *is nonsingular. In particular, if there exists a negative definite common Lyapunov solution for* $\mathbf{X}$, *then* $\mathscr{C}(\mathbf{X})$ *is stable.*

This corollary provides us with a very simple, but rather strong, necessary condition for the existence of common Lyapunov solutions. This necessary condition is in general not sufficient. This gap will be examined in Theorem 4.2 and Propositions 5.2, 5.3 below.

IV. LYAPUNOV EQUATION AND HERMITIAN CICS

It was shown in Proposition 2.3(iii) that the traditional linear preservers of inertia—similarity, transposition and complex conjugation—also preserve cic structure. As indicated, this agrees well with Proposition 2.6(a), which states that in a nonsingular cic all elements have the same regular inertia.

In this section we examine the special case of nonsingular cics of Hermitian matrices. Here, congruence substitutes for similarity in the classi-

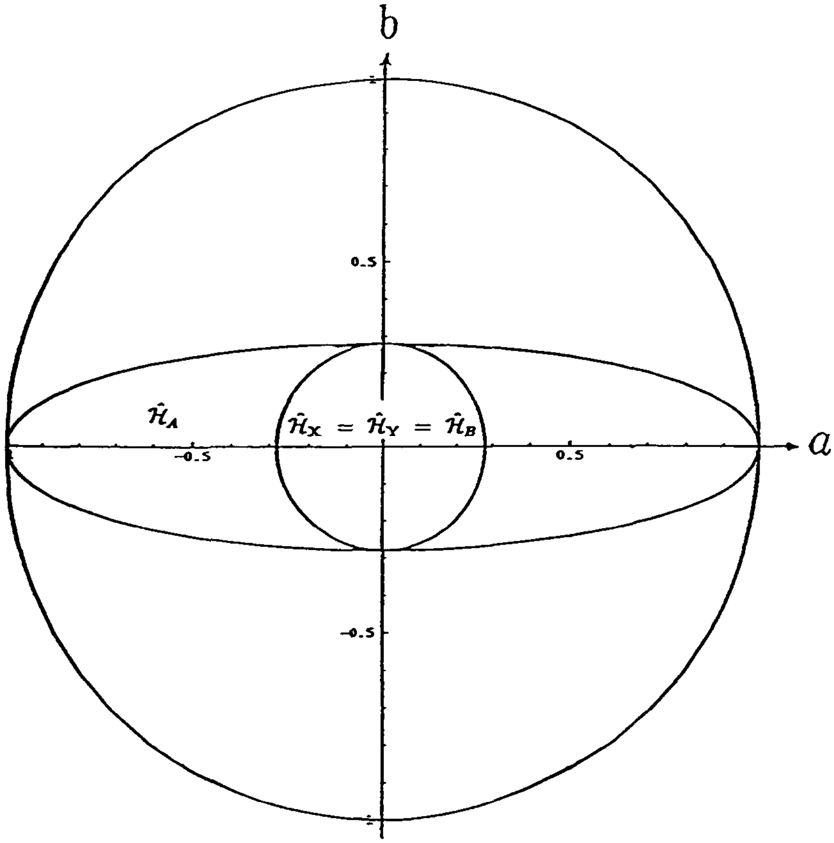

FIG. 1. The set of all Lyapunov solutions to **X** and **Y** in the $\{a, b\}$ parameter plane.

cal inertia preserver results; see [20]. Concerning congruence, we shall need the following definitions based on [17]:

DEFINITION.

(1) A nonsingular $n \times n$ Hermitian matrix is said to be *inertia explicit* if it can be partitioned as

$$\begin{pmatrix} -P_\nu & F \\ F^* & P_{n-\nu} \end{pmatrix}$$

where the matrices P_ν, $P_{n-\nu}$ are respectively $\nu \times \nu$ and $(n-\nu)\times(n-\nu)$ positive definite for some ν where $0 \leqslant \nu \leqslant n$.

(2) For a fixed ν, $0 \leqslant \nu \leqslant n$, we shall denote by $\mathscr{H}(\nu, n)$ the set of all $n \times n$ inertia explicit Hermitian matrices with the above pattern.

(3) A set of nonsingular matrices $\mathbf{X} \subset \mathscr{H}$ is said to be *simultaneously congruently inertia explicit* if there exists a nonsingular matrix $M \in \mathbb{C}^{n\times n}$ such that $(M^*\mathbf{X}M) \subseteq \mathscr{H}(\nu, n)$ for some ν, where $0 \leqslant \nu \leqslant n$.

Note for example that $\mathscr{H}(0, n) = \mathscr{P}$. Using this notation, we can now present a special analog of Proposition 3.7.

PROPOSITION 4.1. *Given an integer ν with $0 \leqslant \nu \leqslant n$, then:*

(i) $\mathscr{H}(\nu, n) = \mathscr{H} \cap \mathscr{A}_{\mathrm{diag}\{-I_\nu, I_{n-\nu}\}}$.

(ii) *Let U be a unitary matrix. Then $U^*\mathscr{H}(\nu, n)U$ is a maximal open nonsingular Hermitian cic.*

(iii) *For an arbitrary matrix A with regular inertia, the set $\mathscr{H}_A$ is simultaneously congruently inertia explicit.*

Proof. (i) trivially follows from Observation 3.3. Now, (ii) is obtained from (i) by first using Lemma 3.4(i) with a unitary M, and then applying Proposition 2.3(i) and 3.7.

(iii): First, for some nonsingular M and an arbitrary $0 \leqslant \nu \leqslant n$ let $\mathrm{Sign}(A) = M\,\mathrm{diag}\{-I_\nu, I_{n-\nu}\}M^{-1}$. Then, from the proof of parts (i), (ii) together with Equation (3.7) we have that $\mathscr{H}_{\mathrm{Sign}(A)} = M^*\mathscr{H}(\nu, n)M$. Now from Proposition 3.9(ii) we have in particular that $\mathscr{H}_A \subset \mathscr{H}_{\mathrm{Sign}(A)}$, so the proof is complete. ∎

From Corollary 3.11, it is obvious that the nonsingularity of $\mathscr{C}(\mathbf{X})$ is a necessary condition for $\mathscr{H}_{\mathbf{X}}$ to be nonempty. Below we provide a special case of interest where it is also sufficient.

THEOREM 4.2. *For arbitrary A, $B \in \mathscr{H}$ the following are equivalent:*

(i) *The matrix AB^{-1} does not have real negative eigenvalues.*

(ii) $\mathrm{conv}(A, B)$ *is nonsingular.*

(iii) $\mathscr{C}(A, B)$ *is nonsingular.*

(iv) $\mathscr{H}_A \cap \mathscr{H}_B \neq \varnothing$.

(v) *The pair $\{A, B\}$ is simultaneously congruently inertia explicit.*

Proof. First, (v) $\Rightarrow$ (iv), since up to congruence $\operatorname{diag}\{-I_\nu, I_{n-\nu}\} \in \mathscr{H}_A \cap \mathscr{H}_B$. The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are easy. The implication (ii) $\Rightarrow$ (v) was proved in [14, Theorem 5], so the equivalence of the last four statements is established. Finally, the equivalence of (i) and (ii) is well known (e.g. [14]) and holds for any pair of nonsingular matrices (not necessary Hermitian), so the proof is complete. ∎

A critical part of the proof is to show that Theorem 4.2(ii) implies (v), and this was done in [17]. In Section 5 of that paper it was demonstrated by an example that this implication does not extend to a triple of real symmetric matrices. As a direct consequence we have the following,

COROLLARY 4.3. *Let $A, H \in \mathscr{H}$ be such that $A \in \mathscr{A}_H$. Then the matrices A and H are simultaneously congruently inertia explicit.*

Proof. Trivially, $H \in \mathscr{H}_A \cap \mathscr{H}_H$. Now apply Theorem 4.2. ∎

Corollary 4.3 goes beyond the inertia theorem (Theorem 3.1), which only guarantees that A and H have the same inertia. The converse of Corollary 4.3 does not hold even in the case of definite inertia. A counterexample is provided by

$$A = \operatorname{diag}\{1, 6\} \quad \text{and} \quad H = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

both positive definite.

V. TRIANGULAR MATRICES

In this section we study cics of upper triangular matrices and examine in this context the gap between the existence of a common Lyapunov solution and the nonsingularity of a cic. Obviously, any set **X** which can be brought to a triangular form by a common similarity can be subjected to the same analysis. For simplicity of exposition we state the results for upper triangular matrices.

We find it convenient to denote by $\hat{S}$ a *diagonal* involution, i.e. a signature matrix. Clearly, there are 2^n diagonal $n \times n$ involutions. First, we define the set of all upper triangular matrices sharing the same Sign along their diagonal, i.e.:

$$\mathscr{T}_{\hat{S}} := \{T \mid T \text{ upper triangular with regular inertia, } \hat{S} = \operatorname{Sign}(\operatorname{diag}(T))\}.$$

The sets $\mathscr{T}_{\hat{S}}$ are all nonsingular cics: see Proposition 5.3(ii) below. However, these sets differ substantially, in particular with respect to their involutions, $\operatorname{Sign}(\mathscr{T}_{\hat{S}})$: the more complex the sign pattern along $\hat{S}$, the richer the structure of the associated set of involutions. In the simplest case, the stable set $\mathscr{T}_{-I}$ contains a single involution $-I$: see Propositions 2.5 and 2.6(a). In the following example, we consider the case of one and two sign changes along the diagonal.

EXAMPLE 5.1. Let ν and n, where $0 \leqslant \nu \leqslant n$, be given. Consider the set $\mathscr{T}_{\operatorname{diag}\{-I_\nu, I_{n-\nu}\}}$. All involutions in this cic have the form

$$\begin{pmatrix} -I_\nu & M \\ 0 & I_{n-\nu} \end{pmatrix},$$

where $M \in \mathbb{C}^{\nu \times (n-\nu)}$ is arbitrary. Hence, the set of involutions in $\mathscr{T}_{\operatorname{diag}\{-I_\nu, I_{n-\nu}\}}$ is affine.

If $\hat{S}$ has more than one sign change along its diagonal, then the set of involutions in $\mathscr{T}_{\hat{S}}$ is not convex. For example, consider the set $\mathscr{T}_{\operatorname{diag}\{I_\pi, -I_\nu, I_{n-\pi-\nu}\}}$ where $\pi, \nu \geqslant 1$ and $n - \pi - \nu \geqslant 1$. All involutions in this cic are of the form

$$S(A, B) := \begin{pmatrix} I_\pi & A & -\frac{1}{2}AB \\ 0 & -I_\nu & B \\ 0 & 0 & I_{n-\pi-\nu} \end{pmatrix},$$

where $A \in \mathbb{C}^{\pi \times \nu}$, $B \in \mathbb{C}^{\nu \times (n-\pi-\nu)}$ are arbitrary. From Lemma 2.3 we deduce that the convex hull of two such involutions, say $S(A_1, B_1)$ and $S(A_2, B_2)$, contains other involutions if and only if $(A_1 - A_2)(B_1 - B_2) = 0$.

In spite of the above, the different sets $\mathscr{T}_{\hat{S}}$ are still isomorphic. The following observation, in the spirit of Lemma 3.6, is straightforward to verify.

LEMMA 5.2. *Let $\hat{S}_1$, $\hat{S}_2$ be two diagonal involutions. Then*

$$\hat{S}_1 \mathscr{T}_{\hat{S}_2} = \mathscr{T}_{\hat{S}_2} \hat{S}_1 = \mathscr{T}_{\hat{S}_1 \hat{S}_2}.$$

In particular, the set $\mathscr{T}_{\hat{S}}$ is isomorphic to $\mathscr{T}_{-I}$, the cic of stable upper triangular matrices.

Indeed, taking $\hat{S}_1 = -\hat{S}_2 = \hat{S}$ establishes the second part of the claim. Note that the above isomorphism need not take an involution to an involution.

We are to compare now two families of cics: $\mathscr{A}_H$, studied in Section III, and $\mathscr{T}_{\hat{S}}$. First, note that the set $\mathscr{T}_{-I}$ overlaps nontrivially with any of the stable cics $\mathscr{A}_H$, where $(-H) \in \mathscr{P}$: by Propositions 2.5 and 2.6 the matrix $-I$ belongs to both. Unlike $\mathscr{A}_H$, the set $\mathscr{T}_{\hat{S}}$ is not open in the usual matrix topology. In particular, these cics are not similar.

PROPOSITION 5.3. *Let the diagonal involution $\hat{S}$ and the nonsingular Hermitian matrix H be arbitrary.*

(i) *The set $\mathscr{T}_{-I}$ is a maximal stable cic open with the set of upper triangular matrices.*

(ii) *The set $\mathscr{T}_{\hat{S}}$ is a maximal nonsingular cic open with the set of upper triangular matrices.*

(iii) *The set $\mathscr{A}_H$ is a maximal open (in the usual matrix topology) nonsingular cic.*

(iv) *The set $\mathscr{A}_H$ neither contains nor is contained in $\mathscr{T}_{\hat{S}}$.*

Proof. Clearly $\mathscr{T}_{-I}$ is a stable cic, so let $A = [a_{jk}]$ be an arbitrary matrix not in $\mathscr{T}_{-I}$, and we are to show that $\mathscr{C}(A, \mathscr{T}_{-I})$ contains an unstable matrix. If A is unstable we are done, so let A be stable and $a_{j_0 k_0} \neq 0$ for some $j_0 > k_0$. We shall construct a matrix $T \in \mathscr{T}_{-I}$ such that $A + T$ is singular. Let $\tilde{T} = [\tilde{t}_{jk}]$ be an upper triangular matrix defined by

$$\operatorname{Re} \tilde{t}_{jj} = -|\operatorname{Re} a_{jj}|, \ \operatorname{Im} \tilde{t}_{jj} = -\operatorname{Im} a_{jj}, \quad 1 \leqslant j \leqslant n, \quad \tilde{t}_{k_0 j_0} = \frac{a^*_{j_0 k_0} d_{j_0 j_0} d_{k_0 k_0}}{|a_{j_0 k_0}|^2},$$

where $d_{jj} := \operatorname{Re} a_{jj} - |\operatorname{Re} a_{jj}| - 1$, and all other entries are zero. Let A_U denote the strict upper triangular part of A. It is easy to see that $T := \tilde{T} -$

$A_U - I$ is in $\mathscr{T}_{-I}$, but direct calculation shows that $\det(A + T) = 0$, so the construction is complete.

(ii): The claim follows from part (i) together with Lemma 5.2

(iii) is a restatement of Proposition 3.7.

(iv): $\mathscr{A}_H \not\subseteq \mathscr{T}_{\hat{S}}$; namely, for a fixed $H \in \mathscr{H}$ one can find $A \in \mathscr{A}_H$ so that $A \notin \mathscr{T}_{\hat{S}}$. Note that the set $\mathscr{A}_H$ is open in the sense that if $A \in \mathscr{A}_H$, then so is an arbitrary perturbation of A, provided it is sufficiently small. Hence, in particular, the set $\mathscr{A}_H$ contains matrices which are not upper triangular.

$\mathscr{T}_{\hat{S}} \not\subseteq \mathscr{A}_H$; namely, for a fixed $H \in \mathscr{H}$ and for each $\hat{S}$ we construct $A \in \mathscr{T}_{\hat{S}}$ so that $A \notin \mathscr{A}_H$. Clearly, if $\hat{S} \neq \text{Sign}(\text{diag}(H))$ then it does not belong to the set $\mathscr{A}_H$, so assume that $\hat{S} = \text{Sign}(\text{diag}(H))$. First, assume that H is diagonal and partitioned as $H = \text{diag}\{h_1, h_2, \tilde{H}\}$, where $\tilde{H}$ is an $(n-2) \times (n-2)$ diagonal matrix. Then the matrix

$$T = \begin{pmatrix} h_1 & ah_2 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & \tilde{H} \end{pmatrix}$$

is within $\mathscr{T}_{\hat{S}}$, but belongs to $\mathscr{A}_H$ only for $|a| < 2$.

Assume now that H is not diagonal, say $H = \text{diag}(H) + \tilde{T} + \tilde{T}^T$, where $\tilde{T}$ is a nonzero strictly upper triangular matrix. For all $\alpha \geqslant 0$ the matrix $T(\alpha) := \text{diag}(H) - \alpha\tilde{T}$ is within the set $\mathscr{T}_{\hat{S}}$. Note now that $\frac{1}{2}\,\text{diag}(HT(\alpha) + T^T(\alpha)H) = [\text{diag}(H)]^2 - \alpha\,\text{diag}(\tilde{T}^T\tilde{T})$. Hence, for α sufficiently large $T(\alpha) \notin \mathscr{A}_H$, so the proof is complete. ■

Note that Proposition 5.3(iv) in particular states that for an arbitrary $\hat{S}$ there is no common Lyapunov solution for the set $\mathscr{T}_{\hat{S}}$. In the rest of this section we show that is "almost" the case.

In the context of upper triangular matrices $\mathscr{T}_{\hat{S}}$, it is natural to consider *diagonal* Lyapunov solutions of the following form:

$$H = \text{diag}\{h_1, \dots, h_n\}, \qquad \text{Sign}(H) = \hat{S}, \quad |h_{j+1}| \geqslant |h_j|, \; 1 \leqslant j \leqslant n-1. \tag{5.1}$$

This definition is justified by the following result:

PROPOSITION 5.4. *A nonsingular upper triangular matrix, and more generally a finitely generated subcic within* $\mathscr{T}_{\hat{S}}$, *has a common Lyapunov solution of the form* (5.1).

Proof. First we consider a single matrix $A \in \mathcal{T}_{\hat{S}}$, and a candidate for Lyapunov solution of the form (5.1). We shall prove the claim by induction on n, the dimension of $\hat{S}$. If $n = 1$ we may simply choose $H = \hat{S}$. Assume now that $n > 1$. We may conformably partition A, H, Q as

$$A = \begin{pmatrix} T_a & x \\ 0 & t_b \end{pmatrix}, \qquad H = \begin{pmatrix} H_a & f \\ f^* & b_h \end{pmatrix},$$

$$Q = \begin{pmatrix} Q_a & y \\ y^* & q_b \end{pmatrix}, \qquad \hat{S} = \begin{pmatrix} \hat{S}_a & 0 \\ 0 & \hat{s}_b \end{pmatrix},$$

so that t_b, h_b, q_b, and $\hat{s}_b$ are scalars. The Lyapunov equation $Q = HA + A^*H$ now splits as

$$Q_a = H_aT_a + T_a^*H_a, \qquad y = H_ax + T_a^*f + t_bf,$$

$$q_b = 2\,\mathrm{Re}(f^*x + h_bt_b).$$

By the induction hypothesis, we may choose H_a diagonal and modulus nondecreasing, so that $Q_a \in \mathcal{P}$. Since $f = 0$, y is determined. Next choose $h_b = s_b\alpha$ for some α positive and large. Then $q_b = |t_b|\alpha$ can be made positive and arbitrarily large, making the entire matrix Q positive definite. Also, if α is sufficiently large, H will be modulus nondecreasing.

The same technique enables us to find a common Lyapunov solution for a finite set $\mathbf{X} \subset \mathcal{T}_{\hat{S}}$. Simply choose at each step α big enough to satisfy the positive definiteness requirement uniformly in $\mathbf{X}$. By Proposition 3.9(i), the same H will in fact be valid for all of $\mathscr{C}(\mathbf{X})$. ■

This proposition implies that if $\mathbf{X}$ is a finite set of matrices and there exists a nonsingular matrix M such that $\{M\mathbf{X}M^{-1}\} \subset \mathcal{T}_{\hat{S}}$, then $\mathcal{H}_{\mathbf{X}} \neq \varnothing$. In this case, the common Lyapunov solution is not necessarily diagonal; see (3.7). For a general treatment of diagonal Lyapunov solutions see [2]. Proposition 5.4 is applied to the study of exponential stability of triangular differential inclusions.

Proposition 5.3(iii) states that the set $\mathcal{T}_{\hat{S}}$ has no common Lyapunov solution. On the other hand, Proposition 5.4 guarantees that any finitely generated subcic of $\mathcal{T}_{\hat{S}}$ does have a common Lyapunov solution (in fact a diagonal one). The gap between these two results is quite narrow, since $\mathcal{T}_{\hat{S}}$ itself has countable generating sets, as illustrated below. For simplicity of exposition we present the stable case only.

OBSERVATION 5.5. *The cic of stable matrices $\mathscr{T}_{-I}$ can be generated by the set*

$$\left.\begin{aligned} A(j,l,m) &:= -I + (1-l+im)E_{jj} \\ B(j,k,l,m) &:= -I + (l+im)E_{jk} \end{aligned}\right\},$$

$$1 \leqslant j < k \leqslant n,\ l,m = 1,2,\dots .$$

The verification of this fact is straightforward and thus is left for the reader; it is based on the facts that

$$A^{-1}(j,l,m) = A\left(j, \frac{l}{l^2+m^2}, \frac{-m}{l^2+m^2}\right) \quad \text{and}$$

$$B^{-1}(j,k,l,m) = B(j,k,-l,-m).$$

The construction for general inertia is similar, but slightly more involved.

Our analysis leaves open the problem of characterization of countable sequences in $\mathscr{T}_{\hat{S}}$ which admit a common Lyapunov solution. The following result is a practical sufficient condition for the existence of a common Lyapunov solution.

COROLLARY 5.6. *A set* $\mathbf{X}$ *of* $n \times n$ *matrices within* $\mathscr{T}_{\hat{S}}$ *has a common Lyapunov solution if the value*

$$\tilde{t} := \sup_{A \in \mathbf{X}} \frac{\max_{1 \leqslant j < k \leqslant n} |a_{jk}|}{\min_{1 \leqslant j \leqslant n} |\operatorname{Re} a_{jj}|}$$

is finite.

Indeed, if $\tilde{t}$ is bounded, one can easily find in Observation 5.5 an integer m_0 large enough so that the stable sets $\{\hat{S}\mathbf{X}\}$ and $\{\mathbf{X}\hat{S}\}$ are contained within the finitely generated cic $\mathscr{C}(\mathbf{Y})$, where

$$\mathbf{Y} := \left\{A(k,l,m),\ B(j,k,l,m) \mid 1 \leqslant k < j \leqslant n,\ l,m \leqslant m_0\right\}.$$

Now use Proposition 5.4. Finally, by Lemma 5.2 the claim is established to any regular inertia.

We conclude by remarking that some of the above derivations can be extended to *block* upper triangular matrices; see Proposition 2.2(ii), [3], and [4] for relevant material. In particular, Proposition 5.4 is then extended so that the matrix H has a block diagonal structure of compatible dimensions.

VI. CONCLUDING REMARKS

In this paper the structure of a general matrix cic has been studied, and in particular in conjuction with the Lyapunov equation. To simplify the exposition, we associated the cic structure only with a strict form of the Lyapunov equation where the right hand side matrix Q was assumed to be positive definite. Alternatively, for a give $H \in \mathscr{H}$ one could define sets of the form $\mathscr{A}_H$ under a well-known extended version of the Lyapunov equation (see e.g. [Theorem 2]):

$$\{A \mid HA + A^*H := Q,\ Q \in \bar{\mathscr{P}},\ (A, Q) \text{ controllable}\}.$$

(Similarly, the set $\mathscr{H}_X$ can be generalized as well.) Under this generalization one obtains a nonsingular cic, which strictly contains the open cic $\mathscr{A}_H$ and in turn is strictly contained within $\bar{\mathscr{A}}_H$, the closure of $\mathscr{A}_H$. Most of the previous results still apply to this new set.

Many of the properties of cics are yet to be explored. Moreover, applications of this structure to other fields such as systems and control theory (see [2, 3]) were merely touched upon.

REFERENCES

1 L. A. Baltzer, Accelerated convergence of the matrix sign function method of solving Lyapunov, Riccati and other matrix equations, *Internat. J. Control* 32(6):1057–1078 (1980).

2 G. P. Barker, A. Berman, and R. J. Plemmons, Positive diagonal solutions to the Lyapunov equations, *Linear and Multilinear Algebra* 5:249–256 (1978).

3 S. Boyd and Q. Yang, Structured and simultaneous Lyapunov functions for system stability problems, *Internat. J. Control* 49(6):2215–2240 (1989).

4 D. Carlson, D. Hershkowitz, and D. Shasha, Block diagonal semistability factors and Lyapunov semistability of block triangular matrices, *Linear Algebra Appl.* 172:1–25 (1992).

5 C.-T. Chen, A generalization of the inertia theorem, *SIAM J. Appl. Math.* 25(2):158–161 (Sept. 1973).

6 N. Cohen and I. Lewkowicz, A necessary and sufficient criterion for the stability of a convex set of matrices, *IEEE Trans. Automat. Control* AC-38(4):611–615 (Apr. 1993).
7 N. Cohen and I. Lewkowicz, Convex invertible cones of matrices in systems and control theory, submitted for publication.
8 N. Cohen and I. Lewkowicz, Convex Invertible Cones of State Space Systems, submitted for publication.
9 N. Cohen and I. Lewkowicz, Convex Invertible Cones, Positive Real and the Sign Algorithm, submitted for publication.
10 N. Cohen, I. Lewkowicz and L. Rodman, Exponential Stability of Triangular Differential Inclusions, submitted for publication.
11 E. D. Denman and A. N. Beavers, Jr., The matrix sign function and computations in systems, *Appl. Math. Comput.* 2:63–94 (1976).
12 C. A. Eschenbach, F. J. Hall, and C. R. Johnson, Self-Inverse Sign Pattern, Inst. for Math. Appl., Preprint Ser., No. 1005, Aug. 1992.
13 K. Fan, On real matrices with positive definite symmetric component, *Linear and Multilinear Algebra* 1:1–4 (1973).
14 M. Fu and B. R. Barmish, Stability of convex and linear combinations of polynomials and matrices arising in robustness problems, in *Proceedings of the Conference on Information Sciences and Systems*, Johns Hopkins University, Baltimore, MD, pp. 16–21, 1987.
15 R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.
16 R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge U.P., 1991.
17 C. R. Johnson and L. Rodman, Convex sets of Hermitian matrices and constant inertia, *SIAM J. Algebraic Discrete Methods* 6:351–359 (July 1985).
18 P. Lancaster and L. Rodman, *Algebraic Riccati Equations*, Oxford University Press, 1995.
19 R. Loewy, On ranges of real Lyapunov transformation, *Linear Algebra Appl.* 13:79–89 (1976).
20 R. Loewy, Inertia preservers, *Linear and Multilinear Algebra* 33:23–30 (1992).
21 R. Mathias, Matrices with positive definite Hermitian part: Inequalities and linear systems, *SIAM J. Matrix Anal.* 13(2):640–654 (Apr. 1992).
22 O. Taussky, Matrices C with $C^n \to 0$, *J. Algebra* 1:5–10 (1964).
23 H. K. Wimmer, Inertia Theorems for Matrices, Controllability and Linear Vibrations, *Lin. Alg. and its Appl.*, Vol. 8, pp. 337–343, 1974.

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