JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **224,** 255–272 (1998)
ARTICLE NO. AY986002

Geometric Stability Conditions for Higher Order Difference Equations

Hassan Sedaghat*

Department of Mathematical Sciences, Virginia Commonwealth University, Richmond,

Received September 23, 1997

1. INTRODUCTION

The equation,

$$x_n = f(x_{n-1}, \ldots, x_{n-m}), \qquad n = 1, 2, 3, \ldots \tag{1}$$

is a higher order (or delay) autonomous difference equation if the order m is a fixed positive integer larger than 1. The real-valued function f is assumed continuous on $\mathbf{R}^m$, and $x_{1-m}, \ldots, x_0 \in \mathbf{R}$ are the initial values. In addition to the fact that discretizations of scalar differential equations result in equations of type (1), direct applications of difference equations of order greater than 1 date back to more than half a century ago and nowadays these equations are increasingly encountered in both social and natural sciences; see, [4] and [6] for some applications and additional references.

A solution $\bar{x}$ of $f(x, \ldots, x) = x$ is a fixed point (or equilibrium) of Eq. (1). Like its differential analogs, (1) can be (and often is) translated into a first-order vector equation; the instability or the asymptotic stability of $\bar{x}$ may then be determined by linearizing the associated vector function,

$$V_f(u_1, \ldots, u_m) \doteq (f(u_1, \ldots, u_m), u_1, \ldots, u_{m-1}),$$

at its fixed point $\bar{X} \doteq (\bar{x}, \ldots, \bar{x})$ and determining the maximum possible modulus for eigenvalues—or the spectral radius—of the derivative of V_f.

* E-mail address: hsedaghat@ruby.vcu.edu.

0022-247X/98 $25.00

This is adequate for many purposes; on the other hand, in many applied problems (e.g., when unusual dynamical behavior occurs at certain parameter values in a model, or information is needed on the extent of the basin of attraction of a sink) as well as in theoretical questions on the nonlinear aspects, one or more of the following may occur:

(a) f (hence also V_f) is not C^1-smooth at $\overline{X}$;

(b) f is C^1-smooth at $\overline{X}$ but the derivative of V_f has eigenvalues of modulus one;

(c) Some information is needed on how close to $\overline{X}$ a vector of initial conditions must be selected in order to ensure convergence to $\bar{x}$ (e.g., it is desired to show that $\bar{x}$ is *globally* asymptotically stable).

To discuss the stability of $\bar{x}$ when any one of these conditions applies, we need a deeper understanding of the properties of f than is required for linearization (when the latter is applicable). In this article, we take a new step in this direction by considering functions f that satisfy certain norm inequalities on $\mathbf{R}^m$. The origin of the inequalities to be considered in the following text, is found in the case $m = 1$ and the inequality,

$$|f(x) - \bar{x}| < |x - \bar{x}|. \tag{2}$$

If (2) holds for all x in some deleted neighborhood U of $\bar{x}$ then $\bar{x}$ is easily seen to be asymptotically stable, although if the reverse of (2) holds on U then $\bar{x}$ is strongly unstable (or repelling). Note that (2) generalizes linearization on the line because it implies the inequality $|f'(\bar{x})| \leq 1$ when f is C^1-smooth at $\bar{x}$. On the other hand, inequality (2) is not necessary for asymptotic stability of $\bar{x}$ for $m = 1$; indeed, several conditions that are both necessary and sufficient for asymptotic stability on the line are now known—see [7] and [8]. The salient feature of (2) for our purposes here is that it admits direct generalization to higher dimensions if the absolute value on the right is replaced by a norm on $\mathbf{R}^m$ to yield

$$|f(X) - \bar{x}| < \|X - \overline{X}\|, \qquad X \in \mathbf{R}^m. \tag{3}$$

We present results that explore the consequences of (3) and related inequalities. Among the issues discussed here are the following:

(I) Demonstrate asymptotic stability relative to the invariant sets of V_f whose elements satisfy (3) relative to the sup-norm—see Theorem 1;

(II) For C^1-smooth maps condition (I) applies in some cases where a unit eigenvalue exists for the derivative $DV_f(\overline{X})$;

(III) The invariant sets in (I) cannot be contained in certain regions of $\mathbf{R}^m$ having $\overline{X}$ on their boundary (this establishes a connection to the Liapunov theory)—see Theorem 2 and related discussions;

(IV) Inequality (3) has no useful vector analogs for V_f in any l^p-norm. Further, (3) itself in l^p-norms with $p < \infty$ does not imply asymptotic stability;

(V) Demonstrate exponential stability relative to the invariant sets of V_f whose elements satisfy a stronger version of (3) with a positive coefficient less than unity on the right side—see Theorem 3;

(VI) For C^1-smooth maps and small neighborhoods of $\bar{X}$, establish the equivalence of the condition in (V) to a well-known linearization condition—see Theorem 4;

(VII) Demonstrate instability with the aid of a general class of sets (that include invariant sets of V_f) whose elements satisfy the reverse of (3) relative to the sup-norm—see Theorem 5.

We address these and several other matters in the sequel. A few examples illustrate the applicability of our results to specific problems for which some of (a)–(c) hold.

2. PRELIMINARIES

In the sequel, we assume that f is continuous and that $m \geq 2$ unless otherwise noted.

General notation. In addition to V_f defined in the Introduction, the following notation is used in the sequel:

$\|X\| = \max\{|\xi_1|, \dots, |\xi_m|\}$ is the sup-norm of $X = (\xi_1, \dots, \xi_m) \in \mathbf{R}^m$;

$B(\bar{X}; r) = \{X \in \mathbf{R}^m: \|X - \bar{X}\| < r\} = (\bar{x} - r, \bar{x} + r)^m$ is the open ball in the sup-norm, centered at the fixed point $\bar{X}$ and of radius $r > 0$;

p_i: $\mathbf{R}^m \to \mathbf{R}$ denotes the projection into the ith coordinate, i.e., $p_i(u_1, \dots, u_m) = u_i$, $i = 1, \dots, m$;

$f_i = \partial f / \partial u_i$ is the ith partial derivative of f: $\mathbf{R}^m \to \mathbf{R}$;

$\|\nabla f(X)\|_1 = \sum_{i=1}^{m} |f_i(X)|$ for all $X \in \mathbf{R}^m$;

∂S and S^- denote the boundary and the closure of a set S, respectively;

$f \circ g$ is the composition of two functions f and g;

F^n is the composition of a function F: $\mathbf{R}^m \to \mathbf{R}^m$ with itself n times, $n = 1, 2, 3, \dots$ with F^0 defined to be the identity map.

Some basic definitions. 1. A nonempty subset S of $\mathbf{R}^m$ is positively invariant under a map F: $\mathbf{R}^m \to \mathbf{R}^m$ if $F(S) \subset S$. In the sequel, by an *invariant set* we mean a set that is positively invariant under $F = V_f$.

2. A fixed point $\overline{X} \in \mathbf{R}^m$ of a continuous map F of $\mathbf{R}^m$ is *asymptotically stable relative to a set* $M \subset \mathbf{R}^m$ if $\overline{X} \in M$ is Liapunov stable in the relative topology on M and every point of M is attracted to $\overline{X}$; i.e., $F^n(X) \to \overline{X}$ as $n \to \infty$ for all $X \in M$. If $M = \mathbf{R}^m$, then $\overline{X}$ is *globally asymptotically stable*.

3. If $M = B(\overline{X}; r)$ then we may say that a fixed point $\bar{x}$ of (1) is asymptotically stable relative to the interval $(\bar{x} - r, \bar{x} + r)$; in this case, the initial values are (unordered) sets of m real numbers in $(\bar{x} - r, \bar{x} + r)$.

Other basic definitions and concepts such as asymptotic stability, Liapunov stability, linearization, etc. that are used here are well known and are readily found in the literature; see, e.g., [4, Chap. 1] and [5, Chap. 1]. The next two results complete our preliminary discussion.

LEMMA 1. *For any function f: $\mathbf{R}^m \to \mathbf{R}$, and for all integers $k > m$,*

$$p_i \circ V_f^k = f \circ V_f^{k-i}, \qquad i = 1, \ldots, m, \tag{4}$$

although for every $1 \leq k \leq m$,

$$p_i \circ V_f^k(u_1, \ldots, u_m) = \begin{cases} f \circ V_f^{k-i}(u_1, \ldots, u_m), & i = 1, \ldots, k-1, \\ f(u_1, \ldots, u_m), & i = k, \\ u_{i-k}, & i = k+1, \ldots, m. \end{cases} \tag{5}$$

Proof. For each point $X = (u_1, \ldots, u_m) \in \mathbf{R}^m$ note that

$$V_f^2(X) = V_f(f(X), u_1, \ldots, u_{m-1}) = \big(f(V_f(X)), f(X), u_1, \ldots, u_{m-2}\big).$$

Proceeding in this way, it is clear that we obtain inductively,

$$V_f^m(X) = \big(f\big(V_f^{m-1}(X)\big), f\big(V_f^{m-2}(X)\big), \ldots, f(X)\big). \tag{6}$$

The previous process in particular establishes (5). However, as for (4), note that (6) implies

$$V_f^{m+1}(X) = \big(f\big(V_f^m(X)\big), f\big(V_f^{m-1}(X)\big), \ldots, f\big(V_f(X)\big)\big),$$

so that (4) holds for $m + 1$. Suppose now that (4) for some $k > m$. Then

$$\begin{aligned} V_f^{k+1}(X) &= \big(f\big(V_f^k(X)\big), p_1 \circ V_f^k(X), \ldots, p_{m-1} \circ V_f^k(X)\big) \\ &= \big(f \circ V_f^k(X), f \circ V_f^{k-1}(X), f \circ V_f^{k-2}(X), \ldots, f \circ V_f^{k-m+1}(X)\big). \end{aligned}$$

Therefore, (4) holds for all $k > m$ by induction. ∎

LEMMA 2. *If the sequence* $\{x_n\}$ *is a solution of* (1) *generated by a vector of initial values* $X_0 = (x_0, \ldots, x_{1-m})$, *then*

$$x_n = f\left(V_f^{n-1}(X_0)\right), \qquad n = 1, 2, 3, \ldots .$$

3. ASYMPTOTIC STABILITY

THEOREM 1. *Let* $\bar{x} \in \mathbf{R}$ *be a fixed point of* (1) *and let* T *be a closed, invariant set containing* $\overline{X}$. *Define the set*,

$$A \doteq A(f; \bar{x}) \doteq \left\{X \in \mathbf{R}^m : |f(X) - \bar{x}| < \|X - \overline{X}\|\right\} \cup \{\overline{X}\}.$$

Then $\overline{X}$ *is asymptotically stable relative to each invariant subset* S *of* $T \cap A$ *that is closed in* T; *in particular,* $\bar{x}$ *attracts every trajectory with a vector of initial values* $(x_0, \ldots, x_{1-m}) \in S$.

Proof. Let S be a T-closed, invariant subset of $T \cap A$. Then

$$\left|f\left(V_f^k(X)\right) - \bar{x}\right| < \left\|V_f^k(X) - \overline{X}\right\|, \tag{7}$$

for all positive integers k and for every $X \in S$. Next, for $1 \leq k \leq m - 1$, observe that by (5),

$$V_f^k(u_1, \ldots, u_m)$$
$$= \left(f\left(V_f^{k-1}(u_1, \ldots, u_m)\right), \ldots, f(u_1, \ldots, u_m), u_1, \ldots, u_{m-k}\right), \tag{8}$$

and further,

$$\left\|V_f(X) - \overline{X}\right\| \leq \|X - \overline{X}\|, \tag{9}$$

for all $X \in A$. Now (9), (8) and induction on k imply that

$$\left\|V_f^k(X) - \overline{X}\right\| \leq \|X - \overline{X}\|, \tag{10}$$

for $k = 1, \ldots, m - 1$. Therefore, from (7), (10), and (6) we may conclude that

$$\left\|V_f^m(X) - \overline{X}\right\| = \max\left\{\left|f\left(V_f^{m-1}(X)\right) - \bar{x}\right|, \ldots, |f(X) - \bar{x}|\right\}$$
$$< \|X - \overline{X}\|, \tag{11}$$

for all $X \in S$. Next, let $X_0 = (x_0, \ldots, x_{1-m})$ be any vector of initial values for (1) in S. Then (11) implies that

$$\left\|V_f^{mn}(X_0) - \overline{X}\right\| = \left\|V_f^m\left(V_f^{m(n-1)}(X_0)\right) - \overline{X}\right\| < \left\|V_f^{m(n-1)}(X_0) - \overline{X}\right\|,$$

for every positive integer n. Therefore, the sequence,

$$\left\{\left\| V_f^{mn}(X_0) - \overline{X} \right\|\right\}, \qquad n = 1, 2, 3, \ldots$$

is strictly decreasing to a limit $r_0 \geq 0$ as $n \to \infty$. If $r_0 > 0$ and Ω_0 is the (forward) limit set of the vector sequence $\{V_f^{mn}(X_0)\}$, then

$$\Omega_0 \subset \partial B\left(\overline{X}; r_0\right) \cap S \subset T \cap A,$$

where the first inclusion holds because S is closed in T. Therefore, by the invariance of Ω_0 under V_f^m, for any $Y \in \Omega_0$, (11) implies

$$r_0 = \left\| V_f^m(Y) - \overline{X} \right\| < \|Y - \overline{X}\| = r_0,$$

which is impossible. Hence, for every $X_0 \in S$, $V_f^{mn}(X_0) \to \overline{X}$ as $n \to \infty$. By Lemmas 1 and 2,

$$V_f^{mn}(X_0) = \left(x_{mn-1}, \ldots, x_{m(n-1)}\right),$$

so it may be concluded that for all $i = 1, \ldots, m$,

$$|x_{mn-i} - \bar{x}| \leq \left\| V_f^{mn}(X_0) - \overline{X} \right\| \to 0,$$

as $n \to \infty$. It follows that $x_n \to \bar{x}$; thus $\overline{X}$ attracts every point of S, and S is closed, so $\overline{X} \in S$. With (9) implying the stability of $\overline{X}$ in the relative topology on A, the proof is complete. ∎

Theorem 1 is, of course, valid with $T = \mathbf{R}^m$; on the other hand, in many applied models, the positive orthant $T = [0, \infty)^m$ is the relevant invariant set, to which attention may be restricted. It should be emphasized that if $T \neq \mathbf{R}^m$, then it is possible that $\overline{X} \in \partial T$, as in the next example.

EXAMPLE 1. The third-order equation,

$$x_n = ax_{n-1} + bx_{n-3}\exp(-cx_{n-1} - dx_{n-3}), \qquad a, b, c, d \geq 0, c + d > 0 \tag{12}$$

represents a special case of the flour beetle population model; see [3] where it is shown in particular that when $a + b \leq 1$ and $a, b > 0$, then the origin is asymptotically stable relative to the nonnegative octant in $\mathbf{R}^3$ (i.e., all nonnegative solutions are attracted to the origin and the beetles become extinct). We now show that the same conclusion holds under the conditions,

$$a + b \leq 1, \qquad b > 0 \tag{13}$$

using Theorem 1. Before proceeding, it is worth noting that the linearization of (12) at the origin has a unit eigenvalue when $a + b = 1$; this resulted in a separate proof in [3] for $a + b \leq 1$ that uses the detailed properties of (12). This is not necessary; we need only observe that if (13) holds, then for all $(x, y, z) \in [0, \infty)^3$, $(x, y, z) \neq (0, 0, 0)$,

$$\begin{aligned} ax + bz \exp(-cx - dz) &\leq [a + b \exp(-cx - dz)] \max\{x, z\} \\ &< (a + b) \max\{x, y, z\} \\ &\leq \max\{x, y, z\}, \end{aligned}$$

so Theorem 1 implies that the origin is a stable global attractor of nonnegative solutions. If $a + b < 1$, then Theorem 3 in the following text shows that the origin is exponentially stable for all nonnegative solutions; clearly, this global information could not be obtained by linearization alone.

The remarkable laboratory research on the flour beetle population patterns is still continuing, with experiments under way at the time of the writing of this article; see [2].

EXAMPLE 2. However, as a simple application of Theorem 1, with $S = A = \mathbf{R}^m$ it is easy to check that the equation,

$$x_n = a x_{n-k} \exp\left[-c\left(x_{n-1}^2 + \cdots + x_{n-m}^2\right)\right], \qquad 1 \leq k \leq m, |a| \leq 1, c > 0 \tag{14}$$

has a globally asymptotically stable fixed point at the origin. Note that when $k = m$ and $|a| = 1$, then the characteristic polynomial equation for (14) is $\lambda^m - a = 0$. Therefore, *every* eigenvalue of the linearization of (14) lies on the unit circle in this case. For $|a| < 1$ the global nature of asymptotic stablity could not be inferred from linearization alone.

Theorem 1 applies more generally to any continuous, real-valued function g with the property,

$$|g(X)| < 1, \qquad X \neq (0, \ldots, 0),$$

in place of the function $a \exp[-c(u_1^2 + \cdots + u_m^2)]$ in (14).

COROLLARY 1. *Let A be open in Theorem 1. Then $\bar{x}$ is asymptotically stable relative to $(\bar{x} - r, \bar{x} + r)$, where $r > 0$ is the largest real number such that $B(\overline{X}; r) \subset A$. In particular, if $A = \mathbf{R}^m$, then $\bar{x}$ is globally asymptotically stable.*

Proof. Although the ball in the statement of the corollary is not closed, note that for r_0 in the proof of Theorem 1, $\partial B(\overline{X}; r_0) \subset B(\overline{X}; r)$ if $X_0 \in B(\overline{X}; r)$. Hence, $\Omega_0 \subset A$ and the rest of the proof of Theorem 1 is applicable. ∎

Remarks. (1) In spite of its finite-dimensional settings, *Theorem* 1 *is not norm-blind* and may be false if a norm other than the sup-norm (say, the Euclidean norm) is used in the definition of A. Counterexamples involving linear functions are easy to construct.

Another way of looking at this metric-sensitivity of Theorem 1 is to note that if the graph of f in $\mathbf{R}^{m+1}$ lies within the *complement* of the closed polyhedral cone,

$$C_{m+1}(\bar{x}) \doteq \left\{(u_1, \dots, u_{m+1}) : |u_{m+1} - \bar{x}| \geq \max\left\{|u_1 - \bar{x}|, \dots, |u_m - \bar{x}|\right\}\right\},$$

for all $(u_1, \dots, u_m)$ in a deleted neighborhood of $\overline{X}$, then A is open and Corollary 1 applies. However, if the graph of f lies within the complement of the Euclidean cone,

$$|u_{m+1} - \bar{x}| \geq \left[\sum_{i=1}^{m} (u_i - \bar{x})^2\right]^{1/2},$$

in a deleted neighborhood of $\overline{X}$, then Theorem 1 need not apply, because the Euclidean cone is properly contained in the polyhedral one. This preference for the polyhedral cones is to some extent due to the interesting geometric fact that the set $\{C_1, \dots, C_m\}$ of all polyhedral cones in $\mathbf{R}^m$ centered at any point, say, the origin, provides a symmetric covering (with nonoverlapping interiors) of each sup-norm ball centered at that point. This is not true of other norms; for example, Euclidean cones do not have a similar property with respect to the spherical balls of the Euclidean norm when $m > 2$.

(2) It is evident that the set,

$$\left\{X \in \mathbf{R}^m : \left\|V_f(X) - \overline{X}\right\| < \|X - \overline{X}\|\right\} \tag{15}$$

is contained in $A(f; \bar{x})$. Because the set (15) is easily generalizable to arbitrary maps of $\mathbf{R}^m$ (not just V_f), how much would we lose by using the set (15) in Theorem 1? For an answer, consider a linear map $f(u_1, \dots, u_m) = \sum_{i=1}^m a_i u_i$ with $\sum_{i=1}^m |a_i| < 1$; then $A(f; \bar{0}) = \mathbf{R}^m$ where $\bar{0}$ is the origin. Even for such f, the set (15) plus $\{\bar{0}\}$ cannot contain any neighborhoods of the origin, not just in the sup-norm, but also in the l^p-norm for all $0 < p < \infty$ (assuming $a_1 \neq 0$, consider the vectors $(a, 0, \dots, 0)$ where $a \to 0$). In particular, we would not be able to use the restricted version of Theorem 1 based on (15) to prove the global asymptotic stability of the origin for the aforementioned linear map!

(3) The functional $\sigma(X) \doteq \|X - \overline{X}\|$ on $\mathbf{R}^m$ is not a Liapunov function for V_f in the traditional or strict sense, because as inequality (9)

shows, the Liapunov–LaSalle difference $\sigma(V_f(X)) - \sigma(X)$ is not negative definite on A. Nevertheless, the general theory in [5] is applicable to σ, and raises the question as to whether Theorem 1 can alternately be proved using the aforementioned general theory (of course, because of the sup-norm, we need not look for a Liapunov function). Because the latter theory does not make explicit use of the sup-norm, and even with the sup-norm, its conclusions involve the maximal invariant subset of

$$E \doteq E(f;\bar{x}) \doteq \left\{X \in \mathbf{R}^m : \left\| V_f(X) - \bar{X} \right\| = \|X - \bar{X}\|\right\},$$

rather than the invariant singleton $\{\bar{X}\} \subset E$, to prove Theorem 1 it is necessary to show that $\{\bar{X}\}$ is the maximal invariant set of V_f in $E \cap A$. We begin by observing in the next lemma that $E \cap A$ has a simple geometric structure; for $m \geq 2$, define the polyhedral set $P_m \subset \mathbf{R}^m$ as

$$P_m \doteq P_m(\bar{x}) \doteq \left\{(u_1, \dots, u_m) : |u_m - \bar{x}| \leq \max\left\{|u_1 - \bar{x}|, \dots, |u_{m-1} - \bar{x}|\right\}\right\},$$

(so that P_m is just the complement of the interior of the polyhedral cone C_m).

LEMMA 3. $E(f;\bar{x}) \cap A(f;\bar{x}) = P_m(\bar{x}) \cap A(f;\bar{x})$.

Proof. Let f and the fixed point $\bar{x}$ be given so that their explicit mention is not necessary. $X = (u_1, \dots, u_m) \in E \cap A$ implies that

$$\begin{aligned} |f(X) - \bar{x}| &< \left\| V_f(X) - \bar{X} \right\| \\ &= \max\left\{|f(X) - \bar{x}|, |u_1 - \bar{x}| \dots, |u_{m-1} - \bar{x}|\right\}, \end{aligned}$$

so $|u_m - \bar{x}|$ cannot exceed $|u_k - \bar{x}|$ for all $k = 1, \dots, m-1$; therefore, $X \in P_m$. Conversely, if $X \in P_m \cap A$, then

$$|f(X) - \bar{x}| < \max\left\{|u_1 - \bar{x}|, \dots, |u_{m-1} - \bar{x}|\right\},$$

which implies that $X \in E$. ∎

In particular, if $A = \mathbf{R}^m$ then $E = P_m$ so that E can be quite large with a nonempty interior. However, there is also the next result.

THEOREM 2. *Each trajectory $\{V_f^n(X_0)\}$ with a vector of initial values $X_0 \in P_m \cap A$, $X_0 \neq \bar{X}$, must exit P_m in at most $m - 1$ steps. In particular, $\{\bar{X}\}$ is the largest invariant subset of $E \cap A$.*

Proof. Let $X_0 = (x_1^0, \dots, x_m^0) \in P_m \cap A$, $X_0 \neq X$. Then using the definitions of A and P_m we easily conclude that

$$|f(X_0) - \bar{x}| < \max\left\{|x_1^0 - \bar{x}|, \dots, |x_{m-1}^0 - \bar{x}|\right\}. \tag{16}$$

Given that the number of arguments inside the "max" in (16) is now reduced by one, let us assume inductively that for some $k \in \{1, 2, \dots, m - 1\}$ it was shown that

$$\left| f(X_j) - \bar{x} \right| < \max\left\{ |x_1^0 - \bar{x}|, \dots, |x_{m-k}^0 - \bar{x}| \right\}, \qquad X_j \doteq V_f^j(X_0) \in P_m \cap A, \quad (17)$$

for all $j = 0, 1, \dots, k - 1$. If

$$X_k = V_f^k(X_0) \in P_m \cap A, \tag{18}$$

then using Lemma 1 and (17) we obtain

$$|f(X_k) - \bar{x}| < \max\left\{ |x_1^0 - \bar{x}|, \dots, |x_{m-k}^0 - \bar{x}| \right\}. \tag{19}$$

If $|x_{m-k}^0 - \bar{x}| > |x_i^0 - \bar{x}|$ in (19) for all $i = 1, 2, \dots, m - k - 1$, then (17) implies that

$$|x_{m-k}^0 - \bar{x}| > \left| f(X_j) - \bar{x} \right|, \qquad j = 0, 1, \dots, k - 1. \tag{20}$$

However, (18) and Lemma 1 imply that

$$|x_{m-k}^0 - \bar{x}| \leq \max\{ |f(X_{k-1}) - \bar{x}|, \dots, |f(X_0) - \bar{x}| \}, \tag{21}$$

which contradicts (20). Hence, inequality (19) reduces to

$$|f(X_k) - \bar{x}| < \max\left\{ |x_1^0 - \bar{x}|, \dots, |x_{m-k-1}^0 - \bar{x}| \right\},$$

and because of (21), (17) in turn reduces to

$$\left| f(X_j) - \bar{x} \right| < \max\left\{ |x_1^0 - \bar{x}|, \dots, |x_{m-k-1}^0 - \bar{x}| \right\}, \tag{22}$$

as long as $X_j \in P_m \cap A$ for all $j = 0, 1, \dots, k$. It follows by induction that (17) holds for each $k = 1, 2, \dots, m - 1$ so long as X_j remains in $P_m \cap A$ for $j \leq k - 1$. In particular, for $k = m - 1$, (17) gives

$$\left| f(X_j) - \bar{x} \right| < |x_1^0 - \bar{x}|, \qquad j = 0, 1, \dots, m - 2. \tag{23}$$

Now, if $X_{m-1} \in P_m$, then because of (23),

$$|x_1^0 - \bar{x}| \leq \max\{ |f(X_{m-2}) - \bar{x}|, \dots, |f(X_0) - \bar{x}| \} < |x_1^0 - \bar{x}|,$$

which is impossible. Therefore, if $X_k \in P_m \cap A$ for $k = 1, 2, \dots, m - 2$ then $X_{m-1} \notin P_m$. The proof of the last assertion is now immediate from Lemma 3. ∎

4. EXPONENTIAL STABILITY

In this section we obtain a stronger asymptotic stability result by restricting the set A of the previous section. We also discuss the precise relationship of the results to be obtained with linearized asymptotic stability, which is also of exponential type.

DEFINITIONS. A fixed point $\bar{x}$ of (1) is *exponentially stable* (or an exponential sink) if there is $\gamma \in (0,1)$ such that for every solution $\{x_n\}$ of (1) with initial conditions $x_0, \ldots, x_{1-m}$ in some nontrivial interval containing $\bar{x}$ we have for all $n \geq 1$,

$$|x_n - \bar{x}| \leq c\gamma^n,$$

where $c = c(x_0, \ldots, x_{1-m}) > 0$ is independent of n. Exponential stability for $\overline{X}$ is similarly defined with obvious modifications (see, e.g., [1]) and exponential stability relative to a set M is defined in the same way as the analogous asymptotically stable case.

An exponentially stable fixed point is asymptotically stable with the added distinction that convergence near an exponential sink is faster and more easily noticed than near a nonexponential one.

THEOREM 3. *Let $\bar{x} \in \mathbf{R}$ be a fixed point of* (1) *and for fixed $\alpha \in (0,1)$, define the* (*closed*) *set*,

$$A_\alpha \doteq A_\alpha(f; \bar{x}) \doteq \left\{X \in \mathbf{R}^m : |f(X) - \bar{x}| \leq \alpha \|X - \overline{X}\|\right\}.$$

Then $\overline{X}$ is exponentially stable relative to the largest invariant subset of A_α.

Proof. Let S be the largest invariant subset of A_α, and note that S must be closed. A straightforward modification of the proof of Theorem 1 shows that

$$\left\|V_f^{mn}(X_0) - \overline{X}\right\| \leq \alpha \left\|V_f^{m(n-1)}(X_0) - \overline{X}\right\|,$$

for all positive integers n and for every $X_0 \in S$. Therefore, for all $i = 1, \ldots, m$,

$$|x_{mn-i} - \bar{x}| \leq \left\|V_f^{mn}(X_0) - \overline{X}\right\| \leq \alpha^n \|X_0 - \overline{X}\|,$$

i.e., for every $n \geq 1$,

$$|x_n - \bar{x}| \leq \alpha^{n/m} \|X_0 - \overline{X}\|,$$

and the exponential stability of $\bar{x}$ follows immediately. ∎

COROLLARY 2. *Let* $\alpha \in (0,1)$, *and assume that* $\{\overline{X}\}$ *is in the interior of* A_α. *Then* $\bar{x}$ *is exponentially stable relative to* $[\bar{x} - r, \bar{x} + r]$, *where* $r > 0$ *is the largest real number such that* $B(\overline{X}; r) \subset A_\alpha$. *In particular, if* $A_\alpha = \mathbf{R}^m$, *then* $\bar{x}$ *is globally exponentially stable.*

Remarks. Because linearized asymptotic stability is also of exponential type, a relationship between Theorem 3 and the linearization theorem can be expected at the local level when f is continuously differentiable. Some remarks are in order before we discuss this relationship. First, we note that if f has continuous partial derivatives on a deleted neighborhood $\hat{U}$ of $\overline{X}$ then the inequality,

$$|f(X) - \bar{x}| \leq \alpha \|X - \overline{X}\| \tag{24}$$

can hold on $\hat{U}$ (i.e., $\hat{U} \subset A_\alpha$) for some $\alpha \in (0,1)$ even if the partial derivatives are unbounded on $\hat{U}$, as in Example 3. Second, if the partial derivatives of f exist but are not continuous at $\overline{X}$, then (24) may hold on some neighborhood of $\overline{X}$ although in every neighborhood of $\overline{X}$ there is X such that

$$\|\nabla f(X)\|_1 > 1, \tag{25}$$

(see Example 4). On the other hand, when f is continuously differentiable, it is a well-known consequence of Rouchè's theorem that the reverse of the inequality in (25) implies asymptotic stability for the linearization of (1) at $\bar{x}$; see, e.g., [4]. We now show that for a C^1 map f, the reverse of (25) for all X sufficiently near $\overline{X}$ is equivalent to condition (24) holding in a neighborhood of $\overline{X}$.

THEOREM 4. *Let* $\bar{x}$ *be a fixed point of* (1) *and assume that* f *is continuously differentiable at* $\overline{X}$. *Then* $\overline{X}$ *is in the interior of* A_α *for some* $\alpha \in (0,1)$ *if and only if,*

$$\|\nabla f(\overline{X})\|_1 < 1. \tag{26}$$

In particular, Corollary 2 *generalizes condition* (26) *to continuous maps.*

Proof. Assume that (26) holds. Then due to the continuity of the f_i, there is $\delta > 0$ such that

$$\alpha \doteq \sup\left\{\|\nabla f(u_1, \ldots, u_m)\|_1 : |u_j - \bar{x}| \leq \delta,\ j = 1, \ldots, m\right\} < 1.$$

Now for all $X = (u_1, \ldots, u_m)$ in the closed ball $B = B(\overline{X}; \delta)^-$, the mean value theorem for real-valued functions on $\mathbf{R}^m$ implies that

$$f(X) - \bar{x} = \int_0^1 \sum_{i=1}^m f_i\left[tX + (1-t)\overline{X})\right](u_i - \bar{x})\, dt,$$

so that

$$|f(X) - \bar{x}|$$
$$\leq \|X - \bar{X}\| \int_0^1 \sum_{i=1}^{m} \left| f_i\left[(tu_1 + (1-t)\bar{x}, \ldots, tu_m + (1-t)\bar{x})\right]\right| dt,$$

and (24) holds on B; hence, $\bar{X} \in B \subset A_\alpha$. Conversely, suppose that

$$\left\| \nabla f(\bar{X}) \right\|_1 \geq 1.$$

Define the translation $g(X) \doteq f(X + \bar{X}) - \bar{x}$ and note that $g(0, \ldots, 0) = 0$ with

$$\| \nabla g(0, \ldots, 0) \|_1 \geq 1. \tag{27}$$

Define $\mathbf{c} \doteq (c_1, \ldots, c_m)$, where

$$c_i = \begin{cases} 1, & \text{if } g_i(0, \ldots, 0) \geq 0, \\ -1, & \text{if } g_i(0, \ldots, 0) < 0, \end{cases}$$

and observe that for every $t \in \mathbf{R}$, $\max\{|c_1 t|, \ldots, |c_m t|\} = |t|$. Next define φ: $\mathbf{R} \to \mathbf{R}$ as

$$\varphi(t) \doteq g(t\mathbf{c}),$$

so that φ is continuously differentiable at $t = 0$ and

$$\varphi'(0) = \sum_{i=1}^{m} g_i(0, \ldots, 0) c_i = \sum_{i=1}^{m} |g_i(0, \ldots, 0)| = \| \nabla g(0, \ldots, 0) \|_1.$$

Let $\alpha \in (0, 1)$ be fixed. By (27) and the mean value theorem of calculus, there is $\delta_\alpha > 0$ such that for $|t| < \delta_\alpha$ we have

$$|\varphi(t)| = |\varphi'(t_0)| |t| > \alpha |t|,$$

where t_0 is between t and 0. Hence, for $X = (u_1, \ldots, u_m) = t\mathbf{c} + \bar{X}$ and $t \in (-\delta_\alpha, \delta_\alpha)$,

$$|f(X) - \bar{x}| = |g(t\mathbf{c})| > \alpha |t| = \alpha \max\left\{|u_1 - \bar{x}|, \ldots, |u_m - \bar{x}|\right\},$$

no matter what constant $\alpha \in (0, 1)$ is chosen. Hence, (24) cannot hold in any neighborhood of $\bar{X}$; i.e., $\bar{X}$ cannot be in the interior of A_α. ∎

Remark. Note that inequality (26) is a generalization of the one-dimensional inequality $|f'(\bar{x})| < 1$, though it is not as strong for $m \geq 2$ as the conclusion of linearization theorem, which may be abbreviated as $\rho(\bar{X}) < 1$, with ρ denoting the spectral radius.

5. INSTABILITY

When f is continuously differentiable but $DV_f(\overline{X})$ has eigenvalues of unit modulus, then neither linearization nor Theorem 3 apply, as stability (if it exists) may be of nonexponential type. In this case, one may consider applying Theorem 1 (as in Examples 1 and 2), or if instability is suspected, the next result in which invariant sets are convenient but not necessary.

THEOREM 5. *Let $\bar{x}$ be a fixed point of* (1), *and define the* (*open*) *set*,

$$A' \doteq A'(f; \bar{x}) \doteq \left\{X \in \mathbf{R}^m : |f(X) - \bar{x}| > \|X - \overline{X}\|\right\}.$$

Assume that there is an open subset $S \subset A'$ such that:

(a) $\partial S \cap \partial A' \cap B(\overline{X}; \delta)^- = \{\overline{X}\}$ *for some* $\delta > 0$;

(b) $V_f^k(S \cap B(\overline{X}; \delta)) \subset S$ *for some* $k \in \{1, \dots, m\}$;

Then for every solution $\{x_n\}$ of (1) *with $X_0 \in S \cap B(\overline{X}; \delta)$, there is q such that $|x_q - \bar{x}| \geq \delta$. In particular, $\bar{x}$ is unstable.*

Proof. Let $X_0 = (x_0, \dots, x_{1-m})$ be a vector of initial conditions in $S \cap B$, where $B = B(\overline{X}; \delta)$, and note that there are two possible cases:

(I) $V_f^{kn}(X_0) \in S \cap B$ for all positive integers n, or:

(II) there is a least integer $j \geq 1$ such that $V_f^{kj}(X_0) \notin S \cap B$.

To show that case (I) cannot occur, we observe that

$$\left\|V_f^{kn}(X_0) - \overline{X}\right\| \geq \left|f\left(V_f^{k(n-1)}(X_0)\right) - \bar{x}\right| > \left\|V_f^{k(n-1)}(X_0) - \overline{X}\right\|, \quad (28)$$

for all $n \geq 1$, where the first inequality in the foregoing text holds because $k \leq m$ and Lemma 1 applies. If (I) holds, the real sequence $\{\|V_f^{kn}(X_0) - \overline{X}\|\}$, which is strictly increasing by (28), must have limit $r_0 \in (0, \delta]$ as $n \to \infty$. The (forward) limit set Ω_0 of the vector sequence $\{V_f^{kn}(X_0)\}$ is then contained in $A' \cap \partial B(\overline{X}; r_0)$. To see that $\Omega_0 \subset A'$, we note that $\Omega_0 \subset (S \cap B)^-$. If $X \in \Omega_0 \cap \partial(S \cap B)$ but $X \notin \partial S$, then $X \in S \subset A'$; if $X \in \partial S$, then $X \in B^- \cap \partial S$ so $X \notin \partial A'$ and because X is in the closure of A', it follows that $X \in A'$ and the claim about Ω_0 is proved. But now, as in the proof of Theorem 1, the definition of A' and the invariance of Ω_0 under V_f^k imply that for any $Y \in \Omega_0$ we have

$$r_0 = \left\|V_f^k(Y) - \overline{X}\right\| \geq \left|f(Y) - \overline{X}\right| > \|Y - \overline{X}\| = r_0,$$

which is impossible. Therefore, case (I) cannot occur.

In case (II), the definition of j implies that $V_f^{k(j-1)}(X_0) \in S \cap B$ so that by our hypothesis $V_f^{kj}(X_0)$ is in S but not in B. Therefore,

$$|x_{kj+1} - \bar{x}| = \left|f\left(V_f^{kj}(X_0)\right) - \bar{x}\right| > \left\|V_f^{kj}(X_0) - \overline{X}\right\| \geq \delta. \quad (29)$$

Because case (I) cannot occur, for each $X_0 \in S$ there is always a j such that (29) holds, regardless of how small $\|X_0 - \overline{X}\|$ is. It follows that $\bar{x}$ is not stable. ∎

Remark. It is easy to see that

$$A'(f;\bar{x}) = \left\{X \in \mathbf{R}^m : \left\|V_f(X) - \overline{X}\right\| > \|X - \overline{X}\|\right\}. \tag{30}$$

Thus, unlike A, the expansive set A' can be defined in terms of V_f without loss; however, because the right side of (30) does not properly contain A', there is also no significant gain in adopting such a definition in this article.

EXAMPLE 3. Consider the equation,

$$x_n = a x_{n-1}^p x_{n-2}^q, \qquad a, p, q > 0,\ x_{-1}, x_0 \geq 0. \tag{31}$$

Clearly, the origin is always a fixed point of (31) and if $p + q \neq 1$, then there is also a positive fixed point,

$$\bar{x} = a^{1/(1-p-q)}.$$

We only analyze the stability of origin where linearization is not applicable when $p < 1$ or $q < 1$ (by contrast, the sets of parameter values in Examples 1 and 2 where linearization fails are "thin"). In addition, if $f(x,y) = ax^p y^q$ and either of the last two inequalities hold, then the left-hand side of (25) will be unbounded in each neighborhood of the origin, even when the origin can be shown to be exponentially stable.

There are three cases to consider: First, assume that $p + q > 1$, where for every $\delta \in (0, \bar{x})$ and $x, y \in [0, \delta]$ we have

$$f(x,y) \leq a\left(\max\{x,y\}\right)^{p+q} \leq a\delta^{p+1-q} \max\{x,y\},$$

with

$$a\delta^{p+q-1} < a\bar{x}^{p+q-1} = 1.$$

Letting $\delta \to \bar{x}$, Theorem 3 implies that the origin is exponentially stable in (31) relative to $[0, \bar{x})^2$, regardless of the value of a.

Next, assume that $p + q < 1$ and define the bounded open set,

$$S = \{(x,y) : 0 < y < \bar{x},\ y < x < ay^{p+q}\}.$$

Notice that $S \subset A'$ and it is easy to see that S is invariant under V_f (which is more than is required by Theorem 5). Hence, the origin is unstable in this case, again regardless of the value of a. In Fig. 1, S is graphed for $a = 1$, $p = 1/4$, and $q = 1/3$, together with A' and A. The latter two sets are separated in this case by the zero level curve of the function,

$$|f(X) - \bar{x}| - \|X - \overline{X}\| = x^{1/4} y^{1/3} - \max\{x,y\},$$

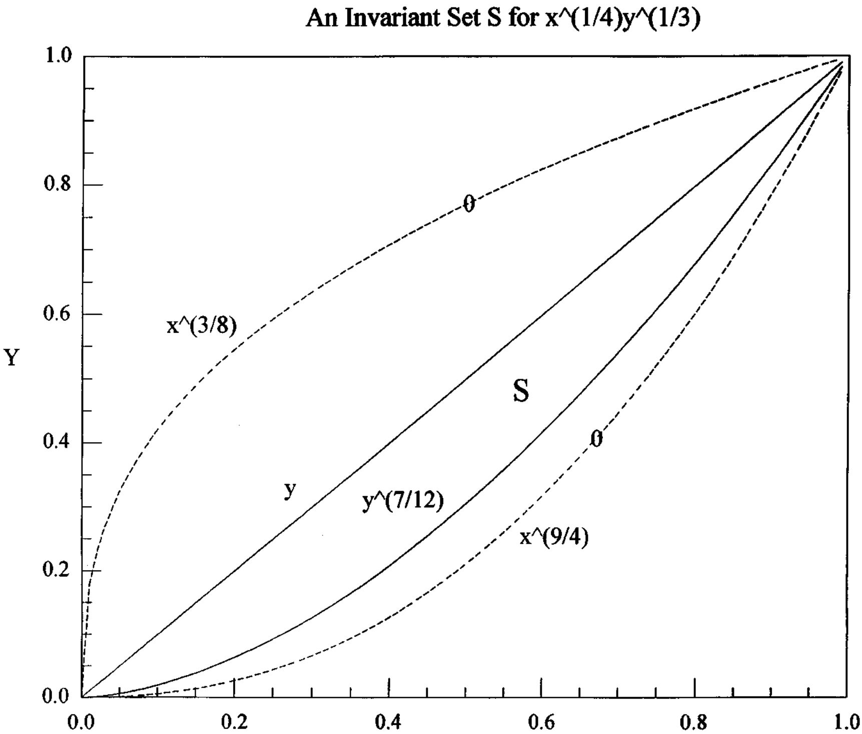

FIG. 1. An invariant set S for $x\hat{}(1/4)y\hat{}(1/3)$.

whose equation (as well as the equation for the lower boundary of S) is easy to compute. Of course, a rather quick way of visually identifying the sets A and A' is by drawing at least one nonzero level curve.

Finally, consider the case $p + q = 1$. This is the only case where the value of a affects stability, and the only case where the origin is the unique fixed point, if to avoid degeneracy, we assume that $a \neq 1$. Then we have two cases: (i) $a < 1$ and (ii) $a > 1$. In case (i), for all $x, y \geq 0$,

$$f(x, y) \leq a(\max\{x, y\})^{p+q} = a \max\{x, y\},$$

so Theorem 3 implies that the origin is globally exponentially stable. In case (ii), the open wedge,

$$S = \{(x, y): y > 0,\ y < x < ay\}$$

is contained in A' and it is invariant under V_f. The instability of origin now follows by Theorem 5.

EXAMPLE 4. For our final example, consider the one-parameter, second-order equation,

$$x_n = \frac{a x_{n-1} x_{n-2}}{\sqrt{x_{n-1}^2 + x_{n-2}^2}}, \qquad a > 0, a \neq \sqrt{2}, \tag{32}$$

where we assume for continuity that the right-hand side is zero at the origin. Then the only fixed point of (32) occurs at the origin, where the function on the right-hand side, namely, $f(x, y) = axy(x^2 + y^2)^{-1/2}$ has zero, but discontinuous partial derivatives. Hence, linearization is not applicable; but with the aid of Theorems 3 and 5 we can show that the origin is globally exponentially stable if $a < \sqrt{2}$, and is unstable if $a > \sqrt{2}$. Note that for $|x| \geq |y|$,

$$|f(x, y)| = a\sqrt{\frac{y^2}{x^2 + y^2}}\,|x| \leq \frac{a}{\sqrt{2}} \max\{|x|, |y|\}. \tag{33}$$

The same is true for $|x| \leq |y|$, as seen by switching x and y in (33). Hence, if $a < \sqrt{2}$, then Theorem 3 establishes the global exponential stability of 0. Next, for $x, y > 0$ and $a > \sqrt{2}$, define

$$S = \left\{(x, y) : y > 0,\ y < x < \sqrt{a^2 - 1}\,y\right\}.$$

The linear wedge S is an invariant subset of $A' \cap \{x > y\}$. Hence, by Theorem 5 the origin is unstable.

With regard to the inequality (25) and Theorems 3 and 4, we mention that the partial derivatives of f are constant on lines $y = cx$ through the origin, and on each such line,

$$\|\nabla f(x, cx)\|_1 = \frac{a(c^3 + 1)}{(c^2 + 1)^{3/2}}, \qquad x > 0.$$

It is easily seen that the supremum of the preceding ratio over all $c > 0$ is a; thus if $1 < a < \sqrt{2}$ then (25) holds in every neighborhood of the globally exponentially stable origin.

REFERENCES

1. R. P. Agarwal, "Difference Equations and Inequalities," Dekker, New York, 1992.
2. B. A. Cipra, Chaotic bugs make the leap from theory to experiment, *SIAM News* **30**, July/August (1997).
3. J. M. Cushing and Y. Kuang, Global stability in a nonlinear difference-delay equation model of flour beetle population growth. *J. Difference Equations Appl.* **2** (1996), 31–37.

4. V. L. Kocic and G. Ladas, ‘‘Global Behavior of Nonlinear Difference Equations of Higher Order with Applications,’’ Kluwer Academic, Boston, MA, 1993.
5. J. P. La Salle, ‘‘The Stability of Dynamical Systems,’’ SIAM, Philadelphia, PA, 1976.
6. H. Sedaghat, A class of nonlinear second order difference equations from macroeconomics, *J. Nonlinear Anal.* **29** (1997), 593–603.
7. H. Sedaghat, Inverse map characterization of asymptotic stability on the line, *Rocky Mountain J. Math.*, in press.
8. A. N. Sharkovsky, Y. L. Maistrenko, and E. Y. Romanenko, ‘‘Difference Equations and Their Applications,’’ Kluwer Academic, Dordrecht, 1993.