



Matrix differential equations and scalar polynomials satisfying higher order recursions [☆]

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ABSTRACT

We show that any scalar differential operator with a family of polynomials as its common eigenfunctions leads canonically to a matrix differential operator with the same property. The construction of the corresponding family of matrix valued polynomials has been studied in [A. Durán, A generalization of Favard's theorem for polynomials satisfying a recurrence relation, *J. Approx. Theory* 74 (1993) 83–109; A. Durán, On orthogonal polynomials with respect to a positive definite matrix of measures, *Canad. J. Math.* 47 (1995) 88–112; A. Durán, W. van Assche, Orthogonal matrix polynomials and higher order recurrence relations, *Linear Algebra Appl.* 219 (1995) 261–280] but the existence of a differential operator having them as common eigenfunctions had not been considered. This correspondence goes only one way and most matrix valued situations do not arise in this fashion. We illustrate this general construction with a few examples. In the case of some families of scalar valued polynomials introduced in [F.A. Grünbaum, L. Haine, Bispectral Darboux transformations: An extension of the Krall polynomials, *Int. Math. Res. Not.* 8 (1997) 359–392] we take a first look at the algebra of all matrix differential operators that share these common eigenfunctions and uncover a number of phenomena that are new to the matrix valued case.

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1. Introduction

The last few years have witnessed some progress in the problem of finding explicit families of matrix valued orthogonal polynomials that are joint eigenfunctions of some fixed differential operator with matrix coefficients, and a (possibly matrix valued) eigenvalue that depends on the degree of the polynomial. Most of these examples involve differential operators of order two, an issue raised in the case of symmetric operators in [7]. The subject of matrix valued orthogonal polynomials, without any reference to differential operators, was initiated in two papers by M.G. Krein [24,25]. The search of situations where these polynomials satisfy extra properties, such as the one above, makes it more likely that they will be useful in applications.

There are by now four main ways to search for cases with this particular extra property: appealing to some group representation structure, as in [16–19,26], solving (in the case of symmetric differential operators) an appropriate set of differential equations as in [8,10,11,13,18] or a set of moment equations as in [7], and finally and so far much less successfully, solving the so-called *ad-conditions* as in [3]. One can see that these *ad-conditions* are necessary and sufficient for the

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existence of a differential operator, but finding all solutions remains a huge task. Most of the examples obtained by this route correspond most likely to an “orthogonality functional” that is not given by a positive definite matrix as envisaged by M.G. Krein. It appears reasonable at this point to consider in some detail any method that gives explicit examples.

One such path is taken in [1] where one finds instances where there is a differential operator of order one, and in [2] where one explores the algebra of all differential operators going along with a fixed family of differential operators.

In this paper we take the initial steps in yet another path. This comes about by putting together two previously unrelated lines of work.

The first one comes from studying the relation between scalar valued polynomials satisfying higher order recursion relations and matrix valued polynomials. This has been considered in [5,6,9], and is reviewed here in Section 2. These papers do not look into the issue of possible differential equations satisfied by these polynomials.

The second one comes from an extension of the well-known Krall–Laguerre and Krall–Jacobi orthogonal polynomials. This has been considered in [12] to produce, via Darboux factorization, families of (scalar) polynomials that (in general) satisfy five term recursion relations and fourth order differential operators with polynomial coefficients. The relevant parts of this construction are recalled in Section 4.

These two tools are combined in this paper as follows: we first establish a very general result that produces one matrix valued differential operator every time that we start with a scalar one (but not necessarily the other way around, as spelled out later). This general result is first illustrated in the case of the Laguerre polynomials where the orders of all operators is equal to two. This completes the material in Section 3. We then illustrate, in Section 4, the case of higher order operators by looking at the extended scalar valued Krall–Laguerre and Krall–Jacobi polynomials.

A slightly more detailed description of the contents of Sections 3 and 4 follows. Given specific families of scalar valued polynomials we consider the corresponding family of matrix valued polynomials as in [5,6,9]. We find that in the Krall–Laguerre case there is (up to scalars) only one differential operator with matrix valued coefficients (and of order four) going along with each one of these examples. In particular, there is no lower order operator with this property. By going up to order six we see that the corresponding algebra cannot be generated by the fourth order operator constructed in the canonical fashion considered in the first part of the paper. The case of the extended Krall–Jacobi polynomials leads to a richer variety of examples. In particular we get a third order differential operator in the matrix case which cannot yield a third order scalar differential operator. All of this is discussed in several subsections of Section 4.

To better compare the results of this paper with previously known material notice that the situations uncovered so far feature an algebra that includes differential operators of low order, typically one or two. There are even situations where one finds a nontrivial (i.e. not a scalar multiple of the identity) differential operator of order zero. One could wonder as to the existence of situations where the lowest order operators in the algebra needs to be of order higher than two. The group theoretical line of attack mentioned above, is not likely to produce examples of this kind, since in this case there is a Laplace–Beltrami operator around. Solving the appropriate differential equations for the higher order case appears to be a difficult task, while solving the corresponding ad-conditions appears entirely hopeless at this point.

As a by-product of the results in this paper we produce examples of the type envisaged above, which as far as we know have not been seen before.

Another new result is the observation that we have here a one way street; while the results in [5,6,9] establish a reversible link between scalar valued and matrix valued polynomials we see here that if the scalar family is made up of common eigenfunctions of some differential operators this property extends to the matrix valued polynomials but not (necessarily) the other way round.

We close this introduction with some remarks, aimed at a broad audience, about some of the possible uses of matrix valued orthogonal polynomials.

One of the many applications of the theory of scalar orthogonal polynomials is to a detailed study of a special kind of Markov chains with state space given by the non-negative integers and with a one step transition probability matrix that allows only moves to the nearest neighbours. The basic paper here is [20]. These authors derive an expression (based on the spectral analysis of the corresponding difference operator) for the n -step transition probability between states i and j in terms of the orthogonal polynomials going along with the one step transition probability matrix. There are several advantages to such a formula. For instance if one knows the orthogonality measure one can compute the probability of going from states i, j with $i \leq j$ in any number of steps from the knowledge of the first j rows of the one step transition matrix. Other uses of this connection with orthogonal polynomials lie a bit deeper: by using the Stieltjes transform of the orthogonality measure one can see that (in principle) the knowledge of the probability of going from state 0 to state 0 in an arbitrary number of steps determines the full n -step transition probability matrix for any n .

In the last section of the paper by Karlin and McGregor mentioned above, these authors consider the problem of using their methods, based on a three term recursion relation that is semi-infinite, to the case of a doubly infinite three term recursion. Although they stop short of introducing explicitly the notion of matrix valued orthogonal polynomials they compute in detail the orthogonality matrix measure that is relevant to this problem. The full blown analysis of the problem of a random walk (with nearest neighbour transitions) on the set of all integers that was the concern of the last section in [20] has been done in detail later in [4,14]. This extension from the study by means of scalar valued orthogonal polynomials of birth-and-death processes to the study of so-called quasi-birth-and-death processes by means of matrix valued orthogonal polynomials is already a concrete example of a good motivation for this study.

For a situation where the matrix valued orthogonal polynomials happen to satisfy matrix valued differential equations which along with the orthogonality measure are an outgrowth of the work started in [16], see [15].

The best known families of scalar valued orthogonal polynomials, connected with the names of Hermite, Laguerre, and Jacobi owe their importance to the differential equations they satisfy and thus they appear in connection with simple situations involving the Schrödinger equation. In the scalar case the only possible examples of orthogonal polynomials with a differential operator floating around are the ones mentioned above. On the other hand the situation in the matrix valued case is much more complicated and it opens the door to an embarrassment of riches in terms of examples. The study carried out here is an effort to produce more examples of this novel situation as a step towards other applications.

2. Scalar and matrix valued polynomials

The relation between scalar polynomials satisfying higher order recursions and matrix valued ones satisfying three term recursions was found by one of us (see [5] and [6]) and it is well described in [9].

The procedure shows a way to produce a vector polynomial from an scalar one $p(x)$ by considering the decomposition of $p(x)$ congruent with the residues of the exponent of x in $p(x)$ modulo N .

This is done by using the operators $R_{N,m}$, $m = 0, \dots, N - 1$, defined by

$$R_{N,m}(p)(x) = \sum_n \frac{p^{(nN+m)}(0)}{(nN+m)!} x^n, \tag{2.1}$$

i.e., the operator $R_{N,m}$ takes from p those powers congruent with m modulo N , then removes the common factor x^m and changes x^N to x .

From a sequence of scalar polynomials $(p_n)_n$ we can then define a sequence of matrix valued polynomials $P_n(x)$, $n = 0, 1, \dots$, by the recipe

$$P_n(x) = \begin{pmatrix} R_{N,0}(p_{nN})(x) & \dots & R_{N,N-1}(p_{nN})(x) \\ R_{N,0}(p_{nN+1})(x) & \dots & R_{N,N-1}(p_{nN+1})(x) \\ \vdots & & \vdots \\ R_{N,0}(p_{nN+N-1})(x) & \dots & R_{N,N-1}(p_{nN+N-1})(x) \end{pmatrix}. \tag{2.2}$$

This process can be reversed since for any polynomial $p(x)$ we have

$$p(x) = R_{N,0}(p)(x^N) + xR_{N,1}(p)(x^N) + \dots + x^{N-1}R_{N,N-1}(p)(x^N).$$

It turns out that if $p_n(x)$, $n = 0, 1, 2, \dots$, satisfy the $(2N + 1)$ -term recurrence relation

$$x^N p_n(x) = c_{n,0} p_n(x) + \sum_{k=1}^N [c_{n,-k} p_{n-k}(x) + c_{n,k} p_{n+k}(x)] \tag{2.3}$$

($p_n(x)$ vanishes if n is negative), then the matrix valued ones P_n , $n = 0, 1, 2, \dots$, satisfy the following three term matrix recurrence relation

$$xP_n(x) = A_n P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x), \tag{2.4}$$

where the matrix coefficients are nothing but the $N \times N$ blocks of the $(2N + 1)$ banded matrix associated to the scalar family $(p_n)_n$ featuring in the higher order recursion that defines the $p_n(x)$. This semi-infinite banded matrix has rows that look as follows:

first row

$$c_{0,0}, c_{0,1}, \dots, c_{0,N}, 0, 0, 0, \dots,$$

second row

$$c_{1,-1}, c_{1,0}, c_{1,1}, c_{1,2}, \dots, c_{1,N}, 0, 0, 0, \dots,$$

third row

$$c_{2,-2}, c_{2,-1}, c_{2,0}, c_{2,1}, c_{2,2}, c_{2,3}, \dots, c_{2,N}$$

and for i large enough, the $(i + 1)$ th row looks like

$$0, 0, \dots, 0, c_{i,-N}, c_{i,-N+1}, \dots, c_{i,-1}, c_{i,0}, c_{i,1}, \dots, c_{i,N}, 0, 0, \dots$$

Given a block-tridiagonal matrix (that defines, by using (2.4), a sequence of matrix valued polynomials) one can always assume, by using an n dependent unitary matrix, that the off-diagonal blocks are triangular, and thus the matrix becomes a

banded matrix with scalar entries. This shows that using the operators $R_{N,m}$, $m = 0, \dots, N - 1$, one can also go from matrix valued polynomials satisfying a three term recurrence relation like (2.4) to scalar valued polynomials satisfying a higher recurrence relation as in (2.3).

The polynomial x^N in the formula (2.3) does not play any special role. One can replace it by $(x - a)^N$ for an appropriate choice of the constant a ; in this case, one has to use the operators

$$R_{N,m,a}(p)(x) = \sum_n \frac{p^{(nN+m)}(a)}{(nN + m)!} x^n,$$

instead of the $R_{N,m}$ defined in (2.1) (see [9, p. 269], for the general case when in (2.3), the polynomial x^N is substituted by a generic polynomial $h(x)$).

We say that a higher order recursion (2.3) for $(p_n)_n$ is symmetric if $c_{n,-k} = \bar{c}_{n-k,k}$, and, analogously, that the matrix three term recurrence relation (2.4) is symmetric if $A_n = C_{n+1}^*$. As it was shown in [9], a symmetric higher order recursion gives a symmetric matrix three term recurrence relation.

The symmetric case corresponds to a Hermitian $(2N + 1)$ -banded matrix and the associated matrix polynomials defined by (2.2) are then orthonormal with respect to a weight matrix (i.e. a positive definite matrix of measures having finite moments of any order).

Given a higher order scalar recursion that is not symmetric there are two ways to try to symmetrize it. Starting from the polynomials $p_n(x)$ defined by the recursion one can define new polynomials $r_n(x) = \gamma_n p_n(x)$, and look for a convenient choice of the sequence of numbers $(\gamma_n)_n$ so that the corresponding higher order recursion for $(r_n)_n$ is symmetric.

A different, and more general approach consists of producing from the scalar polynomials $p_n(x)$ the matrix polynomials $P_n(x)$ as above. Now one considers a new sequence of matrix polynomials $R_n(x) = \Gamma_n P_n(x)$, and looks for a convenient choice of the sequence of matrices $(\Gamma_n)_n$ so that the matrix three term recurrence relation for $(R_n)_n$ is symmetric. It is easy to see that the first attempt is a special case of the second one, which consists of insisting that the matrices Γ_n be diagonal.

3. Producing matrix differential equations out of scalar ones

Our starting point is a sequence $(p_n)_n$ of polynomials, with $\deg p_n = n$, satisfying the differential equation

$$\sum_{l=1}^m a_l(x) p_n^{(l)}(x) = \gamma_n p_n(x), \tag{3.1}$$

where a_l , $l = 1, \dots, m$, are polynomials of degree not bigger than l and γ_n are real numbers.

We split up the polynomials p_n as explained in the previous section to get a sequence of matrix polynomials P_n of size $N \times N$. We now show that these matrix polynomials inherit a matrix differential equation from the differential equation (3.1) for the scalar polynomials p_n . The proof is constructive, so that it produces an explicit expression for the differential coefficients A_k , $k = 0, \dots, m$, in the new differential operator.

For the sake of simplicity, we show the result in full only for the case $N = 2$ (the general case can be proved in the same way).

Theorem 3.1. *Assume that $(p_n)_n$ is a sequence of scalar polynomials satisfying a differential equation of the form (3.1) where a_l , $l = 1, \dots, m$, are polynomials of degree not bigger than l and γ_n are numbers. Then the matrix polynomials P_n defined from p_n by (2.2) satisfy the differential equation*

$$\sum_{k=0}^m P_n^{(k)}(x) A_k(x) = \Gamma_n P_n(x),$$

where the coefficients A_k , $k = 0, \dots, m$, are given by

$$A_k = \left(\sum_{l=k}^m C_{k,l} \right),$$

$$C_{k,l}(x) = \begin{pmatrix} b_{k,l,0}(x) & b_{k,l,1}(x) \\ x b_{k,l,1}(x) + l b_{k,l-1,0}(x) & b_{k,l,0}(x) + l b_{k,l-1,1}(x) \end{pmatrix} \begin{pmatrix} a_{l,0}(x) & a_{l,1}(x) \\ x a_{l,1}(x) & a_{l,0}(x) \end{pmatrix},$$

$$b_{k,l} = \left(\frac{(-1)^k}{k!} \sum_{j=\lceil(l+1)/2\rceil}^k (-1)^j \binom{k}{j} 2j(2j-1) \cdots (2j-l+1) \right) x^{2k-l}$$

(as usual $[x]$ denotes the integer part of x) and we write $p_0 = R_{2,0}(p)$ and $p_1 = R_{2,1}(p)$, where $R_{2,0}$ and $R_{2,1}$ are the operators defined in (2.1) for $N = 2$.

The eigenvalues Γ_n , $n \geq 0$, are given by

$$\Gamma_n = \begin{pmatrix} \gamma_{nN} & 0 & \cdots & 0 \\ 0 & \gamma_{nN+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_{nN+N-1} \end{pmatrix}.$$

Proof. To simplify the expressions, we again write $p_0 = R_{2,0}(p)$ and $p_1 = R_{2,1}(p)$, so that $p(x) = p_0(x^2) + xp_1(x^2)$. A simple computation shows that

$$(f(x^2))^{(l)} = \sum_{k=\lceil(l+1)/2\rceil}^l b_{k,l}(x) f^{(k)}(x^2),$$

where

$$b_{k,l}(x) = \left(\frac{(-1)^k}{k!} \sum_{j=\lceil(l+1)/2\rceil}^k (-1)^j \binom{k}{j} 2j(2j-1)\cdots(2j-l+1) \right) x^{2k-l}, \quad k > l/2.$$

Applying this formula to $p_n(x) = p_{n,0}(x^2) + xp_{n,1}(x^2)$, we get after some simplifications that

$$\begin{aligned} (p_{n,0}(x^2) + xp_{n,1}(x^2))^{(l)} &= \sum_{k=0}^l (b_{k,l,0}(x^2) p_{n,0}^{(k)}(x^2) + (x^2 b_{k,l,1}(x^2) + lb_{k,l-1,0}(x^2)) p_{n,1}^{(k)}(x^2)) \\ &\quad + x \left(\sum_{k=0}^l (b_{k,l,1}(x^2) p_{n,0}^{(k)}(x^2) + (b_{k,l,0}(x^2) + lb_{k,l-1,1}(x^2)) p_{n,1}^{(k)}(x^2)) \right), \end{aligned}$$

where we take $b_{k,k'} = 0$, when $k > k'$. This can be written in the form:

$$\begin{aligned} (p_{n,0}(x^2) + xp_{n,1}(x^2))^{(l)} &= \sum_{k=0}^l \begin{pmatrix} p_{n,0}^{(k)}(x^2) & p_{n,1}^{(k)}(x^2) \end{pmatrix} \begin{pmatrix} b_{k,l,0}(x^2) \\ (x^2 b_{k,l,1}(x^2) + lb_{k,l-1,0}(x^2)) \end{pmatrix} \\ &\quad + x \sum_{k=0}^l \begin{pmatrix} p_{n,0}^{(k)}(x^2) & p_{n,1}^{(k)}(x^2) \end{pmatrix} \begin{pmatrix} b_{k,l,1}(x^2) \\ (b_{k,l,0}(x^2) + lb_{k,l-1,1}(x^2)) \end{pmatrix}. \end{aligned} \tag{3.2}$$

If we write

$$C_{k,l}(x) = \begin{pmatrix} b_{k,l,0}(x) & b_{k,l,1}(x) \\ xb_{k,l,1}(x) + lb_{k,l-1,0}(x) & b_{k,l,0}(x) + lb_{k,l-1,1}(x) \end{pmatrix} \begin{pmatrix} a_{l,0}(x) & a_{l,1}(x) \\ xa_{l,1}(x) & a_{l,0}(x) \end{pmatrix},$$

and take into account that for any polynomials p, q one has $(pq)_0 = p_0q_0 + xp_1q_1$ and $(pq)_1 = p_0q_1 + p_1q_0$, we get from (3.2) that

$$\left((a_l(x) p_n^{(l)}(x))_0 \quad (a_l(x) p_n^{(l)}(x))_1 \right) = \sum_{k=0}^l \begin{pmatrix} p_{n,0}^{(k)}(x) & p_{n,1}^{(k)}(x) \end{pmatrix} C_{k,l}. \tag{3.3}$$

From here, it follows that the differential equation (3.1) gives the following matrix equation for the vector $(p_{n,0}^{(k)}(x) \quad p_{n,1}^{(k)}(x))$:

$$\sum_{k=0}^m \begin{pmatrix} p_{n,0}^{(k)}(x) & p_{n,1}^{(k)}(x) \end{pmatrix} \left(\sum_{l=k}^m C_{k,l} \right) = \gamma_n \begin{pmatrix} p_{n,0}(x) & p_{n,1}(x) \end{pmatrix}.$$

It is now easy to finish the proof. \square

We illustrate the general construction above with an example.

Take the Laguerre polynomials $(L_n^\alpha)_n$, which satisfy the second order differential equation

$$x(L_n^\alpha(x))'' + (\alpha + 1 - x)(L_n^\alpha(x))' = -nL_n^\alpha(x).$$

The proof of Theorem 3.1 gives for the matrix polynomials

$$L_{n,2 \times 2}^\alpha = \begin{pmatrix} \frac{L_{2n}^\alpha(\sqrt{x}) + L_{2n}^\alpha(-\sqrt{x})}{2} & \frac{L_{2n}^\alpha(\sqrt{x}) - L_{2n}^\alpha(-\sqrt{x})}{2\sqrt{x}} \\ \frac{L_{2n+1}^\alpha(\sqrt{x}) + L_{2n+1}^\alpha(-\sqrt{x})}{2} & \frac{L_{2n+1}^\alpha(\sqrt{x}) - L_{2n+1}^\alpha(-\sqrt{x})}{2\sqrt{x}} \end{pmatrix}$$

the matrix differential equation

$$\begin{aligned} & (L_{n,2 \times 2}^\alpha)'' \begin{pmatrix} 0 & 4x \\ 4x^2 & 0 \end{pmatrix} + (L_{n,2 \times 2}^\alpha)' \begin{pmatrix} -2x & 2\alpha + 4 \\ (8 + 2\alpha)x & -2x \end{pmatrix} + L_{n,2 \times 2}^\alpha \begin{pmatrix} 0 & 0 \\ \alpha + 1 & -1 \end{pmatrix} \\ & = \begin{pmatrix} -2n & 0 \\ 0 & -2n - 1 \end{pmatrix} L_{n,2 \times 2}^\alpha. \end{aligned}$$

In this case, the sequence $(L_{n,2 \times 2}^\alpha)_n$ is orthogonal with respect to the weight matrix

$$e^{-\sqrt{x}} \begin{pmatrix} \frac{1}{\sqrt{x}} & 1 \\ 1 & \sqrt{x} \end{pmatrix}.$$

For any size N , the matrix polynomials $(L_{n,N \times N}^\alpha)_n$ defined from the Laguerre polynomials by (2.2) satisfy the matrix second order differential equation

$$(L_{n,N \times N}^\alpha)'' A_2(x) + (L_{n,N \times N}^\alpha)' A_1(x) + L_{n,2 \times 2}^\alpha A_0 = \begin{pmatrix} \gamma_{nN} & 0 & \cdots & 0 \\ 0 & \gamma_{nN+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_{nN+N-1} \end{pmatrix} L_{n,2 \times 2}^\alpha,$$

where

$$\begin{aligned} A_2(x) &= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 4 \\ 4x^2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 4x^2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 4x^2 & 0 \end{pmatrix}, \\ A_1(x) &= \begin{pmatrix} -x & 0 & 0 & \cdots & 0 & 4 + \alpha/2 \\ (N + 2 + \alpha)x & -x & 0 & \cdots & 0 & 0 \\ 0 & (N + 4 + \alpha)x & -x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -x & \vdots \\ 0 & 0 & 0 & \cdots & (N + 2(N - 1) + \alpha)x & -x \end{pmatrix}, \end{aligned}$$

and

$$A_0(x) = \begin{pmatrix} N - 1/N^2 & 0 & 0 & \cdots & 0 & 0 \\ \alpha + 1/N^2 & N - 2/N^2 & 0 & \cdots & 0 & 0 \\ 0 & 2\alpha + 2^2/N^2 & N - 3/N^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1/N^2 & 0 \\ 0 & 0 & 0 & \cdots & (N - 1)\alpha + (N - 1)^2/N^2 & 0 \end{pmatrix}.$$

4. A look at some examples involving higher order operators

In Section 2 we have shown that a scalar valued sequence of polynomials that are common eigenfunctions of a given differential operator gives rise, in a canonical fashion, to a sequence of matrix valued polynomials and to a differential operator that has them as common eigenfunctions.

The purpose of this section is to illustrate a number of issues suggested by this general result in the case of some examples. These examples deal with what are called extended Krall–Laguerre and extended Krall–Jacobi polynomials in [12]. In both cases the scalar polynomials we consider satisfy a unique fourth order differential equation which therefore gives rise to a fourth order differential operator with matrix coefficients. The general theme of this section centers around the existence of other matrix valued differential operators having these same polynomials as common eigenfunctions very much in the spirit of [2].

4.1. The scalar extended Krall–Laguerre polynomials

The scalar valued (extended) Krall–Laguerre polynomials $p_n(x)$ first introduced in [12] are given, for $n = 0, 1, 2, \dots$, by

$$p_n(x) = \frac{1}{x} (q_{n+1}(x) + x_{n+1}q_n(x) + y_nq_{n-1}(x)),$$

where

$$x_n = \alpha + \frac{2n^2 + 2(R - 1)n - R}{(n - 1) + R}, \tag{4.1}$$

$$y_n = \frac{n(n + \alpha)(n + 1 + R)}{n + R} \tag{4.2}$$

and $q_n(x) = L_n^\alpha(x)$ stand for the familiar Laguerre polynomials given by

$$L_n^\alpha(x) = (x - (2n - 1 + \alpha))L_{n-1}^\alpha(x) - (n - 1)(n - 1 + \alpha)L_{n-2}^\alpha(x)$$

and $L_{-1}^\alpha(x) = 0, L_0^\alpha(x) = 1$.

Here R is a free parameter, and so is α , which is taken to be larger than -1 .

The $p_n(x)$ are obtained in [12] by an application of the Darboux process starting from the Laguerre polynomials $L_n^\alpha(x)$ which are orthogonal and satisfy the three term recursion relation given above. For generic values of $\alpha > -1$ the $p_n(x)$ do not satisfy such a three term recursion relation, but they still satisfy the five term recursion relation given by

$$\begin{pmatrix} x_1 & 1 & & & \\ y_1 & x_2 & 1 & & \\ & y_2 & x_3 & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \bar{x}_1 & 1 & & & \\ \bar{y}_1 & \bar{x}_2 & 1 & & \\ & \bar{y}_2 & \bar{x}_3 & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \cdot \\ \cdot \end{pmatrix} = x^2 \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \cdot \\ \cdot \end{pmatrix}, \tag{4.3}$$

where x_n and y_n are given above and \bar{x}_n and \bar{y}_n are given as follows

$$\bar{x}_n = \alpha + \frac{2n^2 + 2(R - 1)n - R}{n + R}, \tag{4.4}$$

$$\bar{y}_n = \frac{n(n + \alpha)(n - 1 + R)}{n + R}. \tag{4.5}$$

The point of the construction in [12] is that the new family of polynomials $p_n(x)$ are common eigenfunctions of a fixed fourth order differential operator in the spectral parameter x , as will be seen shortly.

The method discussed in [12] shows that a proper use of the Darboux process will take us from a trivial bispectral situation, i.e. the Laguerre polynomials $L_n^\alpha(x)$ are eigenfunctions of a fixed second order differential operator and they are the common eigenvectors of a tridiagonal difference operator, to a much less trivial one: the polynomials $p_n(x)$ feature in a higher order bispectral situation that is not a trivial consequence of the original one.

In our case we have

$$\begin{aligned} & [x^2 D^4 - 2x(x - \alpha - 2)D^3 + (x(x - 2(R + \alpha) - 6) + \alpha(\alpha + 3))D^2 + 2((R + 1)x - (\alpha + 1)R - \alpha)D + R(R + 1)]p_n(x) \\ & = (R + n)(R + n + 1)p_n(x). \end{aligned} \tag{4.6}$$

We now produce out of these $p_n(x)$ a sequence of matrix valued polynomials of size 2×2 using the method explained in Section 2.

For concreteness we display below $P_0(x)$ and $P_1(x)$. We have

$$\begin{aligned} P_0(x) &= \begin{pmatrix} 1 & 0 \\ -\frac{\alpha + (\alpha + 1)R}{R + 1} & 1 \end{pmatrix}, \\ P_1(x) &= \begin{pmatrix} x + \frac{(\alpha + 2)((\alpha + 1)R + 2\alpha)}{R + 2} & -\frac{2((\alpha + 2)R + 2\alpha + 3)}{R + 2} \\ -3x \frac{((\alpha + 3)R + 3\alpha + 8)}{R + 3} - \frac{(\alpha + 2)(\alpha + 3)((\alpha + 1)R + 3\alpha)}{R + 3} & x + \frac{3(\alpha + 3)((\alpha + 2)R + 3\alpha + 4)}{R + 3} \end{pmatrix}. \end{aligned}$$

Here we display a fourth order differential operator B , arising by applying the method explained in the previous section, that has the matrix valued polynomials $(P_n(x))_n$ as a set of common eigenfunctions

$$\begin{aligned} B &= D^4 x^3 I + D^3 \begin{pmatrix} (\alpha + 5)x^2 & -x^2 \\ -x^3 & (\alpha + 7)x^2 \end{pmatrix} + D^2 \begin{pmatrix} \frac{x^2 + x(\alpha^2 + 9\alpha + 15)}{4} & -\frac{(R + \alpha + 6)x}{2} \\ -\frac{(R + \alpha + 9)x^2}{2} & \frac{x^2 + (\alpha^2 + 15\alpha + 39)x}{4} \end{pmatrix} \\ &+ D \begin{pmatrix} \frac{(2R + 3)x}{8} + \frac{\alpha(\alpha + 3)}{8} & -\frac{(\alpha + 2)R + 2\alpha + 3}{4} \\ -x \frac{(\alpha + 4)R + 4(\alpha + 3)}{4} & x \frac{2R + 5}{8} + \frac{3(\alpha + 1)(\alpha + 4)}{8} \end{pmatrix} + \begin{pmatrix} -\frac{R + 1}{8} & 0 \\ -\frac{(\alpha + 1)R + \alpha}{8} & 0 \end{pmatrix}. \end{aligned}$$

The matrix valued eigenvalue is read-off from the expression

$$P_n(x)B = \left[\begin{pmatrix} -\frac{R + 1}{8} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{2R + 1}{8} & 0 \\ 0 & \frac{2R + 3}{8} \end{pmatrix} n + \frac{n^2}{4} I \right] P_n(x).$$

Notice that our differential operator acts on the right on its argument. This is also the reason why the eigenvalue matrix appears before the eigenfunction P_n .

Observe that the differential operator above depends both on R and α while the eigenvalue does not. If one were to consider, for a fixed pair R, α the family of matrix valued polynomials, and then in the spirit of [2] the full algebra we would end up getting isomorphic algebras of differential operators with matrix coefficients for different values of α . This follows from the fact that these algebras of differential operators are isomorphic to the algebra of matrix valued eigenvalues.

4.2. Matrix extended Krall–Laguerre and orthogonality

The sequence of matrix polynomials $(P_n)_n$ is not always orthogonal with respect to a weight matrix of measures. This only happens if each polynomial P_n can be normalized using a nonsingular matrix: $Q_n = I_n P_n$ so that $(Q_n)_n$ is an orthonormal sequence with respect to a weight matrix. In that case, the sequence $(Q_n)_n$ satisfies a three term recurrence relation of the form

$$xQ_n = A_{n+1}Q_{n+1} + B_nQ_n + A_n^*Q_{n-1},$$

with A_n nonsingular and B_n Hermitian.

If we were dealing with a monic sequence of polynomials P_n satisfying a three term recursion, such as

$$xP_n = P_{n+1} + B_nP_n + A_nP_{n-1},$$

this symmetrization can be done only when one can find positive definite matrices S_n such that B_nS_n is Hermitian and $A_nS_{n+1} = S_n$.

As we noticed at the end of Section 2, given a scalar family of polynomials $p_n(x)$ we could attempt a scalar symmetrization of the (higher order) recursion relation. Given the Krall–Laguerre polynomials $(p_n)_n$ we can look for a normalization $r_n(x) = \tau_n p_n(x)$ satisfying a symmetric five term recurrence relation of the form

$$x^2 r_n = a_{n+2} r_{n+2} + b_{n+1} r_{n+1} + c_n r_n + b_n r_{n-1} + a_n r_{n-2}, \quad (4.7)$$

where $a_n \neq 0, n \geq 0$.

If we use the parameters $(x_n)_n, (y_n)_n, (\bar{x}_n)_n$ and $(\bar{y}_n)_n$ (see (4.1), (4.2), (4.4) and (4.5)) it turns out that we would need

$$y_{n+1} \bar{y}_n = \frac{(y_n \bar{x}_n + x_{n+1} \bar{y}_n)(y_{n+1} \bar{x}_{n+1} + x_{n+2} \bar{y}_{n+1})}{(x_n + \bar{x}_{n+1})(x_{n+1} + \bar{x}_{n+2})}.$$

This only happens when $\alpha = 0$ and then:

$$\begin{aligned} a_{n+2,R,0} &= (n+1)(n+2) \sqrt{\frac{(n+R)(n+R+3)}{(n+R+1)(n+R+2)}}, \\ b_{n+1,R,0} &= 2(n+1) \frac{2(n+1)R^2 + (2n+1)(2n+3)R + 2n(n+1)(n+2)}{(n+R+1)\sqrt{(n+R)(n+R+2)}}, \\ c_{n,R,0} &= \frac{n^2(n+R+1)^2 + (n+1)^2(n+R)^2 + (2n(n+1) + (2n+1)R)^2}{(n+R)(n+R+1)}. \end{aligned}$$

For $\alpha = 0$, the five terms recurrence relation (4.7) is just an iteration of the three term recurrence relation for the Laguerre type orthogonal polynomials $(L_n^R)_n$ introduced by Koornwinder (see [22] and also [21]). They are orthogonal in $[0, +\infty)$ with respect to the measure $e^{-x} + \frac{1}{R} \delta_0$. The four order differential equation (4.6) reduces to the well-known four order differential equation for this family (see, for instance, [23, p. 112]).

The corresponding matrix polynomials $(P_{n,R,0})_n$ are then orthogonal in $[0, +\infty)$ with respect to the weight matrix

$$e^{-\sqrt{x}} \begin{pmatrix} \frac{1}{\sqrt{x}} + \frac{2}{R} \delta_0 & 1 \\ 1 & \sqrt{x} \end{pmatrix}.$$

In this case one can see that the more general attempt to symmetrize the matrix valued recursion relation fails too, except in the case $\alpha = 0$.

4.3. The scalar extended Krall–Jacobi case

The scalar valued (extended) Krall–Jacobi polynomials $p_n(x)$, first introduced in [12] are given by

$$q_n(x) = \frac{1}{x+1} (y_n p_{n-1}(x) + x_{n+1} p_n(x) + p_{n+1}(x)),$$

where, as in (4.11) and (4.12) in [12], we have

$$x_n = \frac{2(\alpha + \beta + n)(\alpha\beta + \beta^2 + 2n\alpha + 2(n - 1)\beta + 2n(n - 1))}{(\alpha + \beta + n - 1)(\alpha + \beta + 2n - 2)(\alpha + \beta + 2n)} - \frac{2(\alpha + n - 1)R}{(\alpha + \beta + n - 1)(\alpha + \beta + 2n - 2)\theta_{n-1}},$$

$$y_n = a_n \frac{\theta_{n+1}}{\theta_n}$$

and

$$\theta_n = n^2 + (\alpha + \beta)n + R.$$

Here $p_n(x)$ stands for the usual (monic) Jacobi polynomials satisfying

$$xp_n(x) = a_n p_{n-1}(x) + b_{n+1} p_n(x) + p_{n+1}(x)$$

with

$$a_n = \frac{4n(n + \alpha + \beta)(n + \alpha)(n + \beta)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)},$$

$$b_n = \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta - 2)(2n + \alpha + \beta)}.$$

As can be seen in [12], the crux of the construction of these polynomials is that the polynomials $p_n(x)$ and $q_n(x)$ are also related by means of

$$p_n(x) = \frac{1}{x + 1} (\bar{y}_n q_{n-1}(x) + \bar{x}_{n+1} q_n(x) + q_{n+1}(x)).$$

As explained in [12] the expressions for \bar{x}_n and \bar{y}_n can be obtained by replacing b_n by $b_n + 1$ in (3.17) of that paper. The resulting expression for \bar{x}_n is correctly given by the first part of (4.13), but there is an error in the expression for \bar{y}_n reported in [12], and we give the correct expression below, namely

$$\bar{y}_n = x_{n+1}(x_{n+2} - b_{n+1} - b_{n+2} - 2) - y_{n+1} + a_n + a_{n+1} + (b_{n+1} + 1)^2.$$

For completeness we reproduce the expression for \bar{x}_n from [12], namely

$$\bar{x}_n = \frac{2(\alpha + \beta + n - 1)(\alpha\beta + \beta^2 + 2(n - 1)\alpha + 2n\beta + 2n(n - 1))}{(\alpha + \beta + n)(\alpha + \beta + 2n - 2)(\alpha + \beta + 2n)} + \frac{2(\alpha + n)R}{(\alpha + \beta + n)(\alpha + \beta + 2n)\theta_n}.$$

Finally R is a free parameter.

These polynomials $q_n(x)$ satisfy, for generic values of α, β , a five term recursion relation and they are the common eigenfunctions of a fourth order differential operator PQ with

$$P = (x^2 - 1)D^2 + ((\alpha + \beta + 1)x - \beta + \alpha + 1)D + R$$

and

$$Q = (x^2 - 1)D^2 + ((\alpha + \beta + 3)x - \beta + \alpha - 1)D + (R + \alpha + \beta + 1).$$

More explicitly we have

$$PQq_n(x) = \lambda_n q_n(x)$$

with

$$\lambda_n = (R + n(\alpha + \beta + n))(R + (n + 1)(\alpha + \beta + n + 1)).$$

The five term recursion relation expresses the product

$$(x + 1)^2 q_n(x)$$

as a linear combination of the polynomials $q_{n+2}, q_{n+1}, q_n, q_{n-1}, q_{n-2}$.

One could raise again the possibility that this five recursion could be made symmetric by premultiplying the scalar polynomials by appropriate n dependent constants. This would require, as in the case of the modified Krall–Laguerre polynomials that a certain identity should hold among the quantities $x_n, y_n, \bar{x}_n, \bar{y}_n$. An explicit computation using the expressions above shows that this happens only for $\beta = 0$.

If one obtains the matrix valued polynomials out of these scalar ones and tries to symmetrize the resulting three term recursion relation one finds, once again, that in this case nothing is gained by allowing this more general cure to our problem: there is a positive definite orthogonality matrix only in the case of $\beta = 0$.

We consider now (as an example) the special case given by $\alpha = 3/2$, $\beta = 7/8$, $R = 7$. In this case the scalar fourth order operator mentioned earlier becomes

$$(x^2 - 1)^2 D^4 + \frac{(x^2 - 1)(51x + 5)}{4} D^3 + \frac{7}{64} (503x^2 + 98x - 281) D^2 + \frac{35}{32} (83x + 21) D.$$

The general method indicated earlier produces a fourth order differential operator with matrix coefficients satisfied by the matrix valued polynomials obtained from the sequence $p_n(x)$. In the spirit of [2] one can study the algebra of all differential operators associated with this family of matrix valued polynomials.

In this example we have checked that the algebra contains only one (up to scalars) differential operator of order four. If we look for operators of order less than or equal to six, we find only one such operator (modulo operators of order 4), whose highest order terms are

$$D^6 \begin{pmatrix} (x-1)^3 x^3 & 0 \\ 0 & (x-1)^3 x^3 \end{pmatrix} + \frac{3}{16} D^5 \begin{pmatrix} (x-1)^2 x^2 (107x - 40) & 5(x-1)^2 x^2 \\ 5(x-1)^2 x^3 & (x-1)^2 x^2 (123x - 56) \end{pmatrix} + \dots$$

In particular the algebra cannot be generated by the fourth order operator constructed out of the scalar one. As mentioned in the introduction, this also gives examples of situations where the lowest order of a (nontrivial) differential operator in the algebra is higher than two.

Extensive computations suggest that these two observations hold for generic values of α, β, R . Just as in a previous example, and all the examples below, all differential operators with matrix valued coefficients act on the right of their matrix valued arguments.

4.4. Lower order operators connected to the extended Krall–Jacobi case

A remarkable phenomenon takes place when α and β differ by one. Consider the case $\beta = \alpha + 1$. Here, very much as in the generic case, we find that the scalar extended Krall–Jacobi polynomials are the common eigenfunctions of only one fourth order differential operator, namely

$$(x^2 - 1)^2 D^4 + 2(1 - x^2)((2\alpha + 5)x - 1) D^3 - 2(x((2\alpha^2 + 11\alpha + 19)x - 2\alpha - 2) - 3\alpha - 13) D^2 - 2((2\alpha + 3)(2\alpha + 9)x - 7) D.$$

A surprise comes about when we build the matrix valued polynomials going along with the scalar ones alluded to above.

We consider the differential operators with matrix coefficients and order not exceeding three. Notice that for generic values of α, β this gives only the identity operator. The situation now is entirely different. We find four nontrivial operators, which along with identity, give a basis for this five-dimensional space.

The matrix valued polynomials $P_n(x)$ are common eigenfunctions of two linearly independent third order operators. But there are other surprises in store. They also satisfy differential equations of orders one and two, namely, recalling that $\beta = \alpha + 1$,

$$P'_n(x) \begin{pmatrix} 0 & x-1 \\ 0 & 0 \end{pmatrix} + P_n(x) \begin{pmatrix} 1 + \frac{R}{2\beta} & \beta - \frac{R}{2\beta} \\ 0 & 0 \end{pmatrix} = \left[\begin{pmatrix} 1 + \frac{R}{2\beta} & \beta - \frac{R}{2\beta} \\ 0 & 0 \end{pmatrix} + n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] P_n$$

as well as

$$P''_n(x) \begin{pmatrix} 0 & x(x-1) \\ 0 & x(x-1) \end{pmatrix} + P'_n(x) \begin{pmatrix} 0 & \frac{x+\alpha}{(2\alpha+5)x-3} \\ 0 & 0 \end{pmatrix} + P_n(x) A_0 = \left[A_0 + n \begin{pmatrix} 0 & 0 \\ 0 & \frac{2\alpha+3}{2} \end{pmatrix} + n^2 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right] P_n(x),$$

where

$$A_0 = \begin{pmatrix} -\frac{4\alpha+5}{2} & -(\alpha+1)^2 \\ 0 & 0 \end{pmatrix} + \frac{R(3\alpha+4)}{4(\alpha+1)} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

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