Dichotomies in Terms of Lyapunov Functions for Linear Difference Equations

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Necessary and sufficient conditions that a linear difference equation have an exponential dichotomy on the set of the integer numbers are given in terms of Lyapunov functions.

INTRODUCTION

Consider the linear difference equation

\[ x(n+1) = A(n) x(n), \]

where \( A(n) \) is an \( k \times k \) invertible matrix function for \( n \in \mathbb{Z} = \{ \ldots, -1, 0, 1, \ldots \} \).

In what follows we denote by \( | \cdot | \) any convenient norm either of a vector or of a matrix.

Equation (1) is said to possess an exponential dichotomy on the set \( \mathbb{Z} \) if there exist a projection \( P \) (\( P^2 = P \)) and constants \( K > 0, 0 < p < 1 \) such that

\[
|X(n) PX^{-1}(m)| \leq Kp^{n-m}, \quad n \geq m
\]

\[
|X(n)(I-P) X^{-1}(m)| \leq Kp^{m-n}, \quad m \geq n
\]

where \( n, m \in \mathbb{Z} \) and \( X(n) = A(n-1) \ldots A(0) \) is the fundamental matrix of (1) such that \( X(0) = I \).

In the papers [5–10] we have studied dichotomies for difference equations on the set \( \mathbb{N} = \{0, 1, \ldots \} \). However, in the above papers we usually required that the matrix \( A(n) \) be bounded on \( \mathbb{N} \). Now in Propositions 1 and 2 of this paper using Lyapunov functions we give necessary and sufficient conditions in order that a linear difference equation have an...
exponential dichotomy on $Z$. These conditions do not impose the boundedness of the matrix $A(n)$ on the set $Z$. In Propositions 3 and 4 using quadratic forms we give necessary and sufficient conditions for a dichotomy on $Z$. In Proposition 4 the boundedness of $A(n)$ is required. We note that in [10] using quadratic forms we have proved conditions for a dichotomy on the set $N$.

The results of this paper are the discrete analogues of those of Muldowney [3] and Coppel [2].

We also note that in [4] the theory of dichotomies for difference equations is used as the main tool in the dynamical systems theory.

**MAIN RESULTS**

The following definition is used in Propositions 1 and 2.

According to Muldowney [3, p. 466] a pair of functions $V_i(n, x) : Z \times R^k \rightarrow R, i = 1, 2$, which is Lipschitzian in $x$ for every $n \in Z$ is said to be admissible if there exist supplementary projections $Q_1(n), Q_2(n)$ of rank $k_1, k_2$ independent of $n$ such that for any $n \in Z$

$$|Q_i(n)| \leq N, \quad i = 1, 2 \quad (3)$$

and

$$|Q_i(n)x|^r \leq V_i(n, x) \leq b|Q_i(n)x|^r, \quad i = 1, 2, \quad (4)$$

for all $n \in Z, x \in R^k$, where $N > 0, r > 0, b > 1$.

**Proposition 1.** Suppose that (1) has an exponential dichotomy (2) on $Z$. Then there exists an admissible pair $V_i(n, x), i = 1, 2$ such that for every $n \in Z, x \in R^k$

$$V_1(n, x) \leq p V_1(n-1, A^{-1}(n-1)x) \quad (5)$$

$$V_2(n+1, A(n)x) \geq p^{-1} V_2(n, x), \quad (6)$$

where $p$ is the constant in (2).

**Proof:** If $n \in Z, x \in R^k$ consider the functions

$$V_1(n, x) = \sup \{ |X(m)PX^{-1}(n)x| p^n - m, m \in Z : -\infty < n \leq m < \infty \}$$

$$V_2(n, x) = \sup \{ |X(m)(I-P)X^{-1}(n)x| p^m - n, m \in Z : -\infty < m \leq n < \infty \},$$

where $P, X(n)$ are defined in (2). Let $Q_1(n) = X(n)PX^{-1}(n)$ and $Q_2(n) = X(n)(I-P)X^{-1}(n)$. 

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From (2) and arguing as in the proof of Theorem 2.3 [3, p. 471] we can easily prove that $V_i(n, x)$, $i = 1, 2$ are Lipschitz in $x$ for every $n \in \mathbb{Z}$ and

$$|Q_i(n) x| \leq V_i(n, x) \leq K |Q_i(n) x|, \quad i = 1, 2,$$

where $n \in \mathbb{Z}$ and $K$ are defined in (2). So the pair $V_i(n, x)$, $i = 1, 2$ is admissible.

We now prove (5). Let $n \in \mathbb{Z}$, $x \in \mathbb{R}^k$, and $x(s)$, $s \in \mathbb{Z}$ be the solution of (1) such that $x(n) = x$. Since $X^{-1}(n)x(n) = X^{-1}(n-1)x(n-1)$ we get

$$V_i(n, x(n)) = \sup \{|X(m) PX^{-1}(n) x(n)| p^{-m}, m \in \mathbb{Z} : n \leq m\} \leq \sup \{|X(m) PX^{-1}(n-1) x(n-1)| p^{-1-m}, m \in \mathbb{Z} : n \leq m\}.$$

Therefore we obtain

$$V_i(n, x(n)) \leq p \sup \{|X(m) PX^{-1}(n-1) x(n-1)| p^{-1-m}, m \in \mathbb{Z}, n \leq m + 1\} = pV_i(n-1, x(n-1)) = pV_i(n-1, A^{-1}(n-1)x(n)).$$

Since $x(n) = x$ we have that (5) holds. Similarly we can prove (6). Thus the proof of the proposition is completed.

**Example 1.** Consider the system (1) where

$$A(n) = \begin{pmatrix} \left(\frac{1}{2}\right)^{3n^2 + 3n + 1} & 2^{3n^2 + 3n + 1} - \left(\frac{1}{2}\right)^{3n^2 + 3n + 1} \\ 0 & 2^{3n^2 + 3n + 1} \end{pmatrix}, \quad n \in \mathbb{Z}.$$ 

Then the matrix function

$$X(n) = \begin{pmatrix} \left(\frac{1}{2}\right)^{n^3} & 2^n - \left(\frac{1}{2}\right)^{n^3} \\ 0 & 2^n \end{pmatrix}$$

is a fundamental matrix solution of (1) such that $X(0) = I$. Suppose that we use for a vector $x = \text{col}(x_1, x_2)$ the norm $|x| = \sup \{|x_1|, |x_2|\}$, and for a matrix $A$ the induced operator norm. We can now show that (1) has an exponential dichotomy (2) on $\mathbb{Z}$ with projection $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and constants $K = 2$, $p = \frac{1}{2}$. Therefore, from Proposition 1 there must exist the required admissible pair $V_i(n, x)$, $i = 1, 2$ which satisfies (5) and (6). If $x = \text{col}(x_1, x_2)$ we take

$$V_1(n, x) = |x_1 - x_2|, \quad V_2(n, x) = |x_2|.$$ 

Let $Q_1(n) = X(n) PX^{-1}(n) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$, $Q_2(n) = I - Q_1(n)$. Then it is obvious that for every $n \in \mathbb{Z}$ and $x \in \mathbb{R}^2$,

$$|Q_i(n) x| = V_i(n, x) \leq 2 |Q_i(n) x|, \quad i = 1, 2.$$
So the pair $V_i(n, x)$ is admissible and it is easily proved that relations (5) and (6) are satisfied.

**Proposition 2.** Suppose that there exist a function $q(n): \mathbb{Z} \to \mathbb{R}^+$, $q(n) \neq 0$ for all $n \in \mathbb{Z}$, a constant $p$, $0 < p < 1$, and an admissible pair $V_i(n, x)$, $i = 1, 2$ such that for all $n \in \mathbb{Z}$, $x \in \mathbb{R}^k$

\[
V_1(n, x) < q'(n - 1) V_1(n - 1, A^{-1}(n - 1)x), \quad \text{if } V_1(n, x) > l_2 V_2(n, x) \tag{7}
\]

\[
V_1(n, x) < p' V_1(n - 1, A^{-1}(n - 1)x), \quad \text{if } l_1 V_1(n, x) > V_2(n, x) \tag{8}
\]

\[
V_2(n + 1, A(n)x) > p^{-r} V_2(n, x), \quad \text{if } V_1(n, x) \leq l_2 V_2(n, x) \tag{9}
\]

\[
V_2(n + 1, A(n)x) > q'(n) V_2(n, x) \quad \text{if } l_1 V_1(n, x) \leq V_2(n, x) \tag{10}
\]

where $l_i$, $i = 1, 2$ are constants such that $l_i b < 1$, $i = 1, 2$, $b, r$ are the constants in (4). Then (1) has an exponential dichotomy on $\mathbb{Z}$.

**Proof.** Set $\tilde{V}_i(n, x) = \prod_{s=0}^{n-1} q^{-r}(s) V_i(n, x)$, $i = 1, 2$. Let $x(n)$ be an arbitrary solution of (1).

We claim that if

\[
l_1 \tilde{V}_1(m, x(m)) \geq \tilde{V}_2(m, x(m)) \tag{11}
\]

for a $m \in \mathbb{Z}$ then

\[
l_1 \tilde{V}_1(n, x(n)) \geq \tilde{V}_2(n, x(n)) \tag{12}
\]

for $n \in \mathbb{Z}$, $-\infty < n \leq m$. First we prove (12) for $n = m - 1$. Suppose on the contrary that

\[
l_1 \tilde{V}_1(m - 1, x(m - 1)) < \tilde{V}_2(m - 1, x(m - 1)). \tag{13}
\]

From (7) and (10) we can easily show

\[
\tilde{V}_1(n, x(n)) \leq \tilde{V}_1(n - 1, x(n - 1)) \quad \text{if } \tilde{V}_1(n, x(n)) \geq l_2 \tilde{V}_2(n, x(n)) \tag{14}
\]

\[
\tilde{V}_2(n + 1, x(n + 1)) \geq \tilde{V}_2(n, x(n)) \quad \text{if } l_1 \tilde{V}_1(n, x(n)) \leq \tilde{V}_2(n, x(n)). \tag{15}
\]
Since \( l_i < 1, \ i = 1, 2 \) from (11) we get
\[
\bar{V}_1(m, x(m)) \geq l_i^{-1} \bar{V}_2(m, x(m)) \geq l_2 \bar{V}_2(m, x(m)).
\]

Hence from (14) we take
\[
\bar{V}_1(m, x(m)) \leq \bar{V}_1(m - 1, x(m - 1)).
\] (16)

Combining (13) and (15) we obtain
\[
\bar{V}_2(m - 1, x(m - 1)) \leq \bar{V}_2(m, x(m)).
\]

Using (16), (13) and the last relation we have
\[
\bar{V}_1(m, x(m)) \leq \bar{V}_1(m - 1, x(m - 1)) < l_i^{-1} \bar{V}_2(m - 1, x(m - 1))
\leq l_i^{-1} \bar{V}_2(m, x(m))
\]
which contradicts (11). So (12) holds for \( n = m - 1 \). Arguing as above we can easily prove (12) for \( n = m - 1, m - 2, \ldots \). Thus the proof of our claim is completed. So it is obvious that if
\[
l_1 V_1(m, x(m)) \geq V_2(m, x(m))
\] (17)
for a \( m \in \mathbb{Z} \) then
\[
l_1 V_1(n, x(n)) \geq V_2(n, x(n))
\] (18)
for \( n \in \mathbb{Z}, -\infty < n \leq m \).

Let \( x(n) \) be a solution of (1) such that \( x(m) \in \mathcal{Q}_1(m) \mathbb{R}^k, m \in \mathbb{Z} \). Then \( \mathcal{Q}_2(m) x(m) = 0 \). So since from (4) \( V_2(m, x(m)) \leq b |\mathcal{Q}_2(m) x(m)|^\eta \) we have that (17) holds. Therefore, (18) is satisfied. Then arguing as in the proof of Theorem 2.1 [3, p. 469] we can prove that there exists a \( k_1 \) dimensional subspace \( S_1 \) of solutions of (1) such that if \( x(n) \in S_1 \) we have
\[
l_1 V_1(n, x(n)) \geq V_2(n, x(n)), \quad -\infty < \eta < \infty.
\] (19)

Therefore if \( x(n) \in S_1 \) from (8) we take
\[
V_1(n, x(n)) \leq \rho \bar{V}_1(n - 1, x(n - 1)) \leq \cdots \leq \rho^{(n - m)\eta} V_1(m, x(m)), \quad n \geq m.
\]
So from (4) we get
\[
|\mathcal{Q}_1(n) x(n)| \leq \rho^{(n - m)\rho} |\mathcal{Q}_1(m) x(m)|, \quad n \geq m.
\]

So since from (4) and (19) the relation
\[
|\mathcal{Q}_2(n) x(n)| \leq (l_1 b)^{\eta} |\mathcal{Q}_1(n) x(n)|, \quad n \in \mathbb{Z}
\] (20)
holds we can easily prove that
\[ |x(n)| = |Q_1(n) x(n) + Q_2(n) x(n)| \leq b^{1/r}(1 + (b_1^{1/r}) p^n - m |Q_1(m) x(m)|. \]
Then from (3) we have
\[ |x(n)| \leq Nb^{1/r}(1 + (b_1^{1/r}) p^n - m |x(m)|, \quad n \geq m \quad (21) \]
for all solutions \( x(n) \in S_1 \).

Arguing as above we can easily prove that there exists a \( k_2 \) dimensional subspace \( S_2 \) of solutions of (1) such that if \( x(n) \in S_2 \) we have
\[ V_1(n, x(n)) \leq l_2 V_2(n, x(n)) \]
for \( n, -\infty < n < \infty \). So from (4) we get
\[ |Q_1(n) x(n)| \leq (l_2 b)^{1/r} |Q_2(n) x(n)|, \quad n \in Z. \quad (22) \]
Using (9) and the same argument to prove (21) we can easily prove
\[ |x(n)| \leq Nb^{1/r}(1 + (l_2 b)^{1/r}) p^n - m |x(m)|, \quad m \geq n, \quad (23) \]
where \( x(n) \in S_2 \). From (20) and (22) we have that \( S_1, S_2 \) are supplementary. Let \( P \) be a projection such that \( Px(n) = x(n) \), \( x(n) \in S_1 \). Put \( P_1(n) = X(n) PX^{-1}(n), P_2(n) = X(n)(I - P) X^{-1}(n) \). From (20) and (22) we get
\[ |Q_1(n) P_1(n)| \leq (l_1 b)^{1/r} |Q_1(n) P_1(n)|, \quad n \in Z \]
\[ |Q_1(n) P_2(n)| \leq (l_2 b)^{1/r} |Q_2(n) P_2(n)|, \quad n \in Z. \]
So since \( l_1 b < 1, i = 1, 2 \) from (3) and Lemma 2.5 [3, p. 468] there exists a positive number \( L > 0 \) such that
\[ |X(n) PX^{-1}(n)| \leq L, \quad n \in Z. \quad (24) \]
So from (21), (23), and (24) we have that (1) has an exponential dichotomy. Thus the proof of the proposition is completed.

In the remainder of this paper we use the Euclidean norm. The following lemma which is used in the proofs of Propositions 3 and 4 has been proved in [9]. For readers' convenience we state it here

**Lemma 1.** Equation (1) has an exponential dichotomy (2) with a projection \( P \) if and only if the adjoint equation of (1)
\[ x(n + 1) = \tilde{A}^{-1}(n) x(n), \quad n \in Z \quad (25) \]
has an exponential dichotomy with the projection \( I - \tilde{P} \).
Proof. Suppose that (1) has an exponential dichotomy (2). Then since we use the Euclidean norm we get
\[
|\hat{X}^{-1}(m)(1 - \hat{P}) \hat{X}(n)| \leq Kp^{m-n}, \quad m \geq n
\]
\[
|\hat{X}^{-1}(m) \hat{P} \hat{X}(n)| \leq Kp^{n-m}, \quad n \geq m.
\] (26)
Since \(Y(n) = \hat{X}^{-1}(n)\) is a fundamental matrix solution of (25) the proof of necessity of the lemma is completed. The sufficiency can be proved by using the same argument.

**Proposition 3.** Suppose that (1) has an exponential dichotomy (2) on \(Z\). Then there exist symmetric bounded matrix functions \(H(n), G(n), n \in Z\) such that
\[
\hat{A}(n) H(n+1) A(n) - H(n) \leq -\gamma I
\] (27)
\[
A^{-1}(n) G(n+1) \hat{A}^{-1}(n) - G(n) \leq -\gamma I,
\] (28)
where \(\gamma\) is a positive constant.

Proof. Let \(X(n)\) be the fundamental matrix solution of (1) which satisfies (2). Consider
\[
H(n) = 2 \sum_{s = n}^{\infty} \hat{X}^{-1}(n) \hat{P} \hat{X}(s) X(s) P X^{-1}(n)
\]
\[
- 2 \sum_{s = -\infty}^{n-1} \hat{X}^{-1}(n)(1 - \hat{P}) \hat{X}(s) X(s)(1 - P) X^{-1}(n).
\]
Obviously \(H(n)\) is a symmetric matrix and from (2) we can easily prove that \(H(n)\) is a bounded matrix function. Arguing as in [10, p. 63] the relation (27) is satisfied with \(\gamma = 1\).

From Lemma 1 we have that (25) has a fundamental matrix solution \(Y(n) = \hat{X}^{-1}(n)\) which satisfies (26). Let
\[
G(n) = 2 \sum_{s = n}^{\infty} X(n)(1 - P) X^{-1}(s) \hat{X}^{-1}(s)(1 - \hat{P}) \hat{X}(n)
\]
\[
- 2 \sum_{s = -\infty}^{n-1} X(n) P X^{-1}(s) \hat{X}^{-1}(s) \hat{P} \hat{X}(n).
\]
It is obvious that \(G(n)\) is a symmetric matrix and from (26) we can easily show that \(G(n)\) is a bounded matrix function. Applying the proof of Proposition 1 [10, pp. 62-63] to Eq. (25) we can easily prove that the matrix \(G(n)\) satisfies (28) with \(\gamma = 1\). So the proof of the proposition is completed.
Let $Z^+ = \mathbb{N} = \{0, 1, \ldots \}$ and $Z^- = \{\ldots, -2, -1, 0\}$.

Before giving the last result of this paper we need the following lemma.

**Lemma 2.** Suppose that (1) has exponential dichotomies on $Z^+$, $Z^-$ and has no nontrivial bounded solution on $Z$. Then there exist projections $P_+, P_-$ such that

$$P_+ P_- = P_- P_+ = P_+$$

and (1) has an exponential dichotomy on $Z^+$, $Z^-$ with projections $P_+, P_-$, respectively, corresponding to the fundamental matrix solution $X(n)$, $X(0) = I$.

**Proof.** From the hypothesis, Eq. (1) has an exponential dichotomy on $Z^+$, $Z^-$. Let $P_1, P_2$ be the corresponding projections and $X(n)$ be the fundamental matrix solution of (1) such that $X(0) = I$. Let $V_1$ be the range of $P_1$ and $V_2$ be the nullspace of $P_2$. Since (1) has no nontrivial bounded solution on $Z$ we have $V_1 \cap V_2 = \{0\}$. So, we can choose a projection $P_+$ with range space $V_1$ and a projection $P_-$ with nullspace $V_2$ such that the space $V_2$ is contained in the nullspace of $P_+$ and the space $V_1$ is contained in the range of $P_-$. Then (29) follows immediately.

Now since the projections $P_1, P_+$ have the same range and the projections $P_2, P_-$ have the same nullspace, from [1, pp. 16, 17] we can easily prove that (1) has an exponential dichotomy on $Z^+$, $Z^-$ with projections $P_+, P_-$, respectively, corresponding to $X(n)$, $X(0) = I$. Thus the proof of the lemma is completed.

**Proposition 4.** Suppose that there exist symmetric bounded matrix functions $H(n)$, $G(n)$ which satisfy (27) and (28) correspondingly. Let also $A(n)$ be bounded for all $n \in Z$. Then (1) has an exponential dichotomy on $Z$.

**Proof.** Arguing as in [10, p. 62] it is easy to show that (28) is equivalent to

$$A(\dot{y}(n) G(n) y(n)) = \dot{y}(n+1) G(n+1) y(n+1) - \dot{y}(n) G(n) y(n) \leq -\gamma |y(n)|^2,$$

where $y(n)$ is an arbitrary solution of (25).

First we prove that (25) has no nontrivial bounded solution. Suppose on the contrary that (25) has a nontrivial bounded solution $y(n)$. Since $G(n)$ is bounded for $n \in Z$ we have that the function $\phi(n) = \dot{y}(n) G(n) y(n)$ is bounded for $n \in Z$. We show that

$$\phi(0) > 0.$$
Suppose $\varphi(0) \leq 0$. From (30) we have that $\varphi(n)$ is strictly decreasing. So we have

$$\varphi(1) < \varphi(0) \leq 0. \quad (32)$$

Since from hypothesis $|G(n)| \leq \mu$, $n \in \mathbb{Z}$, $\mu > 0$ it holds $|\varphi(n)| = |\hat{y}(n) G(n) y(n)| \leq \mu |y(n)|^2$. So we get $-|y(n)|^2 \leq \mu^{-1} \varphi(n)$. Therefore from (30) we take

$$\varphi(n+1) \leq (1 + \gamma \mu^{-1}) \varphi(n), \quad n \in \mathbb{Z}$$

from which we take

$$-\varphi(n) \geq -(1 + \gamma \mu^{-1})^{n-1} \varphi(1), \quad n \geq 1.$$ 

So from (32) we have $-\varphi(n) \to \infty$ as $n \to \infty$ which contradicts the boundedness of $\varphi(n)$ on $\mathbb{Z}$. Therefore (31) holds. Since $|\varphi(n)| \leq \mu |y(n)|^2$ we get $\mu^{-1} \varphi(n) \leq |y(n)|^2$. Then from (30) we have $\varphi(n+1) \leq (1 - \gamma \mu^{-1}) \varphi(n)$ from which we obtain

$$\varphi(0)(1 - \gamma \mu^{-1})^n \leq \varphi(n), \quad n \leq 0.$$ 

Without loss of generality we may assume $\mu > \gamma$. Hence from (31) we have $\varphi(n) \to \infty$ as $n \to -\infty$ which contradicts again the boundedness of $\varphi(n)$ on $\mathbb{Z}$. So (25) has no nontrivial bounded solution.

From (27) and arguing as in Proposition 2 [10, p. 64–68] we have that (1) has exponential dichotomies on $\mathbb{Z}^+$, $\mathbb{Z}^-$. So from Lemma 1 we have that (25) has an exponential dichotomy on $\mathbb{Z}^+$, $\mathbb{Z}^-$. Then, since (25) has no nontrivial bounded solution on $\mathbb{Z}$, from Lemma 2 there exist projections $Q_1$, $Q_2$ such that

$$Q_1 Q_2 = Q_2 Q_1 = Q_1 \quad (33)$$

and (25) has an exponential dichotomy on $\mathbb{Z}^+$, $\mathbb{Z}^-$ with $Q_1$, $Q_2$, respectively, corresponding to the fundamental matrix solution $Y(n) = \hat{X}^{-1}(n)$, $X(0) = I$. Therefore from Lemma 1, Eq. (1) has an exponential dichotomy on $\mathbb{Z}^+$, $\mathbb{Z}^-$ with projections $Q_+ = I - Q_1^*$, $Q_- = I - Q_2^*$, respectively, and $X(n)$, $X(0) = I$. Relation (33) implies that

$$Q_+ Q_- = Q_- \quad (34)$$

Using (27) and arguing as above to prove that (25) has no nontrivial bounded solution on $\mathbb{Z}$ we can easily prove that (1) has also no nontrivial bounded solution on $\mathbb{Z}$. Then from Lemma 2 there exist projections $P_+$, $P_-$ such that

$$P_+ P_- = P_+ \quad (35)$$
and (1) has an exponential dichotomy on \( Z^+, Z^- \) with \( P_+, P_- \), respectively, corresponding to \( X(n), X(0) = I \).

It is clear that since (1) has an exponential dichotomy on \( Z^+ \), corresponding to the fundamental matrix solution \( X(n), X(0) = I \), we have that the range of the corresponding projection is uniquely determined as the subspace consisting of the initial values of all bounded solutions of (1) on \( Z^+ \). This implies that the projections \( P_+, Q_+ \) have the same range. Therefore we take

\[
P_+ Q_+ = Q_+. \tag{36}
\]

A similar argument shows that the projections \( I - P_- \) and \( I - Q_- \) have the same range. So we obtain

\[
P_- Q_- = P_- . \tag{37}
\]

Multiply both sides of (34) from the left by \( P_+ \). Then using (36) we get

\[
Q_- = P_+ Q_- . \tag{38}
\]

Now multiply both sides of (35) from the right by \( Q_- \). Then from (37) we have

\[
P_+ = P_+ Q_- . \tag{39}
\]

Relations (38) and (39) imply that \( P_+ = Q_+ \). So (1) has an exponential dichotomy on \( Z \). Thus the proof of the proposition is completed.

**Example 2.** Consider the system (1) where

\[
A(n) = \text{diag} \left( \frac{1}{2n+1}, \frac{2n+5}{2n+3} \right), \quad n \in Z.
\]

Taking \( H(n) = \text{diag}(H_1(n), H_2(n)), n \in Z, \) where

\[
H_1(n) = \frac{4}{27} \frac{1}{(2n+1)^2} (36n^2 + 60n + 41)
\]

\[
H_2(n) = - \frac{1}{27(2n+3)^2} (36n^2 + 12n + 17),
\]

we can easily prove that

\[
\dot{A}(n) H(n+1) A(n) - H(n) = -I, \quad n \in Z.
\]
Consider now the adjoint system of (1)

\[ x(n + 1) = A^{-1}(n) x(n) = \text{diag} \left( \frac{2n+1}{2n+3}, \frac{2n+3}{2n+5} \right) x(n), \quad n \in \mathbb{Z}. \]

We set \( G(n) = \text{diag}(G_1(n), G_2(n)) \), \( G_1(n) = -H_1^{-1}(n) \), \( G_2(n) = -H_2^{-1}(n) \). Since the relations

\[ \frac{(2n+1)^2}{36n^2 + 60n + 41} \geq \frac{1}{41} \quad \text{and} \quad \frac{(2n+1)^2}{36n^2 + 132n + 137} \geq \frac{1}{137} \]

hold it is easily proved that

\[ \frac{4(2n+1)^2}{(2n+3)^2} G_1(n+1) - G_1(n) \leq -0.03, \quad n \in \mathbb{Z}. \]

Now, from

\[ \frac{(2n+3)^2}{36n^2 + 12n + 17} \geq \frac{1}{200} \quad \text{and} \quad \frac{(2n+3)^2}{36n^2 + 84n + 65} \geq \frac{1}{50}, \]

and using simple calculations we get

\[ \frac{1}{4(2n+5)^2} G_2(n+1) - G_2(n) \leq -0.01, \quad n \in \mathbb{Z}. \]

Then it is obvious that

\[ A^{-1}(n) G(n+1) A^{-1}(n) - G(n) \leq -0.01 I, \quad n \in \mathbb{Z}. \]

So all the hypotheses of Proposition 4 are satisfied. Therefore (1) must have an exponential dichotomy on \( \mathbb{Z} \). We can easily prove that (1) has an exponential dichotomy on \( \mathbb{Z} \) with projection \( P = \text{diag}(1, 0) \) and constants \( K = 5, \ p = (\frac{1}{2})^{1/2} \).

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