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J. Differential Equations 249 (2010) 2600–2617

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Well-posedness and small data scattering for the generalized Ostrovsky equation ☆

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ARTICLE INFO

Article history:
Received 11 December 2009
Revised 6 May 2010
Available online 21 May 2010

ABSTRACT

We consider the generalized Ostrovsky equation $u_{tx} = u + (u^p)_{xx}$. We show that the equation is locally well posed in H^s, $s > 3/2$ for all integer values of $p \geqslant 2$. For $p \geqslant 4$, we show that the equation is globally well posed for small data in $H^5 \cap W^{3,1}$ and moreover, it scatters small data. The latter results are corroborated by numerical computations which confirm the heuristically expected decay of $\|u\|_{L^r} \sim t^{-(r-2)/(2r)}$.

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1. Introduction

In this work, we consider the initial value problem for the generalized Ostrovsky (gO for short) equation

$$\left| \begin{array}{l} u_{tx} = u + \left(u^p\right)_{xx}, \quad (t,x) \in \mathbf{R}^1_+ \times \mathbf{R}^1, \\ u(0,x) = f(x), \end{array} \right. \tag{1}$$

where $p \geqslant 2$ is an integer and the initial data f is considered in some Sobolev class $H^s(\mathbf{R}^1)$ with sufficiently high s. Let us assume here and henceforth that the data f that we consider (and hence the solution u) are real-valued.

The motivation for this generalized equation stems from a variety of different areas. The case of $p = 2$ arises as a reduced version of the so-called Ostrovsky equation [1], often referred to

☆ Stefanov is supported in part by NSF-DMS-0701802 and NSF-DMS-0908802. Kevrekidis is supported in part by NSF-DMS-0349023, NSF-DMS-0806762 and by the Alexander von Humboldt Foundation.

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0022-0396/$ – see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2010.05.015

as reduced Ostrovsky equation [2,3], short-wave equation [4], Ostrovsky–Hunter equation [5], or Vakhnenko equation [6,7]. This model arises in different settings such as, for instance, the case of small-amplitude long waves in rotating fluids of finite depth, under the assumption of no-high-frequency dispersion. On the other hand, the case of $p = 3$ has gained a considerable momentum in the nonlinear optical community recently, due to its derivation from Maxwell's equations (under appropriate assumptions) as a model for very short pulse propagation in nonlinear media [8]. For this type of pulses, its favorable comparison to Maxwell [compared to the "standard" nonlinear Schrödinger model] [9] rendered it an interesting topic for study from a physical point of view. On the other hand, the short-pulse equation (SPE) proved to be extremely interesting from a mathematical point of view due to the existence of an infinite hierarchy of conserved quantities [10], an ingenious transformation that related it to the integrable sine-Gordon equation and illustrated its complete integrability [11] and which, in turn, allowed the calculation of explicit analytical solutions of loop- and of breather-form for this model [12]. More recently, on the analysis side, the global well-posedness question [13] and wave-breaking phenomena in this equation were studied [16], while interesting generalizations such as the regularized version of the SPE [17] and applications including the emergence of SPE in descriptions of nonlinear metamaterials [18] have also emerged. Notice that while we are not aware of applications presently of this equation for $p \geqslant 4$, we will consider the equation in its generalized form presented above, keeping our results as general as possible in what follows.

Our first result is a local well-posedness result in the Sobolev space[1] $H^{3/2+}$, which improves upon earlier work of Schäfer and Wayne [8]. Before we get to the specifics of it, let us first clarify in what one can expect in our situation. Note that (1) is a *quasilinear wave equation*, in the sense that the highest order spatial derivative comes linearly, but it has a solution dependent coefficient. Thus, one does not expect to produce a solution via a fixed point argument. In fact, it is well known that such equations will in general not have Lipschitz dependence on the initial data (which would be one of the consequences of a fixed point iteration procedure). We will work with the following

Definition 1. We say that Eq. (1) is locally well posed in H^{s_0}, $s_0 \geqslant 0$, if:

1. There exists a sufficiently large s_1, so that for any initial data $f \in H^{s_1}$, there exist a time $T_0 = T(\|f\|_{H^{s_0}})$ and a classical solution $u \in C[(0,T_0), H^{s_1}) \cap C^1[(0,T_0), H^{s_1-1})$.
2. There exists s_2: $0 \leqslant s_2 \leqslant s_0$, so that for any $f, g \in H^{s_1}$, there exists $C = C(\|f\|_{H^{s_0}}, \|g\|_{H^{s_0}})$, so that the corresponding solutions u, v satisfy the estimate

$$\sup_{0\leqslant t\leqslant T_0} \big\|u(t) - v(t)\big\|_{H^{s_2}} \leqslant C\big(\|f\|_{H^{s_0}}, \|g\|_{H^{s_0}}\big)\|f - g\|_{H^{s_2}},$$

where T_0 is the smaller lifespan of the two solutions u, v.

Note that the local well-posedness in the sense of Definition 1 guarantees uniqueness of solutions, constructed as a limit of classical solutions. Indeed, for fixed initial data $f \in H^{s_0}$, take a sequence $f^n \in H^{s_1}$ approximating f in the H^{s_0} norm, in particular we may arrange so that $\sup_n \|f^n\|_{H^{s_0}} \leqslant 2\|f\|_{H^{s_0}}$. The corresponding classical solutions will then all exist for some time $T_0 = T_0(\|f\|_{H^{s_0}})$ by the first requirement of Definition 1. Moreover, by the second requirement of Definition 1, we will have that $\{u^n(t,\cdot)\}$ will be a Cauchy sequence in H^{s_2} for $0 \leqslant t \leqslant T_0$ and hence its limit (i.e. the solution emanating from f) will be well defined and unique in this class.

Another property of solutions of (1) constructed as a limit of classical ones (that is via the procedure outlined above) is that they are weak solutions, provided $s_2 > 1/2$. A reasonable definition of a weak solution is a function $u(t) \in H^{s_2}$, $0 \leqslant t \leqslant T$ satisfying[2]

[1] For precise definition of Sobolev spaces, we refer the reader to Section 2.1 below.
[2] Which is a subset of all L^p, $p \geqslant 2$ by Sobolev embedding.

$$\int_0^T \int_{-\infty}^{\infty} u(t,x)\psi_{tx}(t,x)\,dx\,dt + \int_{-\infty}^{\infty} f(x)\psi_x(0,x)\,dx = \int_0^T \int_{-\infty}^{\infty} \left[u\psi + u^p\psi_{xx}\right] dx\,dt, \tag{2}$$

for any Schwartz function $\psi : \mathbf{R}^1 \times \mathbf{R}^1 \to \mathbf{R}^1$, so that $supp\, \psi \subset [0,T) \times \mathbf{R}^1$. If we have a sequence of smooth initial data $f^n \to f$ in H^{s_0}, then by Definition 1, the corresponding classical solutions $\{u^n(t)\}$ will form a Cauchy sequence in $C[(0,T]; H^{s_2}(\mathbf{R}^1))$ with limit $u(t)$. In particular

$$\left| \int u(t,x)\psi_{tx}(t,x)\,dx\,dt - \int u^n(t,x)\psi_{tx}(t,x)\,dx\,dt \right| \leqslant C_\psi \left\| u^n - u \right\|_{L^\infty_T L^2} \to 0,$$

$$\left| \int \left(u^n\right)^p \psi_{xx}\,dx\,dt - \int u^p \psi_{xx}\,dx\,dt \right| \leqslant C_\psi \left\| u^n - u \right\|_{L^p} \left(\left\| u^n \right\|_{L^p}^{p-1} + \|u\|_{L^p}^{p-1} \right) \to 0,$$

since $\|u^n - u\|_{L^p} \leqslant \|u^n - u\|_{H^{s_2}}$ etc. As a result, we obtain (2) for u. That is, u is a weak solution, if it is a limit of smooth solutions in the sense of Definition 1.

Theorem 1 *(Local well-posedness for the gO equation). Let $p \geqslant 2$ be an integer and $s > 3/2$ be a fixed real number. Then the initial value problem* (1) *is locally well posed in $H^s(\mathbf{R}^1)$ in the sense of Definition* 1*. In particular, for any $f \in H^s(\mathbf{R}^1)$, there exists a time $0 < T_0 = T_0(\|f\|_{H^s}) \leqslant \infty$, so that the problem* (1) *has a unique strong solution in the space $C([0,T_0), H^s(\mathbf{R}^1))$.*

Our next result concerns the existence of global solutions to (1), provided the initial data is small. We refer to the work of Pelinovsky and Sakovich [13] for related results in the case $p = 3$. In addition to the well-posedness, we are able to establish in this paper scattering for small solutions, provided $p \geqslant 4$.

Theorem 2 *(Global well-posedness and scattering for the gO equation). Let $p \geqslant 4$ be an integer. Then, there exists $\varepsilon = \varepsilon(p) > 0$, so that whenever $\|f\|_{H^5} + \|f\|_{W^{3,1}} < \varepsilon$, the gO equation* (1) *has a unique global solution in $C([0,\infty), H^5(\mathbf{R}^1))$. In addition, the solution is small,*

$$\sup_{0<t<\infty} \left\| u(t) \right\|_{H^5} \leqslant 4\varepsilon,$$

and it scatters in the sense that for $\delta = 0.001$ and for all

$$q, r \in (2,\infty): \quad 1/q + 1/r \leqslant 1/2 - \delta; \qquad 1/q < 1/2 - 3\delta; \qquad r < (p-1)/\delta,$$

there is the estimate

$$\|u\|_{L^q_t W^{3/2,r}} \leqslant C_p \varepsilon.$$

Heuristically speaking, we expect $\|u(t)\|_{L^r}$, $r > 2$ to decay like $t^{-(r-2)/(2r)}$.

In what follows, we present the proof of the two theorems stated above (Sections 2 and 4, respectively), with an intermediate section detailing some dispersive estimates for the free Ostrovsky equation (Section 3). Finally in Section 5, we present some numerical computations that support our heuristic expectation for the scattering of small initial data and the decay of the L^r norms, for the cases of $p \geqslant 4$.

2. Proof of Theorem 1

In this section, we show Theorem 1. It is standard that such local well-posedness results may be essentially reduced to *a priori* estimates for certain H^s norms. In order to explain the procedure in some detail, let us write Eq. (1) in the form

$$u_{tx} = u + \left(pu^{p-1}u_x\right)_x.$$

At this point, we consider the following iterative procedure. Set $u_0 = f(x)$, whereas u_N, $N \geqslant 1$, is defined to be the solution to the *linear* equation,

$$\partial_{tx}u_N = u_N + \left(pu_{N-1}^{p-1}\partial_x u_N\right)_x. \tag{3}$$

It is clear right away, that if u_N is a smooth and decaying solution (which will be shown *a posteriori*), then $\int_{-\infty}^{\infty} u_N(t,x)\,dx = 0$. In order to get estimates on $\|u_N\|_{L^2}$, introduce $v = \int_{-\infty}^{x} u_N(t,y)\,dy$, which vanishes at both infinities and moreover

$$v_{txx} = v_x + \left(pu_{N-1}^{p-1}v_{xx}\right)_x.$$

An integration in x yields the equation $v_{tx} = v + pu_{N-1}^{p-1}v_{xx}$. Take dot product with v_x

$$\partial_t\left\|v_x(t)\right\|_{L^2}^2 = -2p(p-1)\int v_x^2 u_{N-1}^{p-2}\partial_x u_{N-1}\,dx, \tag{4}$$

which implies

$$\partial_t\|u_N\|_{L^2}^2 = \partial_t\left\|v_x(t)\right\|_{L^2}^2 \leqslant 2p(p-1)\left\|u_N(t)\right\|_{L^2}^2\left\|u_{N-1}(t)\right\|_{H^{3/2+}}^{p-1}. \tag{5}$$

This allows one to control $\|u_N\|_{L^2}^2$ in terms of the $H^{3/2+}$ norm of u_{N-1}.

The main technical lemma needed for the proof of Theorem 1 is the following *a priori* energy estimate, which gives control of higher Sobolev norms.

Lemma 1. *Let u be a smooth solution of the equation*

$$u_{tx} = u + F(t,x)u_{xx} + G(t,x), \quad t > 0, \tag{6}$$

where F, G are smooth functions. Then, for every $s > 1$, there exist a constant C_s ($C_s \sim 1/(s-1)$), and an absolute constant C, so that

$$\frac{d}{dt}I_s(t) \leqslant C_s\left\|F_x(t,\cdot)\right\|_{L^\infty}I_s(t) + 2\sqrt{I_s(t)}\left(\left\|G(t,\cdot)\right\|_{\dot{H}^{s-1}} + C\|u_x\|_{L^\infty}\left\|F(t,\cdot)\right\|_{\dot{H}^s}\right), \tag{7}$$

where $I_s(t) = \|u(t,\cdot)\|_{\dot{H}^s}^2$.

Assuming Lemma 1 for a moment, let us finish the proof of Theorem 1. In order to get estimates on higher Sobolev norms, we apply Lemma 1 to (3), where $F = pu_{N-1}^{p-1}$, $G = p(p-1)u_{N-1}^{p-2}\partial_x u_{N-1}\partial_x u_N$. Clearly,

$$\|F\|_{\dot{H}^s} \leqslant C_p\|u_{N-1}\|_{\dot{H}^s}\|u_{N-1}\|_{L^\infty}^{p-2}, \qquad \|F_x\|_{L^\infty} \leqslant \|u_{N-1}\|_{H^{3/2+}}^{p-1},$$

$$\left\|G(t,\cdot)\right\|_{\dot{H}^{s-1}} \leqslant C_p\left(\|u_N\|_{\dot{H}^s}\|u_{N-1}\|_{H^{3/2+}}^{p-1} + \|\partial_x u_N\|_{L^\infty}\|u_{N-1}\|_{H^{3/2+}}^{p-2}\|u_{N-1}\|_{\dot{H}^s}\right).$$

By (5) and (7), we obtain[3] the *a priori* estimate for $I_s^N(t) = \|u_N(t,\cdot)\|_{H^s}^2$ in the form

$$\frac{d}{dt} I_s^N(t) \leqslant C_{s,p}\big(\big\|u_{N-1}(t)\big\|_{H^{3/2+}}^{p-1} + \|u_{N-1}\|_{H^{3/2+}}^{p-2}\|u_{N-1}\|_{\dot{H}^s}\big) I_s^N(t).$$

Integrating in time yields

$$\big\|u_N(t,\cdot)\big\|_{H^s}^2 = I_s^N(t) \leqslant \|f\|_{H^s}^2 \exp\left(C_{p,s}\int_0^t \big\|u_{N-1}(\tau)\big\|_{H^s}^{p-1}\,d\tau\right). \tag{8}$$

Clearly (8) is what we need to show the existence of a weak solution of the quasilinear equation (1). Indeed, setting $C_1 := 2\|f\|_{H^s}^2$, we have $\|u_0\|_{H^s} \leqslant C_1$. We then proceed with an inductive argument, which shows that for appropriate T_0 (namely the solution of the equation $2 = \exp(C_{p,s}T_0(2C_1)^{p-1})$, where $C_{p,s}$ is the constant in (8)), the inequality $\sup_{0\leqslant t\leqslant T_0}\|u_{N-1}(t)\|_{H^s} \leqslant C_1$ implies $\sup_{0\leqslant t\leqslant T_0}\|u_N(t)\|_{H^s} \leqslant C_1$. This shows that we have constructed a bounded in the topology of H^s sequence $\{u_N\}$, which satisfies (3). We have thus verified the first part of Definition 1. Standard arguments apply to extract a subsequence, whose (weak) limit[4] will serve as a weak solution of (1). The construction here is similar to the argument presented in [8, Theorem 4.4].

Regarding uniqueness, take two solutions v, w of (1) and set $z = v - w$. It follows that z satisfies

$$z_{tx} = z + pv^{p-1}z_{xx} + p\big(v^{p-1} - w^{p-1}\big)w_{xx} + p(p-1)v^{p-2}v_x^2 - p(p-1)w^{p-2}w_x^2. \tag{9}$$

Setting $F = pv^{p-1}$ and $G = p(v^{p-1} - w^{p-1})w_{xx} + p(p-1)v^{p-2}v_x^2 - p(p-1)w^{p-2}w_x^2$, we apply Lemma 1 to (9). We obtain for $s > 3/2$,

$$\frac{d}{dt}\big\|v(t) - w(t)\big\|_{\dot{H}^s} \leqslant C_{s,p}\big\|v(t) - w(t)\big\|_{\dot{H}^s}\big(\|w\|_{H^{s+1}} + \|v\|_{H^s}\big)\big(\|v\|_{H^s} + \|w\|_{H^s}\big)^{p-2}.$$

Similar estimates hold for $\|v(t) - w(t)\|_{L^2}$, whence

$$\big\|v(t) - w(t)\big\|_{H^s} \leqslant C\big(s, p, \|w\|_{H^{s+1}}, \|v\|_{H^s}\big)\big\|v(0) - w(0)\big\|_{H^s}$$

up to the time t of joint existence for both v, w. This of course implies the Lipschitz property of the solution map in the sense of Definition 1.

As one can see from the last inequality, the uniqueness statement is rather weak, in the sense that the solution map is Lipschitz only from $H^{s+1} \to H^s$. This is a standard loss of smoothness issue with quasilinear wave equations of this form, see for example Chapter II in the book [19], where similar issues are discussed in great detail.

Thus, to complete the proof of Theorem 1, it remains to prove Lemma 1. Before we dispense with that, we shall need

2.1. Some Fourier analysis preliminaries

Define the Fourier transform $\mathcal{F}$ (and its inverse respectively) acting on a function $f \in \mathcal{S}$ ($\mathcal{S}$ is the Schwartz class of test functions)

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[3] Recalling $s > 3/2+$.

[4] Which become strong limits on bounded sets, after subsequences, in $H^{s'}$, $s' < s$.

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx,$$

$$f(x) = \int_{\mathbf{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi.$$

The (homogeneous) Sobolev space $\dot{H}^s$ is defined as the completion of all Schwartz functions in the norm

$$\|f\|_{\dot{H}^s} = \left(\int_{\mathbf{R}^n} \left|\hat{f}(\xi)\right|^2 |\xi|^{2s}\, d\xi\right)^{1/2}.$$

We will also use the inhomogeneous version H^s, defined by the norm

$$\|f\|_{H^s} = \left(\int_{\mathbf{R}^n} \left|\hat{f}(\xi)\right|^2 \left(1+|\xi|^2\right)^s\right). \tag{10}$$

Next, we introduce the Littlewood–Paley decompositions of a given function f. Let $\psi \in C_0^\infty(\mathbf{R}^n)$ be such that $supp\, \psi \subset (0,2)$ and $\psi(\xi) = 1$ for all $|\xi| \leqslant 1$. Let $\varphi(\xi) = \psi(\xi) - \psi(2\xi)$. Clearly

$$\sum_{j\in Z} \varphi\left(2^{-j}\xi\right) \equiv 1 \quad \text{for all } \xi \neq 0,$$

which gives rise to the **Littlewood–Paley operators**, defined by the multipliers $\varphi(2^{-k}\cdot)$, namely

$$\widehat{P_k f}(\xi) := \varphi\left(2^{-k}\xi\right)\hat{f}(\xi).$$

The Littlewood–Paley operators are, roughly speaking, projections with range all functions having Fourier support in the annulus $\{\xi\colon\ 2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}$. We will very often write f_k instead of $P_k f$ and $P_{<k} := \sum_{l<k} P_l$. Note that

$$f = \sum_{k\in Z} f_k = \lim_{k\to\infty} P_{<k} f.$$

We also define the related operators $P_{\sim k} = P_{k-2} + \cdots + P_{k+2}$, as well as $P_{\nsim k} = Id - P_{\sim k} = P_{<k-2} + P_{>k+2}$ both of which will be useful. We wish also to make use of the following elementary observation. Since $supp\, \widehat{fg} \subset supp\, \hat{f} + supp\, \hat{g}$, we have in particular that $P_k[u_{\leqslant k-3} v_{\leqslant k-3}] = 0$[5] and $P_k[u_{\geqslant k+3} v_{\leqslant k-3}] = 0$ and hence

$$P_k[u v_{\leqslant k-3}] = P_k[u_{k-2\leqslant\cdot\leqslant k+2} v_{\leqslant k-3}]. \tag{11}$$

Similarly, since $P_k[u v_{\geqslant k+3}] = \sum_{l\geqslant k+3} P_k[u v_l]$ and $P_k[u_{\leqslant l-3} v_l] = 0$, $P_k[u_{\geqslant l+3} v_l] = 0$, we have

$$P_k[u v_{\geqslant k+3}] = \sum_{l\geqslant k+3} P_k[u_{l-2\leqslant\cdot\leqslant l+2} v_l]. \tag{12}$$

[5] Here the Fourier support of $u_{\leqslant k-3} v_{\leqslant k-3}$ is inside $\{|\xi| \leqslant 2^{k-2}\}$, according to the rule $supp\, \widehat{fg} \subset supp\, \hat{f} + supp\, \hat{g}$ and hence the Littlewood–Paley operator P_k annihilates it.

We recall the standard definition of the L^p Sobolev spaces $W^{s,p}$ with integer s

$$\|f\|_{W^{s,p}} = \sum_{\lambda:\ |\alpha|\leqslant s} \left\|\partial^\alpha f\right\|_{L^p}.$$

However, for noninteger s, one has to resort to the fractional derivatives, defined in terms of the Fourier transform $\widehat{|\nabla|^s f}(\xi) = |\xi|^s \hat{f}(\xi)$ and then set

$$\|f\|_{W^{s,p}} := \|f\|_{L^p} + \left\||\nabla|^s f\right\|_{L^p}.$$

An equivalent way to define a norm is given via the Littlewood–Paley square function characterization of L^p (or more generally $W^{s,p}$ spaces), namely for all $1 < p < \infty$,

$$\|f\|_{\dot{W}^{s,p}} := \left\||\nabla|^s f\right\|_{L^p} \sim \left\|\left(\sum_k 2^{2ks}|f_k|^2\right)^{1/2}\right\|_{L^p}. \tag{13}$$

This will be very useful in what follows. Due to (13) and the inclusion $l^p \subset l^q$ for all $1 \leqslant p < q \leqslant \infty$, we obtain the useful formulae

$$\|f\|_{\dot{W}^{s,p}} \leqslant C_p\left(\sum_k 2^{2ks}\|f_k\|_{L^p}^2\right)^{1/2}, \quad 2 \leqslant p < \infty, \tag{14}$$

$$\left(\sum_k 2^{2ks}\|f_k\|_{L^p}^2\right)^{1/2} \leqslant C_p\|f\|_{\dot{W}^{s,p}}, \quad 1 < p \leqslant 2. \tag{15}$$

We also note that since P_k (and $P_{<k}$) is given by a convolution with $2^{kn}\check{\varphi}(2^k\cdot)$, we have $\|P_k\|_{L^p\to L^p} \leqslant \|2^{kn}\check{\varphi}(2^k\cdot)\|_{L^1} \lesssim 1$ for all $1 \leqslant p \leqslant \infty$. A useful observation along the same lines is

$$\left\||\nabla|^s f_k\right\|_{L^p} \sim 2^{ks}\|f_k\|_{L^p}. \tag{16}$$

Indeed, the smooth function $\tilde{\varphi} := \psi(\xi/8) - \psi(8\xi)$ is identically one on the support of φ and thus,

$$\left\||\nabla|^s f_k\right\|_{L^p} = 2^{ks}\|\tilde{P}_k f_k\|_{L^p} \leqslant 2^{ks}\|f_k\|_{L^p},$$

where $\widehat{\tilde{P}_k g} := 2^{-ks}|\xi|^s\tilde{\varphi}(2^{-k}\xi)\hat{g}(\xi)$ is a bounded[6] operator on L^p. The reverse inequality can be seen in a similar way. The following lemma gives a useful estimate, when one needs to estimate commutators of the Littlewood–Paley operators P_k with smooth functions. Although it is a standard fact, we present it in the form of Lemma 2.1 of [15].

Lemma 2. *(See Lemma 2.1 of [15].) Let f, g be Schwartz functions and $1 \leqslant p, q, r \leqslant \infty$: $1/r = 1/p + 1/q$. Then, there exists a constant $C = C_d$, so that*

$$\left\|[P_k, f]g\right\|_{L^r(\mathbf{R}^d)} \leqslant C2^{-k}\|\nabla f\|_{L^p}\|g\|_{L^q}.$$

[6] With a $B(L^p)$ norm bounded independent of k.

In particular,

$$\left\|[P_k, u]\partial v_{\sim k}\right\|_{L^r} \leqslant C\|\nabla u\|_{L^p}\|v_{\sim k}\|_{L^q}.$$

The following lemma (due to Christ–Weinstein and Kato–Ponce), widely known as the Fractional Leibnitz's rule, will be used repeatedly in our arguments.

Lemma 3. *Let $s > 0$, $1 < r \leqslant p_1, p_2 < \infty$, $r \leqslant q_1, q_2 \leqslant \infty$, so that $1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$. Then there exists $C = C_{s,p_1,p_2,q_1,q_2}$, so that for every pair of functions $f \in W^{s,p_1} \cap L^{q_2}$, $g \in L^{q_1} \cap W^{s,p_2}$,*

$$\|fg\|_{W^{s,r}} \leqslant C_{s,p_1,p_2,q_1,q_2}\big(\|f\|_{W^{s,p_1}}\|g\|_{L^{q_1}} + \|g\|_{W^{s,p_2}}\|f\|_{L^{q_2}}\big).$$

2.2. Proof of Lemma 1

We are now ready to proceed with the presentation of Lemma 1. Let us make the following notational convention – we will use the notation h' to denote a spatial derivative, unless the function h depends explicitly only on time and then, it will denote the time derivative. We turn back on our energy estimates. Take the Littlewood–Paley projection P_k of (6)

$$\partial_t(u_k)_x = u_k + (Fu_{xx})_k + G_k.$$

Next, take a dot product with $(u_k)_x$

$$\frac{1}{2}\partial_t\|\partial_x u_k\|_{L^2}^2 = \int (Fu_{xx})_k\partial_x u_k\,dx + \int G_k\partial_x u_k\,dx. \tag{17}$$

For the second term on the right-hand side, we just apply the Cauchy–Schwartz inequality $|\int G_k\partial_x u_k\,dx| \leqslant \|G_k\|_{L^2}\|\partial_x u_k\|_{L^2}$. For the first term, we need to perform more careful analysis. Write

$$(Fu_{xx})_k = P_k[F\partial_{xx}u_{\sim k}] + (F\partial_{xx}u_{\nsim k})_k.$$

Furthermore, write

$$P_k[F\partial_{xx}u_{\sim k}] = F\partial_{xx}u_k + [P_k, F]\partial_{xx}u_{\sim k}.$$

Denote $Err^k := [P_k, F]\partial_{xx}u_{\sim k} + (F\partial_{xx}u_{\nsim k})_k$. Thus, an integration by parts yields

$$\begin{aligned}\int (Fu_{xx})_k\partial_x u_k\,dx &= \int F\partial_{xx}u_k\partial_x u_k\,dx + \int Err^k(t,x)\partial_x u_k\,dx\\ &= -\frac{1}{2}\int F_x(\partial_x u_k)^2\,dx + \int Err^k(t,x)\partial_x u_k\,dx.\end{aligned}$$

Hence, we estimate

$$\left|\int (Fu_{xx})_k\partial_x u_k\,dx\right| \leqslant \|F_x\|_{L^\infty}\|\partial_x u_k\|_{L^2}^2 + \left\|Err^k\right\|_{L^2}\|\partial_x u_k\|_{L^2}.$$

Next, recall that by (16), we have $\|\partial_x u_k\|_{L^2} \sim 2^k\|u_k\|_{L^2}$, whence (17) and the subsequent estimates may be rewritten (after multiplication by $2^{2k(s-1)}$) as follows

$$\partial_t 2^{2ks}\|u_k\|_{L^2}^2 \leqslant \|F_x(t,\cdot)\|_{L^\infty} 2^{2ks}\|u_k\|_{L^2}^2 + 2^{2k(s-1)}\big(\|G_k\|_{L^2} + \|Err^k\|_{L^2}\big)2^k\|u_k\|_{L^2}.$$

Taking a sum in k and denoting $I_s := \sum_k 2^{2ks}\|u_k\|_{L^2}^2$, we conclude by Cauchy–Schwartz

$$I_s'(t) \leqslant \|F_x(t,\cdot)\|_{L^\infty} I_s(t) + 2\sqrt{I_s(t)}\left(\|G\|_{\dot H^{s-1}} + \left(\sum_k 2^{2k(s-1)}\|Err^k\|_{L^2}^2\right)^{1/2}\right).$$

It thus remains to suitably estimate $\sum_k 2^{2k(s-1)}\|Err^k\|_{L^2}^2$. There are two terms to deal with. For the commutator term, we have by Lemma 2

$$\sum_k 2^{2k(s-1)}\big\|[P_k, F]\partial_{xx}u_{\sim k}\big\|_{L^2}^2 \leqslant C\|F'\|_{L^\infty}^2 \sum_k 2^{2k(s-1)}\|\partial_x u_{\sim k}\|_{L^2}^2 \leqslant C\|F'\|_{L^\infty}^2\|u(t,\cdot)\|_{\dot H^s}^2.$$

For the term $\sum_k 2^{2k(s-1)}\|(F\partial_{xx}u_{\nsim k})_k\|_{L^2}^2$, observe that by (11) and (12), we have the representation

$$\begin{aligned}(F\partial_{xx}u_{\nsim k})_k &= (F\partial_{xx}u_{\leqslant k-3})_k + (F\partial_{xx}u_{\geqslant k+3})_k \\ &= (F_{k-2\leqslant\cdot\leqslant k+2}\partial_{xx}u_{\leqslant k-3})_k + \sum_{l\geqslant k+3}(F_{l-2\leqslant\cdot\leqslant l+2}\partial_{xx}u_l)_k\end{aligned}$$

and hence, we need to estimate the two new terms separately. We have

$$\begin{aligned}\sum_k 2^{2k(s-1)}\big\|(F_{k-2\leqslant\cdot\leqslant k+2}\partial_{xx}u_{\leqslant k-3})_k\big\|_{L^2}^2 &\leqslant C\|u'\|_{L^\infty}^2\sum_k 2^{2ks}\|F_{k-2\leqslant\cdot\leqslant k+2}\|_{L^2}^2 \\ &\leqslant C\|u'\|_{L^\infty}^2\|F\|_{\dot H^s}^2.\end{aligned}$$

For the other term, we have

$$\sum_k 2^{2k(s-1)}\left(\sum_{l\geqslant k+3}\big\|(F_{l-2\leqslant\cdot\leqslant l+2}\partial_{xx}u_l)_k\big\|_{L^2}\right)^2 \leqslant C_s\|F'\|_{L^\infty}^2\sum_k 2^{2k(s-1)}\left(\sum_{l\geqslant k+3}\|\partial_x u_l\|_{L^2}\right)^2.$$

It now remains to appropriately estimate the last double sum for $s>1$. We have

$$\begin{aligned}\sum_k 2^{2k(s-1)}\left(\sum_{l\geqslant k+3}\|\partial_x u_l\|_{L^2}\right)^2 &= \sum_k 2^{2k(s-1)}\sum_{l_1,l_2\geqslant k+3}\|\partial_x u_{l_1}\|_{L^2}\|\partial_x u_{l_2}\|_{L^2} \\ &\leqslant C_s\sum_{l_1,l_2} 2^{2\min(l_1,l_2)(s-1)}\|\partial_x u_{l_1}\|_{L^2}\|\partial_x u_{l_2}\|_{L^2} \leqslant 10C_s\|u\|_{\dot H^s}^2,\end{aligned}$$

where $C_s = \sum_{j<0} 2^{2j(s-1)}$.

All in all, entering all the estimates that we have obtained

$$I_s'(t) \leqslant C_s\|F'(t,\cdot)\|_{L^\infty} I_s(t) + 2\sqrt{I_s(t)}\big(\|G(t,\cdot)\|_{\dot H^{s-1}} + C\|u'\|_{L^\infty}\|F(t,\cdot)\|_{\dot H^s}\big),$$

as claimed.

3. Dispersive estimates for the free Ostrovsky equation

In this section, we show that an appropriate decay and Strichartz estimates hold for the initial value problem for the free Ostrovsky evolution

$$\begin{cases} u_{tx} = u + F(t,x), & (t,x) \in \mathbf{R}^1_+ \times \mathbf{R}^1, \\ u(0,x) = f(x). \end{cases} \tag{18}$$

We have the following

Theorem 3 *(Decay and Strichartz estimates for the linear Ostrovsky equation). For the homogeneous Ostrovsky equation (i.e. $F = 0$), we have:*

- (*Energy conservation*) *For every s,*

$$\|u(t,\cdot)\|_{\dot{H}^s} = \|f\|_{\dot{H}^s}. \tag{19}$$

- (*Decay estimates*) *For $2 < p < \infty$, there exists C_p, so that for all initial data*[7] *f,*

$$\|u(t,\cdot)\|_{L^p(\mathbf{R}^1)} \leqslant C_p t^{-(1/2-1/p)} \|f\|_{\dot{W}^{3/2-3/p,p'}}. \tag{20}$$

As a consequence, the solutions to the inhomogeneous Ostrovsky equation (18) *obey Strichartz estimates. More precisely, there exists an absolute constant C, so that whenever $2 \leqslant q, r, \tilde{q}, \tilde{r} < \infty$, and $2/q + 1/r \leqslant 1/2$; $2/\tilde{q} + 1/\tilde{r} \leqslant 1/2$, then*

$$\|u\|_{L^q_t \dot{W}^{\alpha,r}_x} \leqslant C\left(\|f\|_{\dot{H}^{1/2+1/q-1/r+\alpha}} + \|F\|_{L^{\tilde{q}'}_t \dot{W}_x^{1/r+1/\tilde{r}-1/q-1/\tilde{q}+\alpha,\tilde{r}'}}\right),$$

for all α.

Proof. For the proof of Theorem 3, it suffices to restrict to the case $f, F \in \mathcal{S}$, since the general case follows by density. Next, we point out that it is a classical result by now,[8] that the Strichartz estimates are a direct consequence of the decay and energy estimates, such as (19), (20). We thus concentrate on those for the rest of the section.

We use the Fourier transform to solve the homogeneous Ostrovsky equation. Namely $\partial_t \widehat{u_x}(t,\xi) = \hat{u}(t,\xi)$, whence one integrates the ODE in t to the formula

$$\hat{u}(t,\xi) = \hat{f}(\xi) \exp\left(-i\frac{t}{2\pi\xi}\right) := T(t)f(\xi). \tag{21}$$

We have to point out that such a formula holds in a classical sense for $\xi \neq 0$, but note that we may still define it rigorously via (21) for all $f \in \mathcal{S}$, so that $\hat{f}(0) = 0$ (which is still a dense subspace in L^p, where $1 < p < \infty$).

The energy conservation is obvious, since $|\exp(-i\frac{t}{2\pi\xi})| = 1$ and hence by Plancherel's

$$\|u(t)\|_{L^2} = \|\hat{u}(t)\|_{L^2} = \left\|\hat{f}(\cdot)\exp\left(-i\frac{t}{2\pi\cdot}\right)\right\|_{L^2} = \|\hat{f}\|_{L^2} = \|f\|_{L^2},$$

[7] From now, we will make the following assignment p': $\frac{1}{p} + \frac{1}{p'} = 1$.

[8] Of Keel and Tao in [14].

and similarly $\|u(t)\|_{\dot{H}^s} = \|f\|_{\dot{H}^s}$. In particular, formula (21) is well defined for L^2 data f. The decay estimate (20) is more complicated. We need a series of reductions. First, note that by (14) and (15), we may reduce (20) to the proof of

$$\left\|u_k(t,\cdot)\right\|_{L^p} \leqslant C_p t^{-(1/2-1/p)} 2^{(3/2-3/p)k} \|f_k\|_{L^{p'}}, \tag{22}$$

for all integer k, where $u_k = P_k u$, $f_k = P_k f$. Indeed, note that (22) is just an instance of (20) for frequency localized data f_k. On the other hand, assuming (22) for all k (and with a constant C_p, independent on k), we have, after squaring and summing,

$$\begin{aligned}\|u\|_{L^p} &\lesssim \left(\sum_k \left\|u_k(t,\cdot)\right\|_{L^p}^2\right)^{1/2} \leqslant C_p t^{-(1/2-1/p)} \left(\sum_k 2^{2(3/2-3/p)k} \|f_k\|_{L^{p'}}^2\right)^{1/2} \\ &\leqslant C_p t^{-(1/2-1/p)} \|f\|_{\dot{W}^{3/2-3/p,p'}},\end{aligned}$$

where we have used (14) in the first step above and (15) at the last step above.[9]

Thus, it remains to show (22). However, note that the decay estimate (20) (and hence (22), which is just a particular case of it) is a scale invariant estimate, which respects the natural scaling of the problem $u \to u_\lambda = u(\lambda t, x/\lambda)$. Thus, (22) itself is reduced to the case $k = 0$

$$\left\|u_0(t,\cdot)\right\|_{L^p} \leqslant C_p t^{-(1/2-1/p)} \|f_0\|_{L^{p'}}. \tag{23}$$

Note that (23) follows by interpolation from the energy conservation (i.e. $p = 2$, which holds with $C_p = 1$) and an $L^1 \to L^\infty$ estimate (i.e. the case $p = \infty$), which reads

$$\left\|u_0(t,\cdot)\right\|_{L^\infty} \leqslant \frac{C}{\sqrt{t}} \|f_0\|_{L^1}. \tag{24}$$

For the rest of the section, we will be concerned with (24). Using the inverse Fourier transform, we obtain from (21) the following explicit form

$$u_0(x) = \int \hat{f}(\xi) e^{2\pi i x\xi} e^{-it/(2\pi\xi)} \varphi(\xi)\, d\xi = \int \left(\int e^{2\pi i(x-y)\xi} e^{-it/(2\pi\xi)} \varphi(\xi) d\xi\right) f(y)\, dy.$$

It is now clear that for the proof of (24), it suffices to check

$$\sup_x \left|\int e^{2\pi i x\xi} e^{-it/(2\pi\xi)} \varphi(\xi)\, d\xi\right| \leqslant C t^{-1/2} \tag{25}$$

for $t > 0$. Rewrite the oscillatory integral as

$$\int e^{-it(1/2\pi\xi + \tilde{x}\xi)} \varphi(\xi)\, d\xi,$$

where $\tilde{x} := -2\pi x/t$. We are now in a position to apply the Van Der Corput lemma with $k = 2$ (see Lemma 4 in Appendix A) with a phase function $\mu(\xi) = 1/(2\pi\xi) + \tilde{x}\xi$. We clearly have $\mu''(\xi) = \frac{1}{\pi\xi^3}$, which implies that on the support of $\varphi(\xi) \subset \{\xi\colon\ 1/2 < |\xi| < 2\}$, we have that $|\mu''(\xi)| \gtrsim 1$ and hence (25) holds, as a consequence of (31). □

[9] Recall $p' < 2 < p$.

4. Proof of Theorem 2

Before we proceed with the specifics of the proof, let us outline our strategy. The first step will be to use Lemma 1 to produce a bound on (any!) Sobolev norms of the solution. Such a bound will be dependent upon a mixed $L^q_t W^{s,r}_x$ norm (for appropriate q, r, s) of the solution, see (26) below. The next step will be to control the mixed norm described above, by using the decay estimates of Theorem 3, back in terms of Sobolev norms of the solution and the initial data. In conclusion, we run a persistence argument that states that all norms remain small (if we start with small data) for all times.

4.1. Energy estimates for the solution of (1)

We have essentially performed this argument after the proof of Lemma 1. Indeed, for a classical solution of (1), the L^2 estimate (4) applies as well (with $u^N = u^{N-1} = u$), so we get

$$\partial_t \|u(t)\|^2_{L^2} \leqslant 2p(p-1)\|u(t)\|^2_{L^2}\|u(t)\|^{p-2}_{L^\infty}\|u'(t)\|_{L^\infty}.$$

Next, we may apply Lemma 1 with $F = pu^{p-1}$ and $G = p(p-1)u^{p-2}u_x^2$. We have

$$\|F\|_{\dot H^s} \leqslant C_p\|u\|_{\dot H^s}\|u\|^{p-2}_{L^\infty}, \qquad \|F'\|_{L^\infty} \leqslant \|u'\|_{L^\infty}\|u\|^{p-2}_{L^\infty},$$
$$\|G(t,\cdot)\|_{\dot H^{s-1}} \leqslant C_p\big(\|u\|_{\dot H^s}\|u'\|_{L^\infty}\|u\|^{p-2}_{L^\infty} + \|u'\|^2_{L^\infty}\|u\|_{\dot H^{s-1}}\|u\|^{p-3}_{L^\infty}\big).$$

We have for $I_s(t) = \|u(t)\|^2_{\dot H^s}$,

$$\begin{aligned} I_s'(t) \leqslant{} & C_s\|u'\|_{L^\infty}\|u\|^{p-2}_{L^\infty} I_s(t) + C_{p,s}\sqrt{I_s}\big(\|u\|_{\dot H^s}\|u'\|_{L^\infty}\|u\|^{p-2}_{L^\infty} + \|u'\|^2_{L^\infty}\|u\|_{\dot H^{s-1}}\|u\|^{p-3}_{L^\infty}\big) \\ & + C_{p,s}\sqrt{I_s}\|u'\|_{L^\infty}\|u\|_{\dot H^s}\|u\|^{p-2}_{L^\infty}. \end{aligned}$$

Thus, we get the following estimate for the time derivative of $J_s(u) = \|u(t)\|^2_{\dot H^s} + \|u(t)\|^2_{L^2}$,

$$J_s'(t) \leqslant C_{p,s} J_s(t)\big(\|u'(t)\|^{p-1}_{L^\infty_x} + \|u(t)\|^{p-1}_{L^\infty_x}\big).$$

By Gronwall's inequality

$$J_s(T) \leqslant J_s(0)\exp\left(\int_0^T \big(\|u'(t)\|^{p-1}_{L^\infty} + \|u(t)\|^{p-1}_{L^\infty}\big)\,dt\right) \leqslant J_s(0)\exp\big(c_p\|u\|^{p-1}_{L^{p-1}_t(0,T)W^{1,\infty}_x}\big). \tag{26}$$

In particular, we will have

$$\|u\|_{L^\infty_T H^s} \leqslant 3\|f\|_{H^s} \tag{27}$$

as long as $\|u\|^{p-1}_{L^{p-1}_T W^{1,\infty}_x} < 1/c_p$.

4.2. Decay estimates for (1)

In this section, we show how to control certain mixed norms of the solution in terms of the Sobolev norms. Write the solution to (1) in the equivalent integral formulation

$$u = T(t)f + \int_0^t T(t-s)\partial_{xx}\big[u^p(s)\big]\,ds, \tag{28}$$

where $T(t)$ is the semigroup generator for the linear Ostrovsky equation $u_{tx} - u = 0$. At this point, we use the estimates that we have proved for the operator $T(t)$ in Theorem 3. Fix $0 < \delta < 1/100$. We will consider a set of indices $\mathcal{A}$, which will consist of all $q, r \in (2,\infty)$, so that

$$\mathcal{A} = \big\{(q,r)\colon\ 1/q + 1/r \leqslant 1/2 - \delta;\ 1/q < 1/2 - 3\delta;\ r < (p-1)/\delta\big\}.$$

For any $(q,r) \in \mathcal{A}$, we will now proceed to show that one can control $\|u\|_{L^q_t(0,T)W^{\alpha,r}}$ in terms of Sobolev norms $\|u\|_{L^\infty_t(0,T)H^\gamma}$ and other norms in the form $\|u\|_{L^{\tilde q}_t(0,T)W^{\tilde\alpha,\tilde r}}$, where $(\tilde q,\tilde r) \in \mathcal{A}$. The smoothness index α will be chosen judiciously in the course of the argument. We first estimate the (more straightforward) time-local norm $\|u\|_{L^q_t(0,\min(1,T))W^{\alpha,r}}$. We have by (28)

$$\|u\|_{L^q_t(0,\min(1,T))W^{\alpha,r}} \leqslant \sup_t \big\|T(t)f\big\|_{W^{\alpha,r}} + \sup_{0<s<t<T}\big\|T(t-s)\partial_{xx}\big[u^p(s)\big]\big\|_{W^{\alpha,r}}.$$

The Sobolev embedding[10] $H^{\alpha+1} \hookrightarrow W^{\alpha,r}$ and the energy estimate (19) yield

$$\sup_t \big\|T(t)f\big\|_{W^{\alpha,r}} \leqslant C\sup_t \big\|T(t)f\big\|_{H^{\alpha+1}} = C\|f\|_{H^{\alpha+1}};$$

$$\sup_{0<s<t<T}\big\|T(t-s)\partial_{xx}\big[u^p(s)\big]\big\|_{W^{\alpha,r}} \leqslant C\sup_{0<s<T}\big\|\partial_{xx}\big[u^p(s)\big]\big\|_{H^{\alpha+1}} \leqslant C\|u\|_{L^\infty_T H^{\alpha+3}_x}\|u\|^{p-1}_{L^\infty_T L^\infty_x},$$

where in the last inequality, we have used the Leibnitz rule in Lemma 3.

If $T > 1$, we also wish to have an estimate for the norm $\|u\|_{L^q_t(1,T)W^{\alpha,r}}$. For that, we first use the decay estimate (20)

$$\big\|u(t)\big\|_{W^{\alpha,r}} \leqslant C_r|t|^{-(1/2-1/r)}\|f\|_{W^{\alpha+3/2-3/r,r'}} + C_r\int_0^t \frac{\|\partial_{xx}[u^p(s)]\|_{W^{\alpha+3/2-3/r,r'}}}{|t-s|^{1/2-1/r}}\,ds.$$

Taking $L^q_t(1,T)$ norm and applying the Young's inequality $L^{s_1,\infty} * L^{s_2} \hookrightarrow L^s$ whenever $1 < s, s_1, s_2 < \infty$ and $1 + 1/s = 1/s_1 + 1/s_2$ yields

$$\begin{aligned}\big\|u(t)\big\|_{L^q_t(1,T)W^{\alpha,r}} &\leqslant C\big\||\cdot|^{-(1/2-1/r)}\big\|_{L^q_t(1,T)}\|f\|_{W^{\alpha+3/2-3/r,r'}}\\ &\quad + \big\||\cdot|^{-(1/2-1/r)}\big\|_{L^{2r/(r-2),\infty}_t}\big\|\partial_{xx}\big[u^p\big]\big\|_{L^\beta_t(0,T)W^{\alpha+3/2-3/r,r'}},\end{aligned}$$

where $1 + \frac{1}{q} = \frac{r-2}{2r} + \frac{1}{\beta}$.

[10] Of course, in our setting of one space dimension, one can use the sharper Sobolev embedding $H^{\alpha+1/2} \hookrightarrow W^{\alpha,r}$, but then the constant incurred in that embedding may blow up in $r \to \infty$.

Note that since $1/q + 1/r < 1/2$, it follows that $q > 2r/(r-2)$ and thus $\||\cdot|^{-(1/2-1/r)}\|_{L_t^q(1,T)} \leqslant \||\cdot|^{-(1/2-1/r)}\|_{L_t^q(1,\infty)} \lesssim 1$. Also, $\||\cdot|^{-(1/2-1/r)}\|_{L_t^{2r/(r-2),\infty}} \lesssim 1$. Finally, by the Leibniz rule in Lemma 3 and Hölder's inequality in time,

$$\begin{aligned}\left\|\partial_{xx}\left[u^p\right]\right\|_{L_t^\beta(0,T)W^{\alpha+3/2-3/r,r'}} &\leqslant C\|u\|_{L_t^\infty(0,T)H^{\alpha+2+3/2-3/r}}\left\|u^{p-1}\right\|_{L_t^\beta(1,T)L_x^{\frac{2r}{r-2}}} \\ &\leqslant C\|u\|_{L_t^\infty(0,T)H^{\alpha+2+3/2-3/r}}\|u\|^{p-1}_{L_t^{(p-1)\beta}(1,T)L_x^{(p-1)\frac{2r}{r-2}}}.\end{aligned}$$

Let us now check that the pair $((p-1)\beta, (p-1)\frac{2r}{r-2})$ belongs to the class of indices $\mathcal{A}$, at least for $p \geqslant 4$. Indeed,

$$\frac{1}{(p-1)\beta} + \frac{1}{(p-1)\frac{2r}{r-2}} = \frac{1}{p-1}\left(1 + \frac{1}{q}\right) \leqslant \frac{1}{2} - \delta,$$

where in the last line, we have used $1/(p-1) \leqslant 1/3$ and $1/q < 1/2 - 3\delta$. Next,

$$\frac{1}{(p-1)\frac{2r}{r-2}} \leqslant \frac{1}{2(p-1)} \leqslant \frac{1}{2} - 3\delta$$

since $\delta < 1/100$. Finally,

$$(p-1)\frac{2r}{r-2} \leqslant \frac{p-1}{\delta},$$

since $1/2 - 1/r \geqslant 1/q + \delta \geqslant \delta$ by the restrictions of $\mathcal{A}$.

Introduce the norm

$$\|u\|_{T,\mathcal{A},\alpha} := \sup_{(q,r)\in\mathcal{A}} \|u\|_{L_t^q(0,T)W^{\alpha,r}},$$

which measures the time decay rate of various spatial Sobolev norms of the solution. If we combine the estimates for the local norm $\|u\|_{L_t^q(0,\min(1,T))W^{\alpha,r}}$ of the solution with our estimate for $\|u\|_{L_t^q(1,T)W^{\alpha,r}}$, we obtain

$$\begin{aligned}\|u\|_{T,\mathcal{A},\alpha} &\leqslant C_{p,\delta}\Big(\|f\|_{H^{\alpha+1}} + \sup_{(q,r)\in\mathcal{A}}\|f\|_{W^{\alpha+3/2,r'}}\Big) \\ &\quad + C_{p,\delta}\|u\|_{L_t^\infty(0,T)H^{\alpha+7/2}}\big(\|u\|^{p-1}_{L_T^\infty L_x^\infty} + \|u\|^{p-1}_{L_t^{(p-1)\beta}(1,T)L_x^{(p-1)\frac{2r}{r-2}}}\big).\end{aligned}$$

We have already checked that $((p-1)\beta, (p-1)\frac{2r}{r-2}) \in \mathcal{A}$, whence

$$\|u\|_{L_t^{(p-1)\beta}(1,T)L_x^{(p-1)\frac{2r}{r-2}}} \leqslant \|u\|_{T,\mathcal{A},\alpha}.$$

By Sobolev embedding

$$\|u\|_{L_T^\infty L_x^\infty} \leqslant \|u\|_{L_T^\infty W^{1/2,4}} \leqslant \|u\|_{T,\mathcal{A},\alpha},$$

if $\alpha \geqslant 1/2$, since $(\infty, 4) \in \mathcal{A}$. All in all,

$$\|u\|_{T,\mathcal{A},\alpha} \leqslant C_{p,\delta}\big(\|f\|_{H^{\alpha+3/2}} + \|f\|_{W^{\alpha+3/2,1}} + \|u\|_{L_t^\infty(0,T)H^{\alpha+7/2}}\|u\|_{T,\mathcal{A},\alpha}^{p-1}\big). \tag{29}$$

Clearly, this estimate would be crucial in establishing the global scattering for the Ostrovsky equation.

4.3. Conclusion of the proof

Let us recapitulate what we have shown so far. On one hand, we have a local solution (up to some nontrivial time T_0), in accordance to Theorem 1. In this interval of existence, we have shown the energy estimate (19) and the decay estimate (29). Set $\alpha = 3/2$ and assume $\|f\|_{H^5} + \|f\|_{W^{3,1}} < \varepsilon$. We will show that the solution is classical and global. As a consequence, we will also show that the solution stays small as well.

To show that the solution with initial data f is global, we need to show that $\|u(t)\|_{H^5}$ stays bounded for all t. We argue by contradiction, which implies that $T_0 < \infty$ and $\limsup_{t\to T_0}\|u(t)\|_{H^5} = \infty$. Then, there is

$$T^* = \inf_{t>0}\big\{t\colon \big\|u(t)\big\|_{H^5} = 4\varepsilon\big\}, \quad T^* < T_0 < \infty.$$

Clearly $T^* > 0$ and $\|u\|_{L_t^\infty(0,T^*)H^5} \leqslant 4\varepsilon$. From (29), we have the *a priori* estimate for all $0 < T < T^*$

$$\|u\|_{T,\mathcal{A},\alpha} \leqslant C_{p,\delta}\big(\|f\|_{H^{\alpha+3/2}} + \|f\|_{W^{\alpha+3/2,1}} + \|u\|_{L_t^\infty(0,T)H^{\alpha+7/2}}\|u\|_{T,\mathcal{A},\alpha}^{p-1}\big) \leqslant C_{p,\delta}\big(\varepsilon + \varepsilon\|u\|_{T,\mathcal{A},\alpha}^{p-1}\big).$$

The last estimate implies that $\|u\|_{T,\mathcal{A},\alpha} \leqslant 2C_{p,\delta}\varepsilon$ provided, $\varepsilon\colon C_{p,\delta}\varepsilon < 1$, which will be one restriction on ε that we will require. In particular, by Sobolev embedding

$$\|u\|_{L_T^{p-1}W_x^{1,\infty}}^{p-1} \leqslant C_p\|u\|_{L_T^{p-1}W_x^{3/2,8}}^{p-1} \leqslant C_p\|u\|_{T,\mathcal{A},\alpha}^{p-1} \leqslant C_p(2C_{p,\delta}\varepsilon)^{p-1}.$$

Now, if we select $\varepsilon\colon C_p(2C_{p,\delta}\varepsilon)^{p-1} \leqslant 1/c_p$, where[11] c_p is the constant on the right-hand side of (26), we see that $\|u\|_{L_T^{p-1}W_x^{1,\infty}}^{p-1} \leqslant 1/c_p$. Hence, by (27), we must have

$$\big\|u(t)\big\|_{L_t^\infty(0,T^*)H^5} \leqslant 3\|f\|_{H^5} \leqslant 3\varepsilon$$

– a contradiction with the definition of T^*. All of this shows that $T^* = \infty$, in particular the solution is defined for all times and

$$\sup_{0<t<\infty}\big\|u(t)\big\|_{H^5} \leqslant 4\varepsilon.$$

[11] This is the second and last restriction for the choice of ε, which turns out to depend on δ and p. Since δ is at our disposition and it can generally be fixed, say at $\delta = 0.001$, ε ultimately depends only on p.

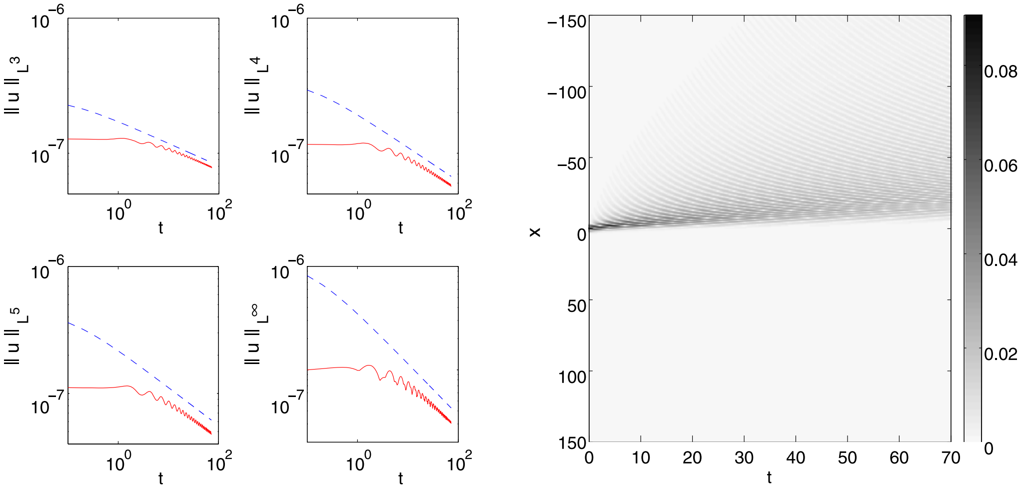

Fig. 1. For the case of $p = 4$, the figure illustrates the evolution of initial data given by Eq. (30) in a domain of size 800 with periodic boundary conditions (only a fraction of the domain is shown). The left panels indicate the decay of $\|u\|_{L^3}$ (top left), $\|u\|_{L^4}$ (top right), $\|u\|_{L^5}$ (bottom left) and $\|u\|_{L^\infty}$ (bottom right). The dashed lines illustrate the theoretically expected power law rates of decay over the same time scales. The right panel shows the space–time evolution of the absolute value of the field, illustrating the scattering of the initial data with $a = 0.1$.

5. Numerical results

In the present section, we incorporate some scattering results for the cases of $p = 4$ and $p = 5$ for which it was heuristically argued above that a decay of L^r norms with $r > 2$ should be expected, according to $\|u\|_{L^r} \sim t^{-(r-2)/(2r)}$. Our numerical method consisted of Fourier transforming the gO equation with respect to x, then solving the ensuing first-order ODE in t (for each wavenumber) for a short time-increment dt, via a fourth-order Runge–Kutta scheme, and then Fourier transforming back to obtain $u(x, t + dt)$. The boundary conditions were periodic by the nature of the spectral approach used, while as initial conditions we used

$$u(x, 0) = a\left(1 - 2bx^2\right)\exp\left(-bx^2\right) \tag{30}$$

motivated by this choice in [16], which ensures exponentially localized initial data but satisfying $\int u\,dx = 0$, as is necessitated by the dynamics [our experience with initial data that did not satisfy this condition indicated that they develop substantially larger amplitude oscillations for early times]. We chose $b = 0.5$ and tried (small) values of a, such as 0.1 or 0.01 with essentially similar results.

We illustrate the results for $a = 0.1$ in what follows, for the cases of $p = 4$ and $p = 5$; see Figs. 1 and 2, respectively. The figures show the decay [via solid (red in the web version) lines] of the norms $\|u\|_{L^3}$, $\|u\|_{L^4}$, $\|u\|_{L^5}$ and $\|u\|_{L^{\mathrm{inf}}}$, during the clearly observed scattering (see the right panel of both figures) of the initial data. For comparison, the heuristically indicated decay of $t^{-(r-2)/(2r)}$ is also shown [by means of dashed (blue in the web version) lines]. Clearly, after a short transient stage in the dynamics, the evolution of the relevant norms, very accurately follows the corresponding decay predictions with exponents $-1/6$, $-1/4$, $-3/10$ and $-1/2$, respectively. Indeed, also, this result seems to be independent of p, as the agreement is excellent both in the case of $p = 4$, as well as in that of $p = 5$.

Given these results for small amplitude data, and the global well-posedness of the gO equation established above, it would be interesting to examine the type of solitary wave solutions that can robustly exist for these higher-p generalizations of the Ostrovsky equation. Naturally, the techniques would have to be different than the ones used herein, but it would, for instance, be of interest to

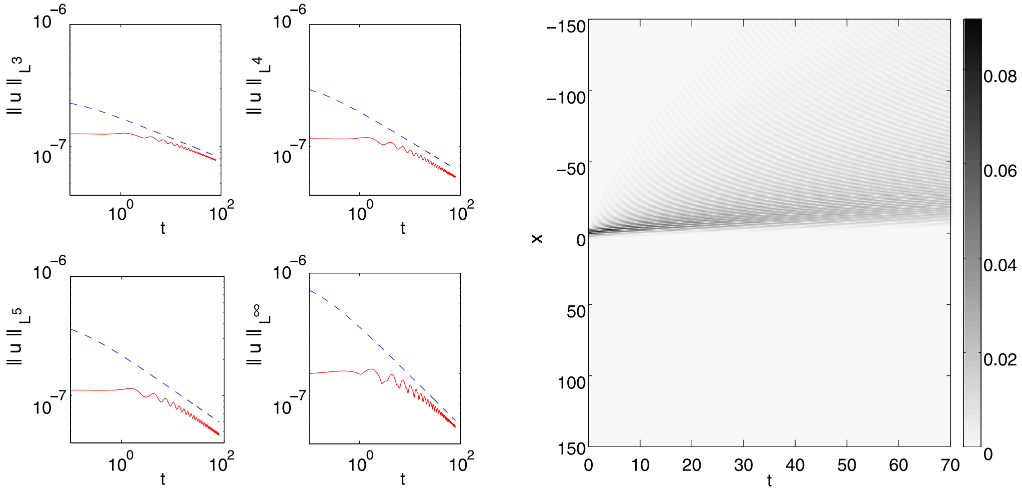

Fig. 2. The figure shows the same diagnostics as Fig. 1 but for the case with $p = 5$. Once again the scattering of the initial data is found to follow the theoretically expected decay rates, after an initial transient stage of the dynamics.

examine whether the breather-like structures of the $p = 3$ case (due to the analogy with the sine-Gordon equation [12,16,18]) would persist in the case of higher p or not. Furthermore, it would certainly be of interest and physical relevance to examine appropriate generalizations of this type of models in higher dimensional settings. Such themes would constitute interesting subjects for future work.

Appendix A. Van Der Corput lemma

Lemma 4 *(Van Der Corput). (See p. 334 in [20].) Suppose μ is a real-valued function, smooth in (a, b), so that $|\mu^{(k)}(x)| \geqslant 1$ for some integer k. (If $k = 1$, we also assume that $\mu'(x)$ is monotonic.) Then,*

$$\left| \int_a^b e^{i\lambda\mu(x)} \psi(x)\, dx \right| \leqslant C_k \lambda^{-1/k} \left[\psi(b) + \int_a^b \left| \psi'(x) \right| dx \right], \tag{31}$$

for some constant C_k, depending only on k.

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