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A Priori Error Analyses of a Stabilized Discontinuous Galerkin Method

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Abstract— We introduce a new stabilized discontinuous Galerkin method within a new function space setting, that is closely related to the discontinuous Galerkin formulation by Oden, Babuška and Baumann [1], but involves an extra stabilization term on the jumps of the normal fluxes across the element interfaces. The formulation satisfies a local conservation property and we prove well posedness of the new formulation. *A priori* error estimates are derived, which are verified by 1D and 2D experiments on a reaction-diffusion type model problem.

Keywords— Discontinuous Galerkin methods, *A priori* error estimation

1. INTRODUCTION

In recent years, several variations of discontinuous Galerkin methods (DGM), for second-order elliptic boundary value problems have been proposed which exhibit special convergence, conservation, and local approximation properties attractive for parallel adaptive *hp*-approximations. An account of several types of DGMs can be found in the book edited by Cockburn, Karniadakis and Shu [2]. Also, Arnold, Brezzi, Cockburn and Marini [3] introduced a general framework to represent various types of DGMs for elliptic problems.

In 1997, Oden, Babuška and Baumann [1] and Baumann [4] introduced a new type of discontinuous Galerkin formulation that is very similar to the GEM formulation by Delves *et al.* [5,6], but sign differences in certain terms result in a positive definite bilinear form. Although a minor difference, the change in sign has remarkable effects. For $p \geq 2$ (where p denotes the minimum order of the polynomial approximations), the DGM by Oden, Babuška and Baumann [1] appears to be unconditionally stable, whereas the GEM formulation requires the inclusion of penalty terms to stabilize the formulation. Moreover, the DGM satisfies a local conservation property, which makes it attractive to use in convection-diffusion problems.

The main drawback with the DGM is that one can observe these advantageous properties *numerically*, but a consistent *theory*, that involves an appropriate function space setting on which

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one can prove the well posedness of the formulation, i.e., existence, uniqueness, and stability of the solution, does not exist.

Rivière et al. [7–9] proposed a method that is an extension of the DGM by Oden, Babuška and Baumann [1], involving a penalty term on the jumps of the solution across the element interfaces. The addition of the penalty term eliminates the local conservation property, but it does stabilize the method (the method is stable for $p = 1$), and enables *a priori* error estimates to be proved that are optimal in h and suboptimal in p. However, again an appropriate function space setting is lacking, on which Inf-Sup and continuity properties of the bilinear forms could be proved. Thus, stability of the solution still remains an open issue.

In this paper, we start in Section 2 by introducing a new DGM formulation of a model linear elliptic boundary value problem, that, in principle, is an extension of the DGM by Oden, Babuška and Baumann [1] but includes an extra stabilization term on the interelement jumps of the fluxes, and that (contrary to other penalty-type DGMs) satisfies a local conservation property. We introduce a new function space setting and prove in Section 3 that the bilinear form, corresponding to this formulation, actually satisfies an Inf-Sup condition. In addition, we establish the existence of *unique stable* solutions to the associated variational boundary value problem. In Section 4, we derive *a priori* error estimates to analyze the rates of convergence of the method. Some 1D and 2D numerical experiments to test our analysis are given in Section 5. Finally, concluding remarks are summarized in Section 6.

2. PRELIMINARIES

2.1. Model Problem and Notations

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with Lipschitz boundary $\partial\Omega$ and let $\{\mathcal{P}_h\}$ be a family of regular partitions of Ω into open elements K, with diameters h_K, such that (see Figure 1)

$$\Omega = \text{int}\left(\bigcup_{K\in\mathcal{P}_h} \bar{K}\right)$$

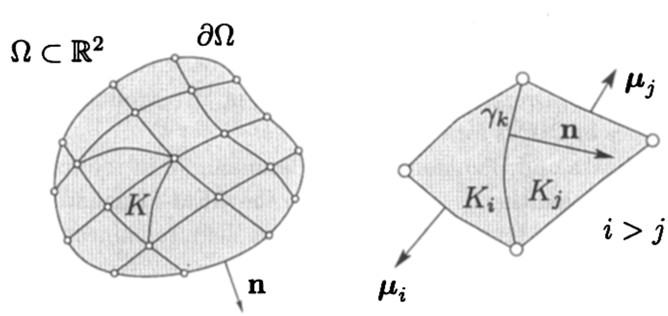


Figure 1. Geometrical definitions.

The maximum diameter in the partition is denoted h. The set of all edges of the partition $\mathcal{P}_h$ is given by $\mathcal{E}_h = \{\gamma_k\}$, $k = 1, \ldots, N_{\text{edge}}$, where N_{edge} represents the number of edges in the partition $\mathcal{P}_h$. The interior interface Γ_{int} is then defined as the union of all common edges shared by elements of partition $\mathcal{P}_h$

$$\Gamma_{\text{int}} = \bigcup_{k=1}^{N_{\text{edge}}} \gamma_k \setminus \partial\Omega.$$

The definition of the unit normal vector $\mathbf{n}$ on each γ_k is related to the numbering of the elements in the partition, such that $\mathbf{n}$ is defined outward w.r.t. the element with the highest index number (see Figure 1). The normal vector μ is defined outward to each element individually. Within this setting, the following reaction-diffusion problem is considered

$$\begin{aligned} -\Delta u + u &= f, \qquad \text{in } \Omega, \\ u &= 0, \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where f is a real-valued function in $L^2(\Omega)$. For the sake of clarity in the notation, the jump and average operators on each $\gamma_k \in \Gamma_{\text{int}}$ are, respectively, defined as (see Figure 1)

$$\begin{aligned} [v] &= v_{|\gamma_k \subset \partial K_i} - v_{|\gamma_k \subset \partial K_j}, \\ \langle v \rangle &= \frac{1}{2}(v_{|\gamma_k \subset \partial K_i} + v_{|\gamma_k \subset \partial K_j}), \qquad \gamma_k = \text{int}(\partial K_i \cap \partial K_j), \qquad i > j. \end{aligned} \tag{2}$$

2.2. The Weak Formulation

First, we introduce the following *broken space*:

$$\mathcal{M}(\mathcal{P}_h) = \left\{ v \in L^2(\Omega) : v_{|K} \in H(\Delta, K),\ \forall\, K \in \mathcal{P}_h, [\nabla v \cdot \mathbf{n}] \in L^2(\Gamma_{\text{int}}) \right\}, \tag{3}$$

where

$$H(\Delta, K) = \{ v \in L^2(K) : \nabla \cdot \nabla v \in L^2(K) \} \subset H^1(K).$$

Notice here, that $v \in H(\Delta, K)$ implies $\nabla v \cdot \mu \in H^{-1/2}(\partial K)$ (see [10,11]). The norm $|||\cdot|||$ on $\mathcal{M}(\mathcal{P}_h)$, is defined as

$$|||v|||^2 = \sum_{K \in \mathcal{P}_h} \left\{ \|v\|^2_{H^1(K)} + \frac{h^\nu}{p^\theta} \|\nabla v \cdot \mu\|^2_{H^{-1/2}(\partial K)} \right\} + \sigma \frac{h^\lambda}{p^\zeta} \|[\nabla v \cdot \mathbf{n}]\|^2_{L^2(\Gamma_{\text{int}})}. \tag{4}$$

The parameter $p \in \mathbb{N}$ that is introduced here represents the minimum of all of the local orders of polynomial approximations p_K in the partition $\mathcal{P}_h$ (see Section 2.3). Notice that the parameters $\nu, \lambda, \theta, \zeta$ are greater than or equal to zero and that the subsequent norms in (4) are defined as

$$\begin{aligned} \|g\|_{H^{-1/2}(\partial K)} &= \sup_{\varphi \in H^{1/2}(\partial K)} \frac{|\langle g, \varphi \rangle_{-1/2 \times 1/2, \partial K}|}{\|\varphi\|_{H^{1/2}(\partial K)}}, \\ \|\varphi\|_{H^{1/2}(\partial K)} &= \inf_{\substack{w \in H^1(K) \\ \gamma_0 w = \varphi}} \|w\|_{H^1(K)}, \end{aligned} \tag{5}$$

where $\langle \cdot, \cdot \rangle_{-1/2 \times 1/2, \partial K}$ denotes the duality pairing in $H^{-1/2}(\partial K) \times H^{1/2}(\partial K)$ and where γ_0 denotes the trace operator

$$\gamma_0 : H^1(K) \to H^{1/2}(\partial K).$$

Now, the choice for the space of test functions, V, is the *completion* of $\mathcal{M}(\mathcal{P}_h)$ with respect to the norm $|||.|||$. The new discontinuous variational formulation, within this new function space setting, is then stated as follows

$$\text{find } w \in V : B(w, v) = L(v), \qquad \forall\, v \in V, \tag{6}$$

where the bilinear form $B(w, v)$ and linear form $L(v)$ are defined as

$$B : V \times V \to \mathbb{R}, \qquad L : V \to \mathbb{R},$$

$$\begin{aligned} B(w, v) &= \sum_{K \in \mathcal{P}_h} \left\{ \int_K \{\nabla w \cdot \nabla v + wv\}\, d\mathbf{x} - \int_{\partial K} \{v(\nabla w \cdot \mu) - (\nabla v \cdot \mu) w\}\, ds \right\} \\ &\quad + \int_{\Gamma_{\text{int}}} \{\langle v \rangle [\nabla w \cdot \mathbf{n}] - \langle w \rangle [\nabla v \cdot \mathbf{n}]\}\, ds + \int_{\Gamma_{\text{int}}} \sigma \frac{h^\lambda}{p^\zeta} [\nabla w \cdot \mathbf{n}][\nabla v \cdot \mathbf{n}]\, ds, \\ L(v) &= \int_\Omega f v\, d\mathbf{x}, \end{aligned} \tag{7}$$

where $[.]$ and $\langle . \rangle$ denote the jump and average operators, respectively, and where it is understood that we use the following notation to denote the duality pairing in $H^{-1/2}(\partial K) \times H^{1/2}(\partial K)$:

$$\int_{\partial K} (\nabla w \cdot \mu) v \, ds = \langle \nabla w \cdot \mu, v \rangle_{H^{-1/2}(\partial K) \times H^{1/2}(\partial K)} .$$

The formulation, although new, is closely related to the DGM formulation by Oden, Babuška and Baumann [1]. Indeed, if we choose the subspace $W(\mathcal{P}_h)$ of V of functions with fluxes $\nabla v \cdot \mathbf{n} \in L^2(\partial K)$, then by using the following identities:

$$\sum_{K \in \mathcal{P}_h} \int_{\partial K} v(\nabla u \cdot \mu) \, ds = \int_{\Gamma_{\text{int}}} [v(\nabla u \cdot \mathbf{n})] \, ds + \int_{\partial \Omega} v(\nabla u \cdot \mathbf{n}) \, ds \tag{8}$$

and

$$[w(\nabla v \cdot \mathbf{n})] = [w] \langle \nabla v \cdot \mathbf{n} \rangle + \langle w \rangle [\nabla v \cdot \mathbf{n}], \tag{9}$$

we recover the DGM formulation by Oden, Babuška and Baumann [1]. The only difference would then be the addition of the last term in (7)[2]. This term has been incorporated in works by Percell and Wheeler [12] and Hughes *et al.* [13], where it is accompanied by another penalty term on the jumps of the functions $[v]$ across the element interfaces. Our motivation to impose only a stabilization term in the interelement jumps of the fluxes lies in the fact that by adding only this term we can prove *both* continuity and Inf-Sup properties of the bilinear form with respect to the space V, $|||.|||$ (see Section 3). In addition, adding this type of stabilization term, does *not* disrupt the local conservation property of the weak form, whereas adding the penalty on the interelement jumps does eliminate this property. In the case of elliptic problems, it is likely that this latter property is of minor importance; however, for hyperbolic problems we expect this to be of greater importance and to affect the robustness of the method.

REMARK 2.1. The DG formulation (6) satisfies a local conservation of balance laws in an average sense. By taking $v = 1$ in (7) on an element $K \in \mathcal{P}_h$, we obtain

$$\int_K w \, d\mathbf{x} - \int_{\partial K} \langle \nabla w \cdot \mu \rangle \, ds = \int_K f \, d\mathbf{x}.$$

2.3. The Discrete Problem

Let $\{\mathbf{F}_K\}$ be a family of invertible maps defined for a regular partition $\mathcal{P}_h$ such that every element $K \in \mathcal{P}_h$ is the image of $\mathbf{F}_K$ acting on a master element $\hat{K}$, as shown in Figure 2.

$$\mathbf{F}_K : \hat{K} \to K, \qquad \mathbf{x} = \mathbf{F}_K(\hat{\mathbf{x}}). \tag{10}$$

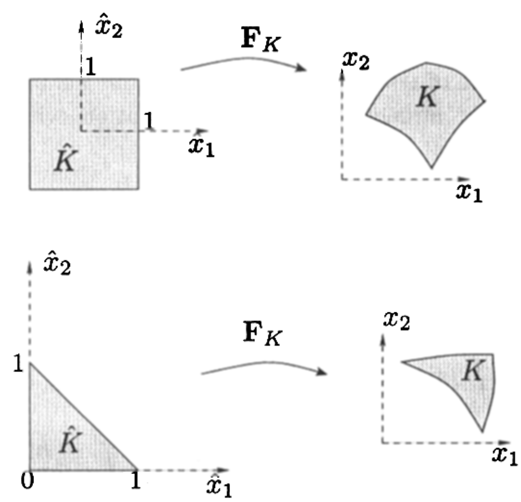


Figure 2. Mapping from the master elements to the physical space.

In the computational model, a finite-dimensional space of real-valued piecewise polynomial functions of degree $\leq p_K$ is introduced, such that

$$V^{hp} = \left\{ v \in L^2(\Omega) : v_{|K} = \hat{v} \circ \mathbf{F}_K^{-1},\ \hat{v} \in P^{p_K}\left(\hat{K}\right),\ \forall K \in \mathcal{P}_h \right\}. \tag{11}$$

We note that V^{hp} is a subspace of V. Now, an approximation u_h of u is sought as the solution of the following discrete problem:

$$\text{find } u_h \in V^{hp} : B(u_h, v_h) = L(v_h), \qquad \forall v_h \in V^{hp}. \tag{12}$$

3. WELL POSEDNESS OF THE VARIATIONAL FORMULATION

In this section, we establish the well posedness of the variational formulation (6). Thus, we show that the solution of the PDE (1) is also a solution to the weak problem. We prove existence and uniqueness of the solution to the weak problem and show its continuous dependence on the input data. Essential in some of these proofs are the continuity and Inf-Sup conditions of the bilinear form in (7), which will be proved in Sections 3.2 and 3.3, respectively.

3.1. Relation between the PDE and the VBVP

We introduce an important lemma [14].

LEMMA 3.1. *If $u \in H(\Delta, \Omega)$, then u and $(\nabla u \cdot n)$ are weakly continuous across the element interface Γ_{int} in the sense that*

$$\int_{\gamma_k} [u]\varphi \, ds = 0, \quad \int_{\gamma_k} [\nabla u \cdot \mathbf{n}]\varphi \, ds = 0, \qquad \varphi \in \mathcal{D}(\gamma_k),$$
$$\gamma_k = \operatorname{int}(\partial K_i \cap \partial K_j) \subset \Gamma_{\text{int}}.$$

THEOREM 3.1. *Let u be the solution of (1). Then, u is a solution to the discontinuous VBVP (6) as well.*

PROOF. If we restrict (1) to an element $K \in \mathcal{P}_h$, multiply this local equation by a test function $\varphi_K \in H^2(K)$, integrate over the element K, and apply Green's identity, we get

$$\int_K \{\nabla u \cdot \nabla \varphi_K + u\varphi_K\} \, d\mathbf{x} - \int_{\partial K} \varphi_K (\nabla u \cdot \mu) \, ds = \int_K f \varphi_K \, d\mathbf{x}.$$

Repeating this for all $K \in \mathcal{P}_h$, extending each φ_K to zero outside of K, and taking the sum, yields

$$\sum_{K \in \mathcal{P}_h} \left\{ \int_K \{\nabla u \cdot \nabla \varphi + u\varphi\} \, d\mathbf{x} - \int_{\partial K} \varphi (\nabla u \cdot \mu) \, ds \right\} = \sum_{K \in \mathcal{P}_h} \int_K f \varphi \, d\mathbf{x}, \qquad \forall \varphi \in H^2(\mathcal{P}_h),$$

where

$$\varphi = \sum_{K \in \mathcal{P}_h} \varphi_K, \qquad H^2(\mathcal{P}_h) = \left\{ v \in L^2(\Omega) : v_{|K} \in H^2(K),\ \forall K \in \mathcal{P}_h \right\}.$$

Since u is the solution to (1), we know that u satisfies the Dirichlet boundary condition on the boundary of Ω. In addition, it is known that u belongs to $H(\Delta, \Omega)$. According to Lemma 3.1, this implies that u and $(\nabla u \cdot \mathbf{n})$ are weakly continuous across the element interfaces. Consequently, we can add these boundary and continuity conditions to the variational formulation in a weak sense, which yields

$$\begin{aligned}
&\sum_{K \in \mathcal{P}_h} \left\{ \int_K \{\nabla u \cdot \nabla \varphi + u\varphi\} \, d\mathbf{x} - \int_{\partial K} \varphi (\nabla u \cdot \mu) \, ds \right\} + \int_{\Gamma_{\text{int}}} \langle \varphi \rangle \left[\nabla u \cdot \mathbf{n}\right] ds \\
&\quad + \int_{\Gamma_{\text{int}}} [u] \left\langle \nabla \varphi \cdot \mathbf{n} \right\rangle \, ds + \int_{\partial\Omega} u (\nabla \varphi \cdot \mathbf{n}) \, ds + \int_{\Gamma_{\text{int}}} \sigma \frac{h^\lambda}{p^\zeta} [\nabla u \cdot \mathbf{n}][\nabla \varphi \cdot \mathbf{n}] \, ds \\
&\quad = \int_\Omega f \varphi \, d\mathbf{x}, \qquad \forall \varphi \in H^2(\mathcal{P}_h).
\end{aligned}$$

Employing now the identities in (8) and (9) gives

$$\sum_{K\in\mathcal{P}_h}\left\{\int_K\{\nabla u\cdot\nabla\varphi+u\varphi\}\,d\mathbf{x}-\int_{\partial K}\{\varphi(\nabla u\cdot\mu)-u(\nabla\varphi\cdot\mu)\}\,ds\right\}$$
$$+\int_{\Gamma_{\text{int}}}\{\langle\varphi\rangle\,[\nabla u\cdot\mathbf{n}]-\langle u\rangle\,[\nabla\varphi\cdot\mathbf{n}]\}\,ds+\int_{\Gamma_{\text{int}}}\sigma\frac{h^\lambda}{p^\zeta}[\nabla u\cdot\mathbf{n}][\nabla\varphi\cdot\mathbf{n}]\,ds=\int_\Omega f\varphi\,d\mathbf{x},$$
$$\forall\varphi\in H^2(\mathcal{P}_h).$$

By applying the density of $H^2(\mathcal{P}_h)$ in V, we conclude the proof. ∎

3.2. Continuity Property

THEOREM 3.2. *Let $B(.,.)$ be the bilinear form as defined in (7). If $\sigma>0$, then*

$$\exists\,M>0:|B(u,v)|\le M|||u|||\,|||v|||,\qquad\forall\,u,v\in V,$$

where

$$M=\max\left\{3,\sqrt{\frac{p^\theta}{h^\nu}},\sqrt{\frac{p^\zeta}{2\sigma h^\lambda}}\right\}.$$

PROOF. From the definition of $B(.,.)$ and observing that for $u,v\in V$ we can write

$$\int_{\Gamma_{\text{int}}}\langle v\rangle\,[\nabla u\cdot\mathbf{n}]\,ds=\frac{1}{2}\sum_{K\in\mathcal{P}_h}\int_{\partial K\cap\Gamma_{\text{int}}}v[\nabla u\cdot\mathbf{n}]\,ds,\tag{13}$$

then we get

$$B(u,v)=\sum_{K\in\mathcal{P}_h}\left\{\int_K\{\nabla u\cdot\nabla v+uv\}\,d\mathbf{x}-\int_{\partial K}v(\nabla u\cdot\mu)\,ds+\int_{\partial K}u(\nabla v\cdot\mu)\,ds\right.$$
$$\left.+\frac{1}{2}\int_{\partial K\cap\Gamma_{\text{int}}}v[\nabla u\cdot\mathbf{n}]\,ds-\frac{1}{2}\int_{\partial K\cap\Gamma_{\text{int}}}u[\nabla v\cdot\mathbf{n}]\,ds\right\}+\sigma\frac{h^\lambda}{p^\zeta}\int_{\Gamma_{\text{int}}}[\nabla u\cdot\mathbf{n}][\nabla v\cdot\mathbf{n}]\,ds.$$

By applying the Schwarz inequality and by using the definition of the $H^{1/2}(\partial K)$ norm (5), we can bound the above as follows

$$B(u,v)\le\max\left\{1,\sqrt{\frac{p^\theta}{h^\nu}},\sqrt{\frac{p^\zeta}{2\sigma h^\lambda}}\right\}$$
$$\cdot\left\{\sum_{K\in\mathcal{P}_h}\left(3\|u\|^2_{H^1(K)}+\frac{h^\nu}{p^\theta}\|\nabla u\cdot\mu\|^2_{H^{-1/2}(\partial K)}\right)+2\sigma\frac{h^\lambda}{p^\zeta}\|[\nabla u\cdot\mathbf{n}]\|^2_{L^2(\Gamma_{\text{int}})}\right\}^{1/2}$$
$$\cdot\left\{\sum_{K\in\mathcal{P}_h}\left(3\|v\|^2_{H^1(K)}+\frac{h^\nu}{p^\theta}\|\nabla v\cdot\mu\|^2_{H^{-1/2}(\partial K)}\right)+2\sigma\frac{h^\lambda}{p^\zeta}\|[\nabla v\cdot\mathbf{n}]\|^2_{L^2(\Gamma_{\text{int}})}\right\}^{1/2},$$

which establishes the assertion. ∎

3.3. The Inf-Sup Condition

3.3.1. The auxiliary problems

Given an arbitrary $u \in V$, find for every $K \in \mathcal{P}_h$ the function z_K, such that

$$\begin{aligned} -\Delta z_K + z_K = 0, \qquad & \text{in } K, \\ \nabla z_k \cdot \mu = \nabla u \cdot \mu, \qquad & \text{on } \partial K. \end{aligned} \tag{14}$$

The equivalent variational formulation is,

$$\begin{gathered} \text{given } u \in V, \text{ find } z_K \in H^1(K) \\ (z_K, v)_{1,K} = \int_{\partial K} (\nabla u \cdot \mu)\gamma_0 v \, ds, \qquad \forall\, v \in H^1(K), \end{gathered} \tag{15}$$

where $(.,.)_{1,K}$ denotes the $H^1(K)$ inner product. By the *generalized Lax-Milgram theorem* it follows easily that the variational BVP has a unique solution $z_K \in H^1(K)$.

REMARK 3.1. By substituting u and z_K for v in (15), we obtain the following two identities:

$$\|z_K\|^2_{H^1(K)} = \int_{\partial K} (\nabla u \cdot \mu)\gamma_0 z_K \, ds,$$

$$(z_K, u)_{1,K} = \int_{\partial K} (\nabla u \cdot \mu)\gamma_0 u \, ds.$$

We also introduce the following theorem, the proof of which is found in [10,11,14].

THEOREM 3.3. *Given $u \in V$, let $z_K = z_K(u)$ be the unique solution to (15), then the following relation holds:*

$$\|z_K\|_{H^1(K)} = \|\nabla u \cdot \mu\|_{H^{-1/2}(\partial K)}.$$

3.3.2. Inf-Sup condition on the space V

In this section, we prove that the bilinear form $B(.,.)$ satisfies the Inf-Sup condition with respect to the norm $|||.|||$, defined in (4). Let us start introducing extension operators Ψ_K

$$\Psi_K : H^1(K) \to V, \qquad \Psi_K(v_K) = \begin{cases} v_K, & \text{in } K, \\ 0, & \text{in } \Omega \setminus K. \end{cases}$$

Hence, given a function $u \in V$, we can solve (15) for a set of functions $z_K(u)$ and construct a function $\hat{u} \in V$, such that

$$\hat{u} = u + \beta \sum_{K \in \mathcal{P}_h} \Psi_K(z_K), \tag{16}$$

where $\beta \in \mathbb{R}$.

LEMMA 3.2. *Given $u \in V$, then for all $\beta \in \mathbb{R}$, there exists a strictly positive $\xi_1 = \xi_1(h,p)$ such that*

$$|||\hat{u}||| \le \xi_1 |||u|||.$$

PROOF. Substitution of the definition of $\hat{u}$ into (4) and recalling from (14) that $(\nabla z_k \cdot \mu) = (\nabla u \cdot \mu)$ on ∂K, we obtain

$$|||\hat{u}||| = \sum_{K \in \mathcal{P}_h} \left\{ \|u\|^2_{H^1(K)} + 2(u, \beta z_K)_{1,K} + \|\beta z_K\|^2_{H^1(K)} + (1+\beta)^2 \frac{h^\nu}{p^\theta} \|\nabla u \cdot \mu\|^2_{H^{-1/2}(\partial K)} \right\}$$
$$+ \sigma \frac{h^\lambda}{p^\zeta} (1+\beta)^2 \|[\nabla u \cdot \mathbf{n}]\|^2_{L^2(\Gamma_{\text{int}})}.$$

By applying the Schwarz and triangle inequality and recalling Theorem 3.3, one gets

$$|||\hat{u}||| \leq \sum_{K\in\mathcal{P}_h} \left\{ 2\|u\|^2_{H^1(K)} + \left((1+\beta)^2 + \frac{2\beta^2 p^\theta}{h^\nu} \right) \frac{h^\nu}{p^\theta} \|\nabla u \cdot \mu\|^2_{H^{-1/2}(\partial K)} \right\} + (1+\beta)^2 \sigma \frac{h^\lambda}{p^\zeta} \|[\nabla u \cdot \mathbf{n}]\|^2_{L^2(\Gamma_{\text{int}})}.$$

Thus, the assertion holds with

$$\xi_1 = \sqrt{\max\left\{2, (1+\beta)^2 + 2\beta^2 \frac{p^\theta}{h^\nu}\right\}}. \qquad (17) \blacksquare$$

LEMMA 3.3. *Given $u \in V$, then there exists $\xi_2 = \xi_2(\sigma, h, p) > 0$, such that*

$$B(u, \hat{u}) \geq \xi_2 |||u|||^2.$$

PROOF. By replacing v by $\hat{u}$ in the definition of $B(u,v)$, and recalling that $(\nabla z_K \cdot \mu) = (\nabla u \cdot \mu)$, we get

$$B(u,\hat{u}) = \sum_{K\in\mathcal{P}_h} \left\{ \|u\|^2_{H^1(K)} + \beta(u, z_K)_{1,K} - \beta \int_{\partial K} z_K (\nabla u \cdot \mu)\, ds + \beta \int_{\partial K} u(\nabla u\cdot\mu)\, ds \right\} + \beta \int_{\Gamma_{\text{int}}} \langle z_K\rangle [\nabla u\cdot\mathbf{n}]\, ds - \beta \int_{\Gamma_{\text{int}}} \langle u\rangle [\nabla u\cdot\mathbf{n}]\, ds + \sigma(1+\beta)\frac{h^\lambda}{p^\zeta} \|[\nabla u\cdot\mathbf{n}]\|^2_{L^2(\Gamma_{\text{int}})}.$$

Notice here, that for simplicity, the traces $\gamma_0 z_K$ and $\gamma_0 u$ have been denoted as z_K and u, respectively. The definition of the average operator $\langle . \rangle$ and the jump operator $[.]$ are given in (2). Now, by using the identities given in Remark 3.1, we can rewrite the above expression as follows

$$B(u,\hat{u}) = \sum_{K\in\mathcal{P}_h} \left\{ \|u\|^2_{H^1(K)} + 2\beta(u, z_K)_{1,K} - \beta \|z_K\|^2_{H^1(K)} \right\} + \beta \int_{\Gamma_{\text{int}}} \langle z_K\rangle [\nabla u\cdot\mathbf{n}]\, ds - \beta \int_{\Gamma_{\text{int}}} \langle u\rangle [\nabla u\cdot\mathbf{n}]\, ds + \sigma(1+\beta)\frac{h^\lambda}{p^\zeta} \|[\nabla u\cdot\mathbf{n}]\|^2_{L^2(\Gamma_{\text{int}})}. \qquad (18)$$

If we take a closer look at the terms involving integrals over Γ_{int}, we see that

$$\beta \int_{\Gamma_{\text{int}}} \langle z_K\rangle [\nabla u\cdot\mathbf{n}]\, ds \geq -\frac{|\beta|}{4} \sum_{K\in\mathcal{P}_h} \|z_K\|^2_{H^1(K)} - \frac{|\beta|}{2} \|[\nabla u\cdot\mathbf{n}]\|^2_{L^2(\Gamma_{\text{int}})},$$

$$-\beta \int_{\Gamma_{\text{int}}} \langle u\rangle [\nabla u\cdot\mathbf{n}]\, ds \geq -\frac{|\beta|}{4} \sum_{K\in\mathcal{P}_h} \|u\|^2_{H^1(K)} - \frac{|\beta|}{2} \|[\nabla u\cdot\mathbf{n}]\|^2_{L^2(\Gamma_{\text{int}})}.$$

A more detailed derivation of the above expressions can be found in [14]. Now, back substitution of these two results into (18), yields

$$B(u,\hat{u}) \geq \sum_{K\in\mathcal{P}_h} \left\{ \left(1 - \frac{|\beta|}{4}\right) \|u\|^2_{H^1(K)} + 2\beta(u, z_K)_{1,K} - \left(\beta + \frac{|\beta|}{4}\right) \|z_K\|^2_{H^1(K)} \right\} + \left\{ \sigma(1+\beta)\frac{h^\lambda}{p^\zeta} - |\beta| \right\} \|[\nabla u\cdot\mathbf{n}]\|^2_{L^2(\Gamma_{\text{int}})}.$$

By using Young's inequality,

$$2(u, z_K)_{1,K} \leq \varepsilon \|u\|^2_{H^1(K)} + \frac{1}{\varepsilon} \|z_K\|^2_{H^1(K)}, \qquad \varepsilon > 0,$$

and applying the Schwarz inequality and Theorem 3.3, one finally obtains

$$B(u,\hat{u}) \geq \sum_{K\in\mathcal{P}_h} \left\{ \left(1 - \varepsilon|\beta| - \frac{|\beta|}{4}\right) \|u\|^2_{H^1(K)} - \left(\beta + \frac{|\beta|}{4} + \frac{|\beta|}{\varepsilon}\right) \|\nabla u \cdot \mu]\|_{H^{-1/2}(\partial K)} \right\}$$
$$+ \left\{ \sigma(1+\beta)\frac{h^\lambda}{p^\zeta} - |\beta| \right\} \|[\nabla u \cdot \mathbf{n}]\|^2_{L^2(\Gamma_{\rm int})}.$$

With $\beta < 0$, this becomes

$$B(u,\hat{u}) \geq \sum_{K\in\mathcal{P}_h} \left\{ \left(1 - \varepsilon|\beta| - \frac{|\beta|}{4}\right) \|u\|^2_{H^1(K)} + \left(\frac{3|\beta|}{4} + \frac{|\beta|}{\varepsilon}\right) \|\nabla u \cdot \mu]\|^2_{H^{-1/2}(\partial K)} \right\}$$
$$+ \left\{ \sigma(1+|\beta|)\frac{h^\lambda}{p^\zeta} - |\beta| \right\} \|[\nabla u \cdot \mathbf{n}]\|^2_{L^2(\Gamma_{\rm int})}.$$

The second term in the RHS is only positive for $\varepsilon > 4/3$. If we take $\varepsilon = 2$, then it is clear that, given the parameters σ, λ, ζ, ν, and θ, we can always find a coefficient β such that there exists a $\xi_2(h,p) > 0$, defined by

$$\xi_2 = \min\left\{1 - \frac{9}{4}|\beta|,\ \frac{|\beta| p^\theta}{4h^\nu},\ 1 - |\beta| - \frac{|\beta| p^\zeta}{\sigma h^\lambda}\right\}, \tag{19}$$

that satisfies the following inequality:

$$B(u,\hat{u}) \geq \xi_2 \|u\|^2.$$

∎

THEOREM 3.4. INF-SUP CONDITION. *Given $\sigma > 0$, there exists $\gamma = \gamma(\sigma, h, p) > 0$ such that*

$$\sup_{v\in V/\{0\}} \frac{|B(u,v)|}{|||v|||} \geq \gamma |||u|||, \qquad \forall\, u \in V.$$

PROOF. By definition of the supremum, one obtains

$$\sup_{v\in V/\{0\}} \frac{|B(u,v)|}{|||v|||} \geq \frac{|B(u,\hat{u})|}{|||\hat{u}|||}, \qquad \forall\, u \in V,$$

where $\hat{u}$ is defined by (16). Next, by applying Lemmas 3.2 and 3.3, we obtain

$$\sup_{v\in V/\{0\}} \frac{|B(u,v)|}{|||v|||} \geq \frac{|B(u,\hat{u})|}{|||\hat{u}|||} \geq \frac{\xi_2(\sigma,h,p)}{\xi_1(\sigma,h,p)} |||u|||, \qquad \forall\, u \in V.$$

Thus, $\gamma = \xi_1/\xi_2$. ∎

COROLLARY 3.1. *If $\lambda = \nu = \theta = \zeta = 0$, i.e., the norm $|||.|||$ and the stabilization term are independent of h and p, then the Inf-Sup coefficient γ is a constant.*

PROOF. For simplicity, we set $\sigma = 1$. Choosing $\beta = 4/10$, for example, (17) and (19) yield $\xi_1 = \sqrt{228}/10$ and $\xi_2 = 1/10$, respectively, and it follows that $\gamma = 1/\sqrt{228}$. ∎

COROLLARY 3.2. *If $\lambda = \nu$ and $\theta = \zeta$, i.e., the h and p dependence of the stabilization term and norm $|||.|||$ are identical, then for $h^\lambda/p^\zeta < 1$ the Inf-Sup coefficient γ is bounded below by a constant $C > 0$.*

PROOF. Again, we set $\sigma = 1$, but now we choose $\beta = 4h^\nu/10p^\theta$. If we take $h^\lambda/p^\zeta < 1$, we obtain the following inequalities from (17) and (19):

$$\xi_1 \leq \frac{\sqrt{228}}{10}, \qquad \xi_2 \geq \frac{1}{10}.$$

Hence, we conclude that $\gamma \geq 1/\sqrt{228}$. ∎

3.3.3. Inf-Sup condition on the discrete space V^{hp}

The following trace theorem [14] is used to establish the Inf-Sup condition on the discrete space V^{hp}.

LEMMA 3.4. TRACE THEOREM. *Let $K \in \mathcal{P}_h$ be characterized by an affine mapping $\mathbf{F}_K$ (see Section 2.3) and $w \in H^2(K)$; then there exists a constant $C > 0$, independent of h_K, such that*

$$\|\nabla w \cdot \mu\|^2_{L^2(\partial K)} \leq C \left\{ \frac{1}{h_K} \|\nabla w\|^2_{L^2(K)} + \|\nabla w\|_{L^2(K)} \|\nabla^2 w\|_{L^2(K)} \right\}.$$

COROLLARY 3.3. *Given $v_h \in P^{p_K}(K)$, a polynomial of degree $\leq p_K$, and given that the mapping between K and $\hat{K}$ is affine, we can state that*

$$\exists C > 0 : \|\nabla v_h\|^2_{L^2(K)} \geq C \frac{h_K}{p_K^2} \|\nabla v_h \cdot \mu\|^2_{H^{-1/2}(\partial K)}.$$

PROOF. Since $v_h \in P^{p_K}(K)$, we know that $v_h \in H^2(K)$. Given this information, we can use the following inverse inequality, obtained from [15, p. 208]:

$$\|\nabla v_h\|_{L^2(K)} \leq C \frac{p_K^2}{h_K} \|v_h\|_{L^2(K)}, \qquad \forall v_h \in P^{p_K}(K).$$

Substitution of this inequality into the inequality in Lemma 3.4 yields

$$\|\nabla v_h \cdot \mu\|^2_{H^{-1/2}(\partial K)} \leq C \frac{p_K^2}{h_K} \|\nabla v_h\|^2_{L^2(K)}.$$ ∎

THEOREM 3.5. INF-SUP CONDITION FOR THE DISCRETE PROBLEM. *Let $\{\mathbf{F}_K\}$ define a family of affine invertible mappings. If $\sigma > 0$, then there exists a $\gamma_h = \gamma_h(\sigma, h, p) > 0$, such that*

$$\sup_{v_h \in V^{hp}/\{0\}} \frac{|B(u_h, v_h)|}{|||v_h|||} \geq \gamma_h |||u_h|||, \qquad \forall u_h \in V^{hp}/\{0\}$$

and

$$\gamma_h = C \min \left\{ 1, \frac{h^{1-\nu}}{p^{2-\theta}} \right\}, \qquad C > 0.$$

PROOF. By definition of the supremum, we get

$$\sup_{v_h \in V^{hp}/\{0\}} \frac{|B(u_h, v_h)|}{|||v_h|||} \geq \frac{B(u_h, u_h)}{|||u_h|||}.$$

By applying Corollary 3.3, it is clear that there exists $C > 0$, such that

$$B(u_h, u_h) \geq C \left\{ \sum_{K \in \mathcal{P}_h} \left(\|u_h\|^2_{H^1(K)} + \frac{h}{p^2} \|\nabla u_h \cdot \mathbf{n}\|^2_{H^{-1/2}(\partial K)} \right) + \sigma \frac{h^\lambda}{p^\zeta} \|[\nabla u_h \cdot \mathbf{n}]\|^2_{L^2(\Gamma_{\text{int}})} \right\},$$

which concludes the proof. ∎

3.4. Existence and Uniqueness of Solutions

LEMMA 3.5. *If $f \in L^2(\Omega)$, then there exists a unique solution $w \in V$ to the discontinuous variational BVP (6) that is a solution to the model problem (1) (in a weak sense).*

PROOF. First, we introduce the classical variational formulation of the model problem (1) in the space $H_0^1(\Omega)$.

$$\text{Find } w \in H_0^1(\Omega) : A(w,v) = L(v), \qquad \forall\, v \in H_0^1(\Omega), \tag{20}$$

where $L(v)$ is defined as in (7) and the bilinear form $A(w,v)$ is defined as follows

$$A : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}, \qquad A(w,v) = \int_\Omega \{\nabla w \cdot \nabla v + wv\}\, d\mathbf{x}.$$

By the generalized Lax-Milgram theorem and by equivalence of this formulation to the model problem (1), we know that if $f \in L^2(\Omega)$ there exists a unique solution $w \in H_0^1(\Omega) \cap H(\Delta,\Omega) \subset V$ to (20) that satisfies the model problem in a distributional sense. Consequently, by Theorem 3.1, we know that $w \in V$ is a solution to the discontinuous VBVP (6) as well.

Thus, existence of a solution to the discontinuous VBVP follows from the existence theory for the continuous variational formulation. Last, the solution is unique because the bilinear form is positive definite, i.e.,

$$B(v,v) = \sum_{K\in\mathcal{P}_h} \|v\|^2_{H^1(K)} + \sigma \frac{h^\lambda}{p^\eta} \|[\nabla v \cdot \mathbf{n}]\|^2_{L^2(\Gamma_{\mathrm{int}})} > 0, \qquad \forall\, v \in V \setminus \{0\}. \qquad \blacksquare$$

3.5. Stability

The last requirement to ensure well posedness of the weak formulation is stability, i.e., *continuous dependence of the solution on the input data.*

PROPOSITION 3.1. *If the stabilization parameter is constant, i.e., $\sigma = 1$, $\lambda = \zeta = 0$, and if the norm parameters are $\nu = \theta = 0$, then the solution u to the variational BVP (6) depends continuously on the input data, i.e., given a small perturbation $\delta f \in L^2(\Omega)$, then there exists unique perturbation $\delta u \in V$ such that*

$$\exists\, C > 0 : \|\delta u\|_{H^1(\mathcal{P}_h)} \le |||\delta u||| \le C\|\delta f\|_{L^2(\Omega)},$$

where C is a constant independent of h and p.

PROOF. Let $\delta f \in L^2(\Omega)$ be a perturbation in the input data f. Consequently, since the problem under consideration is linear, this leads to a perturbation $\delta u \in V$ in the solution u, which satisfies

$$B(\delta u, v) = \int_\Omega \delta f\, v\, d\mathbf{x}, \qquad \forall\, v \in V. \tag{21}$$

By applying the Inf-Sup condition of Theorem 3.4, we obtain

$$|||\delta u||| \le \frac{1}{\gamma} \sup_{v\in V/\{0\}} \frac{|B(\delta u, v)|}{|||v|||}.$$

Substituting (21) and applying the Schwarz inequality, yields

$$|||\delta u||| \le \frac{1}{\gamma} \sup_{v\in V/\{0\}} \frac{\int_\Omega |\delta f v|\, d\mathbf{x}}{|||v|||} \le \frac{1}{\gamma}\|\delta f\|_{L^2(\Omega)}.$$

Since, $\sigma = 1$, $\lambda = \nu = \theta = \zeta = 0$, we know by Lemmas 3.2 and 3.3 and by Theorem 3.4 that the Inf-Sup parameter is constant and is at least $1/\sqrt{228}$. ∎

In addition to stability with respect to the topology on the space V, we also seek stability in the space $H^1(\mathcal{P}_h)$, which is defined as

$$H^1(\mathcal{P}_h) = \{v \in L^2(\Omega) : v_{|K} \in H^1(K),\ \forall\, K \in \mathcal{P}_h\},$$

with norm

$$\|v\|^2_{H^1(\mathcal{P}_h)} = \sum_{K\in\mathcal{P}_h} \|v\|^2_{H^1(K)}. \tag{22}$$

In many applications, this space is referred to as the space of functions with finite energy, even though it admits discontinuities across the element interfaces.

PROPOSITION 3.2. *If $\lambda = \nu$ and $\theta = \zeta$, i.e., if the h and p dependence of the stabilization term and the norm $|||.|||$ are identical, then for $h^\lambda/p^\zeta < 1$ the solution u to the variational BVP (6) depends continuously on the input data; i.e., given a perturbation $\delta f \in L^2(\Omega)$, there exists a unique perturbation $\delta u \in V$ such that*

$$\exists C > 0 : \|\delta u\|_{H^1(\mathcal{P}_h)} \le |||\delta u||| \le C\|\delta f\|_{L^2(\Omega)},$$

where C is a constant independent of h and p.

PROOF. By following the steps in the proof of the previous proposition, we get

$$\|\delta u\|_{H^1(\mathcal{P}_h)} \le \frac{1}{\gamma}\|\delta f\|_{L^2(\Omega)}.$$

Application of Corollary 3.2 finishes the proof. ∎

4. *A PRIORI* ERROR ESTIMATION

In this section, we investigate the convergence properties of solutions $\{u_h\}$ of (12). Let $u \in V$ be the exact solution to the BVP (6), then by using the linearity of $B(\cdot,\cdot)$, it follows easily that the approximation error $e = u - u_h$ is governed by

$$B(e,v) = \underbrace{F(v) - B(u_h,v)}_{\mathcal{R}^h(v)}, \qquad \forall\, v \in V, \tag{23}$$

where $\mathcal{R}^h : V \to \mathbb{R}$ is the *residual functional*. Note that, due to (12), the residual satisfies the following *orthogonality property* on V^{hp}:

$$\mathcal{B}(e, v_h) = 0, \qquad \forall\, v_h \in V^{hp}. \tag{24}$$

We start in Section 4.1 by proving an *interpolation theorem* in the norm $||| \cdot |||$ that we need in Section 4.2 for our proof of the error estimate in this norm. Next, Section 4.3 derives the rates in the norm $\|\cdot\|_{H^1(\mathcal{P}_h)}$. Unless stated otherwise, we assume that the family of partitions $\{\mathcal{P}_h\}$ is regular.

4.1. Interpolation Error in the Norm $|||\cdot|||$

We introduce a family of interpolants $\{\pi_{hp}^K\}$, such that

$$\pi_{hp}^K : H^{r_K}(K) \to P^{p_K}(K),$$
$$\pi_{hp}^K(v_h) = v_h, \qquad \forall\, v_h \in P^{p_K}(K).$$

We will need results for the interpolation errors from the work of Babuška and Suri [16] (in particular, Theorem 4.5 in this reference).

THEOREM 4.1. INTERPOLATION THEOREM. *For $\varphi \in H^{r_k}(K)$, there exists $C > 0$, independent of φ, p_k, and r_k, and a sequence $\pi_{hp}^K(\varphi) \in P^{p_k}(K)$, such that*

$$\left\|\varphi - \pi_{hp}^K(\varphi)\right\|_{L^2(K)} \le C\frac{h^{\mu_k}}{p_k^{r_k}}\|\varphi\|_{H^{r_k}(K)},$$
$$\left\|\nabla\varphi - \nabla\pi_{hp}^K(\varphi)\right\|_{L^2(K)} \le C\frac{h^{\mu_k-1}}{p_k^{r_k-1}}\|\varphi\|_{H^{r_k}(K)}, \qquad r_k \ge 1, \quad p_k \ge 1,$$
$$\left\|\nabla^2\varphi - \nabla^2\pi_{hp}^K(\varphi)\right\|_{L^2(K)} \le C\frac{h^{\mu_k-2}}{p_k^{r_k-2}}\|\varphi\|_{H^{r_k}(K)},$$

where $\mu_k = \min(p_k + 1, r_k)$.

By extending the local interpolants $\pi_{hp}^K(.)$ to zero outside of K for every $K \in \mathcal{P}_h$, we can define the following global interpolant Π_{hp}:

$$\Pi_{hp} : V \to V^{hp}, \qquad \Pi_{hp}(u) = \sum_{K\in\mathcal{P}_h} \pi_{hp}^K(u_{|K}), \qquad u \in V. \tag{25}$$

THEOREM 4.2. INTERPOLATION THEOREM. *Let $u \in V$ and $u_{|K}$ locally be in $H^{r_k}(K)$, $r_k \geq 2$, let the stabilization parameters $\sigma > 0$ and $\lambda, \zeta \geq 0$, and let the norm parameters $\nu, \theta \geq 0$, there exists $C > 0$, independent of u, h, and p such that the interpolation error $\eta = u - \Pi_{hp}u$ can be bounded as follows:*

$$|||\eta||| \leq C\frac{h^{\mu^*}}{p^{r^*}}\sqrt{\sum_{K\in\mathcal{P}_h} \|u\|^2_{H^{r_k}(K)}}, \qquad r_K \geq 2,$$

where

$$\mu^* = \min\left\{\mu - 1, \mu - \frac{3}{2} + \frac{\nu}{2}, \mu - \frac{3}{2} + \frac{\lambda}{2}\right\},$$
$$r^* = \min\left\{r - 1, r - \frac{3}{2} + \frac{\theta}{2}, r - \frac{3}{2} + \frac{\zeta}{2}\right\},$$
$$\mu = \min(p+1, r),$$

and where $r = \min_{K\in\mathcal{P}_h}(r_K)$.

PROOF. By recalling the definition of the norm $|||.|||$ (4), applying the triangle inequality, and using Trace Lemma 3.4, we get

$$|||\eta|||^2 \leq C(\sigma)\sum_{K\in\mathcal{P}_h}\left\{\|\eta\|^2_{L^2(K)} + \left(1 + \frac{h^{\nu-1}}{p^\theta} + \frac{h^{\lambda-1}}{p^\zeta}\right)\|\nabla\eta\|^2_{L^2(K)} + \left(\frac{h^\nu}{p^\theta} + \frac{h^\lambda}{p^\zeta}\right)\|\nabla\eta\|_{L^2(K)}\left\|\nabla^2\eta\right\|_{L^2(K)}\right\}.$$

Now, application of the interpolation Theorem 4.1 gives our final result

$$|||\eta|||^2 \leq C(\sigma)\left(\frac{h^{2\mu-2}}{p^{2r-2}} + \frac{h^{2\mu-3+\nu}}{p^{2r-3+\theta}} + \frac{h^{2\mu-3+\lambda}}{p^{2r-4+\zeta}}\right)\sum_{K\in\mathcal{P}_h}\|u\|^2_{H^{r_K}(K)}.$$ ∎

4.2. Error Estimates in the Norm $|||.|||$

THEOREM 4.3. *Given $\sigma > 0$, let $u \in H^2(\Omega)\cap V$ be the unique solution to the variational BVP (6), $u_h \in V^{hp}$ be an approximation (12) of u, and both the stabilization and norm parameters be of order $O(h/p^2)$ (i.e., $\lambda = \nu = 1$ and $\zeta = \theta = 2$). Then, the error $u - u_h$ satisfies the bound*

$$|||u - u_h||| \leq C(\sigma)\frac{h^{\mu-3/2}}{p^{r-2}}\sqrt{\sum_{K\in\mathcal{P}_h}\|u\|^2_{H^r(K)}}, \qquad p \geq 1, \quad r \geq 2,$$

where $r = \min_{K\in\mathcal{P}_h}(r_K)$ and $\mu = \min(p+1, r)$ and $C(\sigma)$ is a positive constant depending on σ.

PROOF. Given the interpolant $\Pi_{hp}u$ (25), we split e such that $e = \eta - \xi$, where $\eta = u - \Pi_{hp}u$ and $\xi = u_h - \Pi_{hp}u$. Notice that $\xi \in V^{hp}$ and that the interpolation error $\eta \in H^2(\mathcal{P}_h)$. Consequently, by using the triangle inequality, one obtains

$$|||u - u_h||| \leq |||\eta||| + |||\xi|||. \tag{26}$$

By applying the discrete inf-sup condition of Theorem 3.5 and taking $\nu = 1$ and $\theta = 2$, one gets

$$|||\xi||| \leq C \sup_{v_h\in V^{hp}/\{0\}}\frac{|B(\xi, v_h)|}{|||v_h|||}.$$

By using the orthogonality property (24), the inequality can be rewritten as

$$|||\xi||| \leq C \sup_{v_h\in V^{hp}/\{0\}}\frac{|B(\eta, v_h)|}{|||v_h|||}.$$

By taking $\lambda = 1$ and $\zeta = 2$ and applying Theorem 3.2, this becomes

$$|||\xi||| \leq C(\sigma)\frac{p}{\sqrt{h}}|||\eta|||.$$

Thus, returning to (26), we can conclude

$$|||u - u_h||| \leq C(\sigma)\frac{p}{\sqrt{h}}|||\eta|||.$$

With this choice of coefficients we have, we get for the parameters in Theorem 4.2

$$\mu^* = \mu - 1, \qquad r^* = r - 1.$$

Hence, by applying the interpolation Theorem 4.2, we conclude the proof. ∎

Since $\|v\|_{H^1(\mathcal{P}_h)} \leq |||v|||$, the above theorem would imply suboptimal convergence rates for the error in the norm $\|\cdot\|_{H^1(\mathcal{P}_h)}$. However, in the next section, we derive optimal h convergence rates for the error in this specific norm. Unfortunately, these rates are limited to cases where $p \geq 2$. For $p = 1$, we do not prove an optimal convergence rate, but at least we prove the solutions converge. To this date, convergence for $p = 1$ had not been proved for Oden, Babuška and Baumann [1]-type DGMs that do not include penalty terms on the jumps across the element interfaces.

4.3. Error Estimates in the $H^1(\mathcal{P}_h)$ Norm

In the following, we derive optimal h convergence rates for the error in the norm $\|\cdot\|_{H^1(\mathcal{P}_h)}$ for $p \geq 2$. The decisive element in the proof is the use of a specific type of interpolants, that were introduced by Rivière *et al.* [7–9]. The original estimates were then improved in [17,18,22]. By using these interpolants, we succeed at proving optimal h convergence rates for $p \geq 2$, but the p convergence rates appear to be 1/2 order lower than the one predicted in the previous section. So, we succeed at improving the h convergence rate, but not the p rates.

We start our analysis by defining the following norm on $H^2(\mathcal{P}_h)$:

$$|||v|||^2_{H^2(\mathcal{P}_h)} = \sum_{K\in\mathcal{P}_h} \left\{ \|v\|^2_{H^1(K)} + \frac{p^\zeta}{\sigma h^\lambda}\|v\|^2_{L^2(\partial K)} + \frac{\sigma h^\lambda}{p^\zeta}\|\nabla v \cdot \mu\|^2_{L^2(\partial K)} \right\}. \tag{27}$$

LEMMA 4.1. *The bilinear form B is continuous [14] on $H^2(\mathcal{P}_h) \times H^2(\mathcal{P}_h)$ with respect to the norm $|||\cdot|||_{H^2(\mathcal{P}_h)}$, i.e.,*

$$\exists\, K > 0 : |B(u,v)| \leq K|||u|||_{H^2(\mathcal{P}_h)}|||v|||_{H^2(\mathcal{P}_h)}, \qquad \forall\, u, v \in H^2(\mathcal{P}_h),$$

where K is a constant, independent of h and p.

Next, we introduce an important inverse inequality [14] between the spaces $H^1(\mathcal{P}_h)$ and $H^2(\mathcal{P}_h)$ for finite-dimensional functions v_h that are in V^{hp}.

LEMMA 4.2. *Let the parameters in the norm $\|\cdot\|_{H^2(\mathcal{P}_h)}$ be set as $\lambda = \zeta = 1$, then the following inverse relation holds:*

$$\exists\, C > 0 : |||v_h - \bar{v}_h|||_{H^2(\mathcal{P}_h)} \leq C(\sigma)\sqrt{p}\|v_h\|_{H^1(\mathcal{P}_h)}, \qquad \forall\, v_h \in V^{hp},$$

where $\bar{v}_h$ denotes the piecewise average of v_h

$$\bar{v}_h = \sum_{K\in\mathcal{P}_h} \bar{v}_{h|K}, \qquad \bar{v}_{h|K} = \frac{1}{|K|}\int_K v_h \, d\mathbf{x}. \tag{28}$$

Similar to our proofs in the other sections, we need a theorem on the interpolation error in the norm $|||\cdot|||_{H^2(\mathcal{P}_h)}$. As mentioned previously, here we do not use the Babuška and Suri [16] interpolants but rather the interpolants proposed by Rivière *et al.* [7–9].

THEOREM 4.4. INTERPOLATION THEOREM. *Let $\varphi \in H^{r_k}(K)$, $r_k \geq 2$, there exists $C > 0$, independent of φ, p_k, and r_k, and an interpolant $\tilde{\pi}^K_{hp}\varphi \in P^{p_k}(K)$, such that*

$$\int_{\gamma\subset\partial K} \nabla\left(\varphi - \tilde{\pi}^K_{hp}\varphi\right)\cdot \mu\, ds = 0 \tag{29}$$

and

$$\left\|\varphi - \tilde{\pi}^K_{hp}\varphi\right\|_{L^2(K)} \leq C\frac{h^{\mu_k}}{p_k^{r_k-3/2}}\|\varphi\|_{H^{r_k}(K)},$$

$$\left\|\nabla\varphi - \nabla\tilde{\pi}^K_{hp}\varphi\right\|_{L^2(K)} \leq C\frac{h^{\mu_k-1}}{p_k^{r_k-3/2}}\|\varphi\|_{H^{r_k}(K)}, \qquad p_k \geq 2,$$

$$\left\|\nabla^2\varphi - \nabla^2\tilde{\pi}^K_{hp}\varphi\right\|_{L^2(K)} \leq C\frac{h^{\mu_k-2}}{p_k^{r_k-2}}\|\varphi\|_{H^{r_k}(K)},$$

where $\mu_k = \min(p_k + 1, r_k)$.

Again, by extending the corresponding local interpolant $\tilde{\pi}^K_{hp}(.)$ equal to zero outside of each $K \in \mathcal{P}_h$, we can define a global interpolant on V

$$\tilde{\Pi}_{hp} : V \to V^{hp}, \qquad \tilde{\Pi}_{hp}(u) = \sum_{K\in\mathcal{P}_h} \tilde{\pi}^K_{hp}(u_{|K}), \qquad u \in V. \tag{30}$$

THEOREM 4.5. INTERPOLATION THEOREM. *Let $u \in H^2(\Omega)\cap V$, $\tilde{\Pi}_{hp}(u) \in V^{hp}$ be the interpolant of u (30) and let the stabilization parameters $\sigma > 0$ and $\lambda, \zeta \geq 0$, then there exists $C(\sigma) > 0$, independent of u, h, and p such that the interpolation error can be bounded as follows:*

$$\left|\left|\left|u - \tilde{\Pi}_{hp}(u)\right|\right|\right|_{H^2(\mathcal{P}_h)} \leq C(\sigma)\frac{h^{u^{**}}}{p^{r^{**}}}\sqrt{\sum_{K\in\mathcal{P}_h} \|u\|^2_{H^{r_K}(K)}}, \qquad r_K \geq 2, \quad p \geq 2,$$

where

$$\mu^{**} = \min\left\{\mu - 1, \mu - \frac{1}{2} - \frac{\lambda}{2}, \mu - \frac{3}{2} + \frac{\lambda}{2}\right\},$$

$$r^{**} = \min\left\{r - \frac{3}{2}, r - \frac{3}{2} - \frac{\zeta}{2}, r - \frac{7}{4} + \frac{\zeta}{2}\right\},$$

$$\mu = \min\{p + 1, r\},$$

and where $r = \min_{K\in\mathcal{P}_h}(r_K)$.

PROOF. By recalling the definition of the norm $|||.|||_{H^2(\mathcal{P}_h)}$ (27), substituting trace Lemma A.3 in [17] and Lemma 3.4, one gets

$$|||\eta|||^2_{H^2(\mathcal{P}_h)} = \sum_{K\in\mathcal{P}_h}\left\{\|\eta\|^2_{H^1(K)} + \frac{h^{-\lambda-1}}{\sigma p^{-\zeta}}\left(\|\eta\|^2_{L^2(K)} + h\|\nabla\eta\|_{L^2(K)}\|\eta\|_{L^2(K)}\right)\right.$$
$$\left. + \frac{\sigma h^{\lambda-1}}{p^{\zeta}}\left(\|\nabla\eta\|^2_{L^2(K)} + h\|\nabla\eta\|_{L^2(K)}\left\|\nabla^2\eta\right\|_{L^2(K)}\right)\right\}.$$

By applying the interpolation Theorem 4.4, we conclude the proof. ∎

THEOREM 4.6. ERROR ESTIMATE. *Given $\sigma > 0$. Let $u \in H^2(\Omega) \cap V$ be the exact solution to the BVP (6), $u_h \in V^{hp}$ be a discrete approximation (12) and let the stabilization parameter be of order $O(h/p)$ (i.e., $\lambda = 1$ and $\zeta = 1$), then there exists $C(\sigma) \geq 0$ such that*

$$\|u - u_h\|_{H^1(\mathcal{P}_h)} \leq C(\sigma)\frac{h^{\mu-1}}{p^{r-5/2}}\sqrt{\sum_{K\in\mathcal{P}_h}\|u\|^2_{H^r(K)}}, \qquad p \geq 2,$$

where $\mu = \min(p+1, r)$.

PROOF. Given the interpolant $\tilde{\Pi}_{hp}u$ in (30), we again split e such that $e = \eta - \xi$, where $\eta = u - \tilde{\Pi}_{hp}u$ and $\xi = u_h - \tilde{\Pi}_{hp}u$. By using the triangle inequality, one obtains

$$\|u - u_h\|_{H^1(\mathcal{P}_h)} \leq \|\eta\|_{H^1(\mathcal{P}_h)} + \|\xi\|_{H^1(\mathcal{P}_h)}. \tag{31}$$

From (7) and (22), follows that

$$\|\xi\|^2_{H^1(\mathcal{P}_h)} \leq B(\xi, \xi).$$

By using the orthogonality property (24) and the linearity of $B(\cdot, \cdot)$, this can be rewritten as

$$\|\xi\|^2_{H^1(\mathcal{P}_h)} \leq B(\eta, \xi) = B\left(\eta, \xi - \bar{\xi}\right) + B\left(\eta, \bar{\xi}\right),$$

where $\bar{\xi}$ denotes the piecewise average (28) of ξ. Now, by applying Lemma 4.1 to the first term in the RHS, we get

$$\|\xi\|^2_{H^1(\mathcal{P}_h)} \leq C\left|\left|\left|\xi - \bar{\xi}\right|\right|\right|_{H^2(\mathcal{P}_h)} |||\eta|||_{H^2(\mathcal{P}_h)} + B\left(\eta, \bar{\xi}\right).$$

By applying the inverse inequality of Lemma 4.2, we can rewrite the above inequality as

$$\|\xi\|^2_{H^1(\mathcal{P}_h)} \leq C(\sigma)\sqrt{p}\|\xi\|_{H^1(\mathcal{P}_h)}|||\eta|||_{H^2(\mathcal{P}_h)} + B\left(\eta, \bar{\xi}\right). \tag{32}$$

As we shall now see, the term $B(\eta, \bar{\xi})$ can be bounded in terms of $\|\xi\|_{H^1(\mathcal{P}_h)}$ as well, due to the special property (29) of the interpolant $\tilde{\Pi}_{hp}$. By expanding the term $B(\eta, \bar{\xi})$, we get

$$B\left(\eta, \bar{\xi}\right) = \sum_{K \in \mathcal{P}_h} \left\{ \int_K \eta\bar{\xi}\, d\mathbf{x} + \int_{\partial K} \bar{\xi}\nabla\eta \cdot \mu\, ds \right\} - \int_{\Gamma_{\text{int}}} \langle\bar{\xi}\rangle \left[\nabla\eta \cdot \mathbf{n}\right] ds.$$

Now, applying the property (29), gives

$$B\left(\eta, \bar{\xi}\right) = \sum_{K \in \mathcal{P}_h} \int_K \eta\bar{\xi}\, d\mathbf{x} \leq \|\eta\|_{L^2(\Omega)} \left\|\bar{\xi}\right\|_{L^2(\Omega)} \leq C\|\eta\|_{L^2(\Omega)}\|\xi\|_{H^1(\mathcal{P}_h)}.$$

Back substitution of this result into (32) and (31), then yields

$$\|u - u_h\|_{H^1(\mathcal{P}_h)} \leq C(\sigma)\sqrt{p}|||\eta|||_{H^2(\mathcal{P}_h)}.$$

Next, by recalling the interpolation Theorem 4.5, we get

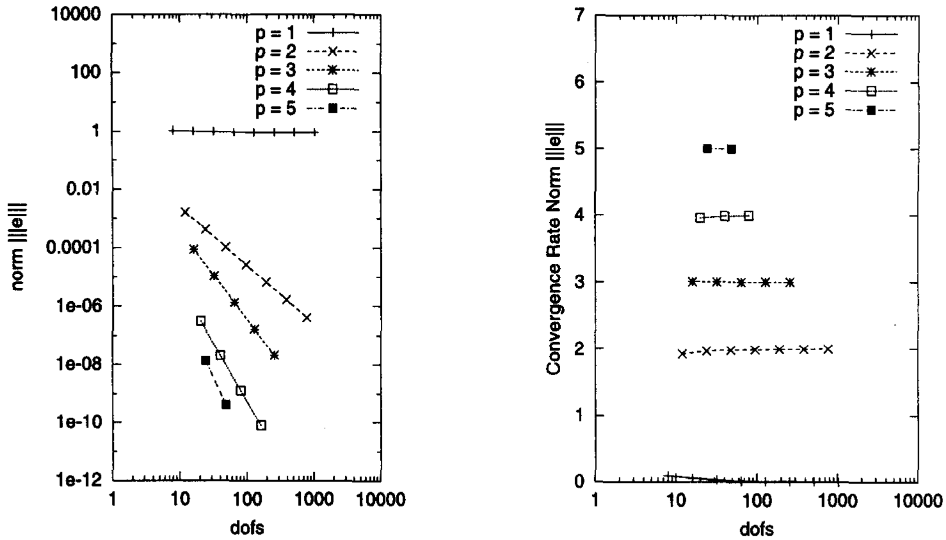
$$\|u - u_h\|_{H^1(\mathcal{P}_h)} \leq C(\sigma)\frac{h^{\mu^{**}}}{p^{r^{**}-1/2}}\sqrt{\sum_{K \in \mathcal{P}_h} \|u\|^2_{H^{r_K}(K)}}.$$

Since $\lambda = \zeta = 1$, we know that $\mu^{**} = \mu - 1$ and $r^{**} = r - 2$. ∎

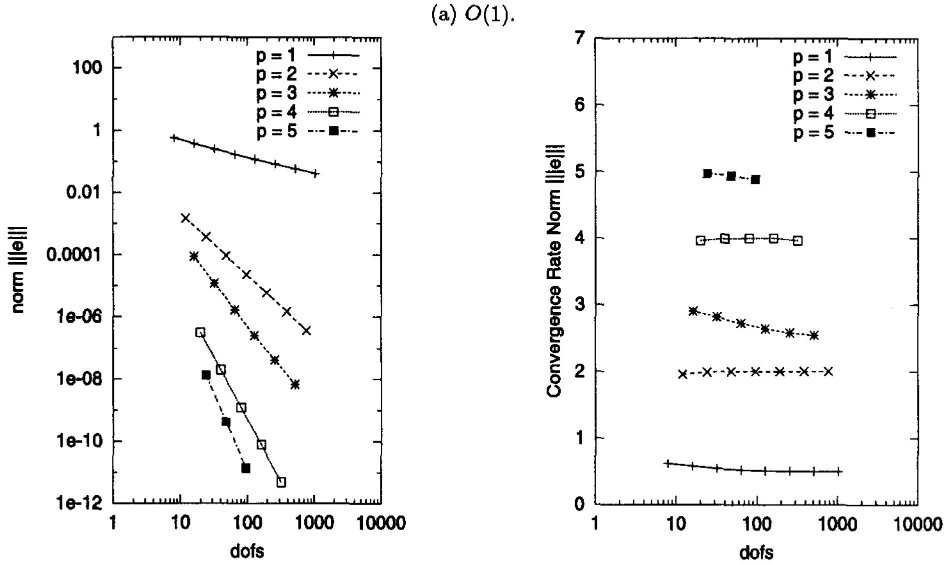
REMARK 4.1. If we combine the results of Theorems 4.3 and 4.6, we can conclude that for a stabilization term of order $O(h/p^2)$ and for $p \geq 2$, the convergence rates are of order $\mu - 1$ and $r - 2$ for h and p convergence, respectively.

5. NUMERICAL RESULTS

Convergence rates are presented in Section 5.1 for the case of a one-dimensional version of the model problem. Section 5.2 shows results for the two-dimensional case.



(a) $O(1)$.



(b) $O(h)$.

Figure 3. Variation of the norm $|||\cdot|||$ of the approximation error with degrees of freedom and for two orders of h of the stabilization and norm parameters.

5.1. One-Dimensional Tests

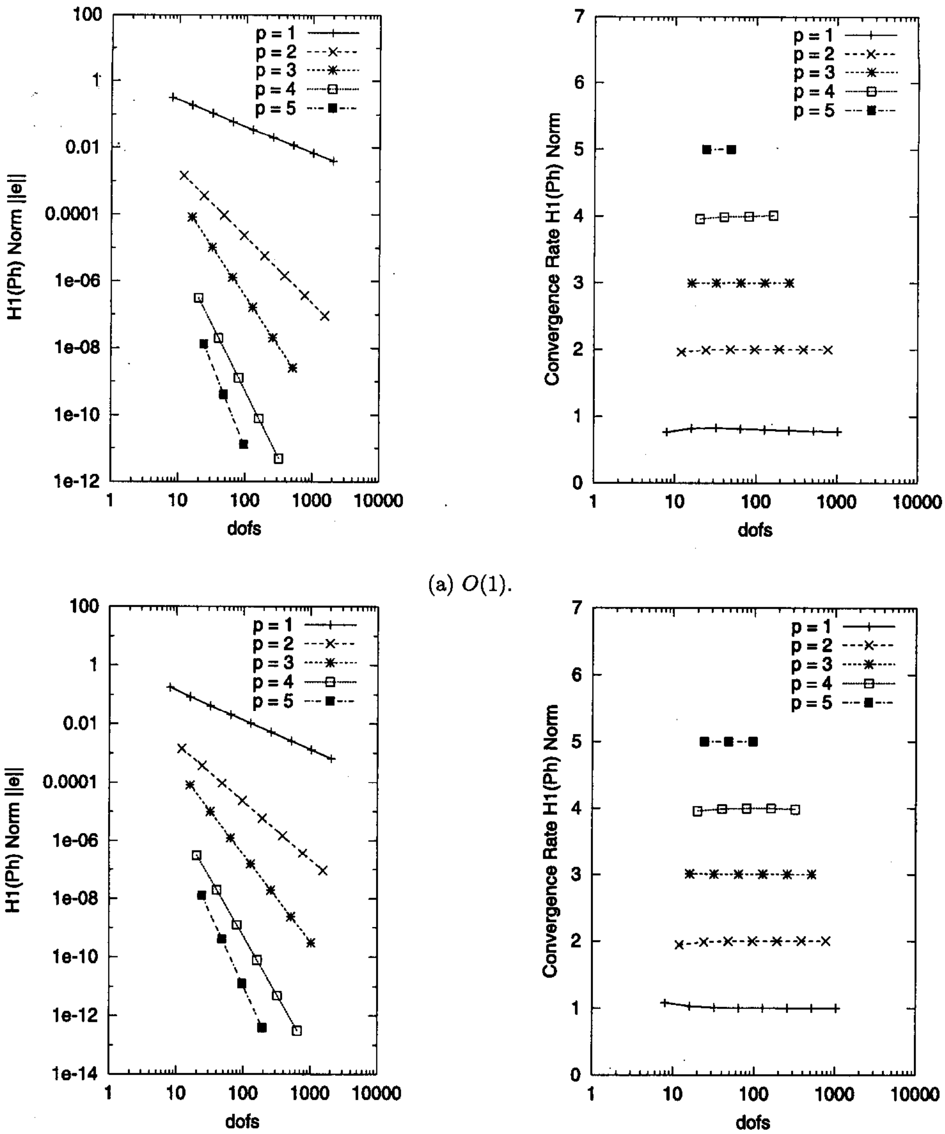
We consider the following *one-dimensional* version of problem (1):

$$-\frac{d^2u}{dx^2} + u = 1, \qquad \text{for } 0 < x < 1, \qquad u(0) = u(1) = 0. \tag{33}$$

The exact solution to this problem is

$$u(x) = 1 - \frac{e^x + e^{1-x}}{1+e}. \tag{34}$$

Using the one-dimensional experiments, we investigate how the convergence of the approximate solutions is affected by the order of the stabilization terms, and we verify the *a priori* error



(a) $O(1)$.

(b) $O(h)$.

Figure 4. Variation of the $H^1(\mathcal{P}_h)$-norm of the approximation error with degrees of freedom for two orders of h of the stabilization and norm parameters.

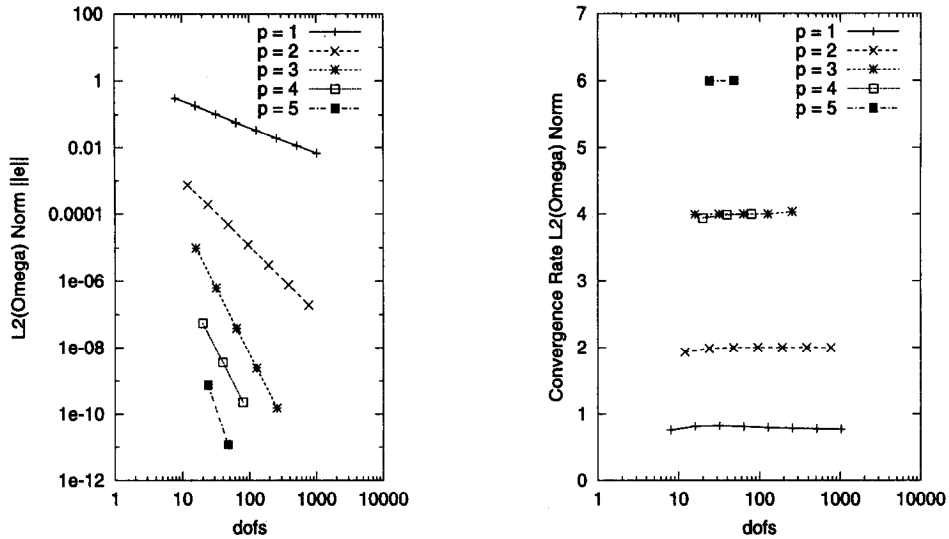
estimates that were derived in Section 4. In Figures 3–5, the computed convergence of the approximation error $u - u_h$ is shown for two separate cases.

1. The stabilization and norm parameter are of order $O(1)$ (i.e., $\lambda = \nu = 0$).
2. The stabilization and norm parameter are of order $O(h)$ (i.e., $\lambda = \nu = 1$).

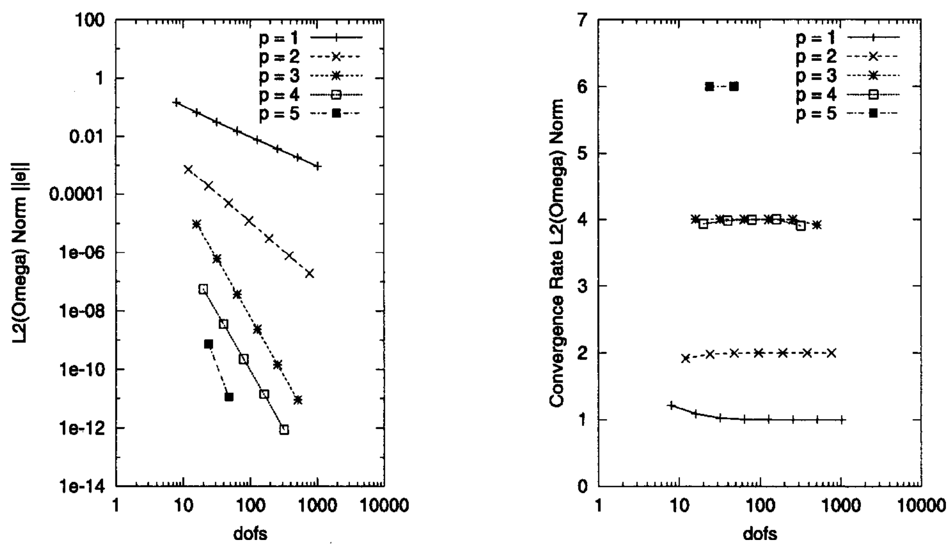
Notice that the uniform h convergence rates are obtained via the following rule:

$$\rho_h = \frac{\log\left(e_h^i / e_h^{i+1}\right)}{\log 2}. \tag{35}$$

In Figure 3, the results are shown with respect to the norm $|||.|||$. It is clear that changing the order of the stabilization term from $O(1)$ to $O(h)$ does improve the convergence, especially for the case where $p = 1$. For $p \geq 2$, a lesser effect is observed, for $p = 3$ the rate gets slightly worse. Also, for $p = 1$ and $p = 3$, rates of convergence of $1/2$ and 2.5 for a stability term of order $O(h)$ are observed in agreement with the predicted rate by Theorem 4.3. For even-order approximations, higher rates by an order of $1/2$ are observed.



(a) $O(1)$.



(b) $O(h)$.

Figure 5. Variation of the $L^2(\Omega)$-norm of the approximation error with degrees of freedom for two orders of h of the stabilization and norm parameters.

In Figure 4, the results are shown for the approximation error in the $H^1(\mathcal{P}_h)$ norm. Indeed, optimal convergence rates are obtained for $p \geq 2$ when the stabilization term is of order $O(h)$ (see Theorem 4.6). Even for $p = 1$, the solutions converge with an optimal rate, although we can only prove a lower rate (Theorem 4.3).

The results for the approximation error in the $L^2(\Omega)$ norm are shown in Figure 5. Changing the order of the stabilization term from $O(1)$ to $O(h)$ only improves the convergence rate for the case where $p = 1$. Furthermore, as is also observed for the DGM formulation by Oden, Babuška and Baumann [1], suboptimal convergence rates are achieved for convergence in the $L^2(\Omega)$ norm when even-order polynomials are used. Apparently, the stabilization on the fluxes is not sufficient to stabilize the method to optimal convergence in $L^2(\Omega)$ for these cases.

5.2. Two-Dimensional Convergence Tests

For our 2D example problem, we consider the following BVP, given on the unit square $\Omega = (0, 1) \times (0, 1)$ with prescribed Dirichlet boundary conditions on $\partial\Omega$

$$\begin{aligned} -\Delta u + u &= 0, && \text{in } \Omega, \\ u(0,y) = u(1,y) &= 0, && y \in [0,1], \\ u(x,0) = 0, \qquad u(x,1) &= \frac{1}{2}\sin(\pi x)\sinh\sqrt{1+\pi^2}, && x \in [0,1]. \end{aligned} \tag{36}$$

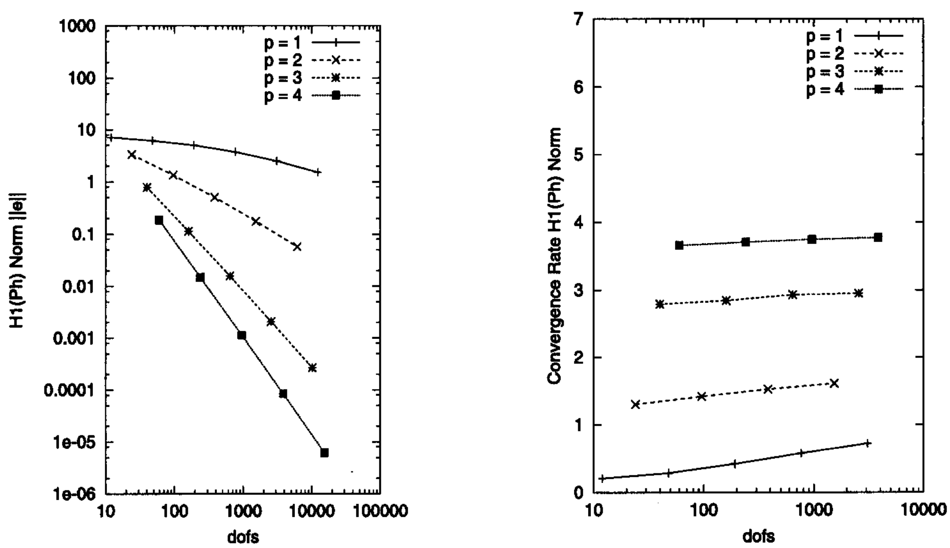
The exact solution to this problem is

$$u(x,y) = \frac{1}{2}\sin(\pi x)\sinh\left(\sqrt{1+\pi^2}y\right). \tag{37}$$

In the convergence analyses performed here, the orders of the norm and stabilization parameters are set equal to $O(1)$ and $O(h)$, as is done in the experiments of the previous section.

In Figure 6, the results are shown for the approximation error in the $H^1(\mathcal{P}_h)$ norm. Figures 6a show the results for stabilization and norm parameters of order $O(1)$, whereas Figures 6b show the results when the order of these parameters are of $O(h)$. It is evident that for a constant stabilization parameter, suboptimal convergence rates are obtained. By setting the stabilization parameter to $O(h)$, the rates become optimal, as is predicted in Theorem 4.6. Notice though, that the results for $p = 1$ again are of order $1/2$ better than the predicted rates. Theorem 4.3 indicates that for a stabilization parameter of order $O(h)$, the convergence rate for $p = 1$ should be at least equal to $1/2$, whereas a rate equal to 1 is obtained here.

In Figure 7, the convergence results in the $L^2(\Omega)$ norm are shown. For $p = 1$ and $p = 3$, the convergence rates improve when the order of the stabilization term changes from $O(1)$ to $O(h)$.



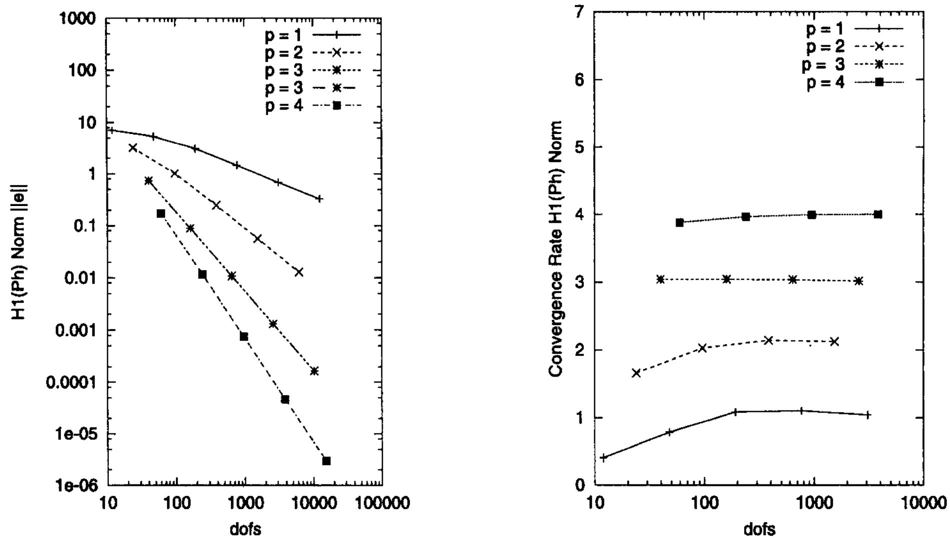
(a) $O(1)$.

Figure 6. Variation of the $H^1(\mathcal{P}_h)$-norm of the approximation error with degrees of freedom for two orders h of the stabilization and norm parameters (2D computations).

For $p = 3$, the observed experimental convergence rate is optimal. The results for the even-order approximations again remain suboptimal and these rates appear to be insensitive to the change in order of the stabilization term. Tests in which we changed the order of the stabilization parameter to $O(h^2)$ and $O(h^3)$ were performed but are not presented here as there was no noticeable change in the results.

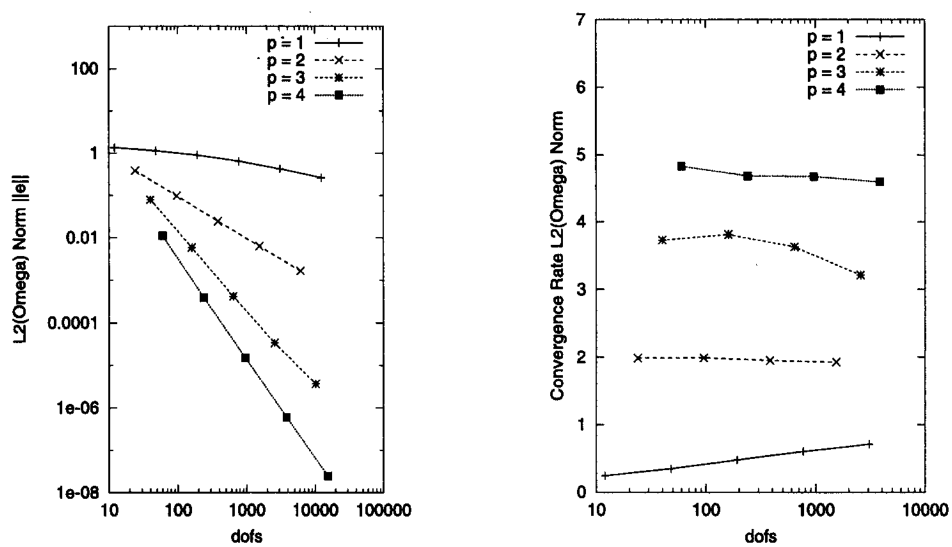
6. CONCLUDING REMARKS

In this work, a new DG formulation is presented and analyzed for the case of a two-dimensional reaction-diffusion problem with Dirichlet boundary conditions. The method is similar to the DGM



(b) $O(h)$.

Figure 6. (cont.)



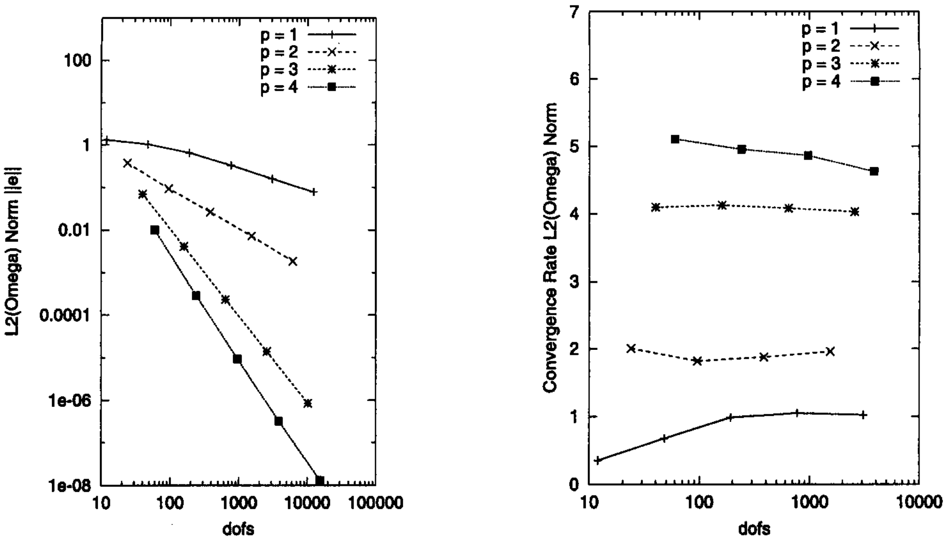
(a) $O(1)$.

Figure 7. Variation of the $L^2(\Omega)$-norm of the approximation error with degrees of freedom for two orders h of the stabilization and norm parameters (2D computations).

of [1], but involves an extra stabilization term on the jumps of the fluxes across the element interfaces.

Also, new in this approach is the unconventional choice for our space of test functions. Instead of choosing the conventional $H^2(\mathcal{P}_h)$, which is predominantly used in discontinuous Galerkin methods, we relax the constrains on the space and choose functions that are locally in $H(\Delta, K)$ and whose jumps in the fluxes across the element interfaces are in $L^2(\Gamma_{\text{int}})$. We summarize the results and conclusions of our approach as follows.

WELL POSEDNESS OF THE FORMULATION. We proved that the solution to the strong form of the PDE (1) is a solution to the discontinuous weak form (6) as well. Continuity and Inf-Sup properties of the corresponding variational forms are established as well as the existence of *unique, stable* solutions to the established DGM.



(b) $O(h)$.

Figure 7. (cont.)

A Priori ERROR ESTIMATES. *A priori* estimates of the approximation error are derived that show that if the stabilization term is of order $O(h/p^2)$, optimal h convergence rates and suboptimal p convergence rates are obtained in the $H^1(\mathcal{P}_h)$ norm for $p \geq 2$. The highest suboptimal p convergence rate we can prove is of order $r - 2$. It is noteworthy that h convergence for the case where $p = 1$ is proven. The corresponding convergence rate is suboptimal of order $1/2$, but we prove that the method does indeed converge.

NUMERICAL CONVERGENCE TESTS. The one- and two-dimensional numerical convergence tests performed in this paper, confirm the stabilizing effect of a stabilization term of order $O(h)$. Overall, the observed convergence rates are higher or equal to the rates predicted by the error analysis performed in Section 4. Furthermore, the convergence rates in $L^2(\Omega)$ still remain suboptimal of order μ for even-order approximations and optimal of order $\mu + 1$ for odd-order approximations (with the exception of the case where $p = 1$). This behavior was also observed for the DGM of Oden, Babuška and Baumann [1].

COMPARISON OF THE STABILIZED DGM WITH OTHER DGMS. If we compare our stabilized DGM (SDGM) with the DGM of [1] and the NIPG methods of [8,19] and [20,21], it is observed that only the SDGM is shown to be well posed. Next, the Inf-Sup condition is proved for the DGM on the space V, $|||.|||$, thus guaranteeing stability of the *exact solution* with respect to the data. To date, stability of the DGM version of Oden, Babuška and Baumann and the NIPG has not been proved.

In addition, compared to the DGM of Oden, Babuška and Baumann, the SDGM is proved to converge for linear approximations. Numerical experiments had shown that the DGM converges for $p = 1$, but no theoretical work has been advanced to confirm these observations until the present work.

Finally, an attractive feature of SDGM over NIPG is that the former satisfies a local conservation property that the latter does not.

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