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Construction of Fixed Points of Nonlinear Mappings in Hilbert Space

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INTRODUCTION

Many of the most important nonlinear problems of applied mathematics reduce to finding solutions of nonlinear functional equations (e.g., nonlinear integral equations, boundary value problems for nonlinear ordinary or partial differential equations, the existence of periodic solutions of nonlinear partial differential equations) which can be formulated in terms of finding the fixed points of a given nonlinear mapping of an infinite dimensional function space X into itself. For mappings satisfying compactness conditions, a general existence theory of fixed points based upon topological arguments has been constructed over a number of decades (associated with the names of Brouwer, Poincaré, Lefschetz, Schauder, Leray, and others). More recently, there has begun the systematic study of fixed points of various classes of non-compact mappings, some of which are described in the Discussion below.

It is our object in the present paper to survey, systematize, and extend a number of recent results concerning the existence of fixed points of non-compact mappings of a subset C of a Hilbert space H into H. From the point of view of application, it is essential not only to show the existence of fixed points of such mappings under suitable hypotheses, but also to develop systematic techniques for the construction or calculation of such fixed points. The results presented below for various classes of nonlinear mappings (contractive, strictly pseudocontractive, pseudocontractive) may be considered as somewhat sophisticated, sharpened forms of the classical iteration scheme (the method of successive approximations) of Picard-Banach-Cacciopoli *et al.* that work in contexts in which the classical iteration scheme no longer applies (and in particular, outside the class of strictly contractive mappings). We have restricted the discussion to the case of mappings defined in Hilbert space both to avoid technical complications in the presentation of the results and proofs, and also, more essentially, because many of the

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results are valid only in rather narrow classes of Banach spaces including Hilbert spaces (e.g., the class of Banach spaces X having a weakly continuous duality mapping as in [14], [18], [23], [45], [42]).

There is a cost to the greater generality of our results which we must mention here explicitly. In addition to the greater complication of the approximation schemes, there is the fact that many (though not all) of the convergence proofs are not strictly constructive in the sense of giving explicit estimates for the error made at any given step of the approximation (a fact not uncommon in nonlinear numerical functional analysis). To cut down the damage done by lack of error estimates for any given limiting process, we have strictly enforced the principle that any such limiting process can occur only once in a construction for any given class of operators and only as the final stage of the approximation.

Historical remarks on the sources of the results given here are attached to the detailed discussion of the results below.

Let C be a convex subset of a real Hilbert space H and let U be a (possibly) nonlinear mapping from C into H. We introduce four classes of mappings P_0, P_1, P_2, and P_3 from C to H which (as will be shown) admit iterative methods for the construction of their fixed points.

DEFINITION. *For mappings $U : C \to H$ we define the classes:*

$$P_0 = \begin{cases} U \mid U \text{ is strictly contractive, i.e., there exists a constant } k \text{ with } k < 1 \text{ such} \\ \text{that } \| Ux - Uy \| \leqslant k \| x - y \| \text{ for all } x \text{ and } y \text{ in } C; \end{cases}$$

$$P_1 = \begin{cases} U \mid U \text{ is contractive (or nonexpansive), i.e., } \| Ux - Uy \| \leqslant \| x - y \| \\ \text{for all } x \text{ and } y \text{ in } C; \end{cases}$$

$$P_2 = \begin{cases} U \mid U \text{ is strictly pseudocontractive, i.e., there exists a constant } k < 1 \text{ such} \\ \text{that } \| Ux - Uy \|^2 \leqslant \| x - y \|^2 + k \| (I - U)x - (I - U)y \|^2 \text{ for} \\ \text{all } x \text{ and } y \text{ in } C; \end{cases}$$

$$P_3 = \begin{cases} U \mid U \text{ is pseudocontractive, i.e.,} \\ \| Ux - Uy \|^2 \leqslant \| x - y \|^2 + \| (I - U)x - (I - U)y \|^2 \text{ for all } x \\ \text{and } y \text{ in } C. \end{cases}$$

It follows from the definition of these classes that P_3 contains P_2 which contains P_1 which contains P_0.

In what follows we denote *strong* and *weak* convergence in H by "$\to$" and "$\rightharpoonup$," respectively.

THE PICARD PRINCIPLE FOR MAPPINGS U IN P^0. The classical method

of successive approximations shows that if U lies in P_0, and maps the closed set C into itself, then for any x_0 in C, the sequence $\{x_n\}$ determined by

$$x_n = Ux_{n-1} = U^n x_0, \qquad n = 1, 2, 3, \ldots, \tag{1}$$

converges strongly in H and its limit $x_\infty = \lim_n U^n x_0$ is the unique fixed point of U in C, with an error estimate given by

$$\| x_n - x_\infty \| \leqslant \frac{k^n}{1-k} \| x_1 - x_0 \|. \tag{2}$$

If U does not lie in P_0, then in general the sequence $\{x_n\}$ determined by (1) does not converge. However, noting that for any U and for any $\lambda \neq 1$ the transformation $U_\lambda = I + (1 - \lambda) U$ has the same fixed points as U, we shall see that for U in P_1 or even in P_2 it is possible to choose λ so that the sequence $\{U_\lambda^n x_0\}$ converges weakly (or even strongly in some cases) to a fixed point of U (Krasnoselski [38], Schaefer [46], Browder-Petryshyn [24], Petryshyn [43], Opial [42]).

Furthermore, if U is a contractive mapping which maps C into itself, then for any given v_0 in C and each s with $0 < s < 1$, the map V_s of C into itself given by

$$V_s(x) = sU(x) + (1 - s) v_0 \tag{3}$$

also maps C into itself and is a strict contraction with ratio s. Hence, it has a unique fixed point u_s in C. As we shall show below, u_s converges *strongly* in X as $s \to 1$ to a fixed point of U in C. Similar conclusions are obtained for U in the wider class P_2 (Browder [17], [18]).

Further approximation arguments under more general hypotheses will be presented in detail below.

REMARK. All the arguments of the present paper are given in complete detail to avoid reference for essential facts to the somewhat scattered literature. We have also tried to avoid the more general or more abstract formulations of the same arguments given in the preceding literature in order to make the paper more self-contained and more serviceable to applied mathematicians.

1. STRUCTURE OF THE CLASSES OF MAPPINGS P_0, P_1, P_2, AND P_3

In this section we shall connect up the theory of mappings of contractive type as defined in the Introduction with the parallel theory of operators on Hilbert space satisfying the following "monotonicity" conditions:

DEFINITION. *For mappings $T : C \to H$ we define the classes:*

$$M_0 = \begin{cases} T \mid T \text{ is strongly monotone, i.e., there exists a constant } c > 0 \text{ such that} \\ (Tx - Ty, x - y) \geqslant c \| x - y \|^2 \text{ for all } x \text{ and } y \text{ in } C; \end{cases}$$

$$M_1 = \begin{cases} T \mid T \text{ is such that there exists a continuous strictly increasing function } c(r) \\ \text{on } \{r \mid r \geqslant 0\} \text{ with } c(0) = 0 \text{ such that } (Tx - Ty, x - y) \geqslant c(\| x - y \|) \\ \text{for all } x \text{ and } y \text{ in } C; \end{cases}$$

$$M_2 = \begin{cases} T \mid T \text{ is such that there exists a constant } \alpha \text{ with } 0 < \alpha < 1 \text{ such that} \\ (Tx - Ty, x - y) \geqslant \alpha \| Tx - Ty \|^2 \text{ for all } x \text{ and } y \text{ in } C; \end{cases}$$

$$M_3 = \{T \mid T \text{ is monotone, i.e., } (Tx - Ty, x - y) \geqslant 0 \text{ for all } x \text{ and } y \text{ in } C.$$

DEFINITION. *A mapping W will be said to lie on the ray from the identity mapping I generated by U (denoted by Ray (U)), if there exists a constant $t > 0$ such that*

$$W = I + t(U - I), \qquad \text{or} \qquad W = tU + (1 - t) I. \tag{4}$$

Note that if $W \in \text{Ray}(U)$ then U and W have the same fixed points and that if $t \leqslant 1$ then W is a convex linear combination of I and U. Furthermore, if $W \in \text{Ray}(U)$, then $U \in \text{Ray}(W)$ and $\text{Ray}(W) = \text{Ray}(U)$.

The basic connection between the P-classes of operators and the M-classes is given by the following theorem.

THEOREM 1. (1) *U lies in P_3 if and only if $T = I - U$ lies in M_3.*

(2) *U lies in P_2 if and only if $T = I - U$ lies in M_2.*

(3) *U lies in P_0 (or U lies in P_1) implies that $T = I - U$ lies in M_2, which is contained in the class M_3.*

(4) *U lies in P_3 implies that* $\text{Ray}(U) \subset P_3$; *$U$ lies in P_2 implies that* $\text{Ray}(U) \subset P_2$.

PROOF. (1a) Suppose that $T = I - U \in M_3$, i.e., $(Tx - Ty, x - y) \geqslant 0$. Then

$$\begin{aligned} \| Ux - Uy \|^2 &= \| (I - T) x - (I - T) y \|^2 \\ &= \| x - y \|^2 + \| Tx - Ty \|^2 - 2(Tx - Ty, x - y) \\ &\leqslant \| x - y \|^2 + \| Tx - Ty \|^2 \\ &= \| x - y \|^2 + \| (I - U) x - (I - Uy) \|^2, \end{aligned}$$

i.e., $U \in P_3$.

(1b) Conversely, suppose $T = I - U$, with $U \in P_3$, i.e.,

$$\| Ux - Uy \|^2 \leqslant \| x - y \|^2 + \| Tx - Ty \|^2.$$

Then, since $T = I - U$, we have

$$\begin{aligned} \| Ux - Uy \|^2 &= \| (I - T) x - (I - T) y \|^2 \\ &= \| x - y \|^2 + \| Tx - Ty \|^2 - 2(Tx - Ty, x - y) \\ &\leqslant \| x - y \|^2 + \| Tx - Ty \|^2, \end{aligned}$$

from which it follows that $(Tx - Ty, x - y) \geqslant 0$ and hence $T = I - U \in M_3$.

(2a) Suppose that $T = I - U \in M_2$, i.e.,

$$(Tx - Ty, x - y) \geqslant \alpha \| Tx - Ty \|^2.$$

Then

$$\begin{aligned} \| Ux - Uy \|^2 &= \| (I - T) x - (I - T) y \|^2 \\ &= \| x - y \|^2 + \| Tx - Ty \|^2 - 2(Tx - Ty, x - y) \\ &\leqslant \| x - y \|^2 + \| Tx - Ty \|^2 - 2\alpha \| Tx - Ty \|^2 \\ &= \| x - y \|^2 + (1 - 2\alpha) \| Tx - Ty \|^2. \end{aligned}$$

Thus $U \in P_2$ with $k = 1 - 2\alpha < 1$.

(2b) Conversely, suppose that $T = I - U$ with $U \in P_2$, i.e.,

$$\| Ux - Uy \|^2 \leqslant \| x - y \|^2 + k \| Tx - Ty \|^2.$$

Then

$$\begin{aligned} \| Ux - Uy \| &= \| (I - T) x - (I - T) y \|^2 \\ &= \| x - y \|^2 + \| Tx - Ty \|^2 - 2(Tx - Ty, x - y) \\ &\leqslant \| x - y \|^2 + k \| Tx - Ty \|^2. \end{aligned}$$

Therefore

$$(Tx - Ty, x - y) \geqslant \left(\frac{1 - k}{2}\right) \| Tx - Ty \|^2.$$

This shows that $T \in M_2$ with $\alpha = (1 - k)/2$.

(3) The assertion (3) follows from (1) and (2).

(4) If $U \in P_3$, then every element W in $\operatorname{Ray}(U)$ is of the form $W = I + t(U - I)$, $t > 0$. Hence $W - I = t(U - I)$. But we know that $W - I = t(U - I)$ is equivalent to $U \in \operatorname{Ray}(U)$ while $T \in M_3$ is equivalent to $tT \in M_3$ for $t > 0$. Hence, by (1) of Theorem 1,

$$U \in P_3 \Leftrightarrow T = I - U \in M_3 \Leftrightarrow t(I - U) = 1 - W \in M_3 \Leftrightarrow W \in P_3 .$$

Similarly, since $T \in M_2 \Leftrightarrow tT \in M_2$ for $t > 0$, by (2) of Theorem 1,

$$U \in P_2 \Leftrightarrow (I - U) \in M_2 \Leftrightarrow t(I - U) = I - W \in M_2 \Leftrightarrow W \in P_2 .$$

THEOREM 2. *U is strictly pseudocontractive if and only if there exists an element W in* Ray (U) *such that W is contractive, i.e., U lies in P_2 if and only if there exists $W \in$* Ray (U) *such that W belongs to P_1.*

PROOF. Suppose there exists $W \in \text{Ray}(U)$ such that $W \in P_1$. Let $T = I - U$ and $T_1 = I - W$. Since $U \in \text{Ray}(W)$, by (4) of Theorem 1, it suffices to show that $W \in P_2$. But $W \in P_1$ and $P_1 \subset P_2$ and, therefore, $U \in P_2$.

CONVERSE. Suppose $U \in P_2$. By Theorem 1, $T = I - U \in M_2$ and hence $(Tx - Ty, x - y) \geqslant \alpha \| Tx - Ty \|^2$ for some $\alpha > 0$. (In fact, as was shown above, in this case, $\alpha = (1 - k)/2$.) Consider the mapping $U_t = I + t(U - I)$ in $R(U)$ for $t > 0$. Since $U_t = I - tT$, it follows that

$$\begin{aligned} \| U_t x - U_t y \|^2 &= \| (I - tT) x - (I - tT) y \|^2 \\ &= \| x - y \|^2 + t^2 \| Tx - Ty \|^2 - 2t(Tx - Ty, x - y) \\ &\leqslant \| x - y \|^2 + (t^2 - 2t\alpha) \| Tx - Ty \|^2. \end{aligned}$$

Hence for any fixed t such that $0 < t \leqslant 2\alpha$, $(2\alpha = 1 - k)$, the map $W = U_t$ has the property that $\| Wx - Wy \| \leqslant \| x - y \|$, i.e., $W \in P_1$.

REMARK. Theorem 2 will play an important role in the iterative computation of fixed points of strictly pseudocontractive mappings U since for any fixed t such that $0 < t \leqslant (1 - k)$ the mapping $U_t = tU + (1 - t) I$ has the same fixed points as U in C and is contractive and therefore, as will be seen below, U_t admits iterative calculation of its fixed points.

DEFINITION. *A mapping U of C into H belongs to the Lipschitz class* Lip *if there exists a constant $L > 0$ such that*

$$\| Ux - Uy \| \leqslant L \| x - y \|, \qquad x, y \in C. \tag{5}$$

THEOREM 3. *Let U be a mapping of C into H. Then there exists W in* Ray $(U) \cap P_0$ *if and only if $T = I - U$ belongs to M_0 and U belongs to* Lip.

PROOF. Suppose that $W \in \text{Ray}(U) \cap P_0$. Then, in particular, $U = I + t(W - I) \in$ Lip. Let $T = I - U$ and $T_1 = I - W$. Then

$$\begin{aligned} (T_1 x - T_1 y, x - y) &= \| x - y \|^2 - (Wx - Wy, x - y) \\ &\geqslant \| x - y \|^2 - \| Wx - Wy \| \, \| x - y \| \\ &\geqslant (1 - k) \| x - y \|^2, \end{aligned}$$

i.e., $T_1 \in M_0$ and therefore $T = tT_1 \in M_0$.

CONVERSE. Suppose that $U \in \mathrm{Lip}$ and $T = I - U \in M_0$. Set

$$U_t = I + t(U - I) = I - tT \qquad \text{for} \qquad t > 0,$$

where $T = I - U$. Then

$$\begin{aligned} \| U_t x - U_t y \|^2 &= \| (I - tT) x - (I - tT) y \|^2 \\ &= \| x - y \|^2 - 2t(Tx - Ty, x - y) + t^2 \| Tx - Ty \|^2 \\ &\leqslant \| x - y \|^2 - 2tc \| x - y \|^2 + t^2 L^2 \| x - y \|^2 \\ &= (1 - 2tc + t^2 L^2) \| x - y \|^2 = k(t) \| x - y \|^2, \end{aligned}$$

where

$$k(t) = 1 - 2tc + t^2 L^2 < 1$$

for any fixed t such that $t < 2c/L^2$.

We note in passing that $k(t)$ assumes its smallest value for $t_0 = c/L^2$ with $k(t_0) = (L^2 - c^2)/L^2$.

HISTORICAL REMARKS. The relationship between conditions of "contractivity" type and those of "monotonicity" type for mappings of a Hilbert space H into itself were first observed and used in Browder [9], where a fixed point theorem for contractive mappings in Hilbert space, based upon the theory of monotone operators, was applied to establish the existence of periodic solutions of a general class of nonlinear equations of evolution in H. The corresponding line of thought in Banach spaces leads to a relation between contractive mappings of a Banach space X into itself and the theory of accretive (J-monotone) mappings of X into X, where $T : X \to X$ is said to be accretive if for all u and v, $(T(u) - T(v), J(u - v)) \geqslant 0$, and J is a duality mapping of X into its conjugate space X^*. (The duality mapping J is defined by the conditions $(J(u), u) = \| Ju \| \| u \|$, $\| Ju \| = \mu(\| u \|)$ for a fixed increasing function μ and each u in X.) The study of contractive mappings in Banach spaces having a weakly continuous duality mapping J has been carried out in Browder [14], [18], and Opial [42]. The general study of accretive operators in such Banach spaces was begun in Browder-Figueiredo [23] and the construction of solutions for equations involving such operators was treated in Petryshyn [45]. Accretive operators in more general Banach spaces have been studied by Vainberg [48] and Browder [20], [21].

The definition of monotone operator was first given by Kachurovski [34], and iterative methods for strongly monotone operators in Hilbert space satisfying a Lipschitz condition were first given by Zarantonello [49] and Vainberg [48]. (For some related results, cf. Koshelev [27], Simeonov [47], Kolodner [36], and Petryshyn [44]).

Theorem 1 of the present section is a variant of the result of Browder [9], Theorem 2 seems to be new, and Theorem 3 is implicitly due to Zarantonello [49].

Strongly monotone operators in Hilbert space which are continuous but do not satisfy a Lipschitz condition were studied by Minty [39] (see also [26]) and further extensions to monotone operators in Hilbert space were given in Browder [3]-[6] with applications to nonlinear elliptic boundary value problems.

The general theory was extended to monotone operators from a Banach soace X to its conjugate space X^* independently by Browder [7] and Minty [40], and an extensive development has since taken place in this direction with applications to various classes of nonlinear partial differential equations. (For references, see Browder [8], [15]).

A related direction of thought inspired by the connection between monotone operators and contractions is the development of a systematic analogy in Hilbert and Banach spaces between fixed point or mapping theorems for compact operators and displacements and corresponding results for contractive and pseudocontractive operators (cf. Browder [9]-[12]).

2. Iterative Computation of Fixed Points of Contractive Mappings

It has been shown independently by Browder [13], Kirk [35], and Gohde [33] that in an uniformly convex Banach space X every contractive (i.e., nonexpansive) map U of a closed bounded convex subset C of X into C must have a fixed point. All these proofs are based on a transfinite argument due to Brodsky and Milman [2]. (Kirk's result in [35] has been formulated for spaces having normal structure in the sense of [2]. A further extension is given by Browder [19]. An elegant reformulation of the argument has been given by De Prima [25]. We note that for isometries, this theorem was already proved by Brodsky-Milman [2] (cf. also [30] for isometries).)

In the case of Hilbert spaces, the same result was obtained earlier in a more constructive way by Browder [9] using the connection with monotone operators that we exploit below. (See also Browder [10], [14], [17], [18], Petryshyn [45], Browder-Petryshyn [24], and Opial [42].)

For the sake of completeness we give a proof of the basic existence result.

Theorem 4. *Let C be a closed bounded convex subset of the Hilbert space H, U a contractive mapping of C into C (i.e., $\| Ux - Uy \| \leqslant \| x - y \|$ for all x and y in C).*

Then U has at least one fixed point in C.

PROOF OF THEOREM 4. For each s with $0 < s < 1$, let

$$V_s(x) = sU(x) + (1 - s)\, v_0$$

for a fixed element v_0 of C. Then V_s is a strict contraction with ratio $s < 1$ and has a unique fixed point u_s in C. Since C is closed, convex and bounded in the Hilbert space H, it is weakly compact. Hence we may find a sequence $s_j \to 1$ as $j \to +\infty$ such that $u_j = u_{s_j}$ converges weakly to an element u_0 of H. Since C is weakly closed, u_0 lies in C. We shall prove that u_0 is a fixed point of U.

If u is any point in H, we note that

$$\begin{aligned} \| u_j - u \| &= \| (u_j - u_0) + (u_0 - u) \|^2 \\ &= \| u_j - u_0 \|^2 + \| u_0 - u \|^2 + 2(u_j - u_0 , u_0 - u), \end{aligned}$$

where

$$2(u_j - u_0 , u_0 - u) \to 0 \qquad (j \to +\infty)$$

since $u_j - u_0$ converges weakly to zero in H. However, since $s_j \to 1$

$$\begin{aligned} Uu_j - u_j &= \{s_j U(u_j) + (1 - s_j)\, v_0\} - u_j + (1 - s_j)\{U(u_j) - v_0\} \\ &= (V_{s_j}(u_j) - u_j) + (1 - s_j)\{U(u_j) - v_0\} \\ &= (1 - s_j)\{U(u_j) - v_0\} \to 0 \qquad (j \to +\infty). \end{aligned}$$

Setting $u = Uu_0$ above, we have

$$\lim_{j \to +\infty} \{\| u_j - Uu_0 \|^2 - \| u_j - u_0 \|^2\} = \| u_0 - Uu_0 \|^2.$$

On the other hand, since U is contractive,

$$\| Uu_j - Uu_0 \| \leqslant \| u_j - u_0 \| .$$

Hence

$$\| u_j - Uu_0 \| \leqslant \| u_j - Uu_j \| + \| Uu_j - Uu_0 \| \leqslant \| u_j - Uu_j \| + \| u_j - u_0 \| .$$

Thus

$$\overline{\lim} \, (\| u_j - Uu_0 \| - \| u_j - u_0 \|) \leqslant 0,$$

and therefore

$$\overline{\lim} \, (\| u_j - Uu_0 \|^2 - \| u_j - u_0 \|^2) \leqslant 0.$$

Finally

$$\| u_0 - Uu_0 \|^2 = 0$$

and u_0 is a fixed point of U.

The present section is devoted to iterative methods for the computation of the fixed points of contractive operators U.

DEFINITION. *Let C be a closed convex subset of H. A mappings U of C into C is called asymptotically regular at x if and only if $\| U^n x - U^{n+1} x \| \to 0$, as $n \to \infty$.*

DEFINITION. *A mapping U of C into C is said to be a reasonable wanderer in C if starting at any x_0 in C, its successive steps $x_n = U^n x_0$ $(n = 1, 2, 3,...)$ are such that the sum of squares of their lengths is finite, i.e.,*

$$\sum_{n=0}^{\infty} \| x_{n+1} - x_n \|^2 < \infty.$$

Obviously every operator which is a reasonable wanderer is asymptotically regular.

THEOREM 5. *If U lies in P_1 and the set F of fixed points of U in C is not empty and if $U_\lambda = I + (1 - \lambda) U$ for any given λ with $0 < \lambda < 1$, then U_λ is a reasonable wanderer from C into C with the same fixed points as U.*

COROLLARY TO THEOREM 5. *If U is in P_1 and the set F of fixed points of U in C is not empty and if $U_\lambda = I + (1 - \lambda) U$ for a given λ with $0 < \lambda < 1$, then U_λ maps C into C, U_λ has the same fixed points as U, and U_λ is asymptotically regular.*

PROOF OF THEOREM 5. For any x in C set $x_n = U_\lambda^n x$ and let y be a fixed point of U and, hence, of U_λ. Then

$$x_{n+1} - y = \lambda x_n + (1 - \lambda) U x_n - y = \lambda(x_n - y) + (1 - \lambda)(U x_n - y).$$

On the other hand, for any constant a,

$$a(x_n - U x_n) = a(x_n - y) - a(U x_n - y).$$

Since

$$\| x_{n+1} - y \|^2 = \lambda^2 \| x_n - y \|^2 + (1 - \lambda)^2 \| U x_n - y \|^2 + 2\lambda(1 - \lambda)(U x_n - y, x_n - y)$$

and

$$a^2 \| x_n - U x_n \|^2 = a^2 \| x_n - y \|^2 + a^2 \| U x_n - y \|^2 - 2a^2(U x_n - y, x_n - y),$$

we see that by adding the corresponding sides and using that fact that U belongs to P_1 and $Uy = y$ we get

$$\| x_{n+1} - y \|^2 + a^2 \| x_n - U x_n \|^2 \leqslant (2a^2 + \lambda^2 + (1 - \lambda)^2) \| x_n - y \|^2 + 2\{\lambda(1 - \lambda) - a^2\}(U x_n - y, x - y).$$

If we assume that a is such that $a^2 \leqslant \lambda(1-\lambda)$, then from the last inequality we obtain

$$\| x_{n+1} - y \|^2 + a^2 \| x_n - Ux_n \|^2$$
$$\leqslant (2a^2 + \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) - 2a^2) \| x_n - y \|^2 = \| x_n - y \|^2.$$

Letting $a^2 = \lambda(1-\lambda) > 0$ and summing up from $n = 0$ to $n = N$ we get

$$\lambda(1-\lambda) \sum_{n=0}^{N} \| x_n - Ux_n \|^2 \leqslant \sum_{n=0}^{N} \{\| x_n - y \|^2 - \| x_{n+1} - y \|^2\}$$
$$= \| x_0 - y \|^2 - \| x_{N+1} - y \|^2 \leqslant \| x_0 - y \|^2.$$

Hence $\sum_{n=0}^{\infty} \| x_n - Ux_n \|^2 < \infty$. Since $x_{n+1} - x_n = (1-\lambda)(Ux_n - x_n)$, we see that

$$\sum_{n=0}^{\infty} \| x_{n+1} - x_n \|^2 \leqslant \frac{(1-\lambda) \| x_0 - y \|^2}{\lambda},$$

i.e., U is a reasonable wanderer in C.

Proof of Corollary to Theorem 5. The proof of this corollary follows from the remark preceeding Theorem 5.

A mapping U of C into H is called *demicompact* (Petryshyn [43]) if it has the property that whenever $\{u_n\}$ is a bounded sequence in H and $\{Ux_n - x_n\}$ is strongly convergent, then there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ which is strongly convergent. For demicompact operators, the iterates of U_λ converge strongly, as we see in the following:

Theorem 6. *Suppose U belongs to P_1 and U maps a bounded closed convex set C into itself. Suppose further that U is demicompact. Then the set F of fixed points of U in C is a nonempty convex set and for any given x_0 in C and any fixed $\lambda > 0$ with $0 < \lambda < 1$ the sequence $\{x_n\} = \{U_\lambda{}^n x_0\}$ determined by the process*

$$x_n = \lambda U x_{n-1} + (1-\lambda) x_{n-1}, \qquad n = 1, 2, 3, \ldots, \tag{6}$$

converges strongly to a fixed point of U in C.

Proof. Since U is a contraction from C into C, by Theorem 4, U has fixed points in C. Thus $F \neq \phi$ and, furthermore, F is convex, i.e., when y_0, $y_1 \in F$ then $y_\lambda = ty_1 + (1-t) y_0$ belongs to F for t with $0 \leqslant t \leqslant 1$. (Indeed, $\| Uy_\lambda - y_0 \| \leqslant \| y_\lambda - y_0 \|$ and $\| Uy_\lambda - y_1 \| \leqslant \| y_\lambda - y_1 \|$ imply that

$\| y_0 - y_1 \| \leqslant \| y_0 - Uy_\lambda \| + \| Uy_\lambda - y_1 \| \leqslant \| y_0 - y_1 \|$. This shows for some a, b with

$$0 \leqslant a, \quad b \leqslant 1, \quad y_0 - Uy_\lambda = a(y_0 - y_\lambda) \quad \text{and} \quad y_1 - Uy_\lambda = b(y_1 - y_\lambda)$$

from which it follows that $Uy_\lambda = y_1 \in F$.)

Now, the sequence $\{x_n\}$ lying in C is bounded and, by the Corollary to Theorem 5, the sequence $\{x_n - Ux_n\} = \{\lambda^{-1}(x_n - x_{n+1})\}$ converges strongly to zero; therefore, the demicompactness of U implies the existence of a strongly convergent subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to y \in F$, since $U \in P_1$, $Ux_{n_i} \to Uy$ and $Uy = y$. The convergence of the entire sequence $\{x_n\}$ to y follows from the inequality $\| x_n - y \| \leqslant \| x_{n-1} - y \|$ valid for each n.

Since, as was shown in [43], the class of demicompact operators contains the compact operators, the results for compact operators in [38, 46] are special cases of Theorem 6.

REMARK. Let us observe in passing that the class of demicompact operators contains also the operators, which often appear in application, of the form $U = S + T$ where T is compact and $(I - S)^{-1}$ exists and is continuous on its range $R(I - S)$. Indeed, if $\{u_n\}$ is a bounded sequence such that $\{Uu_n - u_n\}$ is strongly convergent, then $\{u_n\}$ contains a strongly convergent subsequence. To see this note that, in view of the compactness of T, there exists a subsequence $\{u_{n_i}\}$ such that Tu_{n_i} is strongly convergent and, therefore, $\{g_{n_i}\} = \{(I - S)\, u_{n_i}\} = \{Uu_{n_i} - u_{n_i}\} + \{Tu_{n_i}\}$ converges strongly. Since, $(I - S)^{-1}$ is continuous, $u_{n_i} = (I - S)^{-1} g_{n_i}$ is strongly convergent.

If in Theorem 6 we drop the assumption that U is demicompact, then the next two theorems show that the sequence $\{x_n\}$ determined by the process (6) is at least weakly convergent.

THEOREM 7. *Suppose U belongs to P_1 and $U_\lambda = I + (1 - \lambda)\, U$ for λ with $0 < \lambda < 1$. Suppose further that U maps a bounded closed convex set C into C and that U has exactly one fixed point y in C. Then $U^n x_0 \rightharpoonup y$ for any x_0 in C, i.e., the sequence $\{x_n\}$ determined by (6) converges weakly to y.*

PROOF. It suffices to show that if $x_{n_j} = U^{n_j} x_0 \rightharpoonup y_0$ for an infinite subsequence of integers n_j, then y_0 is a fixed point of U or of U_λ and, therefore, *equals* y.

By our hypothesis U has a fixed point y in C and y is unique. Suppose now that $x_{n_j} \rightharpoonup y_0$. Then

$$\| x_{n_j} - U_\lambda y_0 \| \leqslant \| U_\lambda x_{n_j} - U_\lambda y_0 \| + \| x_{n_j} - U_\lambda x_{n_j} \|$$

$$\leqslant \| x_{n_j} - y_0 \| + \| x_{n_j} - U_\lambda x_{n_j} \|.$$

Since, by the Corollary to Theorem 5, $\| x_{n_j} - U_\lambda x_{n_j} \| \to 0$, as $n_j \to \infty$, the last inequality implies that

$$\overline{\lim} \, (\| x_{n_j} - U_\lambda y_0 \| - \| x_{n_j} - y_0 \|) \leqslant 0. \tag{7}$$

However,

$$\| x_{n_j} - U_\lambda y_0 \|^2 = \| (x_{n_j} - y_0) + (y_0 - U_\lambda y_0) \|^2$$

$$= \| x_{n_j} - y_0 \|^2 + \| y_0 - U_\lambda y_0 \|^2 + 2(x_{n_j} - y_0 , y_0 - U_\lambda y_0).$$

Hence, since $x_{n_j} \rightharpoonup y_0$, the last equality implies that

$$\lim_n \{\| x_{n_j} - U_\lambda y_0 \|^2 - \| x_{n_j} - y_0 \|^2\} = \| y_0 - U_\lambda y_0 \|^2. \tag{8}$$

On the other hand

$$\| x_{n_j} - U_\lambda y_0 \| - \| x_{n_j} - y_0 \|^2$$

$$= (\| x_{n_j} - U_\lambda y_0 \| - \| x_{n_j} - y_0 \|) (\| x_{n_j} - U_\lambda y_0 \| + \| x_{n_j} - y_0 \|). \tag{9}$$

Since $(\| x_{n_j} - U_\lambda y_0 \| + \| x_{n_j} - y_0 \|)$ is bounded, relations (7), (8), and (9) imply that $\| y_0 - U_\lambda y_0 \| \leqslant 0$, i.e., $U_\lambda y_0 = y_0$.

THEOREM 8. *Suppose U lies in P_1 and $U_\lambda = I + (1 - \lambda) U$ for a given λ with $0 < \lambda < 1$. Suppose further that U maps a bounded closed convex set C into C. Then for any x_0 in C, $U_\lambda^n x_0 \rightharpoonup y$ and y is a fixed point of U in C, i.e., for any given x_0 in C the sequence of iterates $\{x_n\}$ determined by the process (6) converges weakly to a fixed point of U in C.*

PROOF. Let F be the set of fixed points of U in C. F is nonempty by Theorem 4. Moreover F is convex. Since for each $y \in F$ and each n,

$$\| x_n - y \| \leqslant \| x_{n-1} - y \| ,$$

the function $g(y) = \lim_n \| x_n - y \|$ is well defined and is a lower semi-continuous convex function on F. Let $d_0 = \inf \{g(y), y \in F\}$. For each $\mathscr{E} > 0$, $F_{\mathscr{E}} = \{y \mid g(y) \leqslant d_0 + \mathscr{E}\}$ is closed, convex, nonempty and bounded and, hence, weakly compact. Hence $\bigcap_{\mathscr{E}>0} F \neq \phi$ and, in fact,

$$\bigcap_{\mathscr{E}>0} F_{\mathscr{E}} = \{y \mid g(y) = d_0\} \equiv F_0 .$$

Moreover, F_0 contains exactly one point. Indeed, since F_0 is convex and closed, if $y_0, y_1 \in F_0$

$$\begin{aligned}
g^2(y_\lambda) &= \lim_n \| y_\lambda - x_n \|^2 = \lim_n \{\| \lambda(y_1 - x_n) + (1-\lambda)(y_0 - x_n) \|^2\} \\
&= \lim_n \{\lambda^2 \| y_1 - x_n \|^2 + (1-\lambda)^2 \| y_0 - x_n \|^2 \\
&\qquad + 2\lambda(1-\lambda)(y_1 - x_n, y_0 - x_n)\} \\
&= \lim_n \{\lambda^2 \| y_1 - x_n \|^2 + (1-\lambda)^2 \| y_0 - x_n \|^2 \\
&\qquad + 2\lambda(1-\lambda) \| y_1 - x_n \| \| y_0 - x_n \|\} \\
&+ \lim_n \{2\lambda(1-\lambda)[(y_1 - x_n, y_0 - x_n) - \| y_1 - x_n \| \| y_0 - x_n \|]\} \\
&= g^2(y) + \lim_n \{2\lambda(1-\lambda)[(y_1 - x_n, y_0 - x_n) \\
&\qquad - \| y_1 - x_n \| \| y_0 - x_n \|]\}.
\end{aligned}$$

Hence,

$$\lim_n \{2\lambda(1-\lambda)[(y_1 - x_n, y_0 - x_n) - \| y_1 - x_n \| \| y_0 - x_n \|]\} = 0$$

Since $\| y_1 - x_n \| \to d_0$ and $\| y_0 - x_n \| \to d_0$, the latter relation implies that

$$\begin{aligned}
\| y_1 - y_0 \|^2 &= \| (y_1 - x_n) + (x_n - y_0) \|^2 \\
&= \| y_1 - x_n \|^2 + \| x_n - y_0 \|^2 \\
&\qquad - 2(y_1 - x_n, y_0 - x_n) \to d_0^2 + d_0^2 - 2d_0 = 0,
\end{aligned}$$

giving a contradiction.

To show that $x_n = U^n x \rightharpoonup y_0$, it suffices to assume that $x_{n_j} \rightharpoonup y$ for an infinite subsequence and then prove that $y = y_0$. By the argument of Theorem 7, y lies in F. First note that, since $x_{n_j} \rightharpoonup y$, the definition of g implies that

$$\begin{aligned}
\| x_{n_j} - y_0 \|^2 &= \| x_{n_j} - y + y - y_0 \|^2 \\
&= \| x_n - y \|^2 + \| y - y_0 \|^2 - 2(x_{n_j} - y, y - y_0) \\
&\qquad \to g^2(y) + \| y - y_0 \| \\
&= g^2(y_0) = d_0^2.
\end{aligned}$$

Since $g^2(y) \geqslant d_0^2$, the last equality implies that $\| y - y_0 \| \leqslant 0$. Hence, $y = y_0$ and Theorem 8 is proved.

Historical Remarks. As we have already remarked in the beginning of Section 2, Theorem 4 was first proved in Hilbert space by Browder [9]

and extended to uniformly convex spaces independently by Browder [13], Kirk [35], and Gohde [33]. The proof which we give is essentially of the same type as in [9] but uses a modification introduced independently by T. Kato (oral communication) and Opial [42].

Theorem 5 is given here explicitly for the first time. Its principal consequence, the Corollary on asymptotic regularity, was first proved in the more general context of uniformly convex spaces by Krasnoselski [38] for $\lambda = \frac{1}{2}$ and for general λ with $0 < \lambda < 1$ by Schaefer [41]. (This was extended to compact maps in strictly convex Banach spaces by Edelstein [31]). Strong convergence for compact contractions U was first proved by Krasnoselski [38] for $\lambda = \frac{1}{2}$ and by Schaefer [41] for general λ. It was extended by Petryshyn [43] to demicompact contractions, which includes the result of Theorem 6.

For weakly continuous contractions, the weak convergence of $\{U_\lambda{}^n x_0\}$ was first proved by Schaefer [41]. The extension of this result to general contractions was carried out in two stages, the proof of Theorem 7 by Browder-Petryshyn [24], and the proof of Theorem 8 by Opial [42]. (We should note that Gohde in [33] argues that the weak convergence result follows from the fixed-point result of the form of Theorem 4, using the literal argument of Schaefer [46]. This remark is not accurate, since the proof by Schaefer uses two facts, the existence of at least one fixed point and the fact that if a subsequence of $\{U_\lambda{}^n x_0\}$ converges weakly to y, then y must be a fixed point. This latter fact follows as in the proof of Theorem 7, and Opial has utilized this argument together with a simplified form of Schaefer's proof.)

We note finally that under the much stronger assumption that the strong limit set of the iterates is nonempty, convergence results in metric and Banach spaces for contractions have been given by Edelstein ([27], [28], [29]).

SECTION 3. RETRACTION-ITERATION METHODS. In the present section we show that the condition that U maps C into C in the preceeding theorem can be replaced by a much weaker condition provided that the method (6) is modified by combining the map U with a suitable retraction R_C of H onto C.

DEFINITION. *Let C be a closed convex subset of H. For each x in H define $R_C x$ as the closest point to x in C.*

REMARK. If $C = B_r(x_0)$, then $R_C : H \to C$ is given by

$$Rx = R_C x = \begin{cases} x & \text{if} \quad \| x - x_0 \| \leqslant r \\ \dfrac{r(x - x_0)}{\| x - x_0 \|} & \text{if} \quad \| x - x_0 \| \geqslant r. \end{cases}$$

LEMMA 1. *If C is a closed convex set of H, then R_C belongs to P_1, i.e., R_C is a contractive mapping of H into C.*

PROOF. For any x and y in H let $x_1 = R_C x$ and $y_1 = R_C y$. Then for every w in C and every t with $0 \leqslant t \leqslant 1$ we have

$$\| x - (tw + (1-t)\, x_1) \|^2 \geqslant \| x - x_1 \|^2. \tag{10}$$

Expanding the inequality (10), we find for $t > 0$ that

$$(x - x_1, x_1 - w) \geqslant -\frac{t}{2} \| w - x_1 \|^2.$$

The latter inequality implies that for every w in C

$$(x - x_1, x_1 - w) \geqslant 0. \tag{11}$$

In a similar way we find that for every u in C

$$(y - y_1, y_1 - u) \geqslant 0. \tag{12}$$

If in (11) and (12) we set $w = y_1$ and $u = x_1$ we obtain the inequalities

$$(x_1, x_1 - y_1) \leqslant (x, x_1 - y_1) \qquad \text{and} \qquad -(y_1, x_1 - y_1) \leqslant (-y_1, x_1 - y_1)$$

from which, by adding corresponding sides, we get

$$\| x_1 - y_1 \|^2 \leqslant (x - y, x_1 - y_1) \leqslant \| x - y \| \, \| x_1 - y_1 \|.$$

Hence, R_C belongs to P_1.

THEOREM 9. *Suppose that U lies in P_1 and maps the bounded closed convex set C of H into H. Suppose further that if u lies on the boundary of C and if $u = R_C(Uu)$, then u is a fixed point of U.*

Then U has a fixed point in C, and for any fixed λ with $0 < \lambda < 1$ and any given x_0 in C, the sequence $\{((R_C U)_\lambda)^n x_0\}$ converges weakly to a fixed point of U in C, i.e. the sequence $\{x_n\}$ defined by

$$x_n = \lambda R_C(U x_{n-1}) + (1 - \lambda)\, x_{n-1} \tag{6A}$$

converges weakly to a fixed point of U in C.

THEOREM 10. *If $C = B_r(0)$ in Theorem 9, the condition that: ($u = R_C U u$ implies that u is a fixed point of U), reduces to the so-called Leray-Schauder condition:*

$$Uu - \lambda u \neq 0 \quad \textit{for} \quad u \quad \textit{in} \quad S_r(0) \quad \textit{and any} \quad \lambda > 1. \tag{LS}$$

PROOF OF THEOREM 9. The map $R_C U$ of C into C is a contraction

since both R_C and U are contractions. $R_C U$ has the same fixed points as U, since if

$$R_C U u = u$$

either u lies on the boundary of C and is a fixed point of U by hypothesis or u lies in the interior of C. However, in the latter case, since R_C maps the exterior of C on the boundary of C, it follows that Uu lies in C and $R_C U u = U u = u$.

We now apply Theorem 6. Q.E.D.

PROOF OF THEOREM 10. If u lies on $C_r(0)$ and $u = R_C U u$ for $C = B_r(0)$, then $Uu = \lambda u$ with $\| Uu \|/r > 1$ if $\| Uu \| > r$. This contradictes the hypothesis (LS). Hence $\| Uu \| \leqslant r$ for such u and $R_C U u = U u = u$ and the theorem is proved.

DEFINITION. *A bounded closed convex set C is called uniformly smooth with smoothing constant $R > 0$ if and only if for each boundary point x_1 in C*

(1) *C has only one supporting hyperplane at x_1, (i.e., $(x_1, u) = c_0$ and $(u, v) \geqslant c_0$ for all u in C).*

(2) *There exists an element u_0 in H such that $\| u_0 - x_1 \| = R$ and $\| u_0 - u \| \leqslant R$ for all u in C (i.e., $C \subset B_R(u_0)$ and $x_1 \in B_R(u_0)$).*

THEOREM 11. *Suppose that C is uniformly smooth and, under the hypotheses of Theorem 9 or 10, that U is a demicompact mapping of C into H. Then*

$$x_n = (R_C U)_\lambda^n x_0$$

converges strongly to a fixed point of U in C.

PROOF OF THEOREM 11. It suffices by the preceding argument and Theorem 8 to show that $R_C U$ is demicompact if U is demicompact. This follows from the following two lemmas:

LEMMA 2. *If C is uniformly smooth with smoothing constant R and $d(x, C) \geqslant r_0$ and $d(y, C) \geqslant r_1$, then*

$$\| R_C x - R_C y \| \leqslant \left(1 + \frac{r_0 + r_1}{2R}\right)^{-1} \| x - y \| .$$

PROOF (We refer to Fig. 1 at the end of the proof). Let $x_1 = R_C x$ and z be any point of C. Then $x - x_1$ is perpendicular to the supporting hyperplane at x_1 and so is the vector $u_0 - x_1$. Let $r = \| x - x_1 \|$ and $d_0 = \| x_1 - z \|$. We construct z_1 with $\| z_1 - u_0 \| = R$ so that $d_0 = \| x_1 - z_1 \|$. Note that the points x, x_1, u_0 and z lie on a single two-dimensional plane and so we

restrict ourselves to this plane. The angle $\varphi = \sphericalangle\, wx_1z_1 \leqslant \sphericalangle\, wx_1z = \theta$ and $0 \leqslant \varphi \leqslant \theta \leqslant \pi/2$. To compute φ note that by the law of sines

$$\frac{\sin 2\varphi}{d_0} = \frac{\sin\left(\frac{\pi}{2} - \varphi\right)}{R},$$

i.e.,

$$\frac{2 \sin \varphi \cos \varphi}{d_0} = \frac{\cos \varphi}{R} \qquad \text{or} \qquad \sin \varphi = \frac{d_0}{2R}.$$

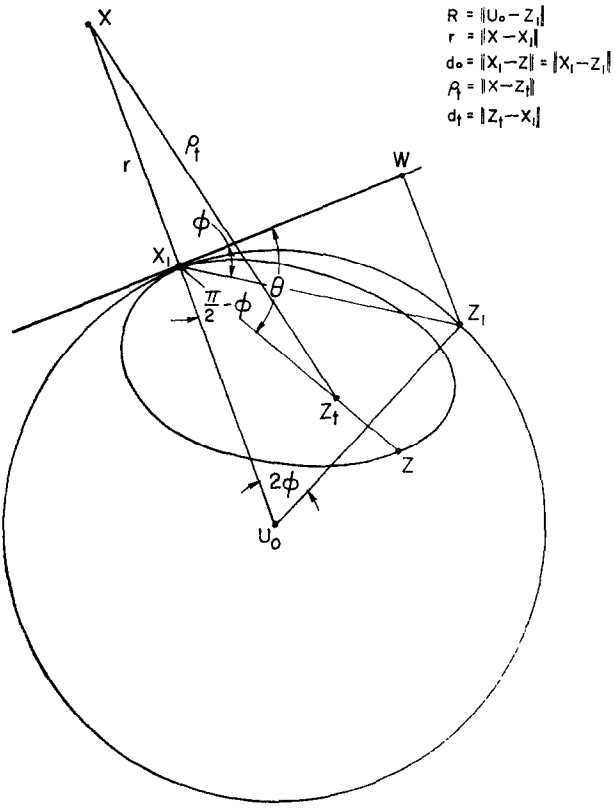

FIG. 1.

Since $\theta \geqslant \varphi$ we have

$$\sin \theta \geqslant \sin \varphi = \frac{d_0}{2R}. \tag{13}$$

Consider now the point $z_t = (1 - t)\, x_1 + tz$ for $0 < t < 1$ and put

$d_t = \| z_t - x_1 \|$ and $\rho_t = \| x - z_t \|$. Then by the law of cosines applied to the triangle $\Delta(x, x_1, z_t)$ and the inequality (13), we have

$$\begin{aligned} \rho_t^2 &= r^2 + d_t^2 - 2r\, d_t \cos\left(\frac{\pi}{2} + \theta\right) \\ &= r^2 + d_t^2 + 2r\, d_t \sin\theta \geqslant r^2 + t^2 d_0^2 + 2t\, d_0 r \frac{d_0}{2R}. \end{aligned} \tag{14}$$

On the other hand,

$$\begin{aligned} \rho_t^2 &= \| x - z_t \|^2 = \| x - x_1 + t(x_1 - z) \|^2 \\ &= \| x - x_1 \|^2 + 2t(x - x_0, x_1 - z) + t^2 \| x_1 - z \|^2 \\ &= r^2 + 2t(x - x_1, x_1 - z) + t^2 d_0. \end{aligned} \tag{15}$$

Hence, on cancellation, (14) and (15) imply that

$$(x - x_1, x_1 - z) \geqslant \frac{r\, d_0^2}{2R} = \frac{r}{2R} \| x_1 - z \|^2.$$

Let $y_1 = R_C y$ and set $z = y_1$. Then the above shows that

$$(x - x_1, x_1 - y_1) \geqslant \frac{r_0}{2R} \| x_1 - y_1 \|^2. \tag{16}$$

Interchange x and y and x_1 and y_1 and using similar argument we obtain for $r_1 = \| y - y_1 \|$ the inequality

$$(y - y_1, y_1 - x_1) \geqslant \frac{r_1}{2R} \| x_1 - y_1 \|^2. \tag{17}$$

Adding the corresponding sides of (16) to (17) we obtain

$$(x - y, x_1 - y_1) - \| x_1 - y_1 \|^2 \geqslant \frac{r_0 + r_1}{2R} \| x_1 - y_1 \|^2$$

or

$$(x - y, x_1 - y_1) \geqslant \left(1 + \frac{r_0 + r_1}{2R}\right) \| x_1 - y_1 \|^2.$$

The Schwarz inequality then implies that

$$\| x_1 - y_1 \| = \| R_0 x - R_C Y \| \leqslant \left(\frac{1}{1 + \dfrac{r_0 + r_1}{2R}}\right) \| x - y \|.$$

LEMMA 3. *If U is a contractive demicompact mapping of C into H, C being uniformly smooth with smoothing constant R, then $R_C U$ is a demicompact contractive mapping of C into C.*

PROOF. Suppose $\{x_n\}$ is a sequence in C such that $g_n = x_n - R_C U x_n \to g$ in H.

(1) If there exists an infinite subsequence $\{x_{n_j}\}$ such that $d(Ux_{n_j}, C) \to 0$, as $j \to \infty$, then $R_C U x_{n_j} - U x_{n_j} \to 0$ and, hence, $U x_{n_j} - x_{n_j} \to g$. Since U is demicompact, there exists a subsequence $\{x_{n_\ell}\}$ of $\{x_{n_j}\}$ which is strongly convergent.

(2) Otherwise, there exists a constant $\delta > 0$ such that $d(Ux_n, C) \geqslant \delta$ for all n. In this case, by Lemma 2,

$$\| R_C U x_n - R_C U x_m \| \leqslant \left(1 + \frac{\delta}{R}\right)^{-1} \| U x_n - U x_m \|$$

$$= k \| U x_n - U x_m \| \leqslant k \| x_n - x_m \|$$

with

$$0 < k = \left(1 + \frac{\delta}{R}\right)^{-1} < 1.$$

Hence

$$\| x_n - x_m \| \leqslant \| (x_n - R_C U x_n) - (x_m - R_C U x_m) \| + \| R_C U x_n - R_C U x_m \|,$$

whence we obtain

$$(1 - k) \| x_n - x_m \| \leqslant \| g_n - g_m \| \to 0,$$

as $n, m \to \infty$, i.e., $\{x_n\}$ is a Cauchy sequence.

REMARK. Note that the Leray-Schauder condition is certainly implied by the condition that U maps B_r into B_r and also by anyone of the following more practical, though less general, conditions

$$(Tx, x) \leqslant \| x \|^2 \quad \text{for all} \quad x \quad \text{in} \quad S_r(0) \tag{18}$$

$$\| Tx \| \leqslant \| x \| \quad \text{for all} \quad x \quad \text{in} \quad S_r(0). \tag{19}$$

Let us also add that under suitable conditions (see [43]) the method of Theorem 9 can also be applied to general equations of the form $Lx - Mx = 0$, where L is a linear continuously invertible mapping and M is some nonlinear mapping, without the necessity of inverting L.

HISTORICAL REMARKS. The mapping R_C is a special case of the notion of proximity mapping studied by Moreau in [41].

The combination of the use of retractions on the unit ball in Hilbert space with theorems on contractions was introduced by Petryshyn in [43]. Theorem 11 for demicompact mappings is an extension of the corresponding result for the case of $C = B_R(0)$ given in [43].

It has been shown by de Figueiredo-Karlovitz [32] that if the mapping R_C for $C = B_1(0)$ is a contraction (i.e., nonexpansive) for a Banach space X of dimension > 2, then X is a Hilbert space. This fact points up the fact that the methods of Section 3 are essentially restricted to Hilbert spaces.

4. Construction of Fixed Points of Mappings U in P_2

Our next two theorems show that, in virtue of Theorem 2, the iterative methods (6) and (6A) are also applicable to computation of fixed points of strictly pseudocontractive mappings.

Theorem 12. *Let C be a bounded closed convex subset of H and let U be a mapping of C into C such that U belongs to P_2, i.e., there exists a constant $k < 1$ such that*

$$\| Ux - Uy \|^2 \leqslant \| x - y \|^2 + k \| (I - U) x - (I - U) y \|^2, \qquad x, y \in C. \tag{20}$$

Then, for any x_0 in C and any fixed γ such that $1 - k < \gamma < 1$, $U_\gamma^n x_0 \rightharpoonup y \in C$ and y is a fixed point of U in C.

If additionally we assume that U is demicompact, then $U_\gamma^n x_0 \to y$.

Proof. Since $U \in P_2$, Theorem 2 shows that for every fixed t such that $0 < t \leqslant k - 1$ the mapping $U_t = tU + (I - t) I$ is contractive. Hence, by Theorem 8, U_t (and therefore U) has a fixed point in C and for any given $x_0 \in C$ and any fixed λ with $0 < \lambda < 1$ the sequence $\{x_n\} = \{(U_t)_\lambda^n x_0\}$ converges weakly to some fixed point y of U in C. But

$$\begin{aligned}(U_t)_\lambda &= \lambda I + (1 - \lambda) U_t \\ &= (1 - \lambda) tU + I - (1 - \lambda) tI = \gamma I + (1 - \gamma) U \equiv U_\gamma,\end{aligned}$$

where $\gamma \equiv 1 - (1 - \lambda) t$. Since, as is easy to see, $\lambda t < k - 1$ for each fixed λ with $0 < \lambda < 1$ if and only if $\gamma > k$, the proof of the first part of our theorem is complete.

To prove the second part of Theorem 12, in virtue of Theorem 6, it is sufficient to show that U_γ is demicompact. But this follows immediately from the demicompactness of U and the equality $U_\gamma x - x = (1 - \gamma)(Ux - x)$ which holds for every x in C.

THEOREM 13. *Suppose U lies in P_2, U maps $B_r(0)$ into H and U satisfies the condition (LS). Let R be the retraction of H onto $B_r(0)$. Then for any $x_0 \in B_r(0)$ and any γ such that $0 < 1 - k < \gamma < 1$, $\{(RU_\gamma)\, x_0{}^n\} \rightharpoonup y$ and y is a fixed point of U in $B_r(0)$. If, in addition, we assume that U is demicompact, then $(RU_\gamma)^n\, x_0 \to y$, as $n \to \infty$.*

PROOF. To prove the first part of Theorem 13, note that, by Theorem 12, it suffices to show that U_γ satisfies the condition (LS) on $S_r(0)$, i.e., $U_\gamma x - \lambda x \neq 0$ for all in $S_r(0)$ and any $\lambda > 1$. Suppose, to the contrary, that $U_\gamma x_0 = \lambda_0 x_0$ for some x_0 in $S_r(0)$ and some $\lambda_0 > 1$. Then

$$U_\gamma x_0 = \gamma x_0 + (1 - \gamma)\, U x_0 = \lambda_0 x_0 \qquad \text{or} \qquad U x_0 = \frac{\lambda_0 - \gamma}{1 - \gamma}\, x_0$$

with $\lambda = (\lambda_0 - \gamma)/(1 - \gamma) > 1$, since $\lambda_0 > 1$, in contradiction to condition (LS).

If in addition we assume that U is demicompact, then as was noted above the operator U_γ is also demicompact and therefore the strong convergence of the sequence $\{x_n\} = \{(RU_\gamma)^n\, x_0\}$ to a fixed point, say, y of U in $B_r(0)$ follows from Theorem 11.

5. STRONG CONVERGENCE OF APPROXIMANTS TO FIXED POINTS OF MAPPINGS IN THE CLASS P_3

In the present section, we show, following Browder [17, 18], that by a different approximation method we can obtain strong convergence of approximants to fixed points for contractive operators U without assumptions of demicompactness, and indeed that this method carries over to mappings U of the broader class P_3.

We carry through the discussion first for U in the class $P_3 \cap \text{Lip}$. The extension to the general class P_3 can be made using the results of the following sections, (and obviously $P_1 \subset P_3 \cap \text{Lip}$).

THEOREM 14. *Let U be a mapping from H to H such that U lies in $P_3 \cap \text{Lip}$ and*

$$(Ux, x) \leqslant \| x \|^2 \qquad \textit{for all} \qquad x \quad \textit{in} \quad S_r(0). \tag{21}$$

Let u_0 be a fixed element in $B_r(0)$, R the radial retraction of H onto $B_r(0)$ and for each fixed s such that $0 < s < 1$ let $V_s(x) = sU(x) + (1 - s)\, u_0$. Then

(1) *V_s has a unique fixed point v_s in $B_r(0)$ which can be calculated by the process*

$$v_s = \lim_n (v_s{}^n), \qquad v_s{}^n = \lim_n \{R(I - \mu(I - V_s))\}^n\, w_0, \qquad (w_0 \textit{ given in } B_r(0)) \tag{22}$$

for any fixed μ *such that* $0 < \mu < 2\ell$, *where*

$$\ell = \frac{1-s}{(1+sL)^2}; \tag{23}$$

with an error estimate $\| v_s - v_s^n \|$ *given by*

$$\| v_s - \{R(I - \mu(I - V_s))\}^n w_0 \| \leqslant \frac{2r\eta^n}{1-\eta}, \tag{24}$$

where $\eta = [1 - 2(1-s) + \mu^2(1+sL)^2]^{1/2}$.

(2) *U has fixed points in $B_r(0)$. Furthermore, as $s \to 1$, v_s converges strongly to a fixed point x_0 of U in $B_r(0)$, namely the fixed point of U in $B_r(0)$ which is nearest u_0.*

PROOF. (1) Let $U \in P_3$. By Theorem 1, the operator $T = I - U \in M_3$, i.e., T is monotone. If we define T_s by $T_s = I - V_s$, then

$$T_s x = x - sUx - (1-s)\, u_0$$

is strongly monotone. In fact,

$$\begin{aligned}(T_s x - T_s y, x - y) &= \| x - y \|^2 - s(Ux - Uy, x - y) \\ &= (1-s) \| x - y \|^2 + s(Tx - Ty, x - y) \\ &\geqslant (1-s) \| x - y \|^2.\end{aligned}$$

In addition, since $U \in \mathrm{Lip}$ and $s < 1$, we have

$$\| V_s x - V_s y \| = s \| Ux - Uy \| \leqslant sL \| x - y \|.$$

Hence, for μ satisfying the inequality (23), it follows from Theorem 3 that the operator $W_{\mu s} = (I - \mu(I - V_s))$ is a strictly contractive mapping of $B_r(0)$ into H, i.e., for all x and y in $B_r(0)$

$$\| W_{\mu s} x - W_{\mu s} y \| \leqslant \eta \| x - y \|$$

with ratio

$$\eta = [1 - 2\mu(1-s) + \mu^2(1+sL)^2]^{1/2} < 1. \tag{25}$$

Observe further that, in view of (21), for all x in $S_r(0)$ and every s we have

$$\begin{aligned}(V_s x, x) &= s(Ux, x) + (1-s)(u_0, x) \\ &\leqslant s \| x \|^2 + (1-s) \| u_0 \| \| x \| \leqslant s \| x \|^2 + (1-s) \| x \|^2 = \| x \|^2.\end{aligned}$$

Hence, as was observed in the Remark following Lemma 3, $W_{\mu s}$ satisfies the condition (LS). Therefore, the mapping $RW_{\mu s}$ of $B_r(0)$ into $B_r(0)$ has the same fixed points as $W_{\mu s}$. The latter, in turn, has the same fixed points as V_s. But Lemma 1 and (25) imply that $RW_{\mu s}$ is a strict contraction with ratio $\eta < 1$ and maps $B_r(0)$ into itself. Thus, by Picard principle, V_s has a unique

fixed point v_s in $B_r(0)$ which is given by the formula (22) and the estimate of the error $\| v_s - v_s^n \|$ is given by (24). This completes the proof of (1).

(2) To prove (2) we first show that U has fixed points in $B_r(0)$ and, more generally, that the set of fixed points of U in $B_r(0)$ is a nonempty closed convex subset of $B_r(0)$.

We know that $T = I - U$ is monotone on H. This implies that x_0 is a fixed point of U if and only if

$$0 \leqslant (Tx, x - x_0) \qquad \text{for all} \qquad x \text{ in } H. \tag{26}$$

Indeed, if x_0 is a fixed point of U, then $Tx_0 = 0$ and hence, for all x in H, $0 \leqslant (Tx - Tx_0, x - x_0) = (Tx, x - x_0)$, i.e., (26) holds. Conversely, suppose (26) holds for all x in H and some fixed x_0. Then setting $x_t = x_0 + tw$ for any $w \in H$ and $t > 0$ in (26) we obtain $t(Tx_t, w) \geqslant 0$ or, cancelling $t > 0$, $(Tx_t, w) \geqslant 0$ for all w in H. Letting $t \to 0$, we get $(Tx_0, w) \geqslant 0$ for all $w \in H$ and replacing w by $-w$ we finally obtain the equality $(Tx_0, w) = 0$ valid for all $w \in H$. This implies that $Tx_0 = 0$, i.e., x_0 is a fixed point of U.

Thus, we have shown that the set F of fixed points of U is equal to the set

$$\{x_0 \in H : (Tx, x - x_0) \geqslant 0 \text{ for all } x \text{ in } H\},$$

i.e.,

$$F = \bigcap_{x \in H} \{x_0 \in H : (Tx, x - x_0) \geqslant 0\}. \tag{27}$$

For each given x in H, the set $\{x_0 \mid (Tx, x - x_0) \geqslant 0\}$ is a closed half space in H which is closed and convex. Hence, F being an intersection of closed convex sets is itself closed and convex and, in particular, so is the set $F_r = F \cap B_r(0)$.

Now we show that $F_r \neq \phi$. For each fixed s let v_s be the unique fixed point of V_s whose existence was proved in (1). Then $v_s = V_s v_s = sUv_s + (1 - s)\, u_0$ and $Tv_s = v_s - Uv_s = (s - 1)\, Uv_s + (1 - s)\, u_0$. Since $U \in \text{Lip}$, there exists a constant $k_0 > 0$ such that $\| Tv_s \| \leqslant k_0(1 - s)$. Since, for each s, $v_s \in B_r(0)$, there exists a sequence $\{s_j\}$ such that $s_j \to 1$ and $v_{s_j} \rightharpoonup x_1 \in B_r(0)$. By the monotonicity of T, for any x in H

$$(Tx, x - v_{s_j}) \geqslant (Tv_{s_j}, x - v_{s_j}). \tag{28}$$

Furthermore, as $j \to \infty$, we have

$$| (Tv_{s_j}, x - v_{s_j}) | \leqslant \| Tv_{s_j} \| \, \| x - v_{s_j} \| \leqslant k_0(1 - s_j) \{\| x \| + r\} \to 0.$$

This and (28) imply that for any x in H

$$(Tx, x - x_1) = \lim_j (Tx, x - x_j) \geqslant 0.$$

Consequently, by the previous argument, $x_1 \in F$ and $F \neq \phi$. Since the set F_r is closed and convex, let x_0 be the point in F_r which is closest to u_0. We claim that $v_s \to x_0$ in H. Indeed, by definition of x_0, for every $w \in F_r$ and t such that $0 \leqslant t \leqslant 1$ we have

$$\| u_0 - (tw + (1-t)\, x_0) \|^2 \geqslant \| u_0 - x_0 \|^2.$$

or, after expansion, cancelling and rearranging,

$$(u_0 - x_0\,,\, w - x_0) \leqslant 0. \tag{29}$$

Let $w \in F$ and $V_s v_s = v_s$. Then, since $Tw = 0$ and $v_s = sUv_s + (1-s)\, u_0$, we get the following two equations

$$(1-s)\, v_s + sTv_s = (1-s)\, u_0 \tag{30a}$$

$$(1-s)\, w + sTw = (1-s)\, w. \tag{30b}$$

Subtracting (30b) from (30a) we obtain

$$(1-s)\,(v_s - w) + s(Tv_s - Tw) = (1-s)\,(u_0 - w).$$

Taking the scalar product of both sides of the last equation by $(v_s - w)$, we get

$$(1-s)\, \| v_s - w \|^2 + s(Tv_s - Tw, v_s - w) = (1-s)\,(u_0 - w, v_s - w),$$

from which, in view of the monotonicity of T, we derive the inequality

$$\| v_s - w \|^2 \leqslant (u_0 - w, v_s - w), \qquad 0 < s < 1. \tag{30}$$

Next we show that if for a sequence $\{s_j\}$ such that $s_j \to 1$ as $j \to \infty$, $v_{s_j} \rightharpoonup x_1$ then $x_1 = x_0$ and $v_{s_j} \to x_0$. We have already shown that $x_1 \in F_r$. If in (30) we set $w = x_0$, then

$$\begin{aligned} \| v_{s_j} - x_0 \|^2 &\leqslant (u_0 - x_0\,,\, v_{s_j} - x_0) \\ &= (u_0 - x_0\,,\, v_{s_j} - x_1) + (u_0 - x_0\,,\, x_1 - x_0). \end{aligned} \tag{31}$$

Now, it follows from (29) with $w = x_1 (\in F_r)$ that $(u_0 - x_0\,,\, x_1 - x_0) \leqslant 0$, while, since $v_{s_j} \rightharpoonup x_1$, the first term $(u_0 - x_0\,,\, v_{s_j} - x_1) \to 0$, as $j \to \infty$. Hence $\| v_{s_j} - x_0 \|^2 \to 0$, i.e., $v_{s_j} \to x_0$, as $j \to \infty$.

Finally, this implies that $v_s \to x_0$ since, if v_s did not converge strongly to x_0, then there would exist $\delta > 0$ and an infinite sequence $s_j \to 1$ such that $\| v_{s_j} - x_0 \| \geqslant \delta$ for all j. Replacing the sequence $\{s_j\}$ by an infinite subsequence and using weak compactness of $B_r(0)$ we may assume that $v_{s_j} \rightharpoonup x_1$ for some x_1 in $B_r(0)$. By the preceding argument, $v_{s_j} \to x_0$ and this contradicts of $\| x_{s_j} - x_0 \| \geqslant \delta > 0$. Hence, $v_s \to x_0$ and Theorem 14 is completely proved.

THEOREM 15. *Suppose U maps H into H and $U \in P_3 \cap$ Lip. Suppose further that U satisfies condition* (LS) *on $S_r(0)$. Then*

(1) *U has fixed points in $B_r(0)$*

(2) *If $V_s = sU$, $0 < s < 1$, then V_s has a unique fixed point v_s in $B_r(0)$ which is given by*

$$v_s = \lim_n v_s{}^n = \lim_n \{R(I - \mu(I - V_s)\}^n w_0 , \qquad w_0 \in B_r(0)$$

for any fixed μ which satisfies the inequality (23).

(3) *As $s \to 1$, $v_s \to x_0$, where x_0 is the fixed point of U nearest the origin* 0.

PROOF. It suffices by the proof of Theorem 14, to show that for μ satisfying (23) the operators V_s and $W_{\mu s} = (I - \mu(I - V_s))$ satisfy the condition (LS) on $S_r(0)$. Suppose that $W_{\mu s}x = \lambda x$ for some x in $S_r(0)$. Then $x - \mu(I - V_s)\, x = \lambda x$, i.e.,

$$\mu V_s x = (\lambda + \mu - 1)\, x \qquad \text{or} \qquad V_s x = \left(1 + \frac{\lambda - 1}{\mu}\right) x_0 .$$

Hence $W_{\mu s}$ satisfies (LS) if and only if V_s does. Now, if $V_s x = \lambda x$ for some x in $S_r(0)$ and some $\lambda > 1$, then $Ux = (\lambda/s)\, x$ and $(\lambda/s) > \lambda > 1$ in violation of condition (LS) satisfied by U. Q.E.D.

6. THE PROJECTION ITERATION METHOD

The only use made in Section 5 of the assumption that U lies in the class Lip was to obtain the fixed points of operator V_s obtained from U and lying in the class P_2 by an iterative method. For U in P_2 which do not satisfy a Lipschitz condition, we can proceed by a combination of projection and iteration methods as developed below.

Let H be a separable real Hilbert space. A mapping U of H into H which maps bounded sets in H into bounded sets is called *bounded.* Let $\{H_k\}$ be an increasing sequence of subspaces of H each of finite dimension $\dim H_k = h_k$ such that $\bigcup_k H_k$ is dense in H. Let $\varphi(t)$ be a non-negative and continuously differentiable function of R^1 into R^+ with compact support. On each space H_k define the function $\varphi_k(w)$ by

$$\varphi_k(w) = \frac{\varphi(\| w \|)}{\int_{H_k} \varphi(\| w \|)\, dw}, \qquad w \in H_k , \qquad dw = dw_1\, dw_2 \cdots d_{w_{h_k}} \tag{32}$$

and for each $\delta > 0$ set

$$\varphi_{k\delta}(w) = \delta^{-h_k} \varphi_k \left(\frac{w}{\delta}\right). \tag{33}$$

THEOREM 16. *Let U be a continuous mapping of H into H such that for some $c_0 > 0$*

$$((I - U)x - I - U)y, x - y) \geqslant c_0 \| x - y \|^2, \qquad x, y \in H. \tag{34}$$

Let P_k be the orthogonal projection of H onto H_k. Then the following is true.

(a) *The map $U_k = P_k U$ of H_k into H_k has exactly one fixed point u_k in H_k which can be calculated in the following way: For each fixed k and each $\delta > 0$ define the mapping $U_{k\delta}$ by*

$$U_{k\delta}(x) = \int_{H_k} \varphi_{k\delta}(x - w)\, U_k(w)\, dw. \tag{35}$$

Then each $T_{k\delta} = I - U_{k\delta}$ is a strongly monotone mapping with the same uniform constant $c_0 > 0$ and each $u_{k\delta}$ can be calculated by Theorems 14 or 15, i.e., by iterations, $u_{k\delta} \to u_k$, as $\delta \to 0$, and the error estimate similar to (24) is also valid.

(b) *$u_k \to x_0$, as $k \to \infty$, and x_0 is a fixed point of U.*

PROOF. We first prove assertion (b). Suppose that, for each k, u_k is given as the unique fixed point of U_k on H_k. Then $\{u_k\}$ is a Cauchy sequence. In fact, if $j \leqslant k$, then $u_j, u_k \in H_k$, $P_k u_j = u_j$, $P_k u_k = u_k$ and therefore by (34),

$$\begin{aligned} c_0 \| u_j - u_k \|^2 &\leqslant ((I - U)\, u_j - (I - U)\, u_k, P_k(u_j - u_k)) \\ &= (u_j - Uu_j, u_j - u_k) - (u_k - U_k u_k, u_j - u_k). \end{aligned}$$

Since $U_k u_k = u_k$, the second term on the right vanishes. This implies that for any fixed j and any k

$$c_0 \| u_j - u_k \|^2 \leqslant (u_j - Uu_j, u_j - u_k), \tag{35_0}$$

and, therefore,

$$\| u_j - u_k \| \leqslant c_0^{-1} \| u_j - Uu_j \|.$$

In particular, the sequence $\{u_k\}$ is bounded. Hence there exists a subsequence $\{u_{k_m}\}$ such that $u_{k_m} \rightharpoonup u_0$. Taking the limit in ($35_0$), we have

$$c_0 \| u_j - u_0 \|^2 \leqslant (u_j - Uu_j, u_j - u_0) = (u_j - Uu_j, u_0).$$

We can find $k_\mathscr{E}$ so large and an element w in $H_{k_\mathscr{E}}$ such that $\| w - u_0 \| < \mathscr{E}$ for a prescribed $\mathscr{E} > 0$. Then

$$(u - Uu_j, u_0) = (u_j - Uu_j, w) + (u_j - Uu_j, u_0 - w).$$

For the second term

$$|(u_j - Uu_j, u_0 - w)| \leqslant \|u_j - Uu_j\| \, \|u_0 - w\| \leqslant C_0 \mathscr{E} \quad \text{for some} \quad \mathscr{C}_0 > 0.$$

For the first, if $j > k_{\mathscr{E}}$,

$$(u_j - Uu_j, w) = (u_j - Uu_j, P_j w) = (T_j u_j, w) = 0.$$

Hence

$$c_0 \|u_j - u_0\|^2 \leqslant C_0 \mathscr{E}, \qquad \text{if} \qquad j \geqslant k.$$

Thus $u_j \to u_0$, and u_0 is a fixed point of U.

To prove (a) first note that, by definition of $\varphi_{k\delta}$, supp $(\varphi_{k\delta}) \subset B_{c\delta}(0)$ if supp $(\varphi_k) \subset B_c(0)$ for some $c > 0$, $\varphi_{k\delta}(w) \geqslant 0$ for all w and

$$\int_{H_k} \varphi_{k\delta}(w)\, dw = \delta^{-h_k} \int_{H_k} \varphi_k\left(\frac{w}{\delta}\right) dw = \int_{H_k} \varphi_k(u)\, du = 1. \tag{36}$$

Furthermore, since $\int_{H_k} \varphi_{k\delta}(x - w)\, dw = 1$ it follows that

$$T_{k\delta}(x) = x - U_{k\delta}(x) = \int_{H_k} \varphi_{k\delta}(x - w)\, T_k(w)\, dw.$$

This implies that

$$(T_{k\delta}(x) - T_{k\delta}(y), x - y) \geqslant c_0 \|x - y\|^2 \qquad \text{for all} \qquad x, y \in H_k.$$

Indeed, changing the variables we see that

$$T_{k\delta}(x) = \int_{H_k} \varphi_{k\delta}(u)\, T_k(x - u)\, du$$

and therefore for all x and y in H_k

$$(T_{k\delta}(x) - T_{k\delta}(y), x - y) = \int_{H_k} \varphi_{k\delta}(u)\, (T_k(x - u) - T_k(y - u), x - y)\, du.$$

But

$$(T_k(x - u) - T_k(y - u), x - y)$$
$$= (T_k(x - u) - T_k(y - u), (x - u) - (y - u)) \geqslant c_0 \|(x - y) - (y - u)\|^2$$
$$= c_0 \|x - y\|^2.$$

Hence, since $\varphi_{k\delta}(u) \geqslant 0$, we can replace the integrant by its lower bound and get the inequality

$$(T_{k\delta}(x) - T_{k\delta}(y), x - y) \geqslant \int_{H_k} c_0 \|x - y\|^2 \varphi_{k\delta}(u)\, du = c_0 \|x - y\|^2.$$

Now we prove that, for each k and each $\delta > 0$, $U_{k\delta}$ belong to a Lipschitz class on $B_r(0)$. In fact, for all x and y in $B_r(0)$

$$\| U_{k\delta}(x) - U_{k\delta}(y) \| = \left\| \int_{H_k} \{\varphi_{k,\delta}(x-u) - \varphi_{k\delta}(y-u)\} U_k(u)\, du \right\|$$

$$\leqslant \sup | \varphi_{k\delta}(x-u) - \varphi_{k\delta}(y-u) | \int_{H_k} \| U_k(u) \| \, du$$

$$\leqslant \sup | \varphi_{k\delta}(x-u) - \varphi_{k\delta}(y-u) | \sup_{\|u\| \leqslant r+c} \| U(u) \|.$$

Now

$$| \varphi_{k\delta}(x-u) - \varphi_{k\delta}(y-u) | \leqslant c_k(\delta) \| (x-u) - (y-u) \| = c_k(\delta) \| x - y \|,$$

where $c_k(\delta) = c_1 \delta^{-h_k}$.
Hence for each $\delta > 0$, $U_{k\delta} \in \mathrm{Lip}$ with the Lipschitz constant

$$\| U_{k\delta} \|_{\mathrm{Lip}} \leqslant \frac{c_2 \cdot c_1}{\delta^{h_k}}, \qquad \text{where} \qquad c_2 \geqslant \sup_{\|u\| \leqslant r+c\delta} \| U_k(u) \|.$$

Thus by previous two theorems the fixed points $u_{k\delta}$ of $U_{k\delta}$ can be constructed by iteration methods, i.e., we can construct $u_{k\delta}$ such that $U_{k\delta} u_k = u_{k\delta}$.

Finally we show that $u_{k\delta} \to u_k$, as $\delta \to 0$. Using the same arguments as in the proof of (b) we show that $\| u_{k\delta} \| \leqslant M$ for some constant $M > 0$. Now

$$u_{k\delta} = \int_{H_k} \varphi_{k\delta}(v)\, U_k(u_{k\delta} - v)\, dv = \int_{\|v\| \leqslant c\delta} \varphi_{k\delta}(v)\, U_k(u_{k\delta} - v)\, dv.$$

Hence

$$\| u_{k\delta} - U_k u_{k\delta} \| \leqslant \int_{\|v\| \leqslant c\delta} \varphi_{k\delta}(v) \| U_k(u_{k\delta} - v) - U_k(u_{k\delta}) \| \, dv.$$

Define the modulus of continuity $w_{Mk}(\delta)$ by

$$w_{Mk}(\delta) = \sup_{\substack{\|v\| \leqslant c\delta \\ \|u\| \leqslant M}} \| U_k(u - v) - U_k(u) \|.$$

Then, since $T_k = I - U_k$ is strongly monotone and

$$\| T_k(u_{k\delta}) \| \leqslant w_{Mk}(\delta),$$

we obtain

$$c \| u_{k\delta} - u_{k\gamma} \|^2 \leqslant (T_k(u_{k\delta}) - T_k(u_{k\gamma}), u_{k\delta} - u_{k\gamma}) \leqslant 2 w_{Mk}(\delta) \| u_{k\delta} - u_{k\gamma} \|,$$

which implies that

$$\| u_{k\delta} - u_{k\gamma} \| \leqslant \frac{2}{c_0} w_{Mk}(\delta) \to 0, \qquad \text{as} \qquad \delta \to 0. \tag{37}$$

Thus, as $\delta \to 0$, $u_{k\delta} \to u_k$ and u_k is the fixed point of U_k since

$$u_k - U_k u_k = \lim_{\delta} T_k(u_{k\delta}) = 0.$$

It also follows from (37) that the error $\| u_{k\delta} - u_k \|$ is given by

$$\| u_{k\delta} - u_k \| \leqslant \frac{2w_{Mk}(\delta)}{c_0}.$$

EXAMPLE. Take

$$\varphi(r) = \begin{cases} 2 - 2r & \text{if} \quad |r| \leqslant 1 \\ 0 & \text{if} \quad |r| \geqslant 1. \end{cases}$$

$$\varphi_k(w) = k^{k/2} \prod_{j=1}^{k} \varphi(\sqrt{k} x_j), \qquad |x_j| \leqslant d, \qquad d = \frac{1}{(k)^{1/2}}.$$

Then

$$\varphi_k(w) = k^{k/2} \prod_{j=1}^{k} \max\{0, 1 - (k)^{1/2} w_j\}, \qquad \dim h_k = k,$$

$$\varphi_{k\delta}(w) = k^{k/2} \prod_{j=1}^{k} \min\left\{0, 1 - (k)^{1/2} \frac{w_j}{\delta}\right\}.$$

Hence

$$U_{k\delta} = \int_{|u_j| \leqslant (1/k)} \cdots \int U_k(x_j - u_1, \ldots, x_j - u) \left(1 - (k)^{1/2} \frac{u_j}{\delta}\right) du_1 \cdots du_k.$$

HISTORICAL REMARKS. For references to the use of projection methods for equations involving operators of monotone type, we refer to Browder [32], and Petryshyn [45].

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