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Necessary Conditions for Uniqueness in L^1-Approximation

A. PINKUS AND B. WAJNRYB

Department of Mathematics, Technion, Haifa, Israel

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Let K be a compact subset of R^m with $K=\overline{\text{int } K}$. Necessary conditions on an n-dimensional subspace U_n of $C(K)$ are given so that for each $f \in C(K)$ there exists a unique best $L^1(w)$-approximation from U_n, for every fixed positive weight function w. © 1988 Academic Press, Inc.

1. NOTATION AND DEFINITIONS

Let K be a compact subset of R^m. For convenience we assume that $K=\overline{\text{int } K}$. W will denote the set of bounded, integrable functions on K for which $\inf\{w(x): x \in K\} > 0$, and $\tilde{W}$ the set of strictly positive continuous functions on K. By $C(K)$ we mean the set of real-valued continuous functions with domain of definition K. U_n will always denote an n-dimensional subspace of $C(K)$. For $w \in W$, the $L^1(w)$-norm of $f \in C(K)$ is defined by

$$\|f\|_w = \int_K |f(x)|\, w(x)\, dx.$$

DEFINITION 1. We say that U_n is a *unicity space for* w, $w \in W$, if to each $f \in C(K)$ there exists a unique best approximation to f from U_n in the $L^1(w)$-norm. Similarly we say that U_n is a *unicity space for* W ($\tilde{W}$) if U_n is a unicity space for w for all $w \in W$ ($w \in \tilde{W}$).

DEFINITION 2. For each $f \in C(K)$, we set $Z(f)=\{x: f(x)=0\}$. Similarly, for a set $F \subseteq C(K)$, we set $Z(F)=\{x: f(x)=0 \text{ for all } f \in F\}$.

DEFINITION 3. For a relatively open subset D of K, we denote by $|D|$ the number (possibly infinite but countable) of the connected components of D. For given $u \in U_n$, we set $M(u)=|K/Z(u)|$. We fix an order on the connected components $A_i = A_i(u)$ of $K/Z(u)$, and set $K/Z(u)=\bigcup_{i=1}^{M(u)} A_i$.

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DEFINITION 4. U_n is said to satisfy *Property A* if for each $u \in U_n/\{0\}$ and every choice of $\varepsilon_i \in \{-1, 1\}$, $i = 1, ..., M(u)$, there exists a $v \in U_n/\{0\}$ satisfying

(a) $v(x) = 0$ a.e. on $Z(u)$,

(b) $\varepsilon_i v(x) \geqslant 0$, $x \in A_i$, $i = 1, ..., M(u)$.

DEFINITION 5. For $u^* \in U_n/\{0\}$, we define

$$U(u^*) = \{u: u \in U_n, u(x) = 0 \text{ a.e. on } Z(u^*)\}.$$

DEFINITION 6. U_n is said to *decompose* on K if there exist disjoint subspaces V_r, W_{n-r} of U_n of dimension r and $n-r$, respectively, $1 \leqslant r \leqslant n-1$ ($V_r \cap W_{n-r} = \{0\}$) and disjoint subsets B and C of K such that each element of V_r vanishes identically off B, and each element of W_{n-r} vanishes identically off C.

2. INTRODUCTION

A classic result of approximation theory is that of Haar [2]. Haar's theorem characterizes those subspaces U_n of $C(B)$, B compact Hausdorff, for which there exists a unique best approximation to each $f \in C(B)$ from U_n in the uniform norm. It is natural to consider this same problem in the $L^1(w)$-norm setting for given $w \in W$. That is, one searches for necessary and sufficient conditions on U_n such that U_n is a unicity space for w. One would, of course, like these conditions to be both easily verifiable and intrinsic for given U_n. Necessary and sufficient conditions were given by Cheney and Wulbert [1], and different (equivalent) conditions were also given by Strauss [11]. Unfortunately these conditions are not at all easily verifiable. One reason for this fact is that the criteria turn out to be weight function (i.e., w) dependent. This is in sharp contrast to the analogous problem in the uniform norm, where the necessary and sufficient conditions as elucidated by Haar are identical if we approximate using any weighted uniform norm with weight $w \in \tilde{W}$.

It is therefore natural to ask for conditions on U_n which are equivalent to the demand that U_n be a uniticity space for W ($\tilde{W}$). A first result in this direction was obtained by Havinson [3] in the case $K = [a, b] \subset R$. Havinson proved that if U_n has the property that no $u \in U_n/\{0\}$ vanishes on a subinterval of $[a, b]$, then U_n is a unicity space for W if and only if U_n is a T-system on (a, b). (The "if" direction is a classic result proven earlier by Krein [4].)

On the basis of work of Strauss [12], Property A was formulated. Strauss showed, for $K = [a, b]$, that if U_n satisfies Property A, then U_n is a unicity space for W. This result has been generalized to any K as above. In fact, however, these two conditions are equivalent, as has been shown by Kroo [6] and Sommer [9]. (Actually Kroo's result holds in a much more general setting.)

THEOREM A (Kroo [6] and Sommer [9]). *For $K \subset R^m$, K compact, $K = \overline{\operatorname{int} K}$, U_n is a unicity space for W if and only if U_n satisfies Property* A.

One may relax the condition that U_n be a unicity space for W to the condition that U_n be a unicity space for $\tilde{W}$ if one imposes a further condition on U_n, namely, $\operatorname{meas}\{Z(u)\} = \operatorname{meas}\{\operatorname{int} Z(u)\}$ for all $u \in U_n$. Theorem A was originally proved for $K = [a, b]$ by Kroo in [5]. Independently, the first author in [8] proved this result, with $K = [a, b]$, for $\tilde{W}$, where the above additional assumption is imposed on U_n. Theorem A for $K \subset R^m$ is a direct generalization of these results.

The verification of Property A for a given subspace U_n is not a simple problem. In the case $K = [a, b]$, the first author went on to obtain more intrinsic conditions on U_n which explicitly characterize all those subspaces U_n which satisfies Property A. He showed that U_n satisfies Property A if and only if it is a "spline-like" space. The explicit conditions are somewhat lengthy to state and may be found in [8]. However, two main results deserve special mention.

THEOREM B (Pinkus [8]). *For $K = [a, b]$, U_n satisfies Property* A *if and only if*

$$|[a, b]/Z(u)| \leqslant \dim U(u)$$

for each $u \in U_n$.

The "only if" part is explicitly stated in [8] as Theorem 4.7. The "if" part is essentially proved, but never explicitly stated. The second result is the following.

THEOREM C (Pinkus [8]). *Let $K = [a, b]$, and assume U_n satisfies Property* A. *If $|[a, b]/Z(U_n)| \geqslant 2$, then U_n decomposes.*

To be more precise, it follows from Theorem C that if $[a, b]/Z(U_n) = \bigcup_{i=1}^r A_i$, where the $\{A_i\}_1^r$ are the relatively open connected components of $[a, b]/Z(U_n)$, then $\dim U_n|_{A_i} = n_i$, $i = 1, ..., r$; $1 \leqslant n_i$; $\sum_{i=1}^r n_i = n$, and there exists a basis for $U_n|_{A_i}$ all of whose elements vanish identically off A_i. Furthermore, by definition, $U_n|_{A_i}$ satisfies Property A on A_i, $i = 1, ..., r$. Thus, as is easily seen, our problem reduces to r independent problems, i.e., U_n satisfies Property A on K if and only if $U_n|_{A_i}$ satisfies Property A on $\bar{A}_i$ for each $i = 1, ..., r$. The best approximation problem reduces to r independent approximation problems.

We wish to generalize Theorems B and C to the multidimensional setting. However, only one direction of Theorem B is valid in more than one dimension. To verify this, consider $U_2 = \text{span}\{x, y\}$, and $K = [-1, 1] \times [-1, 1]$. For each $u \in U_2/\{0\}$, $|K/Z(u)| = 2 = \dim U(u)$. However, there exists no non-negative, non-trivial function in U_2. Thus U_2 does not satisfy Property A. Nonetheless, we will prove the following results.

THEOREM D. *Let $K \subset R^m$, compact, $K = \overline{\text{int } K}$. If U_n satisfies Property* A, *then*

$$|K/Z(u)| \leqslant \dim U(u)$$

for each $u \in U_n$.

THEOREM E. *Let K be as above, and let U_n satisfy Property* A. *If $|K/Z(U_n)| \geqslant 2$, then U_n decomposes.*

Theorem D is a generalization of Theorem 4.7 of [8]. The proof of Theorem 4.7, as given therein, is lengthy and arduous. A simpler proof, which is, however, also only valid for $K \subset R$, has been constructed by Sommer, based on the fact that U_n satisfying Property A must be a WT-system. The proof given here of Theorem D is essentially simpler than the proof in [8] and of course more general than either of these other proofs. Note also that Theorem E together with the results of [8] totally solves the problem of characterizing unicity spaces U_n for W where K is a subset of R (and not necessarily one closed interval) by reducing it to distinct problems on closed intervals.

As a result of Mairhuber's theorem [7], it is known that if U_n is a unicity space in the uniform norm on $C(K)$, and $n > 1$, then K is essentially a subset of R. Thus is no longer true in the situation under consideration. Many examples exist of unicity spaces for W with $K \subset R^m$, $m > 1$. Perhaps the most interesting example so far constructed is that, due to Sommer [10], of certain subspaces of bivariate linear splines in R^2. However, unlike the case where $K \subset R$, an intrinsic characterization of unicity spaces in $C(K)$ for $K \subset R^m$, as above, is a problem yet unresolved.

3. PROOF OF THEOREM D

Our proof is via induction on n. For $n = 1$, the theorem is obvious. Note that if U_n satisfies Property A and $u \in U_n$, then $U(u)$ also satisfies Property A. Thus if $\dim U(u) < n$, then by the induction hypothesis we may assume that our results holds for such u. We therefore assume that there exists a $u \in U_n$ with $M(u) > n$ and will eventually arrive at a contradiction.

For convenience, our proof of Theorem D is divided into a series of lemmas.

LEMMA 1. *Let $u \in U_n/\{0\}$, and $K/Z(u) = \bigcup_{i=1}^{M} A_i$ as in Definition* 3 *(M may be infinite). Let J be a subset of $\{1, ..., M\}$ with $|J|$ elements. Set*

$$U_J = \{v\colon v \in U(u), v = 0 \text{ on } A_j \text{ for } j \notin J\}.$$

If $\dim U_J < |J|$, *then there exists a non-zero sequence $\mathbf{s} = (s_1, ..., s_M)$ for which $s_j = 0$ all $j \notin J$, and*

$$\sum_{j=1}^{M} s_j \int_{A_j} v(x)\,dx = 0$$

for all $v \in U_J$.

Proof. Let $u_1, ..., u_r$ be a basis for U_J. Set

$$c_{ij} = \int_{A_j} u_i(x)\,dx, \qquad i = 1, ..., r;\ j \in J.$$

Since $r < |J|$, there exists an $\mathbf{s} = (s_1, ..., s_M) \neq \mathbf{0}$ with $s_j = 0$, $j \notin J$, which satisfies

$$\sum_{j=1}^{M} c_{ij} s_j = 0, \qquad i = 1, ..., r.$$

Thus

$$\sum_{j=1}^{M} s_j \int_{A_j} v(x)\,dx = 0$$

for all $v \in U_J$. ∎

As an immediate consequence of this lemma we have

COROLLARY 2. *Let the assumptions of Lemma* 1 *hold with some u, J, U_J, and* $\mathbf{s}$. *If $v \in U_J$ and $s_j v(x) \geqslant 0$ for all $x \in A_j$ and $j = 1, ..., M$, then $s_j v(x) = 0$ for all $x \in A_j$ and all $j = 1, ..., M$.*

We shall have frequent recourse to the above corollary with $J = \{1, ..., M\}$. As such we formalize the process.

DEFINITION 7. Let $u \in U_n/\{0\}$, $K/Z(u) = \bigcup_{i=1}^{M} A_i$. A non-zero sequence $\mathbf{s} = (s_1, ..., s_M)$ is said to be an *annihilator* for u if for every function $v \in U(u)$ with $s_i v \geqslant 0$ on A_i, $i = 1, ..., M$, it follows that $s_i v = 0$ on A_i, $i = 1, ..., M$.

If $M(u) > n$, then setting $J = \{1, ..., M\}$, it follows from Lemma 1 and Corollary 2 that there exists an annihilator for u.

Let $u \in U_n/\{0\}$, $K/Z(u) = \bigcup_{i=1}^M A_i$, and assume $\mathbf{s} = (s_1, ..., s_M)$ is an annihilator for u. Set

$$U(u, \mathbf{s}) = \{v\colon v \in U(u), s_i v \geqslant 0 \text{ on } A_i, i = 1, ..., M\}.$$

Let $I^{\mathbf{s}}$ denote the set of indices in $\{1, ..., M\}$ for which some $v \in U(u, \mathbf{s})$ does not identically vanish on A_i.

Set

$$K^{\mathbf{s}} = K/\text{int}\left(\overline{\bigcup_{i \notin I^{\mathbf{s}}} A_i}\right).$$

Note that $K^{\mathbf{s}} = \overline{\text{int } K^{\mathbf{s}}}$, and $I^{\mathbf{s}}$ does not include indices for which $s_i \neq 0$. The important property to remember about $U(u, \mathbf{s})$ is that if $v \in U(u)$ and $s_i v \geqslant 0$ on A_i where $s_i \neq 0$, then $v \in U(u, \mathbf{s})$.

Assume that we are given a $u \in U_n/\{0\}$ with $M(u) > n$. (Because of the induction hypothesis this is the only case of interest.) There then exists an annihilator $\mathbf{s}$ for u, and by Property A, $U(u, \mathbf{s}) \neq \varnothing$. Furthermore, $u \notin U(u, \mathbf{s})$. Thus $1 \leqslant \dim U(u, \mathbf{s}) < n$. Let d denote the *minimal* value for $\dim U(u, \mathbf{s})$ as we vary over all $u \in U_n/\{0\}$ with $M(u) > n$, and all annihilators $\mathbf{s}$ for u.

LEMMA 3. *If $M(u) > n$, $\mathbf{s}$ is an annihilator for u, and* $\dim U(u, \mathbf{s}) = d$, *then*

$$|K^{\mathbf{s}}/Z(u)| \leqslant d.$$

Proof. Assume $|K^{\mathbf{s}}/Z(u)| > d$. Now $K/Z(u) = \bigcup_{i=1}^M A_i$ and $K^{\mathbf{s}}/Z(u) = \bigcup_{i \in I^{\mathbf{s}}} A_i$ with $|I^{\mathbf{s}}| > d$. We apply Lemma 1 with $J = I^{\mathbf{s}}$ to obtain a non-zero sequence $\mathbf{t} = (t_1, ..., t_M)$, with $t_i = 0$ for $i \notin I^{\mathbf{s}}$, such that if $v \in U(u, \mathbf{s})$ and $t_i v \geqslant 0$ on A_i, all i, then $t_i v = 0$ on A_i, all i. Change t_i for $i \notin I^{\mathbf{s}}$ by setting $t_i = s_i$ thereon. It is easily seen that this new $\mathbf{t}$ is an annihilator for u and $U(u, \mathbf{t}) \subsetneqq U(u, \mathbf{s})$ since $t_i \neq 0$ for at least one $i \in I^{\mathbf{s}}$, i.e., $I^{\mathbf{t}} \subsetneqq I^{\mathbf{s}}$. This contradicts the minimality of $\dim U(u, \mathbf{s})$. ∎

From Lemma 3, we have that if $M(u) > n$ and $\mathbf{s}$ is an annihilator for u with $\dim U(u, \mathbf{s}) = d$, then $|K^{\mathbf{s}}/Z(u)| \leqslant d$. Among all such u and $\mathbf{s}$, choose u^* and $\mathbf{s}$ with $|K^{\mathbf{s}}/Z(u^*)|$ *maximal.*

LEMMA 4. *Let u^* and* $\mathbf{s}$ *be as above. If $u \in U(u^*, \mathbf{s})$, then*

$$|K^{\mathbf{s}}/Z(u^* - u)| \leqslant |K^{\mathbf{s}}/Z(u^*)|.$$

Proof. Set $v = u^* - u$, and assume $|K^s/Z(v)| > |K^s/Z(u^*)|$. Now $v = u^*$ on A_i for all $i \notin I^s$. Let $K^s/Z(v) = \bigcup_{j \in J} B_j$. Thus $K/Z(v) = (\bigcup_{j \in J} B_j) \cup (\bigcup_{i \notin I^s} A_i)$. Since $|K^s/Z(v)| > |K^s/Z(u^*)|$ then $M(v) > n$. We define a non-zero annihilator $\mathbf{t}$ for v by setting $t_i = s_i$ for $i \notin I^s$ and $t_j = 0$ for $j \in J$. Clearly $\mathbf{t}$ is an annihilator for v and $I^{\mathbf{t}}$ is a subset of J. Furthermore, by the minimality property of dim $U(u^*, \mathbf{s})$, it follows that $U(v, \mathbf{t}) = U(u^*, \mathbf{s})$. Thus $u \in U(v, \mathbf{t})$.

Assume $I^{\mathbf{t}} \neq J$. There then exists a $k \in J$ such that every element of $U(v, \mathbf{t})$ vanishes identically on B_k. In particular $u = 0$ on B_k. Since v vanishes on the relative boundary of B_k, it follows that u^* vanishes on the relative boundary of B_k. From the definition of the A_i, we see that B_k must contain some A_i with $i \in I^s$. Thus every element of $U(v, \mathbf{t})$ vanishes identically on this A_i, $i \in I^s$, which contradicts the fact that $U(v, \mathbf{t}) = U(u^*, \mathbf{s})$. Hence $I^{\mathbf{t}} = J$.

Because $I^{\mathbf{t}} = J$ we have $K^{\mathbf{t}} = K^s$ and $K^s/Z(v) = K^{\mathbf{t}}/Z(v)$. Since $\dim U(v, \mathbf{t}) = \dim U(u^*, \mathbf{s}) = d$, the maximality of $|K^s/Z(u^*)|$ implies that

$$|K^s/Z(v)| = |K^{\mathbf{t}}/Z(v)| \leqslant |K^s/Z(u^*)|,$$

proving the lemma. ▮

Let u^* and $\mathbf{s}$ be as above. From Property A there exists a $v \in U(u^*)$ such that $s_i v \geqslant 0$ on A_i, $i \notin I^s$, and $u^* v \geqslant 0$ on A_i, $i \in I^s$. Thus, in particular, $v \in U(u^*, \mathbf{s})$, which implies $v = 0$ on A_i, $i \notin I^s$.

LEMMA 5. *Let u^*, $\mathbf{s}$, and v be as above. Then for each $i \in I^s$ there exists an $\alpha_i \geqslant 0$ such that $\alpha_i u^* = v$ on A_i.*

Proof. We first show that if $x \in \bar{A}_i \cap \bar{A}_j$ for some $i \neq j$, then $v(x) = 0$. Suppose to the contrary that there exists an i, j; $i \neq j$; $i, j \in I^s$ (necessarily), and an $x_0 \in \bar{A}_i \cap \bar{A}_j$ such that $v(x_0) \neq 0$. Let $y_i \in A_i$, $i \in I^s$. Since $|I^s| \leqslant d$, then for δ sufficiently small and positive, $|\delta v(y_i)| < |u^*(y_i)|$. From Lemma 4, $|K^s/Z(u^* - \delta v)| \leqslant |K^s/Z(u^*)|$. We contradict this inequality by showing that each of the points $\{y_i\}_{i \in I^s}$ and x_0 are in distinct connected components of $K^s/Z(u^* - \delta v)$.

Let $\varepsilon_i = \operatorname{sgn} u^*$ on A_i, $i \in I^s$. Then $\varepsilon_i v \geqslant 0$ on A_i, $i \in I^s$. Since $\varepsilon_i(u^* - \delta v)(y_i) > 0$, while $\varepsilon_i(u^* - \delta v)(x) = -\varepsilon_i \delta v(x) \leqslant 0$ on the relative boundary of A_i, it follows that the relatively open connected component of $K/Z(u^* - \delta v)$ containing y_i is itself contained in A_i. In particular the components containing different y_i's are distinct and disjoint from the boundaries of the A_i's. Since $x_0 \in \partial A_i$ and $(u^* - \delta v)(x_0) = -\delta v(x_0) \neq 0$, x_0 belongs to still another component of $K^s/Z(u^* - \delta v)$, which is a contradiction to Lemma 4.

Assume now that v is not proportional to u^* on A_j for some $j \in I^s$. We can choose $\alpha > 0$ such that $\alpha u^* - v \not\equiv 0$ on A_i for every $i \in I^s$, and $\alpha u^* - v$

takes both positive and negative values on A_j. Since v vanishes on the relative boundary of A_i for each $i \in I^s$, this implies that $|K^s/Z(\alpha u^* - v)| > |K^s/Z(u^*)|$, contradicting Lemma 4. ∎

On the basis of all of the above, we may assume that there exists a $u^* \in U_n/\{0\}$ with $M = M(u^*) > \dim U(u^*) = n$, associated A_i, $i = 1, ..., M$, and a $v \in U(u^*)/\{0\}$ which satisfies the following:

(i) $v = \alpha_i u^*$ on A_i, $i = 1, ..., M$,

(ii) $\alpha_i \geqslant 0$ and all the α_i are zero except for at most some $d < n$.

LEMMA 6. *Theorem* D *holds.*

Proof. The α_i's take on the distinct values β_j, $j = 1, ..., k$; $2 \leqslant k \leqslant n$. Assume that β_j is taken on n_j times, $j = 1, ..., k$. Thus $\sum_{j=1}^k n_j = M > n$.

Since $\beta_j u^* - v$ vanishes identically on some A_i, we have $u^* \notin U(\beta_j u^* - v)$. Thus $\dim U(\beta_j u^* - v) < n$ and by the induction hypothesis

$$M - n_j = |K/Z(\beta_j u^* - v)| \leqslant \dim U(\beta_j u^* - v),$$

$j = 1, ..., k$. This immediately implies that M cannot be infinite, since all but one n_j is bounded by d.

We claim that $\dim(\bigcap_{j=1}^k U(\beta_j u^* - v)) > 0$. We prove this fact by showing by induction that $\dim(\bigcap_{j=1}^r U(\beta_j u^* - v)) > M - (n_1 + \cdots + n_r)$ for $r = 2, ..., k$. For $r = k$ this gives the desired result. For $r = 2$,

$$\begin{aligned}\dim(U(\beta_1 u^* - v) \cap U(\beta_2 u^* - v)) &= \dim(U(\beta_1 u^* - v)) + \dim(U(\beta_2 u^* - v)) \\ &\quad - \dim(U(\beta_1 u^* - v) + U(\beta_2 u^* - v)) \\ &\geqslant (M - n_1) + (M - n_2) - n \\ &> M - (n_1 + n_2)\end{aligned}$$

since $\dim(U(\beta_1 u^* - v) + U(\beta_2 u^* - v)) \leqslant n < M$. Assume the result holds for $r - 1$, $3 \leqslant r \leqslant k$. Then

$$\begin{aligned}\dim\left(\bigcap_{j=1}^{r} U(\beta_j u^* - v)\right) &= \dim\left(\bigcap_{j=1}^{r-1} U(\beta_j u^* - v)\right) + \dim(U(\beta_r u^* - v)) \\ &\quad - \dim\left(\left(\bigcap_{j=1}^{r-1} U(\beta_j u^* - v)\right) + (U(\beta_r u^* - v))\right) \\ &> (M - (n_1 + \cdots + n_{r-1})) + (M - n_r) - n \\ &> M - (N_1 + \cdots + n_r)\end{aligned}$$

since $\dim((\bigcap_{j=1}^{r-1} U(\beta_j u^* - v)) + (U(\beta_r u^* - v)) \leqslant n < M$. Thus dim

$(\bigcap_{j=1}^{k} U(\beta_j u^* - v)) > 0$. But if $u \in \bigcap_{j=1}^{k} U(\beta_j u^* - v)$, then it is easily seen that $u = 0$. Thus $\dim(\bigcap_{j=1}^{k} U(\beta_j u^* - v)) = 0$. This contradiction proves Theorem D. ∎

4. Proof of Theorem E.

In the proof of Theorem E we shall make use of the following Proposition.

PROPOSITION 7. *Let $K \subset R^m$, K compact, $K = \overline{\operatorname{int} K}$. Let $W = \operatorname{span}\{w_1, \ldots, w_r\}$ be an r-dimensional subspace of $C(K)$. Assume that for all $w \in W$, $|K/Z(w)| \leqslant M$, M finite. Then there exists a $w^* \in W$ of the form*

$$w^* = w_1 + \sum_{i=2}^{r} a_i^* w_i$$

such that if $w \in W$ satisfies

(a) $w(x) = 0$ a.e. *on* $Z(w^*)$,

(b) $w(x)(\operatorname{sgn} w^*(x)) = |w(x)|$ *for all* $x \in K/Z(w^*)$,

then $w = \alpha w^*$ for some $\alpha \geqslant 0$.

Remark. Note that in the statement of the proposition, the coefficient of w_1 is 1.

To prove the proposition, we use the following lemmas. We always assume that the conditions of the proposition hold.

LEMMA 8. *Assume $g_1, \ldots, g_k \in W/\{0\}$, and $\operatorname{int} Z(g_i) \subsetneqq \operatorname{int} Z(g_{i+1})$, $i = 1, \ldots, k-1$. Then $g_1, \ldots, g_k$ are linearly independent.*

Proof. We may assume that $g_1, \ldots, g_{k-1}$ are linearly independent and $g_1, \ldots, g_k$ are linearly dependent. Thus

$$g_k = \sum_{i=1}^{k-1} a_i g_i.$$

On $\operatorname{int} Z(g_k)$, $\sum_{i=1}^{k-1} a_i g_i = 0$. Thus on $\operatorname{int} Z(g_2)$, $0 = \sum_{i=1}^{k-1} a_i g_i = a_1 g_1$. But there exists an $x_1 \in \operatorname{int} Z(g_2)/\operatorname{int} Z(g_1)$ for which $g_1(x_1) \neq 0$. Thus $a_1 = 0$. We continue in this manner to obtain $a_1 = \cdots = a_{k-1} = 0$, a contradiction. ∎

Set

$$V=\left\{w: w=w_1+\sum_{i=2}^{r} a_i w_i\right\}$$

and

$$\tilde{V}=\{w: \alpha w\in V \text{ for some } \alpha\neq 0\}.$$

LEMMA 9. *There exists a $w^*\in V$ which satisfies the following*:

> *If* $w\in V$ *and* $\operatorname{int} Z(w)\supseteq \operatorname{int} Z(w^*)$, *then* $\operatorname{int} Z(w)=\operatorname{int} Z(w^*)$, *and* $|K/Z(w)|\leqslant |K/Z(w^*)|$.

Proof. Choose $v_1\in V$. If there exists a $v_2\in V$ for which $\operatorname{int} Z(v_1)\subsetneqq \operatorname{int} Z(v_2)$, then replace v_1 by v_2. Continue this process. Since $V\subseteq W$ and $\dim W=r$, it follows from Lemma 8 that this process stops after at most r steps. Thus there exists a $\tilde{w}\in V$ such that if $w\in V$, and $\operatorname{int} Z(w)\supseteq \operatorname{int} Z(\tilde{w})$, then $\operatorname{int} Z(w)=\operatorname{int} Z(\tilde{w})$.

Among all $w\in V$ satisfying $\operatorname{int} Z(w)=\operatorname{int} Z(\tilde{w})$, choose $w^*\in V$ for which $|K/Z(w^*)|$ is maximal. Such a choice is possible since $|K/Z(w)|$ is uniformly bounded by M for all $w\in W$. ▮

We shall eventually prove that the $w^*\in V$ of Lemma 9 satisfies the claim of the proposition.

Let $K/Z(w^*)=\bigcup_{i=1}^{k} A_i$, where the A_i are relatively open, connected sets in K, $k\leqslant M$. Let ε_i denote the sign of w^* on A_i, $i=1, ..., k$. Assume, contrary to the claim of the proposition, that there exists a $w\in W/\{0\}$ for which $w\neq \alpha w^*$ for any $\alpha>0$ and

(a) $w(x)=0$ a.e. on $Z(w^*)$,

(b) $\varepsilon_i w(x)\geqslant 0$, all $x\in A_i$, $i=1, ..., k$.

LEMMA 10. *Let w and w^* be as above. For all $x\in \bar{A}_i\cap \bar{A}_j$, $i, j\in\{1, ..., k\}$; $i\neq j$, we have $w(x)=0$.*

The proof of Lemma 10 follows the proof of Lemma 5. The freedom in the choice of δ small and positive in the proof of Lemma 5 allows us to assume that $w^*-\delta w\in \tilde{V}$.

Proof of Proposition 7. Let w^* and w be as above and assume that $w\neq \alpha w^*$ for any $\alpha\geqslant 0$. We divide the proof of the proposition into two cases.

Case I. There exists an A_i, $i\in\{1, ..., k\}$, as above, for which $w\neq \alpha w^*$ on A_i for any $\alpha\geqslant 0$.

In this case (as in the proof of Lemma 5) there exist $x_1, x_2 \in A_i$ and $\beta > 0$ for which $(w^* - \beta w)(x_1) \cdot (w^* - \beta w)(x_2) < 0$. β may be perturbed slightly and the strict inequality maintained. As such we may assume that $w^* - \beta w \in \tilde{V}$. If $w^* - \beta w$ vanishes identically on some A_j, $j \in \{1, ..., k\}$, then we contradict Lemma 9 since then int $Z(w^* - \beta w) \supsetneqq$ int $Z(w^*)$. Otherwise we contradict Lemma 9 because $|K/Z(w^* - \beta w)| > |K/Z(w^*)|$.

Case II. On each A_i, $w = \alpha_i w^*$, $\alpha_i \geqslant 0$, $i = 1, ..., k$.
If all the α_i are equal, then $w = \alpha w^*$. Thus assume that not all the α_i are equal. Let $\alpha_i, \alpha_j \geqslant 0$, $\alpha_i \neq \alpha_j$. Then either $\alpha_i w^* - w \in \tilde{V}$ or $\alpha_j w^* - w \in \tilde{V}$. But int $Z(\alpha w^* - w) \supsetneqq$ int $Z(w^*)$ for $\alpha = \alpha_i, \alpha_j$. This contradicts Lemma 9. ▌

We are now in a position to prove Theorem E.

Proof of Theorem E. Assume $|K/Z(U_n)| \geqslant 2$. From Theorem D we have $|K/Z(U_n)| \leqslant n$. Thus $K/Z(U_n) = \bigcup_{i=1}^k A_i$, where $2 \leqslant k \leqslant n$, and the A_i are the relatively open, connected components of $K/Z(U_n)$. Set $B = A_1$, $C = \bigcup_{i=2}^k A_i$. Let

$$U_n|_B = V_m, \qquad \dim V_m = m \leqslant n$$
$$U_n|_C = W_r, \qquad \dim W_r = r \leqslant n.$$

Now, $m, r \geqslant 1$ and $m + r \geqslant n$. Our aim is to prove that $m + r = n$. This is equivalent to the claim of Theorem E. Assume therefore that $m + r > n$, and set $l = m + r - n > 0$. We shall contradict Property A.

Let $\phi'\colon U_n \to V_m$ and $\phi''\colon U_n \to W_r$ be the restriction maps. Clearly ϕ' and ϕ'' are onto hence dim ker $\phi' = n - m = r - l$ and dim ker $\phi'' = n - r = m - l$. Also ker $\phi' \cap$ ker $\phi'' = \{0\}$ so we can choose a basis $u_1, ..., u_l, v_{l+1}, ..., v_m, w_{l+1}, ..., w_r$ for U_n such that $v_{l+1}, ..., v_m$ span ker ϕ'' and $w_{l+1}, ..., w_r$ span ker ϕ'.

For $u \in U_n$, set $u' = \phi'(u) = u|_B \in V_m$ and $u'' = u|_C \in W_r$. Then

$$V_m = \text{span}\{u_1', ..., u_l', v_{l+1}', ..., v_m'\}$$
$$W_r = \text{span}\{u_1'', ..., u_l'', w_{l+1}'', ..., w_r''\}.$$

The conditions of Proposition 7 hold on $\bar{B}$. There therefore exists a function $v^* \in U_n$ of the form

$$v^* = u_1 + \sum_{i=2}^{l} a_i^* u_i + \sum_{i=l+1}^{m} b_i^* v_i$$

such that if $v \in U_n$ satisfies

$$\begin{aligned} &\text{(i)} \quad v(x) = 0 \text{ a.e. on } Z_B(v^*) \\ &\text{(ii)} \quad v(x)(\operatorname{sgn} v^*(x)) = |v(x)| \text{ for all } x \in \bar{B}/Z_B(v^*) \end{aligned} \tag{1}$$

then there exists an $\alpha \geqslant 0$ such that $(v-\alpha v^*)(x)=0$ on $\bar{B}$. In other words, $v-\alpha v^* \in \ker \phi'$.

Set $u^* = u_1 + \sum_{i=2}^{l} a_i^* u_i$, and $\tilde{W} = \operatorname{span}\{u^*, w_{l+1}, ..., w_r\}$. We now apply Proposition 7 to $\bar{C}$ and $\tilde{W}$. There exists a function $w^* \in \tilde{W}$ of the form

$$w^* = u^* + \sum_{i=l+1}^{r} c_i^* w_i$$

such that if $w \in \tilde{W}$ satisfies

$$\begin{aligned} &\text{(i)} \quad w(x)=0 \text{ a.e. on } Z_{\bar{C}}(w^*) \\ &\text{(ii)} \quad w(x)(\operatorname{sgn} w^*(x)) = |w(x)| \text{ for all } x \in \bar{C}/Z_{\bar{C}}(w^*) \end{aligned} \tag{2}$$

then there exists a $\beta \geqslant 0$ such that $w-\beta w^*=0$ on $\bar{C}$, i.e., $w-\beta w^* \in \ker \phi''$.

Set $\tilde{u} = u^* + \sum_{i=l+1}^{m} b_i^* v_i + \sum_{i=l+1}^{r} c_i^* w_i$. Then $\tilde{u}|_B = v^*|_B$ and $\tilde{u}|_C = w^*|_C$. As a consequence of Property A and the construction of B and C, there exists a function $u \in U_n/\{0\}$ for which

$$\begin{aligned} &\text{(i)} \quad u(x)=0 \text{ a.e. on } Z(\tilde{u}) \\ &\text{(ii)} \quad u(x)(\operatorname{sgn} \tilde{u}(x)) \geqslant 0 \text{ on } B \\ &\text{(iii)} \quad u(x)(\operatorname{sgn} \tilde{u}(x)) \leqslant 0 \text{ on } C. \end{aligned} \tag{3}$$

From (1) and (3) it follows (since $\tilde{u}|_B = v^*|_B$) that $u-\alpha\tilde{u} \in \ker \phi'$ for some $\alpha \geqslant 0$. Since $\tilde{u} \in \tilde{W}$ and $\ker \phi' < \tilde{W}$ we have $u \in \tilde{W}$. Then by (2) and (3), $u-\beta\tilde{u} \in \ker \phi''$ for some $\beta \leqslant 0$. Thus $(\alpha-\beta)\tilde{u} \in \ker \phi' + \ker \phi''$. Since $u \neq 0$ and $\ker \phi' \cap \ker \phi'' = 0$ we have $\alpha \neq 0$ and/or $\beta \neq 0$, hence $\alpha - \beta > 0$. It follows that $\tilde{u} \in \ker \phi' + \ker \phi''$, which contradicts our construction of $\tilde{u}$. ∎

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