Perturbation of spectra of operator matrices and local spectral theory

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Abstract

For $A \in \mathcal{L}(X), B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$ we denote by $M_C$ the operator defined on $X \oplus Y$ by $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$. In this article, we study defect set $D^\Sigma = (\Sigma(A) \cup \Sigma(B)) \setminus \Sigma(M_C)$ for different spectra including the spectrum, the essential spectrum, Weyl spectrum and the approximate point spectrum. We then apply the obtained results to the stability of such spectra ($D^\Sigma = \emptyset$) and the classes of operators $C$ for which stability holds of $M_C$ using local spectral theory.

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1. Introduction

Let $X$ and $Y$ be Banach spaces and let $\mathcal{L}(X, Y)$ denote the space of all bounded linear operators from $X$ to $Y$. When $Y = X$ we write $\mathcal{L}(X, X) = \mathcal{L}(X)$. For $T \in \mathcal{L}(X)$, let $N(T), R(T), \sigma(T), \sigma_r(T), \sigma_l(T), \sigma_{ap}(T), \sigma_{com}(T)$ and $\sigma_s(T)$ denote the null space, the range, the spectrum, the right spectrum, the left spectrum, the approximate point spectrum, the compression spectrum and the surjectivity spectrum of $T$, respectively.

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For $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$ we denote by $M_C$ the operator defined on $X \oplus Y$ by
\[
\begin{bmatrix}
A & C \\
0 & B
\end{bmatrix}.
\]
In the case where $X$ and $Y$ are finite-dimensional, we have obviously $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ and this latter fact is not generally true in the infinite-dimensional case.

The stability of the spectrum and the description of different spectra of $M_C$ have motivated several mathematicians. For instance, if $H$ and $K$ are Hilbert spaces, Du and Pan [8] give a description of $\bigcap_{C \in \mathcal{L}(K,H)} \sigma(M_C)$ by showing that
\[
\bigcap_{C \in \mathcal{L}(K,H)} \sigma(M_C) = \sigma_{ap}(A) \cup \sigma_s(B) \cup \{\lambda \in \mathbb{C}: \alpha(B - \lambda) \neq \beta(A - \lambda)\},
\]
(1)
where $\alpha(B)$ and $\beta(A)$ are the nullity of $B$ and the deficiency of $A$, respectively. Han et al. [11] extended Eq. (1) to the more general setting of Banach spaces. Since clearly, for every $C \in \mathcal{L}(K,H)$ we have $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$, the following auxiliary questions arise naturally from Eq. (1).

**Question 1.** Under which conditions on $A$ and $B$ does $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ (2) for arbitrary $C$?

**Question 2.** Given $A$ and $B$. For which operators $C \in \mathcal{L}(Y, X)$ does equality (2) hold?

The same questions can be stated using different spectra (essential spectrum, approximate point spectrum, Weyl spectrum, ...).

Considerable attention has been devoted the questions above, see, for instance, [3,6–9,11–13, 16].

Let $\Sigma(\cdot)$ be any of the spectra above such that $\Sigma(M_C) \subset \Sigma(A) \cup \Sigma(B)$. Our aim is describe the defect set
\[
D^\Sigma = (\Sigma(A) \cup \Sigma(B)) \setminus \Sigma(M_C).
\]
One observes easily that $D^\Sigma = \emptyset$ precisely when $\Sigma(M_C) = \Sigma(A) \cup \Sigma(B)$.

In this paper, we investigate the stability of different spectra of $M_C$ using tools from local spectral theory. If $\Sigma$ is a specified spectrum, we rely $\Sigma(M_C)$, $\Sigma(A)$ and $\Sigma(B)$.

We recall in Section 2 some basic definitions and known results from local spectral theory.

We devote Section 3 to the defect set for different spectra. It is in particular shown that $D^{\Sigma} \subset \mathcal{S}(A^*) \cap \mathcal{S}(B)$ for $\Sigma \in \{\sigma, \sigma_e\}$, that $D^{\Sigma_{W}} \subset (\mathcal{S}(A) \cap \mathcal{S}(B^*)) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(B))$, where $\sigma_{W}$ denotes the Weyl spectrum. We apply then our results to obtain the stability of $\sigma(M_C)$, $\sigma_{E}(M_C)$ for some classes of operators $A$ and $B$ including quasitriangular operators. Some theorems from [6,7] are extended.

In Section 4, we consider the set $H_{3}(A, B, \Sigma)$ of all $C$ such that $\Sigma(M_C) = \Sigma(A) \cup \Sigma(B)$. We give sufficient conditions to get elements in $H_{3}(A, B, \Sigma)$ for $\Sigma \in \{\sigma, \sigma_e, \sigma_{ap}, \sigma_{W}\}$ and some other spectra.
2. Preliminaries from local spectral theory

Let \( D(\lambda, r) \) be the open disc centered at \( \lambda \in \mathbb{C} \) and with radius \( r > 0 \), the corresponding closed disc is denoted by \( \overline{D}(\lambda, r) \). A bounded linear operator \( T \in \mathcal{L}(X) \) is said to have the single valued extension property (SVEP, for short) at \( \lambda \in \mathbb{C} \) if there exists \( r > 0 \) such that for every open subset \( U \subset D(\lambda, r) \), the constant function \( f \equiv 0 \) is the only analytic solution of the equation

\[
(T - \mu)f(\mu) = 0 \quad \forall \mu \in U.
\]

We use \( S(T) \) to denote the open set where \( T \) fails to have the SVEP and we say that \( T \) has the SVEP if \( S(T) \) is the empty set, \([10]\).

The operator \( T \) is said to satisfy the Bishop’s property \((\beta)\) at \( \lambda \in \mathbb{C} \) if there exists \( r > 0 \) such that for every open subset \( U \subset D(\lambda, r) \) and for any sequence \((f_n)_n\) of analytic \( X \)-valued functions on \( U \) with \( (T - \mu)f_n(\mu) \to 0 \) as \( n \to \infty \) uniformly on compact subsets of \( U \), we have \( f_n(\mu) \to 0 \) as \( n \to \infty \) uniformly on compact subsets of \( U \). Let \( \sigma_\beta(T) \) be the set of all points where \( T \) does not have Bishop’s property \((\beta)\). Then \( T \) is said to satisfy the Bishop’s property \((\beta)\), precisely when \( \sigma_\beta(T) = \emptyset \) \([4]\).

The operator \( T \) is said to have the decomposition property \((\delta)\) at \( \lambda \in \mathbb{C} \) if there exists an open neighborhood \( U \) of \( \lambda \) such that for every finite open cover \( \{U_1, \ldots, U_n\} \) of \( \mathbb{C} \), with \( \sigma(T) \setminus U \subseteq U_1 \), we have

\[
\mathcal{X}_T(\overline{U}_1) + \cdots + \mathcal{X}_T(\overline{U}_n) = X.
\]

Where \( \mathcal{X}_T(F) \) is the vector space of all elements \( x \in X \) for which there exists an analytic function \( f : \mathbb{C} \setminus F \to X \) such that \( (T - \mu)f(\mu) = x \), for \( \mu \in \mathbb{C} \setminus F \).

Similarly we define the \( \delta \)-spectrum to be

\[
\sigma_\delta(T) = \{ \lambda \in \mathbb{C} : T \text{ does not satisfy decomposition property } (\delta) \text{ at } \lambda \}
\]

and we say that \( T \) has the decomposition property \((\delta)\) if \( \sigma_\delta(T) = \emptyset \).

Properties \((\beta)\) and \((\delta)\) are known to be dual to each other in the sense that \( T \) has \((\delta)\) if and only if \( T^* \) satisfies \((\beta)\). It is also known that \((\beta)\) characterizes operators with decomposable extensions \([2]\). Property \((\beta)\) is hence conserved by restrictions while \((\delta)\) is transferred to quotient operators. See also \([15]\) for more details.

Recall that \( T \in \mathcal{L}(X) \) is said to be decomposable provided that for every finite open cover \( \{U_1, \ldots, U_n\} \) of \( \mathbb{C} \), there exists \( X_1, \ldots, X_n \) closed \( T \)-invariant subspaces of \( X \) such that

\[
\sigma(T|X_i) \subseteq U_i \quad \text{for } i = 1, \ldots, n, \text{ and } X_1 + \cdots + X_n = X.
\]

The class of decomposable operators contains all normal operators and more generally all spectral operators. Operators with totally disconnected spectrum are decomposable by the Riesz functional calculus. In particular, compact and algebraic operators are decomposable.

A local version of decomposability can be defined as follows. An operator \( T \) is decomposable at \( \lambda \) if there exists an open neighborhood \( U \) of \( \lambda \) such that for every finite open cover \( \{U_1, \ldots, U_n\} \) of \( \mathbb{C} \), with \( \sigma(T) \setminus U \subseteq U_1 \), there exists \( X_1, \ldots, X_n \) closed \( T \)-invariant subspaces of \( X \) such that Eq. (4) holds.

For \( T \in \mathcal{L}(X) \), we shall denote

\[
\sigma_{\text{dec}}(T) = \{ \lambda \in \mathbb{C} : T \text{ is not decomposable at } \lambda \}.
\]

Note that \( \sigma_\beta(T), \sigma_\delta(T) \) and \( \sigma_{\text{dec}}(T) \) are closed subsets of \( \sigma(T) \) and clearly

\[
S(T) \subseteq \sigma_\beta(T) \subseteq \sigma_{\text{dec}}(T) = \sigma_\beta(T) \cup \sigma_\delta(T) \quad \text{and} \quad \sigma_\beta(T) = \sigma_\delta(T^*).
\]
3. Perturbation of spectra of $M_C$

3.1. The spectrum and the essential spectrum

It is well known that when $X$ and $Y$ are finite-dimensional, then equality (2) holds for every $C \in \mathcal{L}(Y, X)$.

Equality (2) was proved in infinite-dimensional case when $B$ has the SVEP, [12, Theorem 2.1]. The same result can be deduced by duality when $A^*$ has the SVEP. See also [7, Theorem 2.3].

When the equality fails to be true, it is then natural to describe the defect set $D^\sigma = \sigma(A) \cup \sigma(B) \setminus \sigma(M_C)$.

In [9, Theorem 2.5], the spectrum of $M_C$ was relied to the spectrum of $A$ and $B$ and to the sets in which $A^*$ and $B$ do not have the SVEP.

**Theorem 3.1.** Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ be given. Then

$$\sigma(M_C) \cup (S(A^*) \cap S(B)) = \sigma(A) \cup \sigma(B)$$

for every $C \in \mathcal{L}(Y, X)$.

From the following expression

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & C \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$$

it follows that if $M_C$ is invertible, then $A$ is left invertible and $B$ is right invertible. Hence in the previous theorem more is given

$$\sigma(M_C) \cup (\sigma(A) \setminus \sigma_{ap}(A)) \cap (\sigma(B) \setminus \sigma_s(B)) = \sigma(A) \cup \sigma(B).$$

The latter result can be reformulated as

$$D^\sigma \subset (\sigma(A) \setminus \sigma_{ap}(A)) \cap (\sigma(B) \setminus \sigma_s(B)) \subset S(A^*) \cap S(B).$$

In particular, when $A^*$ or $B$ has the SVEP, equality (2) can be deduced from Eq. (8).

**Remark 3.1.** We mention that the inclusion $D^\sigma \subset S(A^*) \cap S(B)$ may be strict. Indeed, let $M_C$ be given by

$$A = S \oplus 0 \in \mathcal{L}(H^2 \oplus H^2), \quad B = S^* \oplus 0 \in \mathcal{L}(H^2 \oplus H^2) \quad \text{and} \quad C = (I - SS^*) \oplus 0.$$

Here and through all the paper $S$ is the usual shift operator on the Hardy space $H^2$. Then

$$\sigma(M_C) = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \cup \{ 0 \},$$

and

$$D^\sigma = \{ \lambda \in \mathbb{C} : 0 < |\lambda| < 1 \} \neq S(A^*) \cap S(B) = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \}.$$

We also derive from Eq. (8) the following result.

**Corollary 3.1.** [3, Proposition 10] Let $A$, $B$ and $C$ be given. Assume that there exists $\lambda \in (\sigma(A) \cup \sigma(B)) \setminus \sigma(M_C)$, then $\sigma_l(A) \cap \sigma_l(B) \setminus (\sigma_l(A) \cup \sigma_l(B))$ contains a neighborhood of $\lambda$. 

**Proof.** Since $S(A^*) \cap S(B)$ is an open set contained in $\sigma_s(A) \cap \sigma_{ap}(B) \subseteq \sigma_l(A) \cap \sigma_r(B)$, we get $(S(A^*) \cap S(B)) \setminus \sigma_l(A) \cup \sigma_r(B)$ is an open set. The corollary follows then from Eq. (8). □

We deduce:

**Corollary 3.2.** If $(\sigma_l(A) \cap \sigma_r(B)) \setminus (\sigma_l(A) \cup \sigma_r(B))$ has no interior point, then Eq. (2) holds for every $C$.

Let $T \in \mathcal{L}(X)$ be with closed range and let $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim X / R(T)$ be the nullity and the deficiency, respectively. We recall that $T \in \mathcal{L}(X)$ is said to be left Fredholm (respectively right Fredholm) if $R(T)$ is a closed complemented subspace such that $\alpha(T) < \infty$ (respectively $N(T)$ is complemented such that $\beta(T) < \infty$). A semi-Fredholm operator is an operator that is either left Fredholm or right Fredholm. The index of a semi-Fredholm operator $T$ is defined as $\text{ind}(T) = \alpha(T) - \beta(T)$. We say that $T$ is Fredholm if it is both left Fredholm and right Fredholm. The left essential spectrum $\sigma_{le}(T)$, the right essential spectrum $\sigma_{re}(T)$ and the essential spectrum $\sigma_e(T)$ are defined by

$$
\sigma_{le}(T) = \{ \lambda \in \mathbb{C}: T - \lambda \text{ is not left Fredholm} \},
$$

$$
\sigma_{re}(T) = \{ \lambda \in \mathbb{C}: T - \lambda \text{ is not right Fredholm} \},
$$

$$
\sigma_e(T) = \{ \lambda \in \mathbb{C}: T - \lambda \text{ is not Fredholm} \}.
$$

From Eq. (6), it is clear that

$$
\sigma_l(A) \cup \sigma_r(B) \subseteq \sigma_e(M_C) \subseteq \sigma_e(A) \cup \sigma_e(B) \quad (9)
$$

and the previous inclusion turns out to be an equality for every $C \in \mathcal{L}(Y, X)$ when $A^*$ or $B$ has the SVEP, see [7, Theorem 2.3].

In the next theorem we extend the previous result. From Eq. (9), it is not hard to see that

$$
D^{\sigma_e} = (\sigma_e(A) \cup \sigma_e(B)) \setminus \sigma_e(M_C) \subseteq \sigma_e(A) \cap \sigma_e(B). \quad (10)
$$

In fact, more can be said:

**Theorem 3.2.** For $A \in \mathcal{L}(X), B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$ we have:

$$
\sigma_e(M_C) \cup (S(A^*) \cap S(B)) = \sigma_e(A) \cup \sigma_e(B) \cup (S(A^*) \cap S(B)). \quad (11)
$$

In particular,

$$
D^{\sigma_e} \subseteq S(A^*) \cap S(B).
$$

**Proof.** Suppose $\lambda_0 \in (\sigma_e(A) \cup \sigma_e(B)) \setminus \sigma_e(M_C)$, then $M_C - \lambda_0$ is Fredholm. It follows that $A - \lambda_0$ is left Fredholm and $B - \lambda_0$ is right Fredholm.

We claim that $\lambda_0 \in S(A^*) \cap S(B)$. Indeed, for the sake of contradiction assume that $\lambda_0 \notin S(A^*) \cap S(B)$.

**Case 1.** If $\lambda_0 \notin S(A^*)$, we derive from [10, Corollary 12], that ind$(A - \lambda_0) \geq 0$. Since $A - \lambda_0$ is left Fredholm then $\alpha(A - \lambda_0) < \infty$. Then it follows from ind$(A - \lambda_0) \geq 0$ and $\alpha(A - \lambda_0) < \infty$ that $\beta(A - \lambda_0) < \infty$. Thus $A - \lambda_0$ is Fredholm and so $B - \lambda_0$ is.

**Case 2.** If $\lambda_0 \notin S(B)$, the proof follows similarly. □
The inclusion in Theorem 3.2 may be strict in general. This can be seen with $M_C$ given by $A = S$, $B = S^*$ and $C = I - SS^*$.

Equation (10) and Theorem 3.2 lead to the next corollary which refines Proposition 3.2 in [6].

**Corollary 3.3.**

\[ D^e \subset S(A^*) \cap S(B) \cap \sigma_e(A) \cap \sigma_e(B). \]

It follows that:

**Corollary 3.4.** If $S(A^*) \cap S(B) \cap \sigma_e(A) \cap \sigma_e(B) = \emptyset$, then for every $C \in \mathcal{L}(Y, X)$

\[ \sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B). \]  \hspace{1cm} (12)

**Corollary 3.5.** If $\sigma_p(A^*)$ or $\sigma_p(B)$ has no interior points, then for every $C \in \mathcal{L}(Y, X)$

\[ \sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B). \]

Using the Calkin homomorphism $\pi$ an immediate adaptation of the proof of Corollary 3.1 is:

**Corollary 3.6.**

\[ S(\pi(A^*)) \cap S(\pi(B)) \subseteq (\sigma_{le}(A) \cap \sigma_{re}(B)) \setminus (\sigma_{re}(A) \cup \sigma_{le}(B)). \]  \hspace{1cm} (13)

**In particular,**

(i) If $\lambda \in D^e$ then $(\sigma_{re}(A) \cap \sigma_{le}(B)) \setminus (\sigma_{re}(A) \cup \sigma_{le}(B))$ contains a neighborhood of $\lambda$.

(ii) If $(\sigma_{re}(A) \cap \sigma_{le}(B)) \setminus (\sigma_{re}(A) \cup \sigma_{le}(B))$ has no interior point, then Eq. (12) holds for every $C$.

The following classes were considered in [6]:

\[ S_+(X) = \{ T \in \mathcal{L}(X) : \alpha(T - \mu) \geq \beta(T - \mu) \text{ if at least one of these quantities is finite} \}, \]

\[ S_-(X) = \{ T \in \mathcal{L}(X) : \alpha(T - \mu) \leq \beta(T - \mu) \text{ if at least one of these quantities is finite} \}. \]

Applying Theorem 3.2, we retrieve the following result from [6].

**Corollary 3.7.** [6, Proposition 3.3] If $A \in S_+(X)$ or $B \in S_-(Y)$, then the equality

\[ \sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B) \]

holds for every $C \in \mathcal{L}(Y, X)$.

**Proof.** Let $\lambda \in \sigma_e(M_C) \setminus \sigma_e(A) \cup \sigma_e(B)$, then $\alpha(A - \lambda) < \infty$. Now if $A \in S_+(X)$ it comes from Theorem 4.4 that $A^*$ does not have the SVEP at $\lambda$. Then using [1, Theorem 2.8] we derive that $\beta(A - \lambda) = +\infty$, which is impossible.

The case $B \in S_-(Y)$ goes similarly. \qed

An operator $A \in \mathcal{L}(H)$ is said to be *quasitriangular* if there exists a sequence $\{P_n\}_{n \geq 0}$ of finite rank projections in $\mathcal{L}(H)$ that converges strongly to the identity and such that $\| P_n A P_n - A P_n \| \to 0$. We deduce the following result in the line of Corollary 4 in [13].
Corollary 3.8. If $A$ is a quasitriangular operator (e.g., $A$ is either compact or cohyponormal) then for every $B$ and $C$, we have

$$\sigma_c(M_C) = \sigma_c(A) \cup \sigma_c(B).$$

Proof. We claim that if $A$ is quasitriangular, then $(\sigma_c(A) \setminus \sigma_l(A)) = \emptyset$. Indeed, Let $\lambda \in (\sigma_c(A) \setminus \sigma_l(A))$, since $A - \lambda I$ is left Fredholm but not Fredholm, we have $\beta(A - \lambda) = \infty$ and $\alpha(A - \lambda) < \infty$. This provides a pseudohole with negative index, which contradicts [18, Theorem 1.31]. Now the corollary derives from (13).

In [22], the authors introduced the notion of the weak decomposition property ($\delta_w$) and gave a new class of operators which extends the class of operators with the decomposition property ($\delta$). For $T \in \mathcal{L}(X)$, we say that $T$ has the weak decomposition property ($\delta_w$), if for every finite open cover $\{U_1, \ldots, U_n\}$ of $\mathbb{C}$, we have

$$X_T(U_1) + \cdots + X_T(U_n)$$

is dense in $X$.

Corollary 3.9. Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ be given. Suppose that $A$ or $B^*$ has the weak decomposition property ($\delta_w$), then

$$\sigma(M_C) = \sigma(A) \cup \sigma(B), \quad \sigma_c(M_C) = \sigma_c(A) \cup \sigma_c(B)$$

for every $C \in \mathcal{L}(Y, X)$.

Proof. If $A$ has the weak decomposition property ($\delta_w$), then it follows from [22, Corollary 2.2] that $S(A^*) = \emptyset$, the result comes now from Theorems 3.1 and 3.2.

3.2. The Weyl spectrum

A bounded linear operator $T$ is said to be Weyl if it is Fredholm of index zero. The Weyl spectrum is defined as

$$\sigma_\omega(T) = \{\lambda \in \mathbb{C}: T - \lambda \text{ is not Fredholm of index zero}\}.$$

It is clear from Eq. (6) that

$$\sigma_\omega(M_C) \subseteq \sigma_\omega(A) \cup \sigma_\omega(B).$$

Theorem 3.3. For $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$ we have

$$D_{\sigma_\omega} \subseteq \left[ S(A) \cap S(B^*) \right] \cup \left[ S(A^*) \cap S(B) \right].$$

Proof. Let $\lambda \in D_{\sigma_\omega}$, then $M_C - \lambda$ is Fredholm of index zero. Hence $A - \lambda$ is left Fredholm and $B - \lambda$ is right Fredholm. If $\lambda \notin S(A^*) \cap S(B)$, then it follows from the proof of Theorem 3.2 that $A - \lambda$ and $B - \lambda$ are Fredholm. Thus $0 = \text{ind}(M_C - \lambda) = \text{ind}(A - \lambda) + \text{ind}(B - \lambda)$ (the second equality follows from (6)). If $\text{ind}(A - \lambda) > 0$ then $\text{ind}(B - \lambda) < 0$. Hence from [10, Corollary 12] we obtain $\lambda \in S(A) \cap S(B^*)$. Now if $\text{ind}(A - \lambda) < 0$ then similarly we show that $\lambda \in S(A^*) \cap S(B)$ which leads a contradiction.
Corollary 3.10. If $A$ and $B$ (or $A^*$ and $B^*$) have the SVEP, then

$$\sigma_\omega(M_C) = \sigma_\omega(A) \cup \sigma_\omega(B).$$

The following corollary applies, in particular, if either $A$ or $B$ is compact or normal and more generally a decomposable operator.

Corollary 3.11. If $A$ and $A^*$ (or $B$ and $B^*$) have the SVEP, then

$$\sigma_\omega(M_C) = \sigma_\omega(A) \cup \sigma_\omega(B).$$

Recall that Weyl’s theorem holds for an operator $T \in \mathcal{L}(X)$ if

$$\sigma(T) \setminus \sigma_\omega(T) = \pi_{00}(T),$$

where $\pi_{00}(T)$ consists of all isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity. And $T$ is called isolid (finite-isolid) if every isolated point of $\sigma(T)$ is an eigenvalues of $T$ (if every isolated point of $\sigma(T)$ is in $\pi_{00}(T)$). It is well known that every hyponormal operator is isolid [21, Theorem 2].

Note that Weyl’s theorem holds for $A$ and $B$ does not imply, in general, Weyl’s theorem holds for $[A \ 0 \ 0 \ B]$, see [16]. Corollary 3.10 together with [16, Lemma 10] yields to the following.

Corollary 3.12. Suppose that $A$ and $B$ are isolid and have the SVEP. If Weyl’s theorem holds for $A$ and $B$ then Weyl’s theorem holds for $[A \ 0 \ 0 \ B]$.

Note that in the last corollary and in the light of Corollary 3.10 the conditions $A$ and $B$ have the SVEP may be changed to $A$ and $B$, $A$ and $A^*$ or $B$ and $B^*$ have the SVEP.

Theorem 3.4. Let $A$ and $B$ have the SVEP. Suppose that $A$ is isolid and $B$ is finite-isolid. Then if Weyl’s theorem holds for $A$ and $B$ then Weyl’s theorem holds for $M_C$ for every $C$.

Proof. It follows from Corollary 3.12 that Weyl’s theorem holds for $[A \ 0 \ 0 \ B]$. Since $B$ has the SVEP and is finite-isolid, then we have from [5, Theorem 2.2] that $\mathcal{V}_B(\{\lambda\})$ is finite-dimensional for each isolated point $\lambda$ of $\sigma(B)$. Now we can deduce the result from [7, Theorem 2.5].

It also may happen that $M_C$ have Weyl’s theorem while $[A \ 0 \ 0 \ B]$ does not satisfy it. Indeed, for $A = S$, $B = S^*$ and $C = I - SS^*$, then $M_C$ is unitary without eigenvalues and hence satisfies Weyl’s theorem, while $[A \ 0 \ 0 \ B]$ does not satisfy it.

Proposition 3.1. Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and let $C \in \mathcal{L}(Y, X)$ be such that $AC = CB$. Then Weyl’s theorem holds for $[A \ 0 \ 0 \ B]$ if and only if it holds for $M_C$.

Proof. We have

$$M_C = \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & C \\
0 & B & 0 \end{bmatrix}.$$

Since $[0 \ C \ 0 \ 0]$ is nilpotent and commute with $[A \ 0 \ 0 \ B]$, then the result follows at once from [17, Theorem 3].
3.3. The approximate point spectrum

The inclusion
\[ \sigma_{ap}(A) \subseteq \sigma_{ap}(MC) \subseteq \sigma_{ap}(A) \cup \sigma_{ap}(B) \]  
(16)
is not hard to obtain and leads to the inclusion
\[ D_{\sigma_{ap}} \subseteq \sigma_{ap}(B). \]  
(17)
In the next theorem, more is obtained:

**Theorem 3.5.** Let \( A \in \mathcal{L}(X) \), \( B \in \mathcal{L}(Y) \) and \( C \in \mathcal{L}(Y, X) \) be given, then
\[ D_{\sigma_{ap}} \subseteq S(A^*) \cap \sigma_{ap}(B). \]  
(18)

**Proof.** Suppose that \( 0 \in D_{\sigma_{ap}} \), hence by Eq. (6) \( A \) is left invertible. We claim that \( A \) is not invertible. Otherwise, using again Eq. (6) will give \( B \) is also left invertible and this provides a contradiction. To conclude now, it suffices to see that \( A^* \) is onto noninvertible, and hence by [10, Theorem 2], \( A^* \) does not have the SVEP at zero. Finally \( 0 \in S(A^*) \) and the proof is complete. \( \square \)

**Corollary 3.13.** If \( A^* \) has the SVEP, then for every \( B \) and \( C \), we have
\[ \sigma_{ap}(MC) = \sigma_{ap}(A) \cup \sigma_{ap}(B). \]

Arguing as in Corollary 3.8 we retrieve a result from [13].

**Corollary 3.14.** [13, Corollary 4] If \( A \) is a quasitriangular operator, then for every \( B \) and \( C \), we have
\[ \sigma_{ap}(MC) = \sigma_{ap}(A) \cup \sigma_{ap}(B). \]

By duality, we have:

**Proposition 3.2.**
\[ D_{\sigma_{ap}} \subseteq \sigma_s(A) \cap S(B). \]

In particular, if \( B \) has the SVEP or if \( B^* \) is quasitriangular, then
\[ \sigma_s(M_C) = \sigma_s(A) \cup \sigma_s(B). \]

Inclusion (18) may be strict as it can be observed for \( M_C \) given by
\[ A = S, \quad B = S^* \quad \text{and} \quad C = 0. \]

Here
\[ \emptyset = D_{\sigma_{ap}} \subset \sigma_{ap}(B) \cap S(A^*) = D(0, 1). \]
3.4. $\beta$-spectrum

**Proposition 3.3.** For $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$ we have

$$\sigma_\beta(A) \subseteq \sigma_\beta(M_C) \subseteq \sigma_\beta(A) \cup \sigma_\beta(B).$$

**Proof.** The first inclusion is trivial since $M_C|X \oplus \{0\} = A$ and the Bishop’s property ($\beta$) is preserved by restriction. For the second inclusion, the proof is the same as that of [12, Proposition 3.3]. □

Proposition 3.3 implies that $\mathcal{D}^{\sigma_\beta} \subset \sigma_\beta(B)$. In fact more can be given:

**Theorem 3.6.** For $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$ we have

$$\sigma_\beta(A) \cup \sigma_\beta(B) \cup \sigma_s(A) = \sigma_\beta(M_C) \cup \sigma_s(A).$$

(19)

Hence

$$\mathcal{D}^{\sigma_\beta} \subseteq \sigma_s(A) \cap \sigma_\beta(B).$$

(20)

**Proof.** It suffices to show that

$$\sigma_\beta(B) \subseteq \sigma_\beta(M_C) \cup \sigma_s(A).$$

Let $\lambda \notin (\sigma_\beta(M_C) \cup \sigma_s(A))$. Then there exists $r > 0$ such that for every open subset $U \subset D(\lambda, r)$ and for any sequence $(h_n)_n$ of analytic $(X \oplus Y)$-valued functions on $U$ with $(M_C - \mu)h_n(\mu) \to 0$ as $n \to \infty$ uniformly on compact subsets of $U$, we have $h_n(\mu) \to 0$ as $n \to \infty$ uniformly on compact subsets of $U$.

We can take $r$ small enough such that $D(\lambda, r) \cap \sigma_s(A) = \emptyset$. Let $(g_n)_{n \geq 0}$ be a sequence of analytic $Y$-valued functions on $U \subset D(\lambda, r)$ such that $(B - \mu)g_n(\mu) \to 0$ as $n \to \infty$ uniformly on compact subsets of $U$.

From Leiterer’s theorem [14], it follows that there exists a sequence $(f_n)_{n \geq 0}$ of analytic $X$-valued functions such that

$$(A - \mu)f_n(\mu) = -Cg_n(\mu), \quad \text{for all } \mu \in U.$$

Therefore, for $h_n = f_n \oplus g_n$ we have

$$(M_C - \mu)(h_n(\mu)) = 0 \oplus g_n(\mu) \to 0 \quad \text{as } n \to \infty$$

uniformly on compact subsets of $U$. It follows from $M_C$ satisfies Bishop’s property ($\beta$) at $\lambda$ that $h_n(\mu)$, and so $g_n(\mu) \to 0$ as $n \to \infty$ uniformly on compact subsets of $U$. This ends the proof. □

**Corollary 3.15.** If $A - \lambda$ is surjective, then $M_C$ satisfies Bishop’s property ($\beta$) at $\lambda$ if and only if $A$ and $B$ satisfy Bishop’s property ($\beta$) at $\lambda$.

**Corollary 3.16.** If $\text{int}(\sigma_s(A)) = \emptyset$, then $M_C$ satisfies Bishop’s property ($\beta$) if and only if $A$ and $B$ satisfy Bishop’s property ($\beta$).

**Example 3.1.** One might expect that to take a smaller subset than $\sigma_s(A)$ in equality (20). But this is not true in general. To see this, let $M_C$ be given by $A = S, B = A^*$ and $C = I - AA^*$, then

$$\sigma_s(A) = \overline{D}(0, 1), \quad \sigma_\beta(A) = \emptyset \quad \text{and} \quad \sigma_\beta(B) = \overline{D}(0, 1).$$

Since $M_C$ is a unitary operator, we get $\sigma_\beta(M_C) = \emptyset$. However, equality holds in (20).
As Bishop’s property ($\beta$) and the decomposition property ($\delta$) are dual, we can deduce from the above results the following.

**Proposition 3.4.** For $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$ we have

$$\sigma_\delta(B) \subseteq \sigma_\delta(MC) \subseteq \sigma_\delta(A) \cup \sigma_\delta(B), \quad \mathcal{D}^\delta \subseteq \sigma_\delta(A) \cap \sigma_{ap}(B).$$

**Theorem 3.7.** For $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$ we have

$$\sigma_\delta(A) \cup \sigma_\delta(B) \cup \sigma_{ap}(B) = \sigma_\delta(MC) \cup \sigma_{ap}(B).$$

We give an immediate consequence of the above results.

**Theorem 3.8.** Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$ be given. Then

(i) $\sigma_\beta(A) \cup \sigma_\delta(B) \subseteq \sigma_{dec}(MC) \subseteq \sigma_{dec}(A) \cup \sigma_{dec}(B)$.

(ii) $\sigma_{dec}(A) \cup \sigma_{dec}(B) \cup (\sigma_{s}(A) \cup \sigma_{ap}(B)) = \sigma_{dec}(MC) \cup (\sigma_{s}(A) \cup \sigma_{ap}(B))$.

(iii) $\mathcal{D}^{\beta}_{\text{dec}} \subseteq \sigma_\delta(A) \cap \sigma_\beta(B)$.

The second inclusion in (i) can be strict. Indeed if we take Example 3.1, then $\sigma_{\text{dec}}(MC) = \emptyset$ and $\sigma_{\text{dec}}(A) = \sigma_{\text{dec}}(B) = \overline{D}(0, 1)$. We notice here that the inclusion in (iii) turns out be an equality.

A direct corollary is:

**Corollary 3.17.** Let $A$ satisfying (\(\delta\)) and $B$ with Bishop’s property (\(\beta\)), then for every $C$, the operator $MC$ is decomposable if and only if $A$ and $B$ are decomposable.

**Remark 3.2.** Using the same argument as in the case of Bishop’s property (\(\beta\)), similar results are valid for the SVEP. For $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$ we have

$$S(A) \cup S(B) \cup \sigma_s(A) = S(MC) \cup \sigma_s(A).$$

If $\text{int}(\sigma_s(A)) = \emptyset$, then

$$S(MC) = \emptyset \iff S(A) = S(B) = \emptyset.$$

**Example 3.2.** Let $MC$ be such that $A = S^*$ and $B = S$. Then,

$$S(MC) = S(A)$$

for every $C \in \mathcal{L}(X)$.

4. For which $C$ does the equality hold

We devote this section to provide operators $C$ such that

$$\Sigma(MC) = \Sigma(A) \cup \Sigma(B),$$

where $\Sigma$ runs over usual spectra.
Let
\[ H_1(A, B) = \{ C \in \mathcal{L}(Y, X) \text{ such that } C \in R(\delta_{A, B}) \}, \]
\[ H_2(A, B) = \{ C \in \mathcal{L}(Y, X) \text{ such that } M_C \text{ is similar to } M_0 \}, \]
\[ H_3(A, B, \Sigma) = \{ C \in \mathcal{L}(Y, X) \text{ such that } \Sigma(M_C) = \Sigma(A) \cup \Sigma(B) \}. \]

It is not difficult to see that
\[ H_1(A, B) \subseteq H_2(A, B) \subseteq H_3(A, B, \Sigma). \] (21)

Moreover, \( H_3(A, B, \Sigma) \) is norm closed, in the case where \( T \to \Sigma(T) \) is upper semi-continuous (for example, when \( \Sigma \in \{ \sigma, \sigma_e \} \)).

Several papers were devoted to the inclusions (21), see for example [3,9,19,20]. These inclusions can be strict, see for instance [9].

4.1. \( \Sigma = \sigma \)

Let \( L_A \) and \( R_B \) denote the left and the right multiplication by \( A \) and \( B \), respectively. The generalized derivation induced by \( A \) and \( B \) is defined by \( \delta_{A, B} = L_A - R_B \).

**Theorem 4.1.** For \( A \in \mathcal{L}(X) \), \( B \in \mathcal{L}(Y) \), \( \lambda \in \mathbb{C} \) and \( n \geq 1 \), we have:

(a) If \( C \in N(R^n_{B - \lambda}) \), then \( S(B) \subseteq S(M_C) \) and hence
\[ \sigma(M_C) = \sigma(A) \cup \sigma(B). \]

(b) If \( C \in N(L^n_{A - \lambda}) \), then \( S(A^*) \subseteq S(M_C^*) \) and hence
\[ \sigma(M_C) = \sigma(A) \cup \sigma(B). \]

**Proof.** Let \( C \in \mathcal{L}(X, Y) \) be such that \( C(B - \lambda)^n = 0 \). Let \( \mu_0 \in S(B) \) and \( f \) be a nonzero analytic \( Y \)-valued function on some neighborhood \( U \) of \( \mu_0 \) such that
\[ (B - \mu)f(\mu) = 0, \quad \text{for all } \mu \in U. \]

Using \( C(B - \lambda)^n = 0 \), we obtain for \( \mu \in U \),
\[ 0 = C(B - \lambda)^n f(\mu) = C \sum_{p=0}^{n} \binom{n}{p} (B - \mu)^p (\mu - \lambda)^{n-p} f(\mu) = (\mu - \lambda)^n C f(\mu). \]

Thus \( C f(\mu) = 0 \) for every \( \mu \in U \) and hence \( (M_C - \mu)(0 \oplus f(\mu)) = 0 \) for all \( \mu \in U \). Therefore \( \mu_0 \in S(M_C) \). Hence \( S(B) \subseteq S(M_C) \) and the result follows from (5).

Assertion (b) is obtained similarly. \( \square \)

**Corollary 4.1.** For \( A \in \mathcal{L}(X) \) and \( B \in \mathcal{L}(Y) \) if \( \sigma_p(R_B) \cup \sigma_p(L_A) \neq \emptyset \), then
\[ \sigma(M_C) = \sigma(A) \cup \sigma(B) \]
for some \( C \in \mathcal{L}(Y, X) \).

Let for \( T \) a bounded operator \( N^\infty(T) = \bigcup_{n \geq 1} N(T^n) \) be the generalized kernel, then
Proposition 4.1. For \( A \in \mathcal{L}(X), B \in \mathcal{L}(Y) \) we have
\[
N(\delta_{A,B}) + H_3(A, B, \sigma) \subseteq H_3(A, B, \sigma),
\]
in particular
\[
N(\delta_{A,B}) + \bigcup_{\lambda \in \mathbb{C}} N^\infty(R_B - \lambda) \cup \bigcup_{\lambda \in \mathbb{C}} N^\infty(L_A - \lambda) \subseteq H_3(A, B, \sigma).
\]

Proof. Let \( C_1 \in N(\delta_{A,B}) \) and \( C_2 \in H_3(A, B, \sigma) \). We have \( M_{C_1+C_2} = M_{C_2} + [\begin{smallmatrix} C_1 & 0 \\ 0 & 0 \end{smallmatrix}] \). Since \( \begin{smallmatrix} C_1 & 0 \\ 0 & 0 \end{smallmatrix} \) is nilpotent and commutes with \( M_{C_2} \), then
\[
\sigma(M_{C_1+C_2}) = \sigma(M_{C_2}) = \sigma(A) \cup \sigma(B).
\]

The next theorem, which can be found in [3], follows from Theorem 4.1, Proposition 4.1 and [3, Lemma 7]. Remark that \( H_3(A, B, \sigma) \) has no linear structure.

Theorem 4.2. [3] For \( A \in \mathcal{L}(X), B \in \mathcal{L}(Y) \) we have
\[
cl \left[ H_3(A, B, \sigma) + N(\delta_{A,B}) + \bigcup_{\lambda \in \mathbb{C}} N(L_{A^{-1}} - \lambda) + \bigcup_{\lambda \in \mathbb{C}} N(R_{B^{-1}} - \lambda) \right] \subseteq H_3(A, B, \sigma).
\]

4.2. \( \Sigma = \sigma_e \)

In a similar way as for the case \( \Sigma = \sigma \) and using the Calkin homomorphism, we get:

Theorem 4.3. For \( A \in \mathcal{L}(X), B \in \mathcal{L}(Y), \lambda \in \mathbb{C} \) and \( n \geq 1 \). If \( C \in N(R^n_{B^{-1}} - \lambda) \cup N(L^n_{A^{-1}} - \lambda) \), then
\[
\sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B).
\]

Corollary 4.2. For \( A \in \mathcal{L}(X), B \in \mathcal{L}(Y) \) we have
\[
cl \left[ H_3(A, B, \sigma_e) + N(\delta_{A,B}) + \bigcup_{\lambda \in \mathbb{C}} N(L_{A^{-1}} - \lambda) + \bigcup_{\lambda \in \mathbb{C}} N(R_{B^{-1}} - \lambda) \right] \subseteq H_3(A, B, \sigma_e).
\]

4.3. \( \Sigma \in \{ S, \sigma_\beta, \sigma_\delta, \sigma_{\text{dec}} \} \)

Arguing as in Theorem 4.1 we get:

Theorem 4.4. For \( A \in \mathcal{L}(X), B \in \mathcal{L}(Y), \lambda \in \mathbb{C} \) and \( n \geq 1 \), we have:

(a) If \( C \in N(R^n_{B^{-1}} - \lambda) \), then
\[
\sigma_\beta(M_C) = \sigma_\beta(A) \cup \sigma_\beta(B), \quad S(M_C) = S(A) \cup S(B).
\]

(b) If \( C \in N(L^n_{A^{-1}} - \lambda) \), then
\[
\sigma_\delta(M_C) = \sigma_\delta(A) \cup \sigma_\delta(B), \quad S(M_C^*) = S(A^*) \cup S(B^*).
\]

(c) If \( C \in N(L^n_{A^{-1}} - \lambda) \cap N(R^n_{B^{-1}} - \lambda) \), then
\[
\sigma_{\text{dec}}(M_C) = \sigma_{\text{dec}}(A) \cup \sigma_{\text{dec}}(B).
\]
References