Necessary and sufficient conditions for the oscillation of systems of difference equations with continuous arguments

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Abstract

In this paper the oscillation of all solutions of system of the form

\[ y(t) + \sum_{i=1}^{n} P_i y(t - \tau_i) = 0, \]

where \( P_i \in \mathbb{R}^{m \times m} \), \( \tau_i \in \mathbb{R} \) \( (i = 1, 2, \ldots, n) \), is studied. Necessary and sufficient conditions in terms of the characteristic equation and explicit conditions in terms of the eigenvalues or the logarithmic norms of the coefficient matrices are established.

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1. Introduction and preliminary

Recently there has been a lot of activity concerning the oscillation of delay difference equation with continuous arguments. See, for example, [1,3,5–15]. In [16], sufficient conditions are obtained for all solutions of the system

\[ x_i(t) - x_i(t - \sigma) + \sum_{k=1}^{l} \sum_{j=1}^{n} p_{ijk} x_j(t - \tau_k) = 0, \]

where \( p_{ijk} \in (-\infty, \infty) \), \( \sigma \) and \( \tau_k \in (0, \infty) \), \( i, j = 1, 2, \ldots, n \), \( k = 1, 2, \ldots, l \), to be oscillatory by comparing with the scalar delay difference equation, which may be the only one paper for oscillation of system with continuous arguments in our known.

In this paper we are interesting in establishing necessary and sufficient conditions for the oscillation of all solutions of the linear autonomous systems of difference equations

\[ y(t) + \sum_{i=1}^{n} P_i y(t - \tau_i) = 0, \] \hspace{1cm} (1)

where \( P_i \in \mathbb{R}^{m \times m} \), \( \tau_i \in \mathbb{R} \) \( (i = 1, 2, \ldots, n) \), in terms of its characteristic equation. Then by utilizing the result, we obtain some explicit conditions for oscillation in terms of the eigenvalues or the logarithmic norms of the coefficient matrices \( P_1, P_2, \ldots, P_n \).

For an \( m \times m \) matrix \( P \) the logarithmic norm of \( P \) is denoted by \( \mu(P) \) and is defined to be

\[ \mu(P) = \max_{\|\xi\| = 1} (P\xi, \xi), \]

where \((,\) is an inner product in \( \mathbb{R}^m \) and \( \|\xi\| = (\xi, \xi)^{1/2} \).

Without loss of generality we can assume that deviating arguments \( \tau_i \ (i = 1, \ldots, n) \) in Eq. (1) are positive because we can always translate the argument of a solution. So we set

\[ \tau_1 > \tau_2 > \cdots > \tau_n > 0. \] \hspace{1cm} (2)

By a solution \( y(t) = [y_1(t), y_2(t), \ldots, y_m(t)]^T \) of Eq. (1) we mean a continuous function \( y \in C[[t_0 - \tau_1, \infty), \mathbb{R}^m] \) which satisfies Eq. (1) for all \( t \geq t_0 \) \( (t_0 \geq 0) \). A solution \( y(t) \) of Eq. (1) is said to be oscillatory if every component \( y_i(t) \) of the solution has arbitrarily large zeros. Otherwise the solution is called non-oscillatory.

This paper is motivated by [2] and [4, Chapter 5]. [4, Chapter 5] deals with the oscillation of the systems of differential equations and [2] deals with the oscillation of the systems of difference equations.

The following four theorems are the results of oscillation about the scalar difference equation and are useful in Sections 2 and 3.

**Theorem A** [6]. The following statements are equivalent.

(a) Every solution of the scalar difference equation

\[ x(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0, \] \hspace{1cm} (3)
where $p_i, \tau_i \in R \ (i = 1, 2, \ldots, n)$, oscillates.

(b) The characteristic equation

$$1 + \sum_{i=1}^{n} p_i e^{-\lambda \tau_i} = 0$$

of Eq. (3) has no real roots.

**Theorem B** [6]. Assume $p, q > 0$, $\sigma > \tau > 0$. Then

$$q^\tau \sigma^\sigma > p^\sigma \tau^\tau (\sigma - \tau)^{\sigma - \tau}$$

is a necessary and sufficient condition for the oscillation of all solutions of the scalar equation

$$x(t) - px(t - \tau) + qx(t - \sigma) = 0.$$ 

**Theorem C** [6]. Assume that $p, \tau, q_i, \sigma_i \in (0, \infty)$ and $\tau < \sigma_i$ for $i = 1, 2, \ldots, n$, and

$$\sum_{i=1}^{n} q_i \left[ \frac{\sigma_i^\sigma_i}{p^\sigma_i \tau^\tau (\sigma_i - \tau)^{\sigma_i - \tau}} \right]^{1/\tau} > 1.$$ 

Then every solution of the scalar difference equation

$$x(t) - px(t - \tau) + \sum_{i=1}^{n} q_i x(t - \sigma_i) = 0$$

oscillates.

**Theorem D.** Assume that $p_1 > p_2 \geq 0$, $\sigma_1 > \sigma_2 > 0$, $p_2(\sigma_1 - \sigma_2) < 1$ and

$$(p_1 - p_2)^\tau \sigma_1^\sigma_1 > \tau^\tau (\sigma_1 - \tau)^{\sigma_1 - \tau}.$$ 

(4)

Then every solution of the difference equation with positive and negative coefficients

$$x(t) - x(t - \tau) + p_1 x(t - \sigma_1) - p_2 x(t - \sigma_2) = 0$$

(5)

oscillates.

**Proof.** For the sake of contradiction, let $x(t)$ be an eventually positive solution of Eq. (5) and

$$z(t) = \int_{t - \tau}^{t} x(s) \, ds - p_2 \int_{t - \sigma_1}^{t - \sigma_2} x(s) \, ds.$$ 

(6)

Then

$$z'(t) = -(p_1 - p_2) x(t - \sigma_1) < 0.$$ 

(7)

We claim $z(t) > 0$ eventually. Otherwise, $z(t) < 0$ eventually, then there exists a $T$ and a positive number $l$ such that $z(t) < -l < 0,$ for $t \geq T.$ From (6), there exists $\xi \in [t - \tau, t]$ such that

$$x(\xi) \tau < -l + p_2(\sigma_1 - \sigma_2) \max_{s \in [t - \sigma_1, \xi]} x(s) \leq -l + p_2(\sigma_1 - \sigma_2) \max_{s \in [t - \sigma_1, \xi]} x(s).$$
By a known result [4, Lemma 1.5.4], the above inequality implies that $x(t)$ cannot be positive. This contradiction proves that the claim is true.

Integrating (7) from $t - \tau$ to $t$, we have

$$z(t) - z(t - \tau) + (p_1 - p_2) \int_{t-\tau}^{t} x(s - \sigma_1) \, ds = 0.$$  

By (6), the above equation reduces the inequality

$$z(t) - z(t - \tau) + (p_1 - p_2) z(t - \sigma_1) \leq 0.$$  

With Lemma 2 [16], we know that the equation

$$z(t) - z(t - \tau) + (p_1 - p_2) z(t - \sigma_1) = 0 \quad (8)$$  

has a positive solution eventually. But from Theorem B and (4), every solution of Eq. (8) is oscillatory, a contradiction. The proof is complete.  

2. Necessary and sufficient conditions for oscillation

With Eq. (1) one associates its characteristic equation

$$\det \left( I + \sum_{i=1}^{n} P_i e^{-\lambda \tau_i} \right) = 0, \quad (9)$$  

where $I$ is the $m \times m$ identity matrix.

**Theorem 1.** Assume that $P_i \in \mathbb{R}^{m \times m}$ ($i = 1, 2, \ldots, n$) and (2) holds. The following statements are equivalent.

(a) Every solution of Eq. (1) oscillates componentwise.

(b) The characteristic equation (9) has no real roots.

**Proof.** (a) $\Rightarrow$ (b) If $\lambda_0$ is a real root of Eq. (9) then there exists a non-zero vector $\xi$ such that

$$\left( I + \sum_{i=1}^{n} P_i e^{-\lambda_0 \tau_i} \right) \xi = 0.$$  

Then clearly, $y(t) = e^{\lambda_0 t} \xi$ is a solution of Eq. (1) with at least one non-oscillatory component.

(b) $\Rightarrow$ (a) We will show, via the Laplace transform, that every solution of Eq. (1) oscillates if Eq. (9) has no real roots. We first show that for some $K > 0$, $\| y(t) \| = O(e^{Kt})$ whenever $y(t)$ is a solution of Eq. (1), so that the Laplace transform of $y(t)$ exists.

Let $\tilde{y}(r) := \max_{0 \leq t \leq r} \| y(t) \|$. Then

(i) $\tilde{y}(r)$ is increasing.
(ii) For \( t \leq r \), \( \|y(t)\| \leq \sum_{i=1}^{n} \|P_i\| \|y(t - \tau_i)\| \leq \left( \sum_{i=1}^{n} \|P_i\| \right) \tilde{y}(r - \tau_n) \) so \( \tilde{y}(r) \leq \left( \sum_{i=1}^{n} \|P_i\| \right) \tilde{y}(r - \tau_n) \).

These two facts imply exponential growth of \( \tilde{y}(r) \) and hence of \( y(t) \).

Assume, for the sake of contradiction, that (b) holds and Eq. (1) has a non-oscillatory solution \( y(t) = [y_1(t), \ldots, y_m(t)]^T \). This means that at least one of the components of \( y(t) \) is non-oscillatory. Without loss of generality we assume that the component \( y_1(t) \) is eventually positive. As Eq. (1) is autonomous, we may (and do) assume that \( y_1(t) > 0 \) for \( t > -\tau_1 \).

By the above proofs, we know that \( y(t) \) is of exponential order and so there exists \( \mu \in \mathbb{R} \) such that the Laplace transform of \( y(t) \), \( Y(s) = \int_{0}^{\infty} e^{-st} y(t) \, dt \), exists for \( \Re s > \mu \).

By taking Laplace transforms of both sides of Eq. (1) we obtain
\[
F(s)Y(s) = \Phi(s), \quad \Re s > \mu, \tag{10}
\]
where
\[
F(s) = I + \sum_{i=1}^{n} P_i e^{-s\tau_i} \tag{11}
\]
and
\[
\Phi(s) = -\sum_{i=1}^{n} P_i e^{-s\tau_i} \int_{-\tau_i}^{0} e^{-st} y(t) \, dt. \tag{12}
\]
By hypothesis, \( \det[F(s)] \neq 0 \) for all \( s \in \mathbb{R} \). Moreover,
\[
\lim_{s \to \infty} \left( \det[F(s)] \right) = 1 \tag{13}
\]
and so
\[
\det[F(s)] > 0, \quad \text{for all } s \in \mathbb{R}. \tag{14}
\]
Let \( Y_1(s) \) be the Laplace transform of the first component \( y_1(t) \) of the solution \( y(t) \). Then, by Cramer’s rule,
\[
Y_1(s) = \frac{\det[D(s)]}{\det[F(s)]}, \quad \Re s > \mu, \tag{15}
\]
where
\[
D(s) = \begin{bmatrix}
\Phi_1(s) & F_{12}(s) & \cdots & F_{1m}(s) \\
\vdots & \vdots & \ddots & \vdots \\
\Phi_m(s) & F_{m2}(s) & \cdots & F_{mm}(s)
\end{bmatrix},
\]
\( \Phi_i(s) \) is the \( i \)th component of vector \( \Phi(s) \) and \( F_{ij}(s) \) is the \( (i, j) \)th component of the matrix \( F(s) \). Clearly, for all \( i, j = 1, 2, \ldots, m \) the functions \( \Phi_i(s) \) and \( F_{ij}(s) \) are entire and hence \( \det[D(s)] \) and \( \det[F(s)] \) are also entire functions.

Let \( \sigma_0 \) be the abscissa of convergence of \( Y_1(s) \), that is \( \sigma_0 = \inf{s \in \mathbb{R}: Y_1(s) \text{ exists}} \). Since \( Y_1(s) \) is the Laplace transform of a positive function, if \( \sigma_0 > -\infty \) then \( Y_1(s) \) must
have a singularity, by Theorem 1.3.1 [4], at \( \sigma_0 \). But \( \Phi(s) \) is an entire function and \( F(s) \neq 0 \), so (15) shows that \( Y_1(s) \) can be analytically continued to a neighborhood of any real \( s \). Thus we must have \( \sigma_0 = -\infty \) and (15) becomes
\[
Y_1(s) = \frac{\det[D(s)]}{\det[F(s)]}, \quad \text{all } s \in \mathbb{R}.
\] (16)

As \( y_1(s) > 0 \), it follow that \( Y_1(s) > 0 \) for all \( s \in \mathbb{R} \) and, by (14) and (15), \( \det[D(s)] > 0 \) for \( s \in \mathbb{R} \). Now one can see from the definition of \( D(s) \) and from (11) and (12) that there are positive constants \( L, \alpha, s_0 \) such that
\[
\det[D(s)] \leq Le^{-\alpha s}, \quad \text{for } s \leq -s_0.
\] (17)

Also from (13), (14) and the fact \( \det[F(s)] \) is polynomial in the variables \( e^{-s\tau_1}, \ldots, e^{-s\tau_n} \), it follows that there exists a positive number \( l \) such that
\[
\det[F(s)] \geq l, \quad \text{for } s \in \mathbb{R}.
\] (18)

From (16)–(18) it follows that
\[
Y_1(s) = \int_{0}^{\infty} e^{-st} y_1(t) \, dt \geq \int_{T}^{\infty} e^{-st} y_1(t) \, dt \geq e^{-sT} \int_{T}^{\infty} y_1(t) \, dt > 0
\]
and so
\[
0 < \int_{T}^{\infty} y_1(t) \, dt \leq \frac{L}{l} e^{s(T-\sigma)} \to 0, \quad \text{as } s \to -\infty.
\]
This implies that \( y_1(t) \equiv 0 \) for \( t \geq T \), which is a contradiction. The proof is complete. \( \square \)

3. Explicit conditions for oscillation

In this section, first we establish the following necessary and sufficient condition for the oscillation of all solutions of the system
\[
y(t) - y(t - \tau) + Py(t - \sigma) = 0 \quad (19)
\]
in terms of the eigenvalues of the coefficient matrices \( P \).

**Theorem 2.** Assume that \( P \in \mathbb{R}^{m \times m} \) and \( \sigma > \tau > 0 \). Then every solution of Eq. (19) oscillates (componentwise) if and only if \( P \) has no eigenvalues in the interval
\[
\left( -\infty, \left( \frac{\tau^\tau}{\sigma^\sigma (\sigma - \tau)^{\sigma - \tau}} \right)^{1/\tau} \right.
\]

**Proof.** The characteristic equation of Eq. (19) is
\[
\det[(1 - e^{-\lambda \tau})I + Pe^{-\lambda \sigma}] = 0,
\] (20)
which can be also written as
\[ \det[e^{\lambda\sigma}(e^{-\lambda\tau} - 1)I - P] = 0. \]

Set
\[ v(\lambda) = e^{\lambda\sigma}(e^{-\lambda\tau} - 1). \]

Note that \( v \) is a continuous function on \((-\infty, \infty)\).

\[ \max_{\lambda \in (-\infty, \infty)} v(\lambda) = \left( \frac{\tau^\tau}{\sigma^\sigma} (\sigma - \tau)^{\sigma - \tau} \right)^{1/\tau}, \quad \lim_{\lambda \to \infty} v(\lambda) = -\infty. \]

Thus the image of \((-\infty, \infty)\) under \( v \) is \((-\infty, \left( \frac{\tau^\tau}{\sigma^\sigma} (\sigma - \tau)^{\sigma - \tau} \right)^{1/\tau}]\). Therefore Eq. (20) has no real roots if and only if \( P \) has no eigenvalues in \((-\infty, \left( \frac{\tau^\tau}{\sigma^\sigma} (\sigma - \tau)^{\sigma - \tau} \right)^{1/\tau}]\). The proof is complete. \( \square \)

The following corollary is an immediate consequence of Theorem 2.

**Corollary 1.** Assume that \( P \in \mathbb{R}^{m \times m} \) and \( \tau \in \mathbb{R} \). Then every solution of the system
\[ y(t) + Py(t - \tau) = 0 \]
oscillates (componentwise) if and only if \( P \) has no eigenvalues in the interval \((-\infty, 0]\).

**Remark 1.** The condition of Theorem 2 can be written as:
\[ \lambda(P) \left( \frac{\sigma^\sigma}{\tau^\tau (\sigma - \tau)^{\sigma - \tau}} \right)^{1/\tau} > 1, \]
where \( \lambda(P) \) stands for any real eigenvalues of \( P \).

**Remark 2.** In the scalar case, that is when \( m = 1 \), the condition of Theorem 2 reduces to
\[ p \left( \frac{\sigma^\sigma}{\tau^\tau (\sigma - \tau)^{\sigma - \tau}} \right)^{1/\tau} > 1, \]
which is the condition of Theorem B.

Next, we obtain sufficient conditions for the oscillation of all solution of the delay difference system
\[ y(t) - y(t - \tau) + \sum_{i=1}^{n} P_i y(t - \sigma_i) = 0. \quad (21) \]

The conditions are given explicitly in terms of logarithmic norm of the matrices \( P_i \). We need the following lemma which is interesting in its own right.

**Lemma 1.** Assume that \( P \in \mathbb{R}^{m \times m} \) and \( \sigma_i > \tau > 0 \) \((i = 1, 2, \ldots, n)\). Then every solution of Eq. (21) oscillates (componentwise) provided that
(ia) \[ \sum_{i=1}^{n} e^{-\gamma \sigma_i} \mu(-P_i) < 0 \quad \text{for } \gamma > 0 \quad \text{or} \quad (22) \]

(b) \[ \sup_{\gamma > 0} \frac{1}{1 - e^{-\gamma \tau}} \sum_{i=1}^{n} e^{-\gamma \sigma_i} \mu(-P_i) < 1 \quad \text{and} \quad \sum_{i=1}^{n} \mu(-P_i) < 0, \quad (23) \]

(ii) \[ \inf_{\gamma < 0} \left[ \frac{1}{1 - e^{-\gamma \tau}} \sum_{i=1}^{n} e^{-\gamma \sigma_i} \mu(-P_i) \right] > 1. \quad (24) \]

**Proof.** Assume, for the sake of contradiction, that Eq. (21) has a non-oscillatory solution. Then by Theorem 1, the characteristic equation of Eq. (21)

\[
\det \left[ (1 - e^{-\lambda \tau})I + \sum_{i=1}^{n} P_i e^{-\lambda \sigma_i} \right] = 0
\]

has a real root \( \gamma \). Therefore, there exists a non-zero vector \( \xi \in \mathbb{R}^m \), with \( \|\xi\| = 1 \), such that

\[
\left[ (1 - e^{-\gamma \tau})I + \sum_{i=1}^{n} P_i e^{-\gamma \sigma_i} \right] \xi = 0.
\]

Hence,

\[
1 - e^{-\gamma \tau} = \sum_{i=1}^{n} e^{-\gamma \sigma_i} (-P_i \xi, \xi)
\]

and so

\[
1 - e^{-\gamma \tau} \leq \sum_{i=1}^{n} e^{-\gamma \sigma_i} \mu(-P_i). \quad (25)
\]

It follows from (22) that (25) cannot hold. Therefore (23) holds.

Now if \( \gamma > 0 \), then from (25) we get

\[
1 \leq \frac{1}{1 - e^{-\gamma \tau}} \sum_{i=1}^{n} e^{-\gamma \sigma_i} \mu(-P_i)
\]

and so

\[
1 \leq \sup_{\gamma > 0} \left[ \frac{1}{1 - e^{-\gamma \tau}} \sum_{i=1}^{n} e^{-\gamma \sigma_i} \mu(-P_i) \right],
\]

which contradicts the first part of (23). Also if \( \gamma = 0 \), then from (25)

\[
0 \leq \sum_{i=1}^{n} \mu(-P_i)
\]

which contradicts the second part of (23). Therefore \( \gamma < 0 \) and (25) yields

\[
1 \geq \frac{1}{1 - e^{-\gamma \tau}} \sum_{i=1}^{n} e^{-\gamma \sigma_i} \mu(-P_i).
\]
Hence
\[ 1 \geq \inf_{\gamma < 0} \left[ \frac{1}{1 - e^{-\gamma \tau}} \sum_{i=1}^{n} e^{-\gamma \sigma_i} \mu(-P_i) \right], \]
which contradicts (24) and the proof is complete. □

**Theorem 3.** Assume that \( P \in \mathbb{R}^{n \times m} \) and \( \sigma_i > \tau > 0 \) (\( i = 1, 2, \ldots, n \)). Suppose that for \( i = 1, 2, \ldots, n \),
\[ \mu(-P_i) \leq 0. \tag{26} \]
Then every solution of Eq. (21) oscillates (componentwise) provided that one of the following two conditions is satisfied:
\[ \text{(i) } \sum_{i=1}^{n} (-\mu(-P_i)) \left( \frac{\sigma_i^{\sigma_i}}{\tau^\sigma (\sigma_i - \tau)^\sigma_i} \right)^{1/\tau} > 1, \tag{27} \]
\[ \text{(ii) } n \left[ \prod_{i=1}^{n} (-\mu(-P_i)) \right]^{1/n} \left( \frac{\sigma^{\sigma}}{\tau^\sigma (\sigma - \tau)^\sigma - \tau} \right)^{1/\tau} > 1, \quad \sigma = \frac{1}{n} \sum_{i=1}^{n} \sigma_i. \tag{28} \]

**Proof.** We employ Lemma 1. As \( \mu(-P_i) \leq 0 \), (22) is satisfied and so it suffices to show that each of (27) and (28) implies (24). First, assume that (27) holds. We have
\[ \sup_{\gamma < 0} \frac{e^{-\gamma \sigma_i}}{1 - e^{-\gamma \tau}} = - \left( \frac{\sigma_i^{\sigma_i}}{\tau^\sigma (\sigma_i - \tau)^\sigma_i} \right)^{1/\tau}. \]
So we get for \( \gamma < 0 \),
\[ \frac{1}{1 - e^{-\gamma \tau}} \sum_{i=1}^{n} e^{-\gamma \sigma_i} \mu(-P_i) = \frac{1}{e^{-\gamma \tau} - 1} \sum_{i=1}^{n} e^{-\gamma \sigma_i} (-\mu(-P_i)) \]
\[ \geq \sum_{i=1}^{n} (-\mu(-P_i)) \left( \frac{\sigma_i^{\sigma_i}}{\tau^\sigma (\sigma_i - \tau)^\sigma_i} \right)^{1/\tau}, \]
so (24) holds.
Next, assume that (28) holds. Then by using the arithmetic mean–geometric mean inequality we see that for \( \gamma < 0 \),
\[ \frac{1}{1 - e^{-\gamma \tau}} \sum_{i=1}^{n} e^{-\gamma \sigma_i} \mu(-P_i) = \frac{1}{e^{-\gamma \tau} - 1} \sum_{i=1}^{n} e^{-\gamma \sigma_i} (-\mu(-P_i)) \]
\[ \geq n \left[ \prod_{i=1}^{n} (-\mu(-P_i)) \frac{e^{-\gamma \sigma_i}}{e^{-\gamma \tau} - 1} \right]^{1/n} = n \left[ \prod_{i=1}^{n} (-\mu(-P_i)) \right]^{1/n} \frac{e^{-\gamma \sigma}}{e^{-\gamma \tau} - 1} \]
\[ \geq n \left[ \prod_{i=1}^{n} (-\mu(-P_i)) \right]^{1/n} \left( \frac{\sigma^{\sigma}}{\tau^\sigma (\sigma - \tau)^\sigma - \tau} \right)^{1/\tau}, \]
so (24) holds. The proof is complete. □
Theorem 3 assume that the logarithmic norms of the negatives of the coefficient matrices in Eq. (21) are all non-positive. In the next theorem, we consider the case where one logarithmic norm is non-positive and the other is non-negative.

**Theorem 4.** Assume that \( P_1, P_2 \in \mathbb{R}^{m \times m} \) and \( \sigma_1, \sigma_2, \tau \in \mathbb{R}^+ \) are such that

\[
-\mu(-P_1) > \mu(P_2) \geq 0, \quad \sigma_1 > \sigma_2 > \tau > 0, \quad \mu(P_2)(\sigma_1 - \sigma_2) < 1
\]

and

\[
-\mu(-P_1) - \mu(P_2) > \left( \frac{\tau^\tau}{\sigma_1^\sigma_1} (\sigma_1 - \tau)^{\sigma_1-\tau} \right)^{1/\tau}.
\] (29)

Then every solution of the system

\[
y(t) - y(t - \tau) + P_1 y(t - \sigma_1) - P_2 y(t - \sigma_2) = 0
\]

oscillates (componentwise).

**Proof.** We employ Theorem D. To do so, it suffices to show that (23) and (24) hold. Set

\[
p_1 = -\mu(-P_1) \quad \text{and} \quad p_2 = \mu(P_2).
\] (30)

Then in order to establish (23) and (24) we must prove, respectively,

\[
sup_{\gamma > 0} \frac{1}{1 - e^{-\gamma \tau}} (-p_1 e^{-\gamma \sigma_1} + p_2 e^{-\gamma \sigma_2}) < 1
\] (31)

and

\[
inf_{\gamma < 0} \frac{1}{1 - e^{-\gamma \tau}} (-p_1 e^{-\gamma \sigma_1} + p_2 e^{-\gamma \sigma_2}) > 1.
\] (32)

Using (30), the hypothesis of theorem becomes

\[
p_1 - p_2 > \left( \frac{\tau^\tau}{\sigma_1^\sigma_1} (\sigma_1 - \tau)^{\sigma_1-\tau} \right)^{1/\tau}.
\]

Thus, by Theorem D, every solution of Eq. (5) oscillates. This implies that the characteristic equation of Eq. (5)

\[
F(\gamma) = 1 - e^{-\gamma \tau} + p_1 e^{-\gamma \sigma_1} - p_2 e^{-\gamma \sigma_2}
\]

has no real roots. Since we find that \( F(0) = p_1 - p_2 > 0 \), so \( F(\gamma) > 0 \), that is

\[
1 - e^{-\gamma \tau} + p_1 e^{-\gamma \sigma_1} - p_2 e^{-\gamma \sigma_2} > 0.
\] (33)

Therefore for \( \gamma > 0 \), we have

\[
\frac{1}{1 - e^{-\gamma \tau}} (-p_1 e^{-\gamma \sigma_1} + p_2 e^{-\gamma \sigma_2}) < 1
\]

and so

\[
sup_{\gamma > 0} \frac{1}{1 - e^{-\gamma \tau}} (-p_1 e^{-\gamma \sigma_1} + p_2 e^{-\gamma \sigma_2}) \leq 1.
\] (34)
Observe that
\[ \lim_{\gamma \to 0^+} \frac{1}{1 - e^{-\gamma \tau}} (-p_1 e^{-\gamma \sigma_1} + p_2 e^{-\gamma \sigma_2}) = -\infty \]
and
\[ \lim_{\gamma \to \infty} \frac{1}{1 - e^{-\gamma \tau}} (-p_1 e^{-\gamma \sigma_1} + p_2 e^{-\gamma \sigma_2}) = 0. \]
So (31) has been established.

Note that for \( \gamma < 0 \), (33) yields
\[ 1 < \frac{1}{1 - e^{-\gamma \tau}} (-p_1 e^{-\gamma \sigma_1} + p_2 e^{-\gamma \sigma_2}). \]
So (32) is satisfied. The proof is complete. \( \square \)

**Remark 3.** When \( m = 1 \), the condition (i) of Theorem 3 is the condition of Theorem C and Theorem 4 is coincident with Theorem D.

**References**


