On the (0,4) lacunary interpolation problem and some related questions

Ezio Venturino*

Dipartimento di Matematica, Città Universitaria, 95125 Catania, Italy

Received 12 September 1994; revised 30 June 1996

Abstract

We consider at first a (0,4) lacunary interpolation problem, and we solve it in an appropriate class of "deficient" splines. Under suitable assumptions, we show existence and uniqueness of the solution. We provide the convergence analysis, showing that the method is of order "almost" five. Some additional considerations are made for problems of similar nature, of arbitrary order and with arbitrary intermediate lacunary conditions. This is a natural method to couple with finite difference schemes for the reconstruction of the solution of boundary value problems. It may constitute a viable alternative to continuous Runge-Kutta methods.

Keywords: Lacunary interpolation; Deficient splines

AMS Classification: 65D05; 65D07

1. Introduction

In this study we find an algorithm to solve a special lacunary interpolation problem, when the function and its fourth derivative are known at a set of nodes, and show its convergence. The importance of this problem in practical applications arises mainly from the "cantilever problem" and related problems, like deformations of beams, see, e.g., [10, Section 3.2], which are governed by a fourth-order nonlinear differential equation. If the initial value problem is solved by reducing it to a system of the first order, its solution by means of any one-step or multistep method provides information on the unknown function and its first four derivatives at a set of (equispaced) nodes. If instead a method based on finite difference techniques is used, only the unknown function is calculated on this mesh but it is not difficult to obtain, via an additional function evaluation, also its fourth derivative. With these methods, the unknown function is calculated only at the grid knots. If we want to reconstruct the solution at some other points, there is a need of interpolating these values.

* E-mail: ezio@dipmat.unict.it
If we want to use the values of the function and its first four derivatives, we need on each subinterval polynomials of degree 9, since there are 5 interpolatory conditions at one endpoint and 5 continuity conditions at the other endpoint. However, notice that it is not possible to obtain continuity of all derivatives up to order 8, as required by the ordinary spline functions. This is because in addition to the function value at the first endpoint we specify there other informations of interpolatory nature, and then in turn we need to decrease the continuity requirements at the other endpoint, if we still want a square system. Thus, “deficient” splines, as they are called, arise naturally.

In the first part of the paper, we consider the simple (0,4) lacunary interpolation problem, solving it by means of splines of degree six. We are able to show that the proposed method converges with order “almost” five. Indeed, some accuracy is lost for our inability to obtain the inverse of a certain matrix. An estimation of its eigenvalues allows only an information on its two norm, and in returning to the supremum norm a half order is lost. Some numerical results are shown in Section 6. In the last section we generalize the problem, allowing for higher-order interpolation coupled with arbitrary lacunary conditions, and investigate how to start the construction of the required interpolant.

2. Problem statement

The problem we investigate in this paper consists in finding a suitable deficient spline \( s(x) \), interpolating data given on the function and its fourth derivative at a given set of nodes.

A local analysis allows us to discuss the class of functions in which we seek the solution. On each subinterval, there are two interpolation conditions at the first node, given by the function value and the value of the fourth derivative. This implies that if we want to use a piecewise polynomial solution, say of degree \( r \), i.e., with \( r + 1 \) unknown coefficients, we need \( r - 1 \) more conditions. These will be continuity conditions at the other node of each subinterval, but since we start from the function, we can match derivatives up to order \( r - 2 \). This means that the piecewise polynomial will not be a spline in the usual sense of the word, since it lacks continuity in the next to the last derivative. It will then be a deficient spline, of order 2, since ordinary splines lack continuity in the highest derivative, i.e., they are deficient of order 1. In our case, we require continuity of the first four derivatives, since we have the fourth derivative and would like continuity up to it, where information from the original problem is available. Hence, we need 5 continuity conditions, i.e., a polynomial of degree 6, with 7 coefficients to be determined. Thus, \( s(x) \) is sought in the class of deficient splines of degree six, \( S_{6,6}^{(2)} \), where the notation emphasizes the deficiency as well as the polynomial degree.

Assume a given nonnecessarily uniform grid \{\( x_k \), \( k = 0(1)n \)\}, of the interval \([a,b] \equiv [0,1]\), with \( h_k = x_k - x_{k-1}, \( k = 0(1)n\)\), satisfies also the mesh requirement, that a constant \( L \) exists, such that

\[
\frac{1}{L} \leq \frac{1}{L} \leq \min_{1 \leq k \leq n-1} h_k. \tag{1}
\]

Then the number of free parameters to be determined is \( 7n \), since in each subinterval \( \Delta_k \equiv [x_{k-1},x_k] \), \( k = 1(1)n \), the sought function is a polynomial of degree 6.

We now take a global point of view in order to determine the additional (initial) conditions to be added to obtain a square system. The number of conditions is counted as follows. There are 2 interpolatory conditions on each node, whether interior or boundary, on \( y \) and \( y^{iv} \), i.e., \( 2(n + 1) \)
conditions. Upon their use the number of unknowns drops to \(5n - 2\). To determine them, the continuity conditions on the function and its first four derivatives at the interior nodes must be used, thus giving another \(5(n - 1)\) equations. Finally, to obtain a square system, we specify the three remaining conditions as the values of the first three derivatives at \(x_0\), to keep the analysis simple.

Mathematically speaking, we seek \(s(x) \in S_{n,6}^{(2)}\), such that

\[
s(x_k) = y_k, \quad s^{(iv)}(x_k) = y^{(iv)}_k, \quad k = 0(1)n,.
\]

together with

\[
y^{(p)}(x_0) = s^{(p)}(x_0) = y^{(p)}_a, \quad p = 1, 2, 3.
\]

Let us consider the inner interval \(A_k\). Let \(s_k(x)\) be the restriction of \(s(x)\) to the interval \(A_k\), \(k = 1(1)n\). We can write the expression for \(s_k(x)\) by using the cubic Hermite polynomial interpolating on the third and fourth derivatives at \(x_{k-1}\) and \(x_k\), and then integrating three times. If we denote the third-order moments by

\[
M^{(3)}_i \equiv s^{(iii)}(x_i) \equiv y^{(iii)}_i, \quad i = 0(1)n,
\]

we have

\[
s^{(iii)}_k(x) = M^{(3)}_{k-1} \left( \frac{x - x_k}{h_k^2} \right)^2 \left( 1 + 2 \frac{x - x_k - 1}{h_k} \right) + M^{(3)}_k \left( \frac{x - x_k}{h_k^2} \right)^2 \left( 1 + 2 \frac{x_k - x}{h_k} \right) + \frac{x - x_k - 1}{h_k^2} (x_k - x)^2 y^{(iv)}_{k-1} - (x - x_k - 1)^2 \frac{x_k - x}{h_k^2} y^{(iv)}_k.
\]

Integrating from \(x_{k-1}\) to \(x\), and denoting by \(M^{(2)}_k\) the second-order moments, \(M^{(2)}_k = s''(x_k), \quad k = 0(1)n - 1\),

\[
s''_k(x) = M^{(3)}_{k-1} \left\{ \frac{(x - x_k)^3}{3h_k^2} \left[ \frac{2}{h_k} \left( x - x_k - 1 - \frac{x - x_k}{4} \right) \right] + \frac{h_k}{2} \right\} + M^{(3)}_k \left\{ \frac{(x - x_k - 1)^3}{3h_k^2} \left[ 1 - \frac{2}{h_k} \left( x - x_k - 1 - \frac{x - x_k}{4} \right) \right] \right\} + y^{(iv)}_{k-1} \left\{ \frac{(x - x_k)^3}{3h_k^2} \left( x - x_k - 1 - \frac{x - x_k}{4} \right) + \frac{h_k^2}{12} \right\}
\]
\[+ y^{(iv)}_k \left\{ \frac{(x - x_k - 1)^3}{3h_k^2} \left( x - x_k - 1 - \frac{x - x_k}{4} \right) + M^{(2)}_{k-1} \right\}.
\]

Since we are assuming that the solution possesses fourth-order moments, \(y^{(iv)}_k\), it is natural to impose that at the nodes \(x_k\) all the previous derivatives of the solution exist and are continuous. For the third derivative this follows from (4). By matching the second derivatives of \(s_{k+1}(x)\) and \(s_k(x)\) at \(x_k\), \(k = 1(1)n - 1\), we obtain

\[
\frac{1}{2} h_k M^{(3)}_{k-1} + \frac{1}{2} h_k M^{(3)}_k + M^{(2)}_{k-1} - M^{(2)}_k = \frac{1}{12} h_k^2 [y^{(iv)}_k - y^{(iv)}_{k-1}].
\]
By iterating the same argument, letting \( M_k^{(1)} \equiv s_k'(x_k), \ k = 0(1), n - 1 \), from the continuity of the first derivative we have

\[
\frac{7}{20} h_k^2 M_k^{(3)} + \frac{3}{20} h_k^2 M_k^{(2)} + h_k M_k^{(2)} + M_k^{(1)} - M_k^{(1)} = \frac{1}{30} h_k^2 s_k^{(iv)} - \frac{1}{20} h_k^2 s_k^{(iv)},
\]

and from the continuity of the function, which for \( k = n \) also gives the rightmost interpolatory condition

\[
\frac{2}{13} h_k^2 M_k^{(3)} + \frac{1}{30} h_k^2 M_k^{(3)} + \frac{1}{2} h_k^2 M_k^{(2)} + h_k M_k^{(1)} = y_k - y_{k-1} + \frac{1}{120} h_k^4 y_k^{(iv)} - \frac{1}{60} h_k^4 y_k^{(iv)}.
\]

We then need to solve the system given by Eqs (7)–(9), together with some appropriate boundary conditions. Upon determination of the unknown moments \( M_j^{(j)}, j = 1, 2, 3 \), the solution can be determined from the formula

\[
s_k(x) = M_k^{(3)} \left\{ (x - x_k)^5 \frac{1}{60 h_k^2} \left[ 1 + \frac{2}{h_k} \left( x - x_{k-1} - \frac{x - x_k}{2} \right) \right] + \frac{h_k^3}{30} \right.
\]

\[+ \frac{h_k}{4} (x - x_{k-1})^2 - \frac{3 h_k^2}{20} (x - x_{k-1}) \right\}
\]

\[+ M_k^{(3)} (x - x_{k-1})^5 \frac{1}{60 h_k^2} \left[ 1 - \frac{2}{h_k} \left( x - x_k - \frac{x - x_{k-1}}{2} \right) \right] \]

\[+ y_k^{(iv)} \left\{ (x - x_k)^5 \frac{1}{60 h_k^2} \left[ x - x_{k-1} - \frac{x - x_k}{2} \right] + \frac{h_k^4}{120} \right. \]

\[ - \frac{h_k^3}{30} (x - x_{k-1}) + \frac{h_k^2}{24} (x - x_{k-1})^2 \right\}
\]

\[+ y_k^{(iv)} (x - x_{k-1})^5 \frac{1}{60 h_k^2} \left[ x - x_k - \frac{x - x_{k-1}}{2} \right] + M_k^{(2)} (x - x_{k-1})^2 \]

\[+ M_k^{(1)} (x - x_{k-1}) + y_{k-1}. \]

(10)

3. Discussion of the linear algebraic system

Before turning to the analysis of the system, we notice that the unknown vector contains \( 3n + 1 \) parameters to be determined, which agrees with the count of the equations; see (7)–(9). Also, observe that since the boundary conditions (3) are explicit, the variables \( M_0^{(3)}, M_0^{(2)}, M_0^{(1)} \) are known. Substitution of their values into the above equations allows the rewriting of the system in block form as discussed below, with suitably modified right-hand sides. In compact form it is

\[
Am = b,
\]

(11)

where the matrix \( A \) and the unknown vector \( m \) have the block structure,

\[
A = [A_{ij}], \quad i, j = 1, 2, 3,
\]

\[
m = [M_0^{(3)}, \ldots, M_n^{(3)}, M_0^{(2)}, \ldots, M_{n-1}^{(2)}, M_0^{(1)}, \ldots, M_{n-1}^{(1)}]^T \equiv [m_1^T, m_2^T, m_3^T]^T.
\]
Note the reversal of the indices in our block notation, to show the order in which variables will be solved for. Except for $A_{33} = O_{n-1}$ each block $A_{pq} \in \mathbb{R}^{(n-1)\times(n-1)}$, $p = 1, 2$, $q = 2, 3$, $A_{p1} \in \mathbb{R}^{n\times(n-1)}$, $p = 1, 2$, $A_{3q} \in \mathbb{R}^{n\times(n-1)}$, $q = 2, 3$, $A_{33} \in \mathbb{R}^{n\times n}$, is either subdiagonal or bidiagonal, as indicated below. In the notation for the zero and the identity matrices, the index emphasizes their size:

$$
A_{11} = \begin{bmatrix}
\frac{h_1}{2} & 0 & \ldots & 0 \\
\frac{h_2}{2} & \frac{h_1}{2} & 0 & \ldots \\
\frac{h_3}{2} & \frac{h_2}{2} & \frac{h_1}{2} & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & \frac{h_{n-2}}{2} & \frac{h_{n-3}}{2} & \ldots & 0 \\
0 & 0 & \frac{h_{n-1}}{2} & \ldots & \frac{h_{n-1}}{2}
\end{bmatrix},
$$

$$
A_{12} = A_{21} = \begin{bmatrix}
-1 & 0 & \ldots & 0 \\
1 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1 \\
\ldots & \ldots & \ldots & \ldots
\end{bmatrix},
$$

$$
A_{21} = \begin{bmatrix}
\frac{3h_1^2}{20} & 0 & \ldots & 0 \\
\frac{7h_2^2}{20} & \frac{3h_1^2}{20} & 0 & \ldots \\
\frac{7h_3^2}{20} & \frac{3h_2^2}{20} & \frac{3h_1^2}{20} & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & \frac{\gamma h_{n-2}^2}{20} & \frac{3h_{n-3}^2}{20} & \ldots & 0 \\
0 & 0 & \frac{\gamma h_{n-1}^2}{20} & \ldots & \frac{3h_{n-1}^2}{20}
\end{bmatrix},
$$

$$
A_{22} = \begin{bmatrix}
0 & 0 & \ldots & 0 \\
h_2 & 0 & 0 \\
0 & h_3 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 \\
0 & 0 & h_{n-1} & 0 \\
0 & 0 & 0 & h_n
\end{bmatrix},
$$

$$
A_{31} = \begin{bmatrix}
\frac{h_1^3}{30} & 0 & \ldots & 0 \\
\frac{2h_2^3}{15} & \frac{h_1^3}{30} & 0 & \ldots \\
\frac{2h_3^3}{15} & \frac{2h_2^3}{15} & \frac{h_1^3}{30} & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & \frac{2h_{n-1}^3}{15} & \frac{h_{n-1}^3}{30} & \ldots & 0 \\
0 & 0 & \frac{2h_{n-1}^3}{15} & \frac{h_{n-1}^3}{30} & \frac{h_n^3}{30}
\end{bmatrix},
$$

$$
A_{32} = \begin{bmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & \frac{h_2^3}{2} \\
0 & 0 & \ldots & \frac{h_3^3}{2} \\
\frac{h_2^3}{2} & 0 & 0 \\
0 & \frac{h_3^3}{2} & 0 & 0 \\
0 & 0 & \frac{h_2^3}{2} & 0 \\
0 & 0 & 0 & \frac{h_3^3}{2}
\end{bmatrix}.
Partitioning also the right-hand side,

\[
b_1 = \left( \frac{h_1^2}{12} \left( y_1^{(iv)} - y_0^{(iv)} \right), \frac{h_2^2}{12} \left( y_2^{(iv)} - y_1^{(iv)} \right), \ldots, \frac{h_{n-1}^2}{12} \left( y_{n-1}^{(iv)} - y_n^{(iv)} \right) \right)^T + \left( -y_0'' h_1 \frac{1}{2} - y_0'', y_0', 0, \ldots, 0 \right)^T,
\]

\[
b_{11} = \left( \frac{h_1^2}{60} \left( 2y_1^{(iv)} - 3y_0^{(iv)} \right), \frac{h_2^2}{60} \left( 2y_2^{(iv)} - 3y_1^{(iv)} \right), \ldots, \frac{h_{n-1}^2}{60} \left( 2y_{n-1}^{(iv)} - 3y_n^{(iv)} \right) \right)^T + \left( -\frac{7h_1^2}{20} y_0''' - h_1 y_0'' - y_0', y_0, 0, \ldots, 0 \right)^T,
\]

\[
b_{111} = \left( \frac{h_1^4}{120} \left( y_1^{(iv)} - 2y_0^{(iv)} \right), \frac{h_2^4}{120} \left( y_2^{(iv)} - 2y_1^{(iv)} \right), \ldots, \frac{h_{n-1}^4}{120} \left( y_{n-1}^{(iv)} - 2y_n^{(iv)} \right) \right)^T + \left( -\frac{2h_1^2}{15} y_0''' - h_1^2 y_0'' - h_1 y_0' + y_1 - y_0, y_2 - y_1, \ldots, y_{n-1} - y_n, y_n - y_{n-1} \right)^T.
\]

System (11) can be easily solved by direct forward substitution, due to its inherent structure. Let us assume to be at the kth stage of the procedure, \( k = 1(1)n - 1 \); by inductive hypothesis, \( M_k(3), M_k(2), M_k(1) \) are known. Then iteratively solve the first current equation in the row block and eliminate the kth column of the vertical block, in the following order:

1. At first use the third row block and the first vertical block;
2. Then the first row block and the second vertical block;
3. Finally the second row block and the third vertical block.

We have thus, in summary

**Theorem 3.1.** The system for the calculation of the required moments in the solution representation is nonsingular.

### 4. Stability results

We would like to show here an estimate on the inverse matrix \( A^{-1} \) of (11). The latter gives a stability result for the method, needed in the next section to show convergence. We begin with a preliminary result, with an easy verification by direct computation.

**Lemma 4.1.** The inverse of the matrix \( A_{12} = A_{23} \) is \( D \in \mathbb{R}^{(n-1) \times (n-1)} \), with

\[
D_{ij} = \begin{cases} 
-1, & j \leq i, \\
0, & i < j.
\end{cases}
\]

Also, \( \|D\|_\infty = n - 1 = O(h^{-1}) \).
Let us define at first the block matrix $U = \text{diag}(I_n, -D, -D)$. Let us further introduce the matrices $Z_j = \text{diag}(h_k)_{k=1}^{(j)}, j = n - 1, n$ and

$$E = Z_{n-1}(-D - I_{n-1}) = \begin{bmatrix} 0 & \ldots & 0 \\ h_2 & 0 \\ \vdots & \ddots & \vdots \\ h_{n-1} & \ldots & h_{n-1} \end{bmatrix}.$$ 

Furthermore, let $N \in \mathbb{R}^{n\times(n-1)}$ be the matrix $E$ bordered with a bottom row as follows:

$$N = \begin{bmatrix} 0 & \ldots & 0 \\ h_2 & 0 \\ \vdots & \ddots & \vdots \\ h_{n-1} & \ldots & h_{n-1} \\ h_n & \ldots & h_n \end{bmatrix}.$$ 

Finally, introduce also the matrix $F \in \mathbb{R}^{n\times(n-1)}$

$$F = \frac{1}{2} Z_n N = \begin{bmatrix} 0 & \ldots & 0 \\ \frac{h_2^2}{2} & 0 \\ \vdots & \ddots & \vdots \\ \frac{h_n^2}{2} & \ldots & \frac{h_n^2}{2} \end{bmatrix}.$$ 

By postmultiplication of the matrix of the system by $U$,

$$B = AU$$

we obtain the block matrix $B$ in the form

$$B_{ii} = A_{ii}, \quad i = 1, 2, 3,$$

$$B_{12} = B_{23} = -I_{n-1}, \quad B_{13} = O_{n-1},$$

$$B_{22} = E, \quad B_{33} = N, \quad B_{32} = F.$$ 

Let us moreover define the block elementary matrices $R^{(k)}, k = 1, 2, 3$ by

$$R^{(k)}_{ii} = I_{n-1}, \quad i = 1, 2, \quad k = 1, 2, 3,$$

$$R^{(1)}_{21} = E, \quad R^{(2)}_{33} = N, \quad R^{(3)}_{31} = F,$$

$$R^{(k)}_{ij} = O_{n-1} \quad \text{otherwise}.$$ 

Then,

$$C = RB,$$
where

\[ \|R\|_2 \leq K_1, \]  

\[ (14) \]

with \( K_i \) representing various constants here and in the next section. The matrix \( C \) is again a block matrix

\[ C_{12} = C_{23} = -I_{n-1}, \quad C_{11} = B_{11} \equiv A_{11}, \quad C_{13} = C_{22} = C_{32} = C_{33} = O_{n-1}. \]

The remaining two nonzero blocks are lower triangular; more specifically,

\[ C_{21} = EA_{11} + A_{21}, \quad C_{31} = (F + NE)A_{11} + NA_{21} + A_{31}. \]

For stability considerations it is important to determine the diagonal elements of the latter block:

\[ (C_{31})_{ii} = \frac{1}{30} b^3, \quad i = 1(1)n. \]

The previous transformations reduce the matrix \( A \) to the matrix \( C \), with a more manageable structure; the equivalent system can be solved again by forward substitution. In this case it is enough indeed to start by solving for the unknowns \( M_1, \ldots, M_{n-1} \) using the lower triangular block \( C_{31} \). The multiplication of these parameters by \( C_{11} \) and \( C_{21} \) allows the results to be moved to the right-hand side, so that the remaining unknowns are easily evaluated since the corresponding matrix is the identity:

\[ C_{31} m_1 = (Rb)_{11}, \quad m_1 = D [C_{11} m_1 - (Rb)_1], \quad m_{11} = D [C_{21} m_1 - (Rb)_1]. \]  

\[ (15) \]

The stability result we need here concerns the matrix \( C_{31} \). Let \( C_{31} = Z_n^3 \tilde{C}_{31} \).

Lemma 4.2. Let us define the matrix \( G = \text{diag}(\delta^{1-k}), k = 1(1)n \). Then the following estimates hold:

\[ 30 \leq \|G \tilde{C}_{31}^{-1}\|_2 \leq 30 + \delta. \]

Proof. This is a restatement of Theorem 1, [5, p.12], saying that if the spectral radius of a matrix is known, then a suitable natural norm can be found, which is arbitrarily close to this value. Observe that the proof is constructive, the starting point being the reduction of the given matrix to upper triangular form by means of a suitable transformation. This step here can be skipped, since \( \tilde{C}_{31} \) is lower triangular and so is its inverse, this of course not causing any alterations in the remainder of the proof. The result follows by noticing that the eigenvalues of \( \tilde{C}_{31} \) are all equal to \( \frac{1}{30} \), and by postmultiplication of its inverse by \( G \). The latter operation gives the two norm in the statement of the lemma, this being the sought norm. \( \square \)

Corollary 4.3. If \( \delta < 1 \), then \( \|G^{-1}\|_2 = 1 \).

Lemma 4.4. If the partition satisfies (1) then

\[ \|Z_n^{-1}\|_2 \leq O(h^{-1}). \]
Proof. Observe that the spectral radius of $Z_n^T Z_n$ is given by $[\min_{1 \leq k \leq n} (h_k^2)]^{-1}$ and then apply (1).

Remark. Since it seems very difficult to give an estimate for the norm of $C_{31}^{-1}$ in the supremum norm, the above estimate and the procedure of the next section will cause an extra factor $\frac{1}{2}$ to be lost in the final error estimate.

5. Error analysis

We begin the discussion by recalling the well-known result, (3.6.15) of [1, p.162]. If the function $f(x) \in C^4[0,h]$, then

$$|f(x) - H(x)| \leq \frac{1}{384} h^4 \|f^{(iv)}\|_\infty.$$  \hspace{1cm} (16)

The function $f(x)$ in this context needs to be replaced by $y'''(x)$; $H(x)$ denotes here the cubic Hermite polynomial constructed on the analytic values of the third and fourth derivatives, $y'''(x_k), y^{(iv)}(x_k)$. Let us recall that the third derivative of the solution obtained by our method, restricted to the interval $A_k$, is denoted by $s'''(x)$. This function is constructed using the values of the moments $M_k^{(3)}$ obtained via the algorithm presented in the previous section. Let us define the error terms

$$e^{(p)}(x) = y^{(p)}(x) - s^{(p)}(x), \quad p = 0(1)3, \quad x \in [0,1],$$

$$e_k^{(p)} = y^{(p)}(x_k) - M_k^{(p)}, \quad p = 1,2, \quad k = 1(1)n - 1, \quad p = 3, \quad k = 1(1)n.$$  

If $x \in A_k$, then we write

$$e_k''(x) = y'''(x) - H(x) + H(x) - s'''(x)$$  \hspace{1cm} (17)

and estimate the first term on the right-hand side by (16), while for the second we need an estimate for $e_k''$. We thus need to analyze the error in the solution of the system (11).

The error equation arising from it can be written as follows. Upon subtraction of the integrated Hermite polynomial on $[x_{k-1},x]$ from (6), denoting as usual by $e_k(x)$ the restriction of $e(x)$ to $A_k$, we have

$$e_k''(x) = e_k^{(3)} \left\{ \frac{(x - x_k)^3}{3h_k^2} \left[ 1 + \frac{2}{h_k} \left( x - x_{k-1} - \frac{x - x_k}{4} \right) \right] + \frac{h_k}{2} \right\},$$

$$e_k''(x) = e_k^{(3)} \left\{ \frac{(x - x_k)^3}{3h_k^2} \left[ 1 + \frac{2}{h_k} \left( x - x_{k-1} - \frac{x - x_k}{4} \right) \right] \right\}$$

$$+ e_k^{(3)} \left\{ \frac{(x - x_{k-1})^3}{3h_k^2} \left[ 1 - \frac{2}{h_k} \left( x - x_k - \frac{x - x_{k-1}}{4} \right) \right] + e_{k-1}^{(2)} + (x - x_{k-1}) K_2 h_k^4 \right\}.$$  \hspace{1cm} (18)

Imposing then the continuity of the second derivatives,

$$\frac{1}{2} h_k e_k^{(3)} + \frac{1}{2} h_k e_k^{(3)} - e_k^{(2)} + e_{k-1}^{(2)} = K_3 h_k^3.$$  \hspace{1cm} (19)
By repeating this procedure twice,
\[
\frac{1}{10} h^2_k e_k^{(3)} + \frac{7}{20} h^2_k e_k^{(3)} - h_k e_k^{(2)} + e_k^{(1)} - e_k^{(1)} = K_4 h^6_k,
\]
and
\[
\frac{1}{40} h^2_k e_k^{(3)} + \frac{1}{12} h^2_k e_k^{(3)} + \frac{1}{2} h^2_k e_k^{(3)} - h_k e_k^{(1)} = K_5 h^7_k.
\]

We notice that the right-hand side of the error equation is then composed of three parts, of orders $h^5$, $h^6$, $h^7$, respectively. For the estimation of the errors at the nodes, the same procedure outlined for solving system (11) can be used, by substituting to the original right-hand side the error vector, found in (19)–(21), and by reducing the matrix as outlined in the previous section.

Then letting $\varepsilon^{(3)}$ denote the third part of the error vector,
\[
\|e_k^{(3)}\|_2 \leq \|Z^{-1}_n\|_2 \|G C^{-1}_{31}\|_2 \|G^{-1}\|_2 \|R\|_2 \|e^{(3)}\|_2.
\]

In view of (14), Lemmas 4.2 and 4.4 and the corollary of the previous section turning to the supremum norm,
\[
\|e_k^{(3)}\|_\infty = O(h^{7/2}).
\]

Notice that substitution of the values of the third-order moments into the first block column of the matrix $C$ gives terms that are of the following orders:
\[
C_{11} e_k^{(3)} = O(h^{9/2}), \quad C_{21} e_k^{(3)} = O(h^{11/2}).
\]

For the remaining moment errors, similar considerations hold from (15). The vector $e_k^{(2)}$ is obtained from the first block row, where upon moving $C_{11} e_k^{(3)}$ to the right-hand side, the error term will be of order given by the minimum of $O(h^{9/2})$ and $O(h^6)$. Hence,
\[
C_{12} e_k^{(2)} = O(h^{9/2}),
\]
from which, in view of Lemma 4.1,
\[
\|e_k^{(2)}\|_\infty = O(h^{7/2}).
\]

Also similarly, from the second block row,
\[
C_{21} e_k^{(3)} = O(h^{11/2}),
\]
so that again,
\[
e_k^{(1)} = O(D) \min\{O(h^6), O(h^{11/2})\}.
\]

In summary we obtain

**Theorem 5.1.** For the moments obtained with the method described in the previous section, we have
\[
\|e_k^{(3)}\|_\infty = \|e_k^{(2)}\|_\infty = O(h^{7/2}), \quad \|e_k^{(1)}\|_\infty = O(h^{9/2}).
\]
From (17), on using the triangular inequality, we then have
\[ \|e''_k(x)\|_\infty = \mathcal{O}(h^{7/2}). \]

From (18), on using the device of successive integrations on \([x_{k-1}, x]\), we have
\[ e''_k(x) = \int_{x_{k-1}}^{x} e'''(t) \, dt + e^{(2)}_{k-1} \leq K h^{7/2}(x - x_{k-1}) + e^{(2)}_{k-1}, \]
so that in view of (22),
\[ e''_k(x) = \mathcal{O}(h^{7/2}). \]

Iterated integrations over the interval \([x_{k-1}, x]\) and use of (22) yield the error estimates for the remaining derivatives of \(s_k(x)\). If the data are smooth enough, we have thus established the convergence of the proposed method.

**Theorem 5.2.** Suppose that \( y \in C^{(vii)}[0, 1] \). Then the algorithm converges to the solution of the problem with rates at least given by
\[ \|y^{(p)}(x) - s^{(p)}_k(x)\|_\infty = \mathcal{O}(h^{11/2-p}), \quad p = 0(1)2. \]

**Remark.** This result tells us also the order of the finite difference method we need to use in the discretization of the original boundary value problem, if we want to solve it via this approach.

6. An example

We have applied the algorithm to the reconstruction of the function \( f(x) = 1 - \exp(x) \) over \([0, 1]\), using different number of nodes. We reproduce in Figs. 1 and 2 the values of the error, in semilogarithmic graphs. We have remarked that the condition number of the system tends to become large even for moderate values of \( n \). In Table 1 we summarize our findings. In the figures this is reflected in the fact that the error graph is skewed, the error being larger toward the right endpoint, this corresponding to the last variables solved for, thus being more contaminated by the growth of the roundoff error.

<table>
<thead>
<tr>
<th>( n )</th>
<th>Condition number</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.571D+04</td>
</tr>
<tr>
<td>3</td>
<td>0.406D+06</td>
</tr>
<tr>
<td>4</td>
<td>0.119D+08</td>
</tr>
<tr>
<td>5</td>
<td>0.293D+09</td>
</tr>
<tr>
<td>6</td>
<td>0.643D+10</td>
</tr>
<tr>
<td>7</td>
<td>0.130D+12</td>
</tr>
<tr>
<td>8</td>
<td>0.249D+13</td>
</tr>
</tbody>
</table>
7. Some extensions

We now turn our attention to some considerations on a more general problem. We consider again at first the $(0,4)$ problem, but adding some intermediate conditions. For the pure $(0,4)$ problem, looking at one subinterval $[x_{k-1}, x_k]$, for some $k = 2, \cdots, n$, there are seven conditions, given by
\( y_{k-1}^{(iv)} \) and \( y_{k-1}^{(v)} \), and by the continuity conditions at the node \( x_k \), for the function and its first four derivatives. Thus, these uniquely determine a polynomial of degree 6. The deficiency is defined by the difference between the degree of the piecewise polynomial interpolating the lacunary data and the most high continuity condition which is required. In this case thus the deficiency is \( d = 2 \). If we require now an extra intermediate condition, say on the second derivative, we will have 8 conditions to be satisfied, since in addition to the former ones, now there is also \( y_{k-1}'' \). No new continuity conditions arise at \( x_k \), since at that node \( s''(x) \) was already imposed be continuous. The polynomial on \( A_k \) is sought now of degree 7. Evidently, here the deficiency is now \( d = 3 \). The same holds if instead of the second derivative we assign the first or the third one.

Now look at the more general case of the \((0,q)\) interpolation. Here there are 2 interpolatory conditions at \( x_{k-1} \), namely \( y_{k-1}^{(i)} \) and \( y_{k-1}^{(q)} \), and \( q + 1 \) continuity conditions at \( x_k \). Thus, the deficiency is \( d = (q + 3 - 1) - q - 2 \) again, since we seek a polynomial of order \( q + 3 \), i.e., with as many coefficients. Again we could start by writing the Hermite cubic interpolant on \( y^{(q)} \) and \( y^{(q-1)} \) and then integrate it \( q - 1 \) times.

What happens with more intermediate conditions? For the \((0,4)\) problem, if we add say the third and second derivatives, there are two extra interpolatory conditions at \( y_{k-1} \), so that we obtain a polynomial of degree 8. The deficiency now becomes \( d = 8 - 4 = 4 \). Notice that this remains 4 even if we require continuity of the fifth derivative; we would have thus one more continuity condition at \( x_k \), but this also increases the degree of the sought polynomial, so that their difference remains constant. Evidently, we can repeat the argument also for demands on higher continuity conditions. For the \((0,q)\) interpolation problem, subject to \( i \) additional conditions on the intermediate derivatives, the polynomial has degree \( q + 2 + i \) and the most high continuity condition at \( x_k \) involves the \( q \)th derivative, giving

\[
d = i + 2.
\]

The problem that arises from these considerations is then the following one. Consider again the \((0,2,4)\) situation discussed above. If we want to represent its solution starting again by the Hermite cubic interpolant constructed on the values of \( y_k^{(3)} \), \( y_{k-1}^{(3)} \) and \( y_k^{(4)} \) and \( y_{k-1}^{(4)} \), we see that integrating again three times we do not obtain the required degree for the polynomial: it will be 6 instead of 7. The problem is thus how to choose correctly the degree of the initial representation from which, by successive integrations, we obtain the deficient spline solving the lacunary interpolation problem.

Consider now the general \((0,q)\) problem, with additional conditions. Suppose then that there are \( i \) intermediate conditions at \( x_k \), beside \( y_k \) and \( y_k^{(q)} \). Suppose also that we need \( k \) integrations to reconstruct the final polynomial. We will start from a generalized Hermite interpolating polynomial, meaning one that matches the functions and all its subsequent derivatives up to a certain order. Here the initial generalized Hermite interpolating polynomial must be constructed then on the derivatives at \( x_{k-1} \) and at \( x_k \) of all the orders from the \( k \)th included up to the \( q \)th one. Its degree thus will be \( 2(q - k) + 1 \), since \( 2(q - k + 1) \) are the conditions given. Integrating it \( k \) times, we obtain a polynomial of degree \( 2(q - k) + 1 + k \), which must be of the correct degree \( q + i + 2 \). Thus, by equating the two, we have

\[
k = q - i - 1.
\]

We can now decide the degree of the starting Hermite interpolating polynomial, which is \( 2i + 3 \), and the lowest derivative on which it is constructed, which is given by the value of \( k \) found above.
References