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Mixed variational principles for dynamic response of thermoelastic and poroelastic continua

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ABSTRACT

The primary objective of the present work is to make further connections between variational methods on the one hand and reversible and irreversible thermodynamics on the other. This begins with the development of a new stationary principle, involving mixed field variables, for continuum problems in infinitesimal dynamic thermoelasticity. By defining Lagrangian and dissipation functions in terms of physically-relevant contributions and invoking the Rayleigh formalism for damped systems, we are able to recover the governing equations of thermoelasticity as the Euler–Lagrange equations. This includes the balance laws of linear momentum and entropy-energy, the constitutive models for elastic response and heat conduction, and the natural boundary conditions. By including energy contributions associated with second sound phenomena, one eliminates the paradox of infinite thermal propagation speeds and the resulting set of governing equations has an elegant symmetry, which is most easily seen in the Fourier wave number domain. A related formulation for dynamic poroelasticity yields two new stationary mixed variational principles. Depending upon the selection of primary field variables, these governing equations can also exhibit an elegant structure, which can deepen our understanding of the underlying phenomena and the thermoelastic–poroelastic analogy. In addition to the theoretical significance, the variational formulations developed here can provide the basis for a class of optimization-based methods for computational mechanics.

1. Introduction

The original variational formulations for dynamical systems by Fermat, Lagrange and Hamilton dealt with conservative phenomena of discrete particle systems. For cases involving known forces derivable from a potential, this approach leads to a principle of least action, while in more general situations, the variational formulations result in only stationary principles. The development of a variational framework for such reversible mechanical systems is now standard with the formulations given in many classical monographs, including those by Lanczos (1949), Goldstein (1950) and Gelfand and Fomin (1963). Meanwhile, the extension to elastodynamic problems of continua is also well established and may be found in Goldstein (1950) and Dym and Shames (1973), among others. On the other hand, the connection of variational approaches to the thermodynamics of reversible and irreversible processes remains under development. In the current paper, we consider that connection within the context of thermoelasticity and poroelasticity and present several mixed variational principles.

Reversible thermodynamics deals with problems that are in thermodynamic equilibrium states. This limits the theory to systems that do not exchange energy with the surroundings, or, in other words, systems isolated from the environment. In a reversible process, the total entropy of the system is unchanged. Of course, many phenomena are associated with energy flow and non-equilibrium processes, including heat diffusion, chemical diffusion, electrical current flows and mass flows. The theory of irreversible or, more appropriately, non-equilibrium thermodynamics was developed to study these dissipative processes. Following the seminal work of Onsager (1931a,b), the theory is limited to non-equilibrium processes that are microscopically reversible (local equilibrium). In these papers, Onsager stated the principle of the least dissipation of energy, which plays an important role in the modern developments of non-equilibrium thermodynamics. Onsager, through statistical mechanics considerations, postulated his fundamental theorem stating that the relationships between effects (fluxes) and causes (forces) must be symmetric. Many examples of such phenomenological laws describing non-equilibrium phenomena and processes exist: Newton's law between force and velocity gradient, Fourier's law between heat flux and temperature gradient, Ohm's law between electrical current and potential gradient; Fick's law between matter flow in a mixture and concentration gradient (Casimir, 1945; de Groot, 1951; Prigogine, 1967). Following the

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approach originally proposed by Rayleigh for mechanical damping of a viscous nature, these phenomena can be brought into a variational formulation by introducing a quadratic dissipation function (Rayleigh, 1877; Marsden and Ratiu, 1994). As an alternative approach, generalized bracket formalisms also have been developed to address a broad range of dissipative processes (Kaufman, 1984; Morrison, 1984; Grmela, 1984; Beris and Edwards, 1994; Grmela and Ottinger, 1997; Ottinger and Grmela, 1997).

Significant contributions to relate the fields of thermoelasticity and irreversible thermodynamics have been made by Biot. In particular, Biot (1954) discusses viscoelastic dissipative processes within the Lagrangian framework by introducing positive definite potential and dissipation functions. He also notes that other dissipative processes governed by the theorem of Onsager (1931a,b) will follow similarly. However, he then introduces operator notation and abandons a direct time-domain extension of Lagrangian dynamics and Hamilton's principle. In Biot (1955), he derives a principle of minimum rate of entropy production, which assumes that the disequilibrium forces are constant. From one perspective, this can be viewed as the dissipative counterpart of the principle of least action. Afterward, Biot (1956a) presents a complete theory of thermoelasticity and irreversible thermodynamics from a Lagrangian viewpoint. However, he then introduces generalized coordinates and, consequently, never actually deals with tractions and heat fluxes within the Lagrangian framework. Two decades later, Biot (1973) provides a more complete presentation of the variational formulation for thermoelasticity with displacement and heat vector as primary variables. However, even in this framework, Biot introduces generalized coordinates to realize the Euler–Lagrange equations. Of course, Biot also made seminal contributions in the analogous field of poroelasticity. This includes his work on the dynamics of porous media (Biot, 1956 b,c; 1962 a,b), which is directly relevant here.

Many other attempts to propose a unified theory for dissipative and non-dissipative systems have been made. For example, Morse and Feshbach (1953) proposed the simultaneous consideration of a mirror-image of the governing equation. For dissipative systems, such as the diffusion and the heat conduction equation, with the mirror-image they defined a mirror-image system that gains the same amount of energy as the real system dissipates. Nevertheless, the calculated momentum densities had nothing to do with the physical phenomena. Vujanovic and Djukic (1972) presented a variational approach based on Hamilton's principle for nonlinear heat transfer problems. The authors assumed a Lagrangian function that, in addition to the field variables and their first derivatives, was also a function of some arbitrary functions that were not subject to variation. From the Lagrangian function, after applying the Euler–Lagrange equations, the one-dimensional heat conduction equation was derived by taking into account finite velocity of heat propagation. Kotowski (1992) reported that application of the Noether theorem to the proposed Lagrangian by Vujanovic and Djukic gave results for internal energy density, internal energy flux and stress that contradicted known theories and experiments.

In a series of papers, Anthony presented theories by means of a Lagrangian formalism for irreversible processes in dislocation dynamics (Anthony and Azirhi, 1995), in thermodynamics of irreversible processes for kinetic equations, such as the Fokker–Planck and Boltzmann equations (Sievers and Anthony, 1996), and thermodynamics of reversible and irreversible processes (Anthony, 2001). The main notion in the Anthony theory is the introduction of complex-valued fields, with which he presented an example of how to construct new fields that describe cross-effects and dissipation. In his most recent paper mentioned above (Anthony, 2001), Anthony stated his opinion that the irreversible thermodynamics can be brought within the framework of a Lagrangian formalism, even though one is faced with the dogma that this is not possible (*e.g.*, Bauer, 1931). In particular, Anthony (2001) presented a unified theory for reversible and irreversible processes by introducing complex-valued fundamental field variables instead of the traditional variables, such as temperature, mass densities and velocities. For irreversible thermodynamics, he introduced the complex-valued field of thermal excitation, or thermion field, and the complex-valued matter field. He proposed that it is possible to contain all the information for the physical irreversible process in one function, the Lagrangian function. Anthony presented one example of reversible material flow, and two examples of irreversible thermodynamical processes that were described as pure phenomena without mutual coupling. With the introduction of non-traditional complex field variables, the full significance of this approach remains unclear.

More recently, Maugin and Kalpakides (2002) presented a Hamiltonian formulation for elasticity and thermoelasticity. They studied a Lagrangian formulation for the Green and Naghdi (1993) dissipation-less thermoelasticity. In contrast to classical thermoelasticity, where heat flow is characterized by the Fourier law, in Green-Naghdi thermoelasticity, the heat flow does not include energy dissipation and the propagation of heat is assumed as thermal waves, traveling at finite speed. However, this dissipation-less model is appropriate only in very special physical circumstances.

The present work focuses on the development of several new mixed variational statements for dynamical continuum problems of thermoelasticity and poroelasticity. It is well-known that dissipative processes cannot be expressed in a Lagrangian function in the time-domain. Instead, in the present paper the irreversible phenomena associated with the above problems are incorporated through the use of dissipation functions, following the approach first introduced by Rayleigh (1877). Although the Rayleigh approach is not strictly a pure variational statement, the resulting formulations offer additional insight into these types of coupled problems and also will provide the basis for the subsequent development of novel optimization-based computational methods. The remainder of the paper is organized as follows. In Section 2, we present the mixed thermoelastic variational formulation by introducing the primary field variables, defining the Lagrangian and dissipation functions, establishing the first variation of the action and extracting the Euler–Lagrange equations. Afterward, we transform the governing equations to the Fourier wave number domain to explore the character from a different perspective. The corresponding mixed variational statement for poroelasticity is the focus of Section 3. Here two different forms are considered, with the second version elucidating further symmetries. This latter form also leads to closer analogies with the thermoelastic formulation, which is discussed more fully in Section 4. Finally, several overall conclusions are provided in Section 5.

2. Mixed Lagrangian Formalism (MLF) for thermoelasticity

2.1. Primary variables

For a general mixed formulation of infinitesimal thermoelasticity, let the mechanical response be represented by the displacement u_i and the impulse of the elastic stresses J_{ij}. Meanwhile, the thermal field is described in terms of θ, the impulse of the temperature T and the heat vector H_i. For consistency, one can view displacement u_i as the impulse of the velocity v_i and the heat vector H_i as the impulse of the heat flux q_i. Thus, in integral form, we may write

$$u_i(t) = \int_0^t v_i(t)dt \tag{1a}$$

$$J_{ij}(t) = \int_0^t (\sigma_{ij}(t) + \beta_{ij}T(t))dt = \int_0^t \sigma^e_{ij}(t)dt = \int_0^t C_{ijkl}\varepsilon_{kl}(t)dt \tag{1b}$$

$$\theta(t) = \int_0^t T(t)dt \tag{1c}$$

$$H_i(t) = \int_0^t q_i(t)dt \tag{1d}$$

or in rate form

$$\dot{u}_i = v_i \tag{2a}$$

$$\dot{J}_{ij} = (\sigma_{ij} + \beta_{ij}T) = \sigma^e_{ij} = C_{ijkl}\varepsilon_{kl} \tag{2b}$$

$$\dot{\theta} = T \tag{2c}$$

$$\dot{H}_i = q_i \tag{2d}$$

where σ_{ij} and ε_{ij} represent the total stress and strain tensors, while C_{ijkl} and β_{ij} are the usual constitutive tensors for anisotropic thermoelastic media. Additionally, in (1b) and (2b), σ^e_{ij} are the purely elastic stresses associated with the total strains. By selecting the primary variables in this manner, we shall find that an elegant symmetry obtains in the governing equations of thermoelasticity. We focus next on the formulation of a mixed variational statement.

2.2. Lagrangian, applied load potential and dissipation functions

In order to extend the mixed variational formulation for mechanical systems by Sivaselvan and Reinhorn (2006) and Sivaselvan et al. (2009) to coupled dynamic thermoelasticity, we must define Lagrangian L, applied load potential V and dissipation F functions. The objective here is to describe these functions in terms of physically meaningful contributions representing the non-dissipative and dissipative processes associated with thermoelastic response. Let L_Ω represent the volumetric Lagrangian. Meanwhile, let V_Ω denote the potential of the applied body forces and heat sources, with V_{Γ_τ} and V_{Γ_q} as the surface potentials due to applied natural boundary conditions of traction and heat flux, respectively. On the other hand, the dissipative processes are assumed to be described exclusively by volumetric contributions defined in F_Ω, although more generally some dissipative surface terms also could be postulated. The specific mixed forms adopted in the present work can be written as follows:

$$L_\Omega = \frac{1}{2}\rho_o \dot{u}_k \dot{u}_k + \frac{1}{2}A_{ijkl}\dot{J}_{ij}\dot{J}_{kl} + (J_{ij} - \beta_{ij}\theta)B_{ijk}\dot{u}_k + \frac{1}{2}\frac{\rho_o c_\varepsilon}{T_o}\dot{\theta}^2 + \frac{1}{2}\lambda_{ij}\tau_o\frac{1}{T_o}\dot{H}_i\dot{H}_j - \frac{1}{T_o}H_i D_i\dot{\theta} \quad \text{in} \quad \Omega \tag{3}$$

$$V_\Omega = \widehat{f}_k u_k + \frac{1}{T_o}\widehat{\Psi}\theta \quad \text{in} \quad \Omega \tag{4a}$$

$$V_{\Gamma_\tau} = \widehat{\tau}_k u_k \quad \text{on} \quad \Gamma_\tau \tag{4b}$$

$$V_{\Gamma_q} = -\frac{1}{T_o}\widehat{q}^{(n)}\theta \quad \text{on} \quad \Gamma_q \tag{4c}$$

$$F_\Omega = \frac{1}{2}\lambda_{ij}\frac{1}{T_o}\dot{H}_i\dot{H}_j \quad \text{in} \quad \Omega \tag{5}$$

Here, T_o is the initial temperature at the free stress state, while T becomes the temperature change from that state. Additionally, ρ_o is the mass density, f_k represents a specified body force density, c_ε is the specific heat coefficient, Ψ is a specified heat source, and τ_o is a relaxation time for the extended Fourier's heat conduction law. Furthermore, $\widehat{\tau}_k$ are the tractions specified on the portion of the surface Γ_τ, while $\widehat{q}^{(n)}$ represent the specified normal heat fluxes on Γ_q. The constitutive tensors A_{ijkl} and λ_{ij} are the inverses of C_{ijkl} and the conductivity κ_{ij}, respectively. Finally, D_i and B_{ijk} are differential operators. The former represents the spatial gradient operator, while the latter third order tensor operator extracts strain rates from the velocity field. Thus, we have

$$\dot{\varepsilon}_{ij} = B_{ijk}\dot{u}_k \tag{6}$$

with

$$B_{ijk} = \frac{1}{2}(\delta_{ik}\delta_{jq} + \delta_{iq}\delta_{jk})\frac{\partial}{\partial x_q} \tag{7}$$

where x_q represent the spatial coordinates.

2.3. Action functional and the first variation

With these definitions, the action functionals I_L and I_V associated with the Lagrangian and potential contributions become

$$I_L = -\int_0^{t_f}\int_\Omega L_\Omega \, d\Omega \, dt \tag{8}$$

$$I_V = I_{V_\Omega} + I_{V_\Gamma} = -\int_0^{t_f}\int_\Omega V_\Omega \, d\Omega \, dt - \int_0^{t_f}\int_{\Gamma_\tau} V_{\Gamma_\tau} \, d\Gamma \, dt - \int_0^{t_f}\int_{\Gamma_q} V_{\Gamma_q} \, d\Gamma \, dt \tag{9}$$

where I_{V_Ω} represents the action due to volumetric applied forces and sources, while I_{V_Γ} is the portion of the action corresponding to applied surface mechanical and thermal loading.

Although the action functional I_F corresponding to the dissipation function cannot be written in explicit form, following the approach originally adopted by Rayleigh (1877), and subsequently by Biot (1954, 1955) within the context of thermoelasticity, the first variation of the total action may be written:

$$\delta I = \delta I_L + \delta I_V + \delta I_F \tag{10}$$

or by considering (5), (8), and (9)

$$\delta I = -\delta\int_0^{t_f}\int_\Omega L_\Omega \, d\Omega \, dt - \delta\int_0^{t_f}\int_\Omega V_\Omega \, d\Omega \, dt + \int_0^{t_f}\int_\Omega \frac{\partial F_\Omega}{\partial \dot{H}_i}\delta H_i \, d\Omega \, dt - \delta\int_0^{t_f}\int_{\Gamma_\tau} V_{\Gamma_\tau} \, d\Gamma \, dt - \delta\int_0^{t_f}\int_{\Gamma_q} V_{\Gamma_q} \, d\Gamma \, dt \tag{11}$$

Then, after substituting the detailed expressions in (4a-c) and (5), this becomes

$$\delta I = -\delta\int_0^{t_f}\int_\Omega L_\Omega \, d\Omega \, dt - \int_0^{t_f}\int_\Omega \widehat{f}_k \delta u_k \, d\Omega \, dt - \int_0^{t_f}\int_\Omega \frac{1}{T_o}\widehat{\Psi}\,\delta\theta \, d\Omega \, dt + \int_0^{t_f}\int_\Omega \lambda_{ij}\frac{1}{T_o}\dot{H}_j\delta H_i \, d\Omega \, dt - \int_0^{t_f}\int_{\Gamma_\tau}\widehat{\tau}_k\delta u_k \, d\Gamma \, dt + \int_0^{t_f}\int_{\Gamma_q}\frac{1}{T_o}\widehat{q}^{(n)}\delta\theta \, d\Gamma \, dt \tag{12}$$

Now all that remains is to address the specific terms in the first integral on the right-hand side of (12). After substituting (3) into this integrand, we may write

$$-\delta\int_0^{t_f}\int_\Omega L_\Omega \, d\Omega \, dt = -\delta\int_0^{t_f}\int_\Omega \left(\frac{1}{2}\rho_o\dot{u}_k\dot{u}_k + \frac{1}{2}A_{ijkl}\dot{J}_{ij}\dot{J}_{kl} + (J_{ij} - \beta_{ij}\theta)B_{ijk}\dot{u}_k\right)d\Omega \, dt - \delta\int_0^{t_f}\int_\Omega \left(\frac{1}{2}\frac{\rho_o c_\varepsilon}{T_o}\dot{\theta}^2 + \frac{1}{2}\lambda_{ij}\tau_o\frac{1}{T_o}\dot{H}_i\dot{H}_j - \frac{1}{T_o}H_i D_i\dot{\theta}\right)d\Omega \, dt \tag{13}$$

The integral on the right-hand side of (13) has six terms. The treatment of two representative terms is presented below in detail, while the derivation of the other terms follows in a straight forward manner.

Term 1: By applying integration by parts, we find

$$-\delta\int_0^{t_f}\int_\Omega \frac{1}{2}\rho_o\dot{u}_k\dot{u}_k d\Omega dt = -\int_0^{t_f}\int_\Omega \rho_o\dot{u}_k\delta\dot{u}_k d\Omega dt = -\left[\int_\Omega \rho_o\dot{u}_k\delta u_k d\Omega\Big|_0^{t_f} - \int_0^{t_f}\int_\Omega \rho_o\ddot{u}_k\delta u_k d\Omega dt\right] = \int_0^{t_f}\int_\Omega \rho_o\ddot{u}_k\delta u_k d\Omega dt \tag{14}$$

where the final form is obtained by assuming δu_k is zero at the two ends of the time interval, as is usually done in Hamilton's principle.

Term 3: This third term is first separated into two contributions. Thus, we let

$$-\delta\int_0^{t_f}\int_\Omega (J_{ij}-\beta_{ij}\theta)B_{ijk}\dot{u}_k d\Omega dt = -\delta\int_0^{t_f}\int_\Omega J_{ij}B_{ijk}\dot{u}_k d\Omega dt + \delta\int_0^{t_f}\int_\Omega \beta_{ij}\theta B_{ijk}\dot{u}_k d\Omega dt \tag{15}$$

From (6) and the symmetry condition $J_{ij}=J_{ji}$, we have

$$J_{ij}B_{ijk}\dot{u}_k = J_{ij}\dot{u}_{i,j} \tag{16}$$

and the first integral on the right-hand side of (15) can be rewritten

$$-\delta\int_0^{t_f}\int_\Omega J_{ij}B_{ijk}\dot{u}_k d\Omega dt = -\int_0^{t_f}\int_\Omega \delta J_{ij}B_{ijk}\dot{u}_k d\Omega dt - \int_0^{t_f}\int_\Omega J_{ij}\delta\dot{u}_{i,j} d\Omega dt \tag{17}$$

Then, applying integration by parts and the divergence theorem to operate on the spatial derivative in the last term in (17), we find

$$-\int_0^{t_f}\int_\Omega J_{ij}\delta\dot{u}_{i,j} d\Omega dt = -\int_0^{t_f}\int_\Omega (J_{ij}\delta\dot{u}_i)_{,j} d\Omega dt + \int_0^{t_f}\int_\Omega J_{ij,j}\delta\dot{u}_i d\Omega dt = -\int_0^{t_f}\int_\Gamma J_{ij}n_j\delta\dot{u}_i d\Gamma dt + \int_0^{t_f}\int_\Omega J_{ij,j}\delta\dot{u}_i d\Omega dt \tag{18}$$

Since this still involves temporal derivatives of the displacement variations (*i.e.*, $\delta\dot{u}_k$), we use another integration by parts, but now over time, to produce

$$-\int_0^{t_f}\int_\Omega J_{ij}\delta\dot{u}_{i,j} d\Omega dt = -\int_\Gamma J_{ij}n_j\delta u_i d\Gamma\Big|_0^{t_f} + \int_0^{t_f}\int_\Gamma \dot{J}_{ij}n_j\delta u_i d\Gamma dt + \int_\Omega J_{ij,j}\delta u_i d\Omega\Big|_0^{t_f} - \int_0^{t_f}\int_\Omega \dot{J}_{ij,j}\delta u_i d\Omega dt \tag{19}$$

After letting the variations of δu_k equal zero at the beginning and end of the time interval and then substituting into (17), we have

$$-\delta\int_0^{t_f}\int_\Omega J_{ij}B_{ijk}\dot{u}_k d\Omega dt = -\int_0^{t_f}\int_\Omega B_{ijk}\dot{u}_k\delta J_{ij} d\Omega dt + \int_0^{t_f}\int_\Gamma \dot{J}_{ij}n_j\delta u_i d\Gamma dt - \int_0^{t_f}\int_\Omega \dot{J}_{ij,j}\delta u_i d\Omega dt \tag{20}$$

A similar manipulation for the second integral on the right-hand side of (15), with symmetric β_{ij}, eventually leads to

$$\delta\int_0^{t_f}\int_\Omega \beta_{ij}\theta B_{ijk}\dot{u}_k d\Omega dt = \int_0^{t_f}\int_\Omega \beta_{ij}B_{ijk}\dot{u}_k\delta\theta d\Omega dt - \int_0^{t_f}\int_\Gamma \beta_{ij}\dot{\theta}n_j\delta u_i d\Gamma dt + \int_0^{t_f}\int_\Omega \beta_{ij}\dot{\theta}_{,j}\delta u_i d\Omega dt \tag{21}$$

Substituting the expressions of all the derived terms above into (13) and then (12) provides the statement for the first variation of the action δI. After grouping terms containing the variations of the primary mixed variables δu_k, δJ_{ij}, δH_i, and $\delta\theta$, we may write:

$$\begin{aligned}\delta I = &\int_0^{t_f}\int_\Omega \left(\rho_o\ddot{u}_k - \dot{J}_{kj,j} + \beta_{kj}\dot{\theta}_{,j} - \widehat{f}_k\right)\delta u_k d\Omega dt\\ &+\int_0^{t_f}\int_\Omega \left(A_{ijk}\ddot{J}_{kl} - B_{ijk}\dot{u}_k\right)\delta J_{ij} d\Omega dt\\ &+\int_0^{t_f}\int_\Omega \left(\beta_{ij}B_{ijk}\dot{u}_k + \frac{\rho_o c_\varepsilon}{T_o}\ddot{\theta} + \frac{1}{T_o}\dot{H}_{i,i} - \frac{1}{T_o}\widehat{\Psi}\right)\delta\theta\, d\Omega dt\\ &+\int_0^{t_f}\int_\Omega \left(\frac{1}{T_o}\dot{\theta}_{,i} + \lambda_{ij}\tau_o\frac{1}{T_o}\ddot{H}_j + \lambda_{ij}\frac{1}{T_o}\dot{H}_j\right)\delta H_i\, d\Omega dt\\ &+\int_0^{t_f}\int_\Gamma \left(\dot{J}_{kj}n_j - \beta_{kj}\dot{\theta}n_j\right)\delta u_k d\Gamma dt - \int_0^{t_f}\int_{\Gamma_\tau}\widehat{\tau}_k\delta u_k d\Gamma dt\\ &-\int_0^{t_f}\int_\Gamma \frac{1}{T_o}\dot{H}_i n_i\delta\theta\, d\Gamma dt + \int_0^{t_f}\int_{\Gamma_q}\frac{1}{T_o}\widehat{q}^{(n)}\delta\theta\, d\Gamma dt\end{aligned} \tag{22}$$

Recall that Γ_τ and Γ_q are the portions of the surface on which tractions and heat flux boundary conditions are specified. Next, define Γ_u and Γ_θ as the portions of the surface with prescribed displacements and temperatures, respectively. More generally, one also could have compliant or convection boundary conditions. Although, this extension is not difficult to accommodate within the proposed framework, for simplicity we take

$$\Gamma_\tau \cup \Gamma_u = \Gamma \tag{23a}$$

$$\Gamma_\tau \cap \Gamma_u = \emptyset \tag{23b}$$

$$\Gamma_q \cup \Gamma_\theta = \Gamma \tag{24a}$$

$$\Gamma_q \cap \Gamma_\theta = \emptyset \tag{24b}$$

2.4. Principle of stationary action

Now we seek the conditions required for stationarity of the thermoelastic action, such that

$$\delta I = 0 \tag{25}$$

along with arbitrary variations of the elastic stress impulse δJ_{ij} and the heat vector δH_i, and kinematically-compatible displacement variations δu_k and thermally-compatible temperature variations $\delta\theta$. Enforcing these conditions on (22) yields the following Euler–Lagrange equations governing thermoelasticity over the domain Ω

$$\rho_o\ddot{u}_k - B_{ijk}(\dot{J}_{ij} - \beta_{ij}\dot{\theta}) = \widehat{f}_k \tag{26a}$$

$$A_{ijk}\ddot{J}_{kl} - B_{ijk}\dot{u}_k = 0 \tag{26b}$$

$$\frac{\rho_o c_\varepsilon}{T_o}\ddot{\theta} + \frac{1}{T_o}D_i\dot{H}_i + \beta_{ij}B_{ijk}\dot{u}_k = \frac{1}{T_o}\widehat{\Psi} \tag{26c}$$

$$\lambda_{ij}\tau_o\frac{1}{T_o}\ddot{H}_i + \lambda_{ij}\frac{1}{T_o}\dot{H}_i + \frac{1}{T_o}D_j\dot{\theta} = 0 \tag{26d}$$

and on the boundary

$$\dot{J}_{kj}n_j - \beta_{kj}\dot{\theta}n_j = \widehat{\tau}_k \Rightarrow \sigma_{ij}n_j = \widehat{\tau}_k \quad \text{on} \quad \Gamma_\tau \tag{27a}$$

$$\delta u_k = 0 \quad \text{on} \quad \Gamma_u \tag{27b}$$

$$\frac{1}{T_o}\dot{H}_i n_i = \frac{1}{T_o}\widehat{q}^{(n)} \quad \text{on} \quad \Gamma_q \tag{27c}$$

$$\delta\theta = 0 \quad \text{on} \quad \Gamma_\theta \tag{27d}$$

Equation (26a) represents linear momentum balance, (26b) is the linear elastic constitutive relation in rate form and (26c) is the entropy-energy balance equation with

$$\dot{S} = -\frac{1}{T_o}\dot{H}_{i,i} + \frac{1}{T_o}\widehat{\Psi} \tag{28}$$

as the entropy density rate. The fourth governing equation, (26d) represents the extended Fourier's law of heat conduction (Chester, 1963).

We also should note that the symmetry of $\dot{J}_{ij}$ and β_{ij} have permitted the reintroduction of the differential tensor operator B_{ijk} in (26a). Similarly, the gradient operator D_i is used to denote spatial derivatives in (26b) and (26d). Additionally, we should mention that due to the presence of these differential operators in the Lagrangian definition, the usual simplified formulas cannot be invoked to construct the Euler–Lagrange equations. Instead, spatial integration by parts operations are needed to create the governing equations and boundary conditions, as was demonstrated above.

To summarize, in this section, we have developed a *mixed variational principle for thermoelasticity*, which recovers all of the governing equations and natural boundary conditions of the extended theory. In the absence of heat conduction, all internal processes are conservative and a true stationary principle for the action $I_L + I_V$ can be developed. When energy dissipation associated with heat conduction is included, we follow the approach of Rayleigh to construct the first variation of the action, even though the action itself cannot be defined in explicit form.

Using the ideas presented in Apostolakis and Dargush (2012), this new stationary principle for thermoelasticity can lead to the development of a minimum principle in discrete form with the proper selection of state variables and temporal action sum operators. Details on this and the corresponding mixed finite element method will be presented elsewhere.

2.5. Fourier domain and symplectic form

An interesting aspect of the character of thermoelasticity can be seen by writing the governing differential equations (26a-d) in terms of the rate variables v_j, σ^e_{mn}, T, q_s and then transforming into Fourier wave number space. The resulting equations can be expressed in the following form:

$$\begin{bmatrix} \rho_o\delta_{ij} & 0 & 0 & 0 \\ 0 & A_{klmn} & 0 & 0 \\ 0 & 0 & \rho_o\bar{c}_\varepsilon & 0 \\ 0 & 0 & 0 & \tau_0\bar{\lambda}_{rs} \end{bmatrix} \begin{Bmatrix} \dot{\tilde{v}}_j \\ \dot{\tilde{\sigma}}^e_{mn} \\ \dot{\tilde{T}} \\ \dot{\tilde{q}}_s \end{Bmatrix} = \begin{bmatrix} 0 & i\tilde{B}_{mni} & -i\tilde{B}_{pqi}\beta_{pq} & 0 \\ i\tilde{B}_{klj} & 0 & 0 & 0 \\ -i\tilde{B}_{pqj}\beta_{pq} & 0 & 0 & -i\tilde{D}_s \\ 0 & 0 & -i\tilde{D}_r & -\bar{\lambda}_{rs} \end{bmatrix} \begin{Bmatrix} \tilde{v}_j \\ \tilde{\sigma}^e_{mn} \\ \tilde{T} \\ \tilde{q}_s \end{Bmatrix} + \begin{Bmatrix} \tilde{f}_j \\ 0 \\ \tilde{\Psi} \\ 0 \end{Bmatrix} \tag{29}$$

where

$$\tilde{B}_{ijk} = \frac{1}{2}(\delta_{ik}k_j + \delta_{jk}k_i) \tag{30a}$$

$$\tilde{D}_i = \frac{k_i}{T_o} \tag{30b}$$

$$\bar{c}_\varepsilon = \frac{c_\varepsilon}{T_o} \tag{30c}$$

$$\bar{\lambda}_{rs} = \frac{\lambda_{rs}}{T_o} \tag{30d}$$

with k_i as the Fourier wave number vector. By introducing Voigt notation for stress quantities, the left-hand side coefficients in (29) can be formatted as a symmetric, block diagonal matrix. Meanwhile, on the right-hand side, the corresponding matrix of coefficients can be decomposed into two sets of components to produce the following set of equations:

$$\begin{bmatrix} \rho_o\mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{A} & 0 & 0 \\ 0 & 0 & \rho_o\bar{c}_\varepsilon\mathbf{I} & 0 \\ 0 & 0 & 0 & \tau_0\bar{\lambda} \end{bmatrix} \begin{Bmatrix} \dot{\tilde{\mathbf{v}}} \\ \dot{\tilde{\boldsymbol{\sigma}}}^e \\ \dot{\tilde{\mathbf{T}}} \\ \dot{\tilde{\mathbf{q}}} \end{Bmatrix} = \begin{bmatrix} 0 & i\tilde{\mathbf{B}}^T & -i\boldsymbol{\beta}\tilde{\mathbf{B}} & 0 \\ i\tilde{\mathbf{B}} & 0 & 0 & 0 \\ -i\tilde{\mathbf{B}}^T\boldsymbol{\beta}^T & 0 & 0 & -i\tilde{\mathbf{D}}^T \\ 0 & 0 & -i\tilde{\mathbf{D}} & 0 \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{v}} \\ \tilde{\boldsymbol{\sigma}}^e \\ \tilde{\mathbf{T}} \\ \tilde{\mathbf{q}} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\boldsymbol{\lambda}} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{v}} \\ \tilde{\boldsymbol{\sigma}}^e \\ \tilde{\mathbf{T}} \\ \tilde{\mathbf{q}} \end{Bmatrix} + \begin{Bmatrix} \tilde{\mathbf{f}} \\ 0 \\ \tilde{\Psi} \\ 0 \end{Bmatrix} \tag{31}$$

Now in (31), the first set of terms on the right-hand side includes all of the conservative thermoelastic processes represented through a skew-Hermitian coefficient matrix, which portrays the symplectic nature of these contributions. Meanwhile, the first set of terms on the second line contains the contribution from the non-conservative process of heat conduction. Notice that this latter process is represented by a negative definite Hermitian matrix that provides the energy dissipation.

3. Mixed Lagrangian Formalism (MLF) for poroelasticity

3.1. Primary variables

For a poroelastic continuum under infinitesimal poroelasticity, let v_i and $\bar{\sigma}_{ij}$ represent the velocity and effective stress of the solid skeleton, respectively. Meanwhile, for the pore fluid, let p and q_i denote the pore pressure and the average velocity relative to the solid skeleton, respectively. Then, the impulses of these four quantities are defined as u_i, J_{ij}, π and w_i, respectively, where

$$u_i(t) = \int_0^t v_i(t)dt \tag{32a}$$

$$J_{ij}(t) = \int_0^t \bar{\sigma}_{ij}(t)dt = \int_0^t C_{ijkl}\varepsilon_{kl}(t)dt \tag{32b}$$

$$\pi(t) = \int_0^t p(t)dt \tag{32c}$$

$$w_i(t) = \int_0^t q_i(t)dt \tag{32d}$$

For example, u_i is the solid skeleton displacement and w_i represents the average pore fluid displacement relative to the solid skeleton. A number of dynamic poroelastic formulations are written in terms of u_i and w_i as primary variables (e.g., Biot, 1962b;

Predeleanu, 1984; Manolis and Beskos, 1989). Here, we consider mixed formulations written in terms of all four variables. Furthermore, in rate form, we have for these variables

$$\dot{u}_i = v_i \tag{33a}$$

$$\dot{J}_{ij} = \dot{\bar{\sigma}}_{ij} = C_{ijkl}\dot{\varepsilon}_{kl} \tag{33b}$$

$$\dot{\pi} = p \tag{33c}$$

$$\dot{w}_i = q_i \tag{33d}$$

where ε_{ij} represents the total strain tensor, while C_{ijkl} is the linear elastic constitutive tensor for the solid skeleton in terms of drained properties. In addition, the total stress σ_{ij} can be written in terms of the effective stress and pore pressure as

$$\sigma_{ij} = \bar{\sigma}_{ij} - \beta_{ij} p \tag{34}$$

with β_{ij} representing a constitutive tensor for anisotropic poroelastic media relating to compressibility of the two-phase mixture. For the isotropic case, $\beta_{ij} = \beta\delta_{ij}$.

3.2. Principle of stationary action

Following an approach similar to that taken for thermoelasticity, we begin by adopting the Lagrangian, potential and dissipation functions in the form:

$$L_\Omega = \frac{1}{2}(1-n)\rho_s \dot{u}_k \dot{u}_k + \frac{1}{2}A_{ijkl}\dot{J}_{ij}\dot{J}_{kl} + (J_{ij} - \beta_{ij}\pi)B_{ijk}\dot{u}_k + \frac{1}{2}\frac{1}{Q}\dot{\pi}^2 + \frac{1}{2}n\rho_f\left(\dot{u}_k + \frac{\dot{w}_k}{n}\right)\left(\dot{u}_k + \frac{\dot{w}_k}{n}\right) - w_i D_i \dot{\pi} \quad \text{in} \quad \Omega \tag{35}$$

$$V_\Omega = \widehat{f}_k u_k + \widehat{\Gamma}\,\pi \quad \text{in} \quad \Omega \tag{36a}$$

$$V_{\Gamma_\tau} = \widehat{\tau}_k u_k \quad \text{on} \quad \Gamma_\tau \tag{36b}$$

$$V_{\Gamma_q} = -\widehat{q}^{(n)}\pi \quad \text{on} \quad \Gamma_q \tag{36c}$$

$$F_\Omega = \frac{1}{2}\lambda_{ij}\dot{w}_i\dot{w}_j \quad \text{in} \quad \Omega \tag{37}$$

where ρ_s and ρ_f represent the mass density of the solid and fluid, respectively, while n is the porosity and Q is the Biot parameter to account for compressibility of the two-phase mixture. Additionally, in (36), $\widehat{f}_k$ represents a specified body force density, $\widehat{\Gamma}$ is a specified volumetric body source rate, $\widehat{\tau}_k$ are the tractions specified on the portion of the surface Γ_τ, while $\widehat{q}^{(n)}$ represents the specified normal relative fluid volume discharge on Γ_q. The constitutive tensors A_{ijkl} in (39) and λ_{ij} in (37) are the inverses of the elastic moduli of the solid skeleton C_{ijkl} and the permeability κ_{ij}, respectively. The permeability, in turn, can be written as $\kappa_{ij} = k_{ij}/\eta$, where k_{ij} and η represent the specific permeability and pore fluid viscosity, respectively. Finally, D_i and B_{ijk} represent the same differential operators that were defined previously for thermoelasticity.

Proceeding as in thermoelasticity, the action functionals I_L and I_V associated with the Lagrangian and potential contributions can be written

$$I_L = -\int_0^{t_f}\int_\Omega L_\Omega d\Omega\, dt \tag{38}$$

$$I_V = I_{V_\Omega} + I_{V_\Gamma} = -\int_0^{t_f}\int_\Omega V_\Omega d\Omega\, dt - \int_0^{t_f}\int_{\Gamma_\tau} V_{\Gamma_\tau} d\Gamma\, dt - \int_0^{t_f}\int_{\Gamma_q} V_{\Gamma_q} d\Gamma\, dt \tag{39}$$

where I_{V_Ω} represents the action due to volumetric applied forces and sources, while I_{V_Γ} is the portion of the action corresponding to applied surface traction and normal fluid discharge loading. As a result, the first variation of the total action may be written:

$$\delta I = \delta I_L + \delta I_V + \delta I_F \tag{40}$$

with δI_F representing the variation associated with the dissipation function F_Ω in (37). By considering (35), (36a–c), (37)–(39), we have

$$\delta I = -\delta\int_0^{t_f}\int_\Omega L_\Omega d\Omega\, dt - \delta\int_0^{t_f}\int_\Omega V_\Omega d\Omega\, dt + \int_0^{t_f}\int_\Omega \frac{\partial F_\Omega}{\partial \dot{w}_i}\delta w_i d\Omega\, dt - \delta\int_0^{t_f}\int_{\Gamma_\tau} V_{\Gamma_\tau} d\Gamma\, dt - \delta\int_0^{t_f}\int_{\Gamma_q} V_{\Gamma_q} d\Gamma\, dt \tag{41}$$

Then, we seek the conditions required for stationarity of the poroelastic action, such that

$$\delta I = 0 \tag{42}$$

First, we substitute (35), (36a–c), (37) into (41) and apply integration by parts and the divergence theorem, where appropriate, as in the thermoelastic case. Then, we assume arbitrary variations of the elastic stress impulse δJ_{ij} and the average relative pore fluid displacement δw_i, along with kinematically-compatible displacement variations δu_k and hydraulically-compatible pore pressure impulse variations $\delta\pi$. As a result, we obtain the Euler–Lagrange equations governing poroelasticity over the domain Ω in the following form:

$$\rho_o \ddot{u}_k + \rho_f \ddot{w}_k - B_{ijk}(\dot{J}_{ij} - \beta_{ij}\dot{\pi}) = \widehat{f}_k \tag{43a}$$

$$A_{ijkl}\ddot{J}_{kl} - B_{ijk}\dot{u}_k = 0 \tag{43b}$$

$$\frac{1}{Q}\ddot{\pi} + D_i\dot{w}_i + \beta_{ij}B_{ijk}\dot{u}_k = \widehat{\Gamma} \tag{43c}$$

$$\frac{\rho_f}{n}\ddot{w}_j + \rho_f\ddot{u}_j + \lambda_{ij}\dot{w}_i + D_j\dot{\pi} = 0 \tag{43d}$$

and over the surface as the conditions:

$$\delta u_k = 0 \quad \text{on} \quad \Gamma_u \tag{44a}$$

$$\dot{J}_{kj}n_j - \beta_{kj}\dot{\pi}n_j = \sigma_{ij}n_j = \widehat{\tau}_k \quad \text{on} \quad \Gamma_{\widehat{\tau}} \tag{44b}$$

$$\delta\pi = 0 \quad \text{on} \quad \Gamma_\pi \tag{44c}$$

$$\dot{w}_i n_i = \widehat{q}^{(n)} \quad \text{on} \quad \Gamma_{\widehat{q}} \tag{44d}$$

where all quantities have been previously defined, except ρ_o which represents the mass density of the solid–fluid mixture. Thus,

$$\rho_o = (1-n)\rho_s + n\rho_f \tag{45}$$

Notice that equation (43a) represents linear momentum balance, (43b) is the linear elastic constitutive relation in rate form and (43c) is the pore fluid balance equation with

$$\dot{\zeta} = -\dot{w}_{i,i} + \widehat{\Gamma} \tag{46}$$

as the fluid content rate. The remaining governing equation (43d) represents an extended Darcy's law for pore fluid flow.

Thus, we have developed a *mixed variational principle for poroelasticity*, which recovers all of the governing differential equations and natural boundary conditions of the dynamical theory. In the absence of pore fluid viscosity, all internal processes are conservative and a true stationary principle for the action $I_L + I_V$ can be developed. When viscosity is included, energy dissipation occurs and the first variation of the action is developed, based upon the approach due to Rayleigh.

We should note, however, that (43a) and (43d) involve second order derivatives with respect to both the solid skeleton displacement u_i and the average relative displacement of the pore fluid w_i. This is unlike the thermoelastic result in (26a-d), in which each equation contains second derivatives of only a single variable. In order to seek a similar formulation in poroelasticity, let us rewrite the action in terms of the average (total) fluid displacement W_i, where

$$W_i = u_i + \frac{w_i}{n} \tag{47}$$

Then, we may introduce Q_i as the average pore fluid velocity, such that

$$W_i(t) = \int_0^t Q_i(t)dt \tag{48a}$$

$$\dot{W}_i = Q_i \tag{48b}$$

The Lagrangian, potential and dissipation functions now become

$$L_\Omega = \frac{1}{2}(1-n)\rho_s \dot{u}_k \dot{u}_k + \frac{1}{2}A_{ijk}\dot{J}_{ij}\dot{J}_{kl} + (J_{ij} - \beta_{ij}\pi)B_{ijk}\dot{u}_k + \frac{1}{2}\frac{1}{Q}\dot{\pi}^2 + \frac{1}{2}n\rho_f \dot{W}_k \dot{W}_k - n(W_i - u_i)D_i\dot{\pi} \quad \text{in} \quad \Omega \tag{49}$$

$$V_\Omega = \widehat{f}_k u_k + \widehat{\Gamma}\,\pi \quad \text{in} \quad \Omega \tag{50a}$$

$$V_{\Gamma_\tau} = \widehat{\tau}_k u_k \quad \text{on} \quad \Gamma_\tau \tag{50b}$$

$$V_{\Gamma_q} = -\widehat{Q}^{(n)}\pi \quad \text{on} \quad \Gamma_q \tag{50c}$$

$$F_\Omega = \frac{1}{2}\lambda_{ij} n^2 (\dot{W}_i - \dot{u}_i)(\dot{W}_j - \dot{u}_j) \quad \text{in} \quad \Omega \tag{51}$$

with $\widehat{Q}^{(n)}$ as the known specific discharge vector across the boundary.

After following what is now a standard procedure, we have an alternative set of Euler–Lagrange equations in the volume

$$(1-n)\rho_s \ddot{u}_k - B_{ijk}(\dot{J}_{ij} - \beta_{ij}\dot{\pi}) - nD_k\dot{\pi} - \lambda_{kj}n^2(\dot{W}_j - \dot{u}_j) = \widehat{f}_k \tag{52a}$$

$$A_{ijk}\ddot{J}_{kl} - B_{ijk}\dot{u}_k = 0 \tag{52b}$$

$$\frac{1}{Q}\ddot{\pi} + D_i(\dot{W}_i - \dot{u}_i) + \beta_{ij}B_{ijk}\dot{u}_k = \widehat{\Gamma} \tag{52c}$$

$$n\rho_f \ddot{W}_j + \lambda_{ij}n^2(\dot{W}_i - \dot{u}_i) + nD_j\dot{\pi} = 0 \tag{52d}$$

and over the boundary

$$\delta u_k = 0 \quad \text{on} \quad \Gamma_u \tag{53a}$$

$$\dot{J}_{kj}n_j - \beta_{kj}\dot{\pi}n_j = \sigma_{kj}n_j = \widehat{\tau}_k \quad \text{on} \quad \Gamma_\tau \tag{53b}$$

$$\delta\pi = 0 \quad \text{on} \quad \Gamma_\pi \tag{53c}$$

$$\dot{W}_i n_i = \widehat{Q}^{(n)} \quad \text{on} \quad \Gamma_q \tag{53d}$$

Thus, we have another form of a *mixed variational principle for poroelasticity*. Notice, however, that the governing differential equations now have a more elegant structure, which can be recognized easily by transforming to the Fourier wave number domain. The final result can be written in matrix form as

$$\begin{bmatrix} (1-n)\rho_s\mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{A} & 0 & 0 \\ 0 & 0 & \mathbf{I}/Q & 0 \\ 0 & 0 & 0 & n\rho_f\mathbf{I} \end{bmatrix} \begin{Bmatrix} \dot{\tilde{\mathbf{v}}} \\ \dot{\tilde{\boldsymbol{\sigma}}}^e \\ \dot{\tilde{\mathbf{p}}} \\ \dot{\tilde{\mathbf{Q}}} \end{Bmatrix} = \begin{bmatrix} 0 & i\tilde{\mathbf{B}}^T & -i(\boldsymbol{\beta}\tilde{\mathbf{B}} - \tilde{\mathbf{D}}) & 0 \\ i\tilde{\mathbf{B}} & 0 & 0 & 0 \\ -i(\tilde{\mathbf{B}}^T\boldsymbol{\beta}^T - \tilde{\mathbf{D}}^T) & 0 & 0 & -i\tilde{\mathbf{D}}^T \\ 0 & 0 & -i\tilde{\mathbf{D}} & 0 \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{v}} \\ \tilde{\boldsymbol{\sigma}}^e \\ \tilde{\mathbf{p}} \\ \tilde{\mathbf{Q}} \end{Bmatrix} + \begin{bmatrix} -n^2\boldsymbol{\lambda} & 0 & 0 & n^2\boldsymbol{\lambda} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ n^2\boldsymbol{\lambda} & 0 & 0 & -n^2\boldsymbol{\lambda} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{v}} \\ \tilde{\boldsymbol{\sigma}}^e \\ \tilde{\mathbf{p}} \\ \tilde{\mathbf{Q}} \end{Bmatrix} + \begin{Bmatrix} \tilde{\mathbf{f}} \\ 0 \\ \tilde{\Gamma} \\ 0 \end{Bmatrix} \tag{54}$$

where $\tilde{B}_{ijk}$ is defined by (30a) and

$$\tilde{D}_i = nk_i \tag{55}$$

with k_i as the Fourier wave number vector. The skew-Hermitian matrix on the first line of the right-hand side of (54) embodies all of the conservative poroelastic processes, while the effect of pore fluid viscosity leads to the negative semi-definite Hermitian matrix on the second line that embodies the energy dissipation. On the other hand, for inviscid pore fluids, the representation can be reduced to symplectic form, which is the characteristic of conservative systems.

4. Discussion

In the previous two sections, we have developed several new mixed variational principles for dynamical problems in thermoelasticity and poroelasticity. As would be expected from the original work of Biot (1956a) and the generalized theories presented in Cheng et al. (1991) and Chen and Dargush (1995), we find that there is a beautiful analogy between the Euler–Lagrange differential equations resulting from these variational principles. By examining the systems of equations more deeply, we can categorize the analogy into mathematical and physical aspects. As noted above, the thermoelastic differential equations (26a-d) have an identical mathematical structure to the poroelastic differential equations (52a-d) written in terms of the average fluid displacement W_i. In the above equations, an extended Fourier's heat conduction law and an extended Darcy's law are used for the thermoelastic and poroelastic systems, respectively. For the thermoelastic formulation, the extended Fourier's heat conduction law includes the term $\lambda_{ij}\tau_o\frac{1}{T_o}\ddot{H}_i$, while Darcy's extended law incorporates the term $n\rho_f\ddot{W}_i$ within the poroelastic formulation. Notice the mathematical analogy of these terms, since both appear in the constitutive equations and are expressed in terms of second order derivatives. Furthermore, if one eliminates the above terms, the result is thermoelastic and poroelastic formulations assuming infinite wave propagation. However, the physical analogy is not complete, since the term $\lambda_{ij}\tau_o\frac{1}{T_o}\ddot{H}_i$ represents ballistic transport within the thermal system, while the term $n\rho_f\ddot{W}_i$ involves an expression of the inertia of the poro system. Notice that the thermoelastic term involves the resistivity λ_{ij} of the medium and is therefore affected by the anisotropy of the system, while the poroelastic term involves the mass density ρ_f of the fluid and is therefore independent of the anisotropy of the system.

The above situation is similar to the case of the mixed formulations for the purely mechanical system, written in terms of either a Kelvin-Voigt or Maxwell model. The mixed formulation for a mechanical continuum with a Kelvin-Voigt model is governed by the differential equations:

$$\rho_o \ddot{u}_k + C_{jk}\dot{u}_j - B_{ijk}\dot{J}_{ij} = \hat{f}_k \tag{56a}$$

$$A_{ijkl}\ddot{J}_{kl} - B_{ijk}\dot{u}_k = 0 \tag{56b}$$

Meanwhile, the governing differential equations for the Maxwell model in mixed variables is written:

$$\rho_o \ddot{u}_k - B_{ijk}\dot{J}_{ij} = \hat{f}_k \tag{57a}$$

$$A_{ijkl}\ddot{J}_{kl} + \Lambda_{ijkl}\dot{J}_{kl} - B_{ijk}\dot{u}_k = 0 \tag{57b}$$

Both sets of equations above are hyperbolic in nature. If one wants to eliminate the terms that make these governing equations hyperbolic, then the term $\rho_o \ddot{u}_k$ in the Kelvin-Voigt mechanical system and the term $A_{ijkl}\ddot{J}_{kl}$ in the Maxwell mechanical system should be eliminated. Physically, there is no analogy, since the term $\rho_o \ddot{u}_k$ captures the inertia effect, while the term $A_{ijkl}\ddot{J}_{kl}$ expresses the solid skeletal flexibility. In addition, notice that the Maxwell model involves the flexibility A_{ijkl} of the medium and is therefore affected by the anisotropy of the system, as with the thermoelastic term identified above. On the other hand, the Kelvin-Voigt term involves the mass density ρ_o of the medium and is therefore independent of the anisotropy of the system, as with the poroelastic term discussed above. Further study of the physical phenomena is needed to examine carefully these subtle differences in the thermoelastic and poroelastic models.

Rather than adopting a Mixed Lagrangian Formalism (MLF), as we have done here, one could develop a generalized bracket formalism following the ideas detailed in Beris and Edwards (1994) or the general equation for the non-equilibrium reversible-irreversible coupling (GENERIC) approach (Grmela and Ottinger, 1997; Ottinger and Grmela, 1997). Both of these alternative approaches involve the definition of the reversible kinematics represented through a symplectic Poisson operator and the irreversible contributions captured by a Ginzburg–Landau dissipative operator. Both of these operators have certain characteristics that provide a common structure to the evolution of a broad range of physical processes. As might be expected, this form is also present in the Euler–Lagrange equations developed here through our Mixed Lagrangian Formalism and is especially evident in the Fourier wave number domain relations of (31) for thermoelasticity and (54) for poroelasticity. However, we favor the MLF approach written in term of impulsive variables, because it also provides underlying scalar potentials to represent the conservative and dissipative aspects of the problem, as well as a weak form that can be an effective starting point for computational algorithms. Furthermore, MLF extensions to plasticity, fracture, contact and softening already have been demonstrated (Sivaselvan and Reinhorn, 2006; Sivaselvan et al., 2009; Lavan et al., 2009; Lavan, 2010; Sivaselvan, 2011).

5. Conclusions

We show that it is possible to incorporate continuum thermoelasticity within the framework of the Lagrangian formalism. A unified Hamiltonian approach that considers both the conservative and dissipative characteristic of thermoelasticity is presented in terms of mixed variables. The weak formulation starts with an appropriate set of primary variables and the selection of the corresponding Lagrangian, potential and dissipation functions, which contain all the information for the reversible and irreversible physical processes. The action integral is introduced and, by applying Hamilton's action principle, as extended by Rayleigh for dissipative systems, one arrives at the Euler–Lagrange equations, which are the coupled thermal–mechanical equations for the continuum. These equations include the dynamic equilibrium equation, the thermoelastic constitutive equation, the entropy-energy balance equation, and the extended Fourier law of heat conduction. As a result, we have a new mixed variational principle for thermoelasticity of general two- or three-dimensional media under infinitesimal theory.

By exploiting the analogy between thermoelastic and poroelastic theories first identified by Biot, we then develop corresponding mixed variational principles for fluid-infiltrated porous bodies. In particular, using the solid skeleton displacement, impulse of the effective stress, impulse of pore pressure and average total fluid displacement, we construct a poroelastic formulation in close mathematical analogy with the thermoelastic system. However, the physical analogy is not complete, as the contributions from the extended Fourier and Darcy laws are shown to be of a different character.

While the results presented here are of theoretical interest, perhaps more importantly these formulations for thermoelasticity and poroelasticity set the stage for the development of new computational approaches in both problem domains. This will be the subject of future work.

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