COMMUNICATION

LINEAR EXTENSIONS OF FINITE POSETS AND A CONJECTURE OF G. KREWERAS ON PERMUTATIONS

Gwihen ETIENNE

Université Pierre et Marie Curie, Paris VI, U.E.R. 48 (E.R. Combinatoire), 75005 Paris, France

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We prove a conjecture of G. Kreweras on the number of solutions of the equation \( xy = z \) for permutations of a given signature.

1. Introduction

Let \( x \) be a permutation of \([n]=\{1, 2, \ldots, n\} \), \( n \) an integer \( \geq 2 \). The signature of \( x \) is the \((n-1)\)-tuple \( e = \{e_1, e_2, \ldots, e_{n-1}\} \) where \( e_i \) is the sign + if \( x_i < x_{i+1} \) and − otherwise.

Let \( P_e \) be the set of permutations of \([n]\) with given signature \( e \). G. Kreweras has conjectured [2] (see also [1]) that the number of solutions \( (x, y) \in P_e \times P_e \) of the equation \( xy = z \) for \( z \in P_e \) does not depend on \( z \) in \( P_e \). We prove this conjecture in a more general form by relating it to linear extensions of posets.

We give an expression of the number of solutions in terms of a Möbius function as a corollary of a theorem of Stanley.

2. Linear extensions

A linear extension of a finite poset \((E, \leq)\) with cardinality \( n \), is a one-to-one mapping \( s:[n] \rightarrow E \) such that \( s_i < s_j \) implies \( i < j \).

Let \( s, t \) be two one-to-one mappings \([n] \rightarrow E \). We denote \( \sigma_s(t) = (\alpha_1, \ldots, \alpha_{n-1}) \) the \((n-1)\)-tuple of signs \( \{+, -\} \) defined by

\[
\alpha_i = \begin{cases} 
+ & \text{if } s^{-1}(t_i) < s^{-1}(t_{i+1}), \\
- & \text{otherwise}.
\end{cases}
\]

Let \( s \) be a linear extension of \((E, \leq)\). Suppose that \( s_i \) and \( s_{i+1} \) are incomparable in \((E, \leq)\). We call allowed transposition \( \tau \) on \( s \) the transposition \( \{s_i, s_{i+1}\} \) leading to
the new linear extension \( t = \tau(s) \) defined by
\[
\begin{align*}
t_j &= s_j & \text{if } j \in [n] \setminus \{i, i + 1\}, \\
t_i &= s_{i+1}, & t_{i+1} &= s_i.
\end{align*}
\]

**Lemma 1.** Let \( \mathcal{S} \) be the set of linear extensions of \((E, \leq)\) and \( s, s' \in \mathcal{S} \). There exists a finite sequence \( \tau_1, \tau_2, \ldots, \tau_k \) of allowed transpositions such that
\[
s' = \tau_k \circ \tau_{k-1} \circ \cdots \circ \tau_1(s).
\]

**Proof.** We denote by \( I(s, s') \) the set of pairs \( \{e, f\} \subset E \) such that \( s^{-1}(e) < s^{-1}(f) \) and \( s'^{-1}(e) < s'^{-1}(f) \).

Suppose that for every \( i \in \{1, \ldots, n-1\} \), \( \{s_i, s_{i+1}\} \notin I(s, s') \). Then by definition,
\[
s'^{-1}(s_1) < s'^{-1}(s_2) < \cdots < s'^{-1}(s_n)
\]
gives \( s'^{-1}s_i = i \) for every \( i \) and \( s' = s \).

This proves that if \( s \neq s' \) there exists \( i \in \{1, \ldots, n-1\} \) and \( \{e, f\} \in I(s, s') \) verifying
\[
s^{-1}(e) = i, \quad s^{-1}(f) = i + 1.
\]

Set \( \tau_1 \) the allowed transposition \( \{e, f\} \) on \( s \). One may verify that \( I(\tau_1(s), s') = I(s, s') \setminus \{e, f\} \). By induction on the cardinality of \( I(s, s') \) it is clear that we can obtain \( s' \) from \( s \) using \( |I(s, s')| \) allowed transpositions. \( \square \)

Let \((\leq, \leq')\) be a couple of partial orders on \( E \). We say that this couple verifies **Property (C)** if for every \( x, y \in E \), \( x \) covers \( y \) for \( \leq' \) implies that \( x \) and \( y \) are comparable for \( \leq \).

Let \((\leq, \leq')\) be a couple of partial orders on \( E \) verifying Property (C). We denote by \( \mathcal{S} \), resp. \( \mathcal{S}' \), the set of linear extensions of \((E, \leq)\), resp. \((E, \leq')\). Let \( s \in \mathcal{S} \), \( \tau \) be an allowed transposition on \( \mathcal{S} \) and \( t = \tau(s) \).

**Lemma 2.** \( \sigma_t(\mathcal{S}') = \sigma_t(\mathcal{S}'') \).

**Proof.** Set \( s = s_1s_2\ldots s_{i-1}abs_{i+2}\ldots s_n \) and \( t = s_1s_2\ldots s_{i-1}bas_{i+2}\ldots s_n \).

**Case 1.** \( a \leq' b \). \( a \) and \( b \) are incomparable for the ordering relation \( \leq \). By property (C), \( b \) does not cover \( a \) for \( \leq' \). Thus we cannot have linear extensions \( s' \in \mathcal{S}' \) with \( s'^{-1}(a) = s'^{-1}(b) - 1 \): in every linear extensions \( s' \in \mathcal{S}' \), \( a \) and \( b \) are separated by other elements of \( E \). Clearly, \( \sigma_t(\mathcal{S}') = \sigma_t(\mathcal{S}'') \).

**Case 2.** \( a \) and \( b \) are incomparable for \( \leq' \). We can partition \( \mathcal{S}' \) into
\[
\mathcal{S}'_1 = \{ \text{linear extensions } s' \in \mathcal{S}' \text{ such that } a, b \text{ are not consecutive in } s' \}
\]
\[
\mathcal{S}'_2 = \{ \text{linear extensions } s' \in \mathcal{S}' \text{ such that } b \text{ is the successor of } a \text{ in } s' \}
\]
\[
\mathcal{S}'_3 = \{ \text{linear extensions } s' \in \mathcal{S}' \text{ such that } a \text{ is the successor of } b \text{ in } s' \}
\]
As in Case 1, $\sigma_s(S_1') = \sigma_r(S_1')$.

The transposition $T = \{a, b\}$ is an allowed transposition on linear extensions $s' \in S_2' \cup S_3'$ and $\tau(S_2') = S_3'$, $\tau(S_3') = S_2'$. We verify that $\sigma_s(S_2') = \sigma_r(\tau(S_2')) = \sigma_r(S_3')$ and $\sigma_r(S_3') = \sigma_r(\tau(S_3')) = \sigma_r(S_2')$.

Therefore $\sigma_s(S') = \sigma_r(S')$. □

The next result immediately follows from Lemmas 1 and 2.

**Proposition 3.** Let $(\leq, \leq')$ be two partial orders on $E$ with property (C), $\mathcal{S}$, resp. $\mathcal{S}'$, the set of linear extensions of $(E, \leq)$, resp. $(E, \leq')$. For every $s, t \in \mathcal{S}$, $\sigma_s(S') = \sigma_r(S')$.

### 3. Permutations with given signature

Let $\varepsilon$ be a $(n-1)$-tuple of signs $\{+, -\}$. We say that a permutation $x$ of the finite set $[n]$ has signature $\varepsilon$ if

$$
\varepsilon_i = + \text{ implies } x_i < x_{i+1} \quad \text{and} \quad \varepsilon_i = - \text{ implies } x_i > x_{i+1}.
$$

We denote by $\sigma(x)$ the signature of a permutation $x$. We recall that $P_\varepsilon$ denote the set of permutations with a given signature $\varepsilon$.

Let $E$ the finite set $\{e_1, e_2, \ldots, e_n\}$ and $e$ be the one-to-one mapping $i \mapsto e_i$.

If $\varepsilon$ is a $(n-1)$-tuple of signs, let $\leq_\varepsilon$ denote the ordering defined by the transitive closure of the couples $\{(e_i, e_{i+1}) \text{ if } \varepsilon_i = +\} \cup \{(e_{i+1}, e_i) \text{ if } \varepsilon_i = -\}$. The set of linear extensions of $(E, \leq_\varepsilon)$ will be denoted by $\mathcal{S}_\varepsilon$.

**Remark 4.** If $\varepsilon$ and $\varepsilon'$ are two not necessarily distinct signatures, the ordering couple $(\leq_\varepsilon, \leq_\varepsilon')$ verifies Property (C).

Clear, since if $e_i$ covers $e_j$ for $\leq_\varepsilon$, then $i = j + 1$ or $i = j - 1$ and $e_i, e_j$ are comparable for $\leq_\varepsilon$.

**Remark 5.** $S : [n] \to E$ is a linear extension of $(E, \leq_\varepsilon)$ if and only if $s^{-1}e \in P_\varepsilon$.

**Proof.**

$$
S^{-1}e \in P_\varepsilon \iff \forall i \in [n-1] \begin{cases} 
S^{-1}e(i) < S^{-1}e(i+1) & \text{if } \varepsilon_i = + \\
S^{-1}e(i) > S^{-1}e(i+1) & \text{if } \varepsilon_i = -
\end{cases}
$$

$$
\iff \forall i \in [n-1] \begin{cases} 
S^{-1}(e_i) < S^{-1}(e_{i+1}) & \text{if } e_i \leq_\varepsilon e_{i+1} \\
S^{-1}(e_i) > S^{-1}(e_{i+1}) & \text{if } e_i \geq_\varepsilon e_{i+1}
\end{cases}
$$

by definition of $(E, \leq_\varepsilon)$

$$
\iff s \in \mathcal{S}_\varepsilon \quad \text{by transitivity.} \quad \Box
$$

**Remark 6.** Let $s, t$ be two one-to-one mappings $[n] \to E$; then $\sigma_s(t) = \sigma(s^{-1}t)$.  

We are now able to prove a conjecture of G. Kreweras [2], generalized by the author to the case of two signatures, and by P. Moszkowski [3] to the case of three signatures.

**Theorem 7.** Let \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) be three \((n-1)\)-tuples of signs \( \{+, -\} \) and \( P_\varepsilon \) be the set of permutations of \([n]\) with given signature \( \varepsilon_i \), \( i \in \{1, 2, 3\} \).

The number of couples \((x, y) \in P_{\varepsilon_1} \times P_{\varepsilon_2} \) solutions of the equation \( xy = z \) does not depend on the choice of \( z \in P_{\varepsilon_3} \).

**Proof.** Let \( z \) be a permutation with signature \( \varepsilon_3 \). Consider the set of signatures \( \{ \sigma(zy^{-1}) \colon y \in P_{\varepsilon_3} \} \).

By Remark 5, \( y \in P_{\varepsilon_3} \Leftrightarrow ey^{-1} \in S_{\varepsilon_2}, \ z \in P_{\varepsilon_3} \Leftrightarrow ez^{-1} \in S_{\varepsilon_3} \). Thus,

\[
\{ \sigma(zy^{-1}) \colon y \in P_{\varepsilon_3} \} = \{ \sigma(ze^{-y^{-1}}) : ey^{-1} \in S_{\varepsilon_2} \}
\]

\[
= \{ \sigma(s^{-1}t) : t \in S_{\varepsilon_2} \}
\]

\[
= \{ \sigma_s(t) : t \in S_{\varepsilon_2} \} \text{ by Remark 6}
\]

\[
= \sigma_s(S_{\varepsilon_2}).
\]

By Remark 4 and Proposition 3, the set of signatures \( \sigma_s(S_{\varepsilon_2}) \) does not depend on the choice of \( s \in S_{\varepsilon_2} \). In other words, \( \{ \sigma(zy^{-1}) : y \in P_{\varepsilon_3} \} \) is the same for every \( z \in P_{\varepsilon_3} \).

Then the number of solutions of \( xy = z \), \((x, y) \in P_{\varepsilon_1} \times P_{\varepsilon_2} \) with \( z \) given in \( P_{\varepsilon_3} \), that is \( |P_{\varepsilon_1} \cup zP_{\varepsilon_2}^3| \) does not depend on the choice of \( z \) in \( P_{\varepsilon_3} \). \( \square \)

4. Average

Let \( (E, \preceq) \) be a finite poset of cardinality \( n \), and \( s \) be a linear extension of \( E \). We call **allowed permutation** on \( s \) a permutation \( \pi \) of \([n]\) such that \( s_0 \pi \) is a linear extension of \( E \).

Now set \( \varepsilon \preceq \varepsilon' \) if the set of signs \( - \) of \( \varepsilon \) is a subset of the set of signs \( - \) of \( \varepsilon' \). For this partial order \( \varepsilon = \{+, -\}^{n-1} \) is a lattice isomorphic to the lattice of subsets of \([n-1]\).

Finally note \( L \) the distributive lattice of ideals of the finite poset \((E, \preceq)\). If \( \varepsilon \in S \), note \( h_1, h_2, \ldots, h_p \) the integers \( i \) such that \( e_i = - \) and note \( L(\varepsilon) \) the subset of elements of \( L \) with height \( 0, h_1, h_2, \ldots, h_p, n \). Denote by \( \mu_{L(\varepsilon)} \) the Möbius function of the ordered set \((L(\varepsilon), \subseteq)\). The following theorem is due to Stanley.

**Theorem** (Stanley [4]). Let \( s \) be a linear extension of \((E, \preceq)\), \( \varepsilon \) a \((n-1)\)-tuple of signs \( \{+, -\} \). The number of allowed permutations with signature \( \varepsilon \) on \( s \) is \( |\mu_{L(\varepsilon)}(\emptyset, E)| \).

**Corollary.** Let \( \varepsilon \in \{+, -\}^{n-1} \). Let \( h_1 < h_2 < \cdots < h_p \) be the places of the signs \( - \) in
Let $L(\varepsilon)$ be the ordered set of ideals of $(E, \leq_{\varepsilon})$ with height $0, h_1, h_2, \ldots, h_p, n$ and $\mu_{L(\varepsilon)}$ be its Möbius function.

The number of solutions $(x, y) \in P_\varepsilon \times P_\varepsilon$ of $xy = z$ for a given $z \in P_\varepsilon$ is $|\mu_{L(\varepsilon)}(\emptyset, E)|$.

References


