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Conic regions and k-uniform convexity

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Dedicated to Professor Haakon Waadeland on the occasion of his 70th birthday

Abstract

Let $\Omega_k \subset \mathbb{C}$ denote a domain, such that $1 \in \Omega_k$ and $\partial\Omega_k$ is a conic section, with eccentricity equal to $1/k$. In this paper authors introduce the class of k-uniformly convex functions k-UCV, with the property that the values of the expression $1+zf''(z)/f'(z)$ lie inside the domain Ω_k. Necessary and sufficient conditions for membership in k-UCV, as well as sharp growth and distortion theorems for k-uniformly convex functions are given. The obtained results generalize the concept of uniform convexity due to A.W. Goodman (Ann. Polon. Math. 56 (1991) 87–92). © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let H denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

analytic in the open unit disk U, and let S denote the class of functions (1.1), analytic and univalent in U. By CV we denote the subclass of convex, univalent functions, defined by the condition

$$\mathrm{CV} = \left\{ f \in S : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in U \right\}. \tag{1.2}$$

In 1991 Goodman [2] investigated the class of functions mapping circular arcs contained in the unit disk, with the center at an arbitrarily chosen point in U, onto convex arcs. Goodman denoted this class by UCV. Recall here his definition.

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Definition 1.1 (Goodman [2]). A function $f \in H$ is said to be uniformly convex in U, if f is convex in U, and has the property that for every circular arc γ, contained in U, with center ζ, also in U, the arc $f(\gamma)$ is convex.

We remark that for $\zeta = 0$, we are back to the class CV, and also that if γ is a complete circle contained in U, it is well known that $f(\gamma)$ is a convex curve also for $f \in \mathrm{CV}$.

An analytic condition for UCV has also been formulated by Goodman [2]. We state this as:

Theorem 1.1 (Goodman [2]). *Let $f \in H$. Then $f \in \mathrm{UCV}$ iff*

$$\operatorname{Re}\left\{1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right\} \geqslant 0, \quad (z,\zeta) \in U \times U. \tag{1.3}$$

However, the above condition contains two variables z and ζ, and for this reason it is not convenient for investigation. Rønning [7], and independently Ma and Minda [6], have given a more applicable one-variable characterization of the class UCV, stated below.

Theorem 1.2 (Rønning [7] and Ma and Minda [6]). *Let $f \in H$. Then $f \in \mathrm{UCV}$ iff*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in U. \tag{1.4}$$

One can describe the domain of values of the expression $p(z) = 1 + zf''(z)/f'(z)$, $z \in U$ geometrically. From (1.4) it follows that $f \in \mathrm{UCV}$ iff p is in the parabolic region

$$\Omega = \{w \in \mathbb{C}: (\operatorname{Im} w)^2 < 2\operatorname{Re} w - 1\}.$$

2. Definition, necessary and sufficient conditions for the class of *k*-uniformly convex functions

Definition 2.1. Let $0 \leqslant k < \infty$. A function $f \in S$ is said to be k-uniformly convex in U, if the image of every circular arc γ contained in U, with center ζ, where $|\zeta| \leqslant k$, is convex.

For fixed k, the class of all k-uniformly convex functions will be denoted by k-UCV. Clearly, 0-UCV = CV, and 1-UCV = UCV.

Definition 2.1 was motivated by the desire to generalize the family of circular arcs contained in U, with center also in U, to a family of circular arcs contained in U with center at any point of the complex plane.

Observe that, the above definition generalizes the idea of convexity, and establishes a continuous passage between the well-known class of convex functions CV and the class UCV.

It is obvious from the definition, that the class k-UCV is invariant under the rotation $e^{i\theta} f(e^{-i\theta} z)$.

Recall that convexity on a curve $z = z(t)$, $t \in [a,b]$, is equivalent to the condition

$$\operatorname{Im}\left\{\frac{z''(t)}{z'(t)} + z'(t)\frac{f''(z)}{f'(z)}\right\} \geqslant 0, \quad t \in [a,b], \tag{2.1}$$

(cf. e.g., [1], vol. I, p. 110).

For the circle with center at any point $\zeta \in \mathbb{C}$, and in view of (2.1), we obtain as an immediate consequence the analogue of Theorem 1.1. We omit the proof.

Theorem 2.1. *Let $f \in H$, and $0 \leqslant k < \infty$. Then $f \in k$-UCV iff*

$$\operatorname{Re}\left\{1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right\} \geqslant 0, \quad z \in U, \quad |\zeta| \leqslant k. \tag{2.2}$$

As with the class UCV it is possible to get a one-variable characterization of the class k-UCV, depending only on the parameter k.

Theorem 2.2. *Let $f \in H$, and $0 \leqslant k < \infty$. Then $f \in k$-UCV iff*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in U. \tag{2.3}$$

Proof. In the case when $k = 0$ inequality (2.3) reduces to the well-known condition of convexity (1.2), and when $k = 1$ we are back to (1.4).

Let $0 < k < \infty$, and assume that $f \in k$-UCV. Then, condition (2.2) is fulfilled for every $z \in U$, and $0 \leqslant |\zeta| \leqslant k$. Choosing $\theta = \operatorname{Arg}[zf''(z)/f'(z)]$, and $\zeta = kz\mathrm{e}^{-\mathrm{i}\theta}$, we obtain from (2.2)

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geqslant \operatorname{Re}\left\{\frac{\zeta f''(z)}{f'(z)}\right\} = k\operatorname{Re}\left\{\frac{\mathrm{e}^{-\mathrm{i}\theta}zf''(z)}{f'(z)}\right\} = k\left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in U. \tag{2.4}$$

Since $1+zf''(z)/f'(z)$ is an analytic function in U, and maps 0 to 1, the Open Mapping Theorem implies that equality in (2.4) is not possible. Thus, we get a necessary condition for f to be in k-UCV.

Next, assume for the proof of the sufficient condition, that for $0 < k < \infty$, condition (2.3) holds. Let $\omega = \zeta/k$. Then we shall prove that

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geqslant k\operatorname{Re}\left\{\frac{\omega f''(z)}{f'(z)}\right\} \tag{2.5}$$

for all z and ω in the unit disk. By the Minimum principle it suffices to prove (2.5) for $1 > |z| = R > |\omega|$. Then (2.3) gives

$$\begin{aligned}\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} &> k\operatorname{Re}\left|\frac{zf''(z)}{f'(z)}\right| > k\left|\frac{\omega f''(z)}{f'(z)}\right| \geqslant k\operatorname{Re}\left\{\frac{\omega f''(z)}{f'(z)}\right\}\\ &= \operatorname{Re}\left\{\frac{\zeta f''(z)}{f'(z)}\right\}\end{aligned}$$

and (2.2) is established.

3. General properties of *k*-uniformly convex functions

Denote by Ω_k with $0 \leqslant k < \infty$, the following set:

$$\Omega_k = \left\{1 + \frac{zf''(z)}{f'(z)} : z \in U, f \in k\text{-UCV}\right\}. \tag{3.1}$$

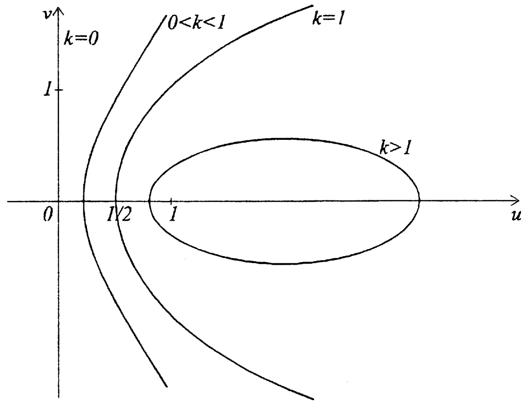

Fig. 1.

Note that, by (2.3), Ω_k is a domain such that, $1 \in \Omega_k$ and $\partial\Omega_k$ is a curve defined by the equality

$$\partial\Omega_k = \{w = u + iv\colon u^2 = k^2(u-1)^2 + k^2v^2\}, \quad (0 \leqslant k < \infty). \tag{3.2}$$

From elementary computations we see that (3.2) represents the conic sections symmetric about the real axis, with eccentricity equal to $1/k$, when $k \neq 0$. In the limiting case when $k = 0$ the curve reduces to the imaginary axis. All of the curves have one vertex at $(k/(k+1), 0)$, and the focus at $(1, 0)$. If the curve is an ellipse the other vertex is at $(k/(k-1), 0)$, see Fig. 1.

This characterization enables us to designate precisely the domain Ω_k, as a convex domain contained in the right half-plane. Moreover, Ω_k is an elliptic region for $k > 1$, parabolic for $k = 1$, hyperbolic for $0 < k < 1$ and finally Ω_0 is the whole right half-plane.

Furthermore, from (3.1) and (3.2) we can also prove that for $f \in k$-UCV with $0 \leqslant k < \infty$,

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{k}{k+1}, \quad z \in U \tag{3.3}$$

and

$$\left|\operatorname{Arg}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \begin{cases} \arctan\dfrac{1}{k} & \text{for } 0 < k < \infty, \\ \dfrac{\pi}{2} & \text{for } k = 0. \end{cases} \tag{3.4}$$

Denote by P the family of analytic and normalized Carathéodory functions, and by $p_k \in P$ the function such that $p_k(U) = \Omega_k$. Denote also by $P(p_k)$, according to Ma and Minda's notation [5], the following family:

$$P(p_k) = \{p \in P\colon p(U) \subset \Omega_k\} = \{p \in P\colon p \prec p_k \text{ in } U\}.$$

We shall specify the functions p_k, which are extremal for the class $P(p_k)$. Obviously,

$$p_0(z) = \frac{1+z}{1-z}, \quad z \in U \tag{3.5}$$

and, in the case of a parabolic domain (cf. [6] or [7]),

$$p_1(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad z \in U. \tag{3.6}$$

Now, we shall give an explicit form of the function which maps U onto the hyperbolic region

$$\Omega_k = \left\{u + \mathrm{i}v\colon\ \frac{(u + k^2/(1-k^2))^2}{k^2/(1-k^2)^2} - \frac{v^2}{1/(1-k^2)} > 1,\ u > 0\right\}, \quad 0 < k < 1. \tag{3.7}$$

The transformation

$$w_1(z) = \left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^A,$$

where $\sqrt{z}$ is such that it takes positive values for positive values of z and $A = (2/\pi)\arccos k$, maps U onto the angular region D of width $A\pi/2 = \arccos k$ in the w_1-plane. Next, the mapping

$$w_2(w_1) = \frac{1}{2}\left(w_1 + \frac{1}{w_1}\right)$$

transforms the region D onto the domain G being the interior of the right branch of the hyperbola whose vertex is at the point $w_2 = k$.

Finally,

$$w(w_2) = \frac{1}{1-k^2}w_2 - \frac{k^2}{1-k^2}$$

maps G onto the interior of hyperbola, given by (3.7).

Summing up, the function $w(w_2) =: p_k(z)$

$$p_k(z) = \frac{1}{2(1-k^2)}\left[\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^A + \left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^A\right] - \frac{k^2}{1-k^2}, \quad z \in U \tag{3.8}$$

and $A = (2/\pi)\arccos k$, gives the desired mappings. This function also has an equivalent form

$$p_k(z) = \frac{1}{1-k^2}\cosh\left\{A\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{k^2}{1-k^2}, \quad z \in U. \tag{3.9}$$

The conformal mapping of the unit disk onto the interior of the ellipse

$$\Omega_k = \left\{u + \mathrm{i}v\colon\ \frac{(u - k^2/(k^2-1))^2}{k^2/(k^2-1)^2} + \frac{v^2}{1/(k^2-1)} < 1\right\} \quad (1 < k < \infty), \tag{3.10}$$

requires the use of Jacobian elliptic functions. (Note that for no choice of the parameter k ($k > 1$), can Ω_k reduce to the disk.) It is known (cf. [3], p. 280, or [4], vol. II, p. 140), that the Jacobian elliptic function $\mathrm{sn}(s, \kappa)$ transforms the upper half-plane (and the upper semidisk of $\{s\colon |s| < 1/\sqrt{\kappa}\}$) onto the interior of a rectangle with vertices $\pm K$, $\pm K + \mathrm{i}K'$. Here, equivalently $K = K(\kappa)$, $\kappa \in (0, 1)$ is Legendre's complete elliptic integral of the first kind

$$K(\kappa) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}},$$

and

$$K'(\kappa) = K(\sqrt{1-\kappa^2})$$

is the complementary integral of $K(\kappa)$. The mapping

$$z(w_1) = \sqrt{\kappa}\,\mathrm{sn}\left(\frac{2K}{\pi}\arcsin\frac{w_1}{c}, \kappa\right)$$

maps conformally the elliptic domain

$$E = \{w\colon |w+c| + |w-c| < 2\alpha\}, \quad 0 < c < \alpha,$$

onto the unit disk U, where $\alpha = c\cosh(\pi K'/4K)$ and $\beta = c\sinh(\pi K'/4K)$ are the semi-axis of the ellipse E. Its inverse

$$\begin{aligned} w_1(z) &= c\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{z/\sqrt{\kappa}} \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}}\right) \\ &= \frac{1}{k^2-1}\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{z/\sqrt{\kappa}} \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}}\right), \end{aligned}$$

where $\kappa \in (0,1)$ is chosen such that $k = \cosh \pi K'(\kappa)/4K(\kappa)$, maps the unit disk U onto the elliptic domain E such that $w_1(0) = 0$. The shift through the distance $k^2/(k^2-1)$ to the right

$$w(w_1) = w_1 + \frac{k^2}{k^2-1}$$

transforms E onto Ω_k, given by (3.10), but with the normalization $w(w_1(0)) = k^2/(k^2-1)$. Combining w and w_1 we see that to obtain the function $p_k = w(w_1)$ with the desired normalization $p_k(0) = 1$ we need to solve the equation

$$p_k(0) = w(w_1(u(0))) = 1,$$

where u is the Möbius transformation of the unit disk onto itself. Hence we get

$$\frac{1}{k^2-1}\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{u(0)/\sqrt{\kappa}} \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}}\right) + \frac{k^2}{k^2-1} = 1,$$

that gives

$$\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{u(0)/\sqrt{\kappa}} \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}}\right) = -1$$

and equivalently

$$\int_0^{u(0)/\sqrt{\kappa}} \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}} = -K(\kappa).$$

From the properties of the Legendre's integral (cf. [4] pp. 127–138) the above equality will be fullfilled if

$$\frac{u(0)}{\sqrt{\kappa}} = -1.$$

The automorphism

$$u(z)=\frac{z-\sqrt{\kappa}}{1-\sqrt{\kappa}z}$$

provides the required self-mapping of U. Finally,

$$p_k(z)=\frac{1}{k^2-1}\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{u(z)/\sqrt{\kappa}}\frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\kappa^2t^2}}\right)+\frac{k^2}{k^2-1}, \tag{3.11}$$

where $z\in U$, is the desired mapping of the disk U onto the elliptic domain Ω_k, given by (3.10), with the normalization $p_k(0)=1$.

Remark 3.1. Note, that $P(p_0)=P$, and $P(p_1)=PAR$ (cf. [5]).

Remark 3.2. Observe that, the functions p_k in the case $k>1$, are regular on the boundary of U.

Property 3.1. (i) It is easy to verify that in the case when $k=0$ we obtain $A=(2/\pi)\arccos 0=1$, and formula (3.8) reduces to $p_0(z)=(1+z)/(1-z)$, $z\in U$.

(ii) Let $k\to 1^-$, then $A\to 0^+$, and by using twice the l'Hospital principle with respect to k to the function p_k given by (3.9), we get that $p_k(z)\to p_1(z)$, $z\in U$.

The characterization of the class k-UCV can be expressed in term of subordination, as follows.

Theorem 3.1. *The function* $f\in k$-UCV *iff* $p(z)=1+zf''(z)/f'(z)\prec p_k(z)$ *in* U.

Note, that for each $0\leqslant k<\infty$, the domain $\Omega_k=p_k(U)$ is symmetric with respect to the real axis, convex, and the function p_k satisfies the condition $p_k'(0)>0$.

Define the function f_k by the following conditions:

$$1+\frac{zf_k''(z)}{f_k'(z)}=p_k(z),\quad z\in U,\quad f_k(0)=f_k'(0)-1=0.$$

Then, by Theorem 3.1, $f_k\in k$-UCV. The function f_k plays the role of the Koebe function for the class k-UCV.

As simple consequences of the above, and the results given in [5] we obtain the following properties for the class k-UCV.

Theorem 3.2. *Let* $0\leqslant k<\infty$, *and* $f\in k$-UCV. *Then*

$$f'(z)\prec f_k'(z)\quad \text{in } U, \tag{3.12}$$

$$f_k'(-r)\leqslant|f'(z)|\leqslant f_k'(r),\quad |z|=r<1, \tag{3.13}$$

$$-f_k(-r)\leqslant|f(z)|\leqslant f_k(r),\quad |z|=r<1. \tag{3.14}$$

Equality in (3.13) *and* (3.14) *occurs for some* $z_0\neq 0$, *if and only if* f *is a rotation of the function* f_k.

The next theorem presents a sufficient condition in terms of the coefficients for the function f to be k-uniformly convex.

Theorem 3.3. *Let $f \in S$. If for some k, $0 \leqslant k < \infty$, the inequality*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leqslant \frac{1}{k+2} \tag{3.15}$$

holds, then $f \in k$-UCV. The number $1/(k+2)$ cannot be increased.

Proof. Suppose that $f \in S$, and that inequality (3.15) holds for a fixed number k, $0 \leqslant k < \infty$. Then, for the same number k

$$\sum_{n=2}^{\infty} n|a_n| \leqslant \frac{1}{k+2}$$

and for $|\zeta| \leqslant k$ we get

$$\begin{aligned}\operatorname{Re}\left\{1+(z-\zeta)\frac{f''(z)}{f'(z)}\right\} &\geqslant 1-\frac{\sum_{n=2}^{\infty} n(n-1)|a_n|\,|z|^{n-2}}{1-\sum_{n=2}^{\infty} n|a_n|\,|z|^{n-1}}\;|z-\zeta| \\ &\geqslant 1-\frac{1/(k+2)}{1-1/(k+2)}\,(k+1)=0.\end{aligned}$$

Thus, from (2.2) and by Theorem 2.1, $f \in k$-UCV. Equality in (3.15) is attained for

$$f(z)=z-\frac{z^2}{2(k+2)}$$

with $z=1$ and $\zeta=-k$.

In the case when $k=1$, Theorem 3.3 reduces to the following result.

Corollary 3.1 (Goodman [2]). *Let $f \in S$. If the inequality*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leqslant \frac{1}{3},$$

holds, then the function $f \in$ UCV.

For $k=0$ we obtain from Theorem 3.3:

Corollary 3.2. *Let $f \in S$. If*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leqslant \frac{1}{2},$$

holds, then $f \in$ CV.

Now, we shall solve a k-UCV radius problem in the class S. The well-known radius of convexity in S is $2-\sqrt{3}$ (see e.g., [1], vol. I, p. 119), and the UCV radius in S is $(4-\sqrt{13})/3$, (cf. [8]).

Theorem 3.4. *The radius of k-uniform convexity in S is*

$$r_0 = \frac{2(k+1) - \sqrt{4k^2+6k+3}}{2k+1} = \frac{1}{2(k+1)+\sqrt{4k^2+6k+3}}. \tag{3.16}$$

Proof. Let $f \in S$. Then the following sharp inequality

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leqslant \frac{4r}{1-r^2}, \quad |z| = r < 1$$

holds (cf. e.g., [1], vol. I, p. 63), or equivalently

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right) - \frac{1+r^2}{1-r^2}\right| \leqslant \frac{4r}{1-r^2}, \quad |z| = r < 1. \tag{3.17}$$

The above condition represents a disk intersecting the real axis at the points $((1+r^2-4r)/(1-r^2), 0)$ and $((1+r^2+4r)/(1-r^2), 0)$. According to Theorem 2.2, and in view of relations (3.1) and (3.2), we search for the largest value of $r = |z|$ such that the disk (3.17) lies completely inside the conic domain Ω_k. Since all the conic sections have one vertex at the point $(k/(k+1), 0)$, it is necessary to fulfill the condition

$$\frac{1+r^2-4r}{1-r^2} \geqslant \frac{k}{k+1}.$$

This inequality is satisfied for $0 \leqslant r \leqslant r_0$, with r_0 given by (3.16). It suffices to check that for this r_0 the disk (3.17) and the conic section $\partial\Omega_k$ have only one common point $(u_1, 0)$, where

$$u_1 = \frac{1+r^2-4r}{1-r^2} = \frac{k}{k+1}. \tag{3.18}$$

In fact, for the value of k determined by (3.18), one can see that the system of equalities

$$u^2 = k^2(u-1)^2 + k^2v^2,$$

$$\left(u - \frac{1+r^2}{1-r^2}\right)^2 + v^2 = \frac{16r^2}{(1-r^2)^2} \tag{3.19}$$

has two solutions $(u_1, 0)$, $(u_2, 0)$, where u_1 is given by (3.18) and $u_2 < 0$. Hence the value u_1 is the only positive solution of (3.19). Thus for $r \leqslant r_0$ the disk (3.17) lies completely inside the domain Ω_k.

Remark 3.3. Since the Koebe function gives equality in (3.17), it follows that r_0 is the k-UCV radius in the class of starlike, univalent functions.

Remark 3.4. In the case, when $k = 0$, $r_0 = 2 - \sqrt{3}$ is the *CV* radius in the class *S*, and if $k = 1$, then $r_0 = (4 - \sqrt{13})/3$, which coincides with a result from [8].

We can also give a characterization of the class k-UCV in terms of convolution. Recall that the Hadamard product, or convolution, of two power series $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined as $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$.

Theorem 3.5. *Let* $0 \leqslant k < \infty$. *A function* $f \in S$ *is in the class* k-UCV *if and only if* $\frac{1}{z}(f * G_t)(z) \neq 0$ *in* U *for all* $t \geqslant 0$, *such that* $t^2 - (kt-1)^2 \geqslant 0$, *where*

$$G_t(z) = \frac{1}{1 - C(t)} \frac{z}{(1-z)^2} \left(\frac{1+z}{1-z} - C(t) \right), \quad z \in U$$

and

$$C(t) = kt \pm \mathrm{i}\sqrt{t^2 - (kt-1)^2}.$$

Proof. Let $0 \leqslant k < \infty$. Assume that $f \in S$ and $p(z) = 1 + zf''(z)/f'(z)$, $z \in U$. Since $p(0) = 1$, it follows that

$$f \in k\text{–UCV} \iff p(z) \notin \partial\Omega_k \quad \text{for all } z \in U.$$

Note that

$$\partial\Omega_k = C(t) = kt \pm \mathrm{i}\sqrt{t^2 - (kt-1)^2}, \quad \text{where } t(k+1) \geqslant 1 \text{ and } (1-k)t \geqslant -1.$$

Also note that

$$\frac{z}{(1-z)^2} * f(z) = zf'(z)$$

and

$$\frac{z(1+z)}{(1-z)^3} * f(z) = \left[z \left(\frac{z}{(1-z)^2} \right)' \right] * f(z) = zf'(z) + z^2 f''(z).$$

Hence,

$$\frac{1}{z}(f * G_t)(z) = \frac{1}{1-C(t)}(f'(z) + zf''(z) - C(t)f'(z)) = \frac{f'(z)}{1-C(t)} \left(1 + \frac{zf''(z)}{f'(z)} - C(t) \right).$$

Thus,

$$\frac{1}{z}(f * G_t)(z) \neq 0 \iff p(z) \notin \partial\Omega_k \iff p(z) \in \Omega_k, \quad z \in U.$$

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