Bifurcation and stability analysis of nonlinear waves in $\mathbb{D}_n$ symmetric delay differential systems

Mingshu Peng

Department of Mathematics, Beijing Jiao Tong University, Beijing 100 044, People’s Republic of China
Received 24 April 2006; revised 11 September 2006
Available online 24 October 2006

Abstract

In this paper, a direct effective computation method is proposed on stability analysis of the bifurcating solutions in symmetric delay differential systems. Mechanism about multistability (there coexist multiple nontrivial equilibria/limit cycles) is analyzed.

MSC: 34K18; 92B20; 37D45; 37N20; 37N25

Keywords: $\mathbb{D}_n$ equivariant bifurcations; Multistability; Spatio-temporal behavior; Lie group action; Delay

1. Introduction

Consider the following network of $n$ identical coupled oscillators (or cells)

$$\dot{x}_j(t) = -x_j(t) + \alpha f(x_j(t - \tau)) + \beta g(x_{j+1}(t - \tau)) + g(x_{j-1}(t - \tau)), \quad (1.1)$$

where $j \mod n$. It is known that system (1.1) models the evolution of a network of $n$ identical resonators with delayed feedback, which can be seen in a more direct way if rewritten in another form,

$$\dot{x}_j(t) = -x_j(t) + h(x_j(t - \tau)) + \beta [g(x_{j+1}(t - \tau)) - 2g(x_j(t - \tau)) + g(x_{j-1}(t - \tau))], \quad (1.2)$$

E-mail address: mshpeng@center.njtu.edu.cn.

0022-0396/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2006.09.010
where $h = \alpha f + 2\beta g$. Suppose that $f(0) = g(0) = 0$, $f'(0) = g'(0) = 1$ (the normalization property).

There has been an increasing interest in equivariant Hopf bifurcations in symmetric delay differential equations since the creative work of Wu [14] (based on the topological methods and the theorem by Golubitsky [6]): There exist three kinds of symmetry breaking oscillation patterns—phase-locked waves, mirror-reflecting waves and standing waves.

For instance, in [2,8,10,15], the symmetric Hopf bifurcation theorem [14] was used to study symmetric delay differential equations. In [3], the perturbation techniques coupled with the Floquet theory were proposed to determine the stability of the phase-locked oscillations. Hopf–Hopf and Hopf-steady state bifurcation interactions were shown to exist and give rise to the coexistence of stable synchronized and desynchronized solutions.

But until now, stability analysis of the bifurcating solutions in symmetric delayed systems is still a very difficult problem [7,8,10]. With the best of our knowledge, there are fewer papers to give a theoretical analysis of equivariant pitchfork bifurcations in delayed systems. As to the numerical simulation, please see [2,3].

Recently, the authors [11–13] develop a direct computation method about how to deal with the symmetric problems in both discrete delay difference equation and delay differential equation, even if the multiplicity of the eigenvalue is not less than 2. In [11], a discrete neural network with $\mathbb{Z}_2$ symmetry was analyzed. A detailed study of $\mathbb{D}_n$-equivariant Hopf/pitchfork/period-doubling bifurcations in discrete-time dynamical systems with delay were given in [12,13]. Multistability, including multiple oscillation patterns, such as the coexistence of multiple stable equilibria, stable limit cycles, stable invariant tori and multiple chaotic attractors, could be determined. Furthermore, the route to chaos in high-dimensional systems was shown [13] by limit cycles (Hopf bifurcation) $\rightarrow$ periodic doubling $\rightarrow$ invariant tori $\rightarrow$ strange (chaotic) attractors.

In this paper, we will develop the methods and techniques proposed in [11–13] and give a detailed analysis of $\mathbb{D}_n$ symmetric bifurcations in delay differential systems. Multistability (Turing patterns) in the neural model (1.1) proposed by Wu et al. [8,10,15] can be analyzed in details and the gap ($j \neq n/4$) in [2] may be filled.

The remaining part of the paper is organized as follows: some properties related to the eigenvalue problem are included in Section 2. The $\mathbb{D}_n$-equivariant structure, eigenvectors, adjoint eigenvectors and the corresponding dual spaces in system (1.1) will be discussed in Section 3. Main results about equivariant Hopf/pitchfork bifurcations are given in Section 4. Finally, we will draw our conclusions in Section 5.

2. Some properties

The linearization of (1.1) at the synchronized equilibrium point $(x^*, \ldots, x^*)$ is

$$
\dot{x}_j(t) = -x_j(t) + \alpha f'(x^*)x_j(t-\tau) + \beta g'(x^*)[x_{j+1}(t-\tau) + x_{j-1}(t-\tau)],
$$

or

$$
\dot{x}_j(t) = -x_j(t) + h'(x^*)x_j(t-\tau) + \beta g'(x^*)[x_{j+1}(t-\tau) - 2x_j(t-\tau) + x_{j-1}(t-\tau)].
$$
In vector form, this can be written as
\[ \dot{x}(t) = -x(t) + \left[ \alpha f'(x^*) I + \beta g'(x^*) M \right] x(t - \tau), \]
where \( I \) is the \( n \times n \) identity matrix and \( M \) is defined by
\[ M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{n \times n}. \] (2.3)

Clearly, \( x^* = 0 \) is the trivial solution of system (1.1). The characteristic matrix [8] becomes
\[ Q_N(0, \lambda, m) = (\lambda + 1) I - (\alpha I + \beta M) \exp(-\lambda \tau). \]

Choose
\[ \chi = \exp(i2\pi/n), \quad v_j = (1, \chi^j, \chi^{2j}, \ldots, \chi^{(n-1)j})^T, \quad j \in \mathbb{N}. \]

Here \( \mathbb{N} \) denotes the set \( \{0, 1, \ldots, n - 1\} \). Then the characteristic equation is
\[ \Delta(\lambda) = \det Q_N(0, \lambda, m) = 0, \]

where
\[ \Delta(\lambda) = \prod_{j=0}^{n-1} \Delta_j(\lambda) = \begin{cases} \Delta_0(\lambda) \prod_{j=1}^{\frac{n-1}{2}} \Delta_j^2(\lambda) & \text{ (n odd)}, \\ \Delta_0(\lambda) \Delta^2_n(\lambda) \prod_{j=1}^{\frac{n-2}{2}} \Delta_j^2(\lambda) & \text{ (n even)}, \end{cases} \] (2.4)

and \( \Delta_j = \lambda + 1 - (\alpha + 2\beta \cos \frac{2j\pi}{n}) \exp(-\lambda \tau) \) \( (j \in \mathbb{N}) \). For more convenience, let
\[ \gamma_j = \alpha + 2\beta \cos \frac{2j\pi}{n}. \]

In the remaining part, the following notations will be used [12]:
\[ \tau_l = \frac{\Theta(\gamma, l)}{\sqrt{\gamma^2 - 1}}, \quad l \in \mathbb{N}_0 \text{ (the set of all nonnegative integers)}, \]

where
\[ \Theta(\gamma, l) = \begin{cases} 2l\pi + \arccos(1/\gamma), & \gamma < -1, \\ (2l + 1)\pi - \arccos(1/\gamma), & \gamma > 1. \end{cases} \] (2.5)

Consider
\[ E(\lambda) = \lambda + 1 - \gamma \exp(-\lambda \tau). \] (2.6)

then it is known [4,8,12] that
• there exists a positive real root for $\gamma > 1$ (independent of $\tau$) and exactly $2(l + 1) + 1$ roots with positive real parts for $\tau_l < \tau < \tau_{l+1}$ whereas if $\gamma < -1$ and $\tau_l < \tau < \tau_{l+1}$, there exist exactly $2(l + 1)$ roots of $E(\lambda)$ with positive real parts.

\[
\frac{d\lambda}{d\gamma} \bigg|_{\lambda=0, \gamma=1} = \frac{1}{1 + \tau} > 0,
\]

\[
\frac{d\Re \lambda_l(\gamma)}{d\tau} \bigg|_{|\gamma| > 1, \tau = \tau_l} = \frac{\omega^2(\gamma)}{(1 + \tau)^2 + \tau^2 \omega(\gamma)^2} > 0,
\]

where $\Re \lambda$ denotes the real part of $\lambda$ respectively;

• there exist the following (locally) asymptotically stable regions of system (1.1):

\[
RD_1 = \left\{ (\alpha, \beta) \left| -1 < \gamma_l = \alpha + 2\beta \cos \frac{2l \pi}{n} < 1, \; l \in \mathbb{N} \right. \right\}; \quad \text{or}
\]

\[
RD_2 = \left\{ (\alpha, \beta) \left| \gamma_l < 1, \; \gamma^* = \min_{l \in \mathbb{N}} \gamma_l < -1, \; 0 < \tau < \tau_{\gamma^*} = \frac{\arccos(1/\gamma^*^2)}{\sqrt{\gamma^*^2 - 1}} \right. \right\}.
\]

Obviously, $RD_1$ is independent of the delay $\tau$, and $RD_2$ is dependent on the time delay $\tau$.

3. Equivariant structure, invariant spaces and dual spaces

The action of $\mathbb{D}_n$ on $\mathbb{R}^n$ is defined as follows:

\[
(\rho x)_j = x_{j+1} = \chi x_j = \exp(i2\pi/n)x_j, \quad (\kappa x)_j = x_{n+2-j},
\]

(j mod $n$). Then both system (1.1) and the linearized system (2.1) are $\mathbb{D}_n$-equivariant [8].

Now, we discuss some properties about the (generalized) eigenspaces associated with the eigenvalues of the linearized system (2.1).

If $\gamma_0 = \alpha + 2\beta = 1$ (or $\gamma_{n/2} = \alpha - 2\beta = 1$ (n even)), zero is a single eigenvalue in Eq. (2.4). Therefore, when $\gamma_0$ ($\gamma_{n/2}$) gets across the critical value 1, the trivial equilibrium point will lose its stability and one-dimensional bifurcation can be expected and nontrivial equilibrium points appear.

If $|\gamma_0| > 1$ and $\tau = \tau_l(\gamma_0)$ (or $|\gamma_{n/2}| > 1, \; \tau = \tau_l(\gamma_{n/2})$), there exist a pair of pure imaginary eigenvalues $\pm i\omega$. If $\tau$ increases through the sequence of critical values $\tau_l(\gamma_0)$ (or $\tau_l(\gamma_{n/2})$), (common) Hopf bifurcations may occur. We will not give a detailed study of such cases (we refer the readers to [8,16]), instead, focus our attention on the critical eigenvalues related to $\Delta_j$ ($j \neq 0, n/2$). It is natural that the methods and techniques proposed in the remaining part, can be applied to the common bifurcation analysis as well.

One can easily check that there exist two linearly independent eigenvectors corresponding to $\lambda_j$ (of $\Delta_j$) as follows:

\[
q_j^0(\theta) = v_j \exp(\lambda_j \theta) = v_j^0 \exp(\lambda_j \theta),
\]

\[
q_{n-j}^0(\theta) = v_{n-j} \exp(\lambda_j \theta) = v_{n-j}^0 \exp(\lambda_j \theta).
\]
with the property
\[ \langle q_j^0(\theta), q_{n-j}^0(\theta) \rangle_1 = 0, \]
where the inner product in the space \( C^1([-\tau, 0], \mathbb{C}^n) \)
\[ \langle \psi, \phi \rangle_1 = \int_{-\tau}^0 \langle \psi(\theta), \phi(\theta) \rangle d\theta \]
and \( \langle u, v \rangle = \sum_{k=1}^n \bar{u}_k v_k \) for \( u, v \in \mathbb{C}^n \) (in what follows, we will allow functions with range \( \mathbb{C}^n \) instead of \( \mathbb{R}^n \) for convenience in computation). Obviously, the eigenspace of \( A \) associated with \( \lambda_j \) is of at least dimension 2 and \( \lambda_j \) is an eigenvalue of \( A \) with multiplicity 2. Therefore \( \{q_j^0, q_{n-j}^0\} \) is a basis of the eigenspace (denoted by \( \Xi(\lambda_j) \)) of \( A \) associated with \( \lambda_j \). Denote \( V^0_j \) as the space spanned by the vector \( q_j^0(\theta) \) and \( V^0_{n-j} \) by the vector \( q_{n-j}^0(\theta) \). Hence
\[ \Xi(\lambda_j) = V^0_j \oplus V^0_{n-j}. \]  (3.1)

Define
\[ q_j^1 = q_j^0 + q_{n-j}^0 \sim \left(1, \cos \frac{2j\pi}{n}, \ldots, \cos \frac{2j(n-1)\pi}{n}\right)^T \exp(\lambda_j \theta) = v_1^1 \exp(\lambda_j \theta), \]
\[ q_{n-j}^1 = q_j^0 - q_{n-j}^0 \sim \left(0, \sin \frac{2j\pi}{n}, \ldots, \sin \frac{2j(n-1)\pi}{n}\right)^T \exp(\lambda_j \theta) = v_2^1 \exp(\lambda_j \theta) \]
and the spaces spanned by \( q_j^1 \) or \( q_{n-j}^1 \) are denoted by \( \mathbb{V}_j^1 \) or \( \mathbb{V}_{n-j}^1 \) respectively. The corresponding conjugate eigenvectors (to \( \bar{\lambda}_j \)) denoted by \( \bar{q}_j^0, \bar{q}_{n-j}^0 \), and \( \bar{q}_j^1, \bar{q}_{n-j}^1 \) respectively, e.g., \( \bar{q}_j^0 = (1, \bar{x}_j, \ldots, \bar{x}^{(n-1)})^T \exp(\bar{\lambda}_j \theta) \). It is easy to verify that
\[ \langle q_j^s, q_{n-j}^s \rangle_1 = 0, \quad \langle q_j^s, \bar{q}_{n-j}^s \rangle_1 = 0, \quad s = 0, 1. \]

Then the following can be induced from the above analysis that

**Proposition 3.1.** \( \mathbb{V}_j^0 (j \neq 0, n/2) \) in Eq. (3.1) is invariant under the rotation generator \( \rho \), \( \mathbb{V}_j^1 \) is invariant under the flip \( \kappa \). The eigenspace corresponding to \( \lambda_j \) (of \( \Delta_j \)) can be represented as
\[ \mathcal{E}(\lambda_j) = \mathbb{V}_j^s \oplus \mathbb{V}_{n-j}^s, \quad s = 0, 1. \]  (3.2)

**Proposition 3.2.**
- If \( \lambda_j (j \neq 0, n/2) \) is a real root of \( \Delta_j \), then Eq. (3.2) (s = 1) is the direct-sum of two one-dimensional real subspaces with \( \mathbb{V}_j^1 \) invariant under the flip \( \kappa \).
• If $\lambda_j (= i\omega(\gamma_j))$ and $\bar{\lambda}_j$ are a pair of pure imaginary roots of $\Delta_j$, then the subspace $\{z^0_j q^0_j + z^0_{n-j} \bar{q}^0_{n-j}\}$ of $\Xi(\pm i\omega(\gamma_j))$ is invariant by the cyclic group $\mathbb{Z}_n$ action, $\{z^1_j q^1_j + z^1_{n-j} \bar{q}^1_{n-j}\}$ invariant under the flip $\kappa$. The generalized eigenspace $\Xi(\pm i\omega(\gamma_j))$ can be represented as

$$\Xi(\pm i\omega(\gamma_j)) = \{z^0_j q^0_j + z^0_{n-j} \bar{q}^0_{n-j}\} \oplus \{z^0_{n-j} q^0_{n-j} + z^0_{n-j} \bar{q}^0_{n-j}\}; \quad (3.3)$$

or (direct-sum of two-dimensional real subspaces)

$$\Xi(\pm i\omega(\gamma_j)) = \{z^1_j q^1_j + z^1_{n-j} \bar{q}^1_{n-j}\} \oplus \{z^1_{n-j} q^1_{n-j} + z^1_{n-j} \bar{q}^1_{n-j}\}. \quad (3.4)$$

In what follows, we will need the following:

**Proposition 3.3.**

$$\sum_{k=0}^{n-1} \cos^2 \frac{2kj\pi}{n} = \begin{cases} \frac{n}{2}, & j \neq 0, n/2, \\ n, & \text{otherwise}; \end{cases}$$

$$\sum_{k=0}^{n-1} \sin^2 \frac{2kj\pi}{n} = \begin{cases} \frac{n}{2}, & j \neq 0, n/2, \\ 0, & \text{otherwise}; \end{cases}$$

$$\sum_{k=0}^{n-1} \cos \frac{4kj\pi}{n} = \begin{cases} n, & j = 0, n/2, \\ 0, & \text{otherwise}. \end{cases}$$

$$\sum_{k=0}^{n-1} \sin \frac{4kj\pi}{n} - \sum_{k=0}^{n-1} \sin \frac{8kj\pi}{n} = 0,$$

$$\sum_{k=0}^{n-1} \cos \frac{8kj\pi}{n} = \begin{cases} 0, & j \neq 0, n/2, n/4, \\ n, & \text{otherwise}. \end{cases} \quad (3.5)$$

**Proof.** It can be directly induced from

$$\sum_{k=0}^{n-1} \chi^{2kj} = \sum_{k=0}^{n-1} \exp \left( i \frac{4kj\pi}{n} \right) = \begin{cases} 0, & j \neq 0, n/2, \\ n, & \text{otherwise}, \end{cases} \quad (3.6)$$

and

$$\sum_{k=0}^{n-1} \chi^{4kj} = \sum_{k=0}^{n-1} \exp \left( i \frac{8kj\pi}{n} \right) = \begin{cases} 0, & j \neq 0, n/2, n/4, \\ n, & \text{otherwise}. \end{cases} \quad (3.7)$$

For simplicity, in addition to the hypothesis $f(0) = g(0) = 0$ and the given normalization property, we further assume that $f, g \in C^1_3$ and $f''(0) = g''(0) = 0$, i.e., the nonlinear functions $f$ and $g$ can be represented (near a neighborhood of the origin) in the form
\[ f(x) = x - d_1 x^3 + O(x^4), \quad d_1 = -f^{(3)}(0)/3!, \]
\[ g(x) = x - d_2 x^3 + O(x^4), \quad d_2 = -g^{(3)}(0)/3!. \]

(3.8)

Let \( C([-\tau, 0], \mathbb{R}^n) \) denote the Banach space of continuous mappings from \([-\tau, 0]\) into \( \mathbb{R}^n \) equipped with the supremum norm \( \|\phi\| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)| \) for \( \phi \in C([-\tau, 0], \mathbb{R}^n) \). In the remainder, if \( \sigma \in \mathbb{R}, T \geq 0 \) and \( x : [\sigma - \tau, \sigma + T] \to \mathbb{R}^n \) is a continuous mapping, then \( x_t \in C([-\tau, 0], \mathbb{R}^n) \) is defined by \( x_t(\theta) = x(t + \theta), -\tau \leq \theta \leq 0 \).

Rewrite Eq. (1.1) as
\[
\dot{x}(t) = Lx(t) + N(x(t - \tau))
\]
with
\[
L\phi = -\phi(0) + (\alpha I + \beta M)\phi(-\tau),
\]
and
\[
N(u) = \frac{1}{6} G'''(0) u^3 + O(\|u\|^4),
\]
where \( G(s) \) is an \( n \times n \) matrix function defined by
\[
G = \begin{pmatrix}
\alpha f(s) & \beta g(s) & 0 & \cdots & \beta g(s) \\
\beta g(s) & \alpha f(s) & \beta g(s) & \cdots & 0 \\
0 & \beta g(s) & \alpha f(s) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta g(s) & 0 & \beta g(s) & \cdots & \alpha f(s)
\end{pmatrix}_{n \times n}.
\]

By the Riesz representation theorem [9], there exist a \( n \times n \) matrix-valued function \( \vartheta(\cdot) : [-\tau, 0] \to \mathbb{R}^{n \times n} \), whose components have bounded variation, such that, for \( \phi \in C([-\tau, 0], \mathbb{R}^n) \), \( L\phi = \int_{-\tau}^{0} d\vartheta(\cdot) \phi(\theta) \). Now we define,
\[
A\phi = \begin{cases}
\frac{d\phi}{d\theta}, & \theta = [-\tau, 0), \\
(\alpha I + \beta M)\phi(-\tau), & \theta = 0,
\end{cases}
\quad R(\phi) = \begin{cases}
0, & \theta = [-\tau, 0), \\
N(\phi), & \theta = 0.
\end{cases}
\]

Then Eq. (3.9) can be rewritten in the following desired form:
\[
\dot{x}_t = A\mathbf{x}_t + R(x_t).
\]

(3.10)

For \( \theta \in [-\tau, 0) \), Eq. (3.10) becomes the trivial case \( dx_t/d\theta = dx_t/dt \), whereas \( \theta = 0 \), it is Eq. (3.9).

Now we introduce the adjoint operator \( A^* \) [9] as follows:
\[
A^*\psi = \begin{cases}
-\frac{d\psi}{dx}, & \xi = (0, \tau], \\
\int_{-\tau}^{0} d\vartheta^T(t) \psi(-t), & \xi = 0,
\end{cases}
\]
where $\vartheta^T$ denotes the transpose of $\vartheta$. Define the bilinear inner product [5] by
\[
\langle\langle \psi, \phi \rangle\rangle = \langle \psi(0), \phi(0) \rangle - \int_{-\tau}^{0} \psi(\xi + \tau) (\alpha I + \beta M) \phi(\xi) \, d\xi.
\]
By direct computation, we find
\[
q_i^{s*}(\xi) = \tilde{\vartheta} v_i^s \exp(-\bar{\lambda} \xi)
\]
and $\langle\langle q^*, q \rangle\rangle = 1$, which implies that $q$ and $q^*$ satisfy the normalization condition, where
\[
Q = \begin{cases} 
\left[ n(1 + \tau \gamma_j \exp(-\lambda \tau)) \right]^{-1}, & s = 0, \\
\left[ n(1 + \tau \gamma_j \exp(-\lambda \tau)) / 2 \right]^{-1}, & s = 1
\end{cases}
\]
(Here Proposition 3.3 is used when $s = 1$.) It is easy to verify that $\bar{\lambda}$ is an eigenvalue for $A^*$, i.e., $A^* q^s = \bar{\lambda} q^s$. As usual, we have $\langle\langle \psi, A \phi \rangle\rangle = \langle\langle A^* \psi, \phi \rangle\rangle$ for $(\psi, \phi) \in D(A) \times D(A^*) = C^1([-\tau, 0], \mathbb{C}^n) \times C^1([0, \tau], \mathbb{C}^n)$.

Since
\[
\bar{\lambda} \langle\langle q^s, \bar{q} \rangle\rangle = \langle\langle q^*, A \bar{q} \rangle\rangle = \langle\langle A^* q^s, \bar{q} \rangle\rangle = \lambda \langle\langle q^*, \bar{q} \rangle\rangle,
\]
which leads to
\[
\langle\langle q_i^{s*}(\theta), \tilde{q}_k^s(\theta) \rangle\rangle = 0, \quad \iota, k = j, n - j, s = 0, 1,
\]
for a pair of complex eigenvalues $\lambda_j, \bar{\lambda}_j$. Furthermore, we obtain
\[
\langle\langle q_i^{s*}(\theta), q_k^s(\theta) \rangle\rangle = 0, \quad \iota \neq k, \iota, k = j, n - j, s = 0, 1. \tag{3.11}
\]
If $\lambda$ is the real nonnegative eigenvalue $\lambda_j^+$ (of $\Delta_j$), then $\tilde{\lambda}_j = \lambda_j^+$, $\tilde{q}_k^1(\theta) = q_k^1(\theta)$ and it is sufficient to consider (3.11) for $s = 1$.

4. Bifurcation results

In order to obtain the explicit expression of the (Poincaré) Normal form at $\gamma_j = 1$ or $\tau = \tau_j(\gamma_j)$ with $|\gamma_j| > 1$, we first compute the dual eigenvectors $q_i^{s*}(\xi)$ with respect to $q_i^s(\theta)$ as in Propositions 3.1 and 3.2,
\[
q_i^{s*}(\xi) = \begin{cases} 
\tilde{\vartheta} v_i^s \exp(i \omega(\gamma_j) \xi), & s = 0, 1, \text{ related to a pair of pure imaginary eigenvalues } \pm i \omega(\gamma_j), \\
v_i^s, & s = 1, \gamma_j = 1, \text{ related to the zero eigenvalue;}
\end{cases}
\]
where $\iota = j, n - j, \xi \in [0, \tau]$.
and if \( |\gamma_j| > 1 \) and \( \tau = \tau_l(\gamma_j) \),

\[
\varrho = \begin{cases} 
\frac{1}{n(1 + \tau \gamma_j \exp(-i\omega(\gamma_j)\tau))}, & s = 0, \\
\frac{2}{n(1 + \tau \gamma_j \exp(-i\omega(\gamma_j)\tau))}, & s = 1.
\end{cases}
\]

Now let us begin the concrete normal form computations and bifurcation analysis. (For more details, also see Appendices A–D.)

Here we will only focus our attention on the case \( j \neq n/4 \). For the case: \( j = n/4 \), the proof is similar (also see that in [12]), and it is omitted.

### 4.1. Equivariant Hopf bifurcations

At \( \tau = \tau_l(\gamma_j) \) (\( |\gamma_j| > 1 \)), there are a (repeated) pair of pure imaginary roots of \( \Delta_j \), denoted by \( \lambda^*_j, \bar{\lambda}^*_j = \pm i\omega(\gamma_j) \), where \( \omega(\gamma_j) = \sqrt{\gamma_j^2 - 1} \). Moreover,

\[
\left. \frac{d\Re \lambda_l(\gamma_j)}{d\tau} \right|_{|\gamma_j| > 1} = \frac{\omega^2(\gamma_j)}{(1 + \tau)^2 + \tau^2 \omega(\gamma_j)^2} > 0.
\]

Restricted to the invariant manifold \( C_0 \) associated with \( \{z^0_{jl}q^0_{jl} + \bar{z}^0_{jl}(n-j)_l \bar{q}^0_{jl}, \bar{z}^0_{jl} \bar{q}^0_{jl} + z^0_{jl}q^0_{jl} \} \), system (1.1) is locally (near the origin) topologically equivalent to one of the following complex normal forms (for detailed computations, please see Appendix A):

\[
\begin{align*}
\dot{z}^0_{jl} &= i\omega(\gamma_j)z^0_{jl} - \frac{3(\alpha d_1 + 2\beta d_2 \cos \frac{2j\pi}{n})}{1 + \tau \gamma_j \exp(-i\omega(\gamma_j)\tau)} \left[ z^0_{jl} e^{-i\omega(\gamma_j)\tau} + z^0_{(n-j)_l} e^{i\omega(\gamma_j)\tau} \right]^2 \\
&\times \left[ z^0_{jl} e^{i\omega(\gamma_j)\tau} + z^0_{(n-j)_l} e^{-i\omega(\gamma_j)\tau} \right] + O(4), \\
\dot{z}^0_{(n-j)_l} &= i\omega(\gamma_j)z^0_{(n-j)_l} - \frac{3(\alpha d_1 + 2\beta d_2 \cos \frac{2j\pi}{n})}{1 + \tau \gamma_j \exp(-i\omega(\gamma_j)\tau)} \left[ z^0_{jl} e^{-i\omega(\gamma_j)\tau} + z^0_{(n-j)_l} e^{i\omega(\gamma_j)\tau} \right]^2 \\
&\times \left[ z^0_{jl} e^{i\omega(\gamma_j)\tau} + z^0_{(n-j)_l} e^{-i\omega(\gamma_j)\tau} \right] + O(4),
\end{align*}
\]

where \( d_1, d_2 \) in (3.8). Truncating the normal form equation to degree three we obtain

\[
\begin{align*}
\dot{z}_1 &= \left( i\omega - \frac{3(\alpha d_1 + 2\beta d_2 \cos \frac{2\pi}{n}) \exp(-i\omega \tau)}{1 + \tau \gamma_j \exp(-i\omega \tau)} \right) (|z_1|^2 + 2|z_2|^2) z_1, \\
\dot{z}_2 &= \left( i\omega - \frac{3(\alpha d_1 + 2\beta d_2 \cos \frac{2\pi}{n}) \exp(-i\omega \tau)}{1 + \tau \gamma_j \exp(-i\omega \tau)} \right) (2|z_1|^2 + |z_2|^2) z_2.
\end{align*}
\]

Let \( z_1 = r_1 \exp(i\theta_1) \) and \( z_2 = r_2 \exp(i\theta_2) \). The phase/amplitude equations [1] corresponding to (4.5) are...
\[ \dot{r}_1 = -3 \left( \alpha d_1 + 2 \beta d_2 \cos \frac{2j\pi}{n} \right) \Re \left( \frac{\exp(-i\omega \tau)}{1 + \tau \gamma_j \exp(-i\omega \tau)} \right) (r_1^2 + 2r_2^2) r_1, \]
\[ \dot{r}_2 = -3 \left( \alpha d_1 + 2 \beta d_2 \cos \frac{2j\pi}{n} \right) \Re \left( \frac{\exp(-i\omega \tau)}{1 + \tau \gamma_j \exp(-i\omega \tau)} \right) (2r_1^2 + r_2^2) r_2, \]
\[ \dot{\theta}_1 = \omega - 3 \left( \alpha d_1 + 2 \beta d_2 \cos \frac{2j\pi}{n} \right) \Im \left( \frac{\exp(-i\omega \tau)}{1 + \tau \gamma_j \exp(-i\omega \tau)} \right) (r_1^2 + 2r_2^2), \]
\[ \dot{\theta}_2 = \omega - 3 \left( \alpha d_1 + 2 \beta d_2 \cos \frac{2j\pi}{n} \right) \Im \left( \frac{\exp(-i\omega \tau)}{1 + \tau \gamma_j \exp(-i\omega \tau)} \right) (2r_1^2 + r_2^2). \]

where \( \Re / \Im \) denotes the real/imaginary part respectively.

Associated with \( \{ z_{jl}^{1} ; q_{jl}^{1}, \bar{q}_{jl}^{1} \} \oplus \{ z_{(n-j)l}^{1} ; q_{(n-j)l}^{1}, \bar{q}_{(n-j)l}^{1} \} \), the restriction of the system to the invariant manifold \( C_0 \) is locally (near the origin) topologically equivalent to one of the following complex Poincaré normal forms (see Appendix B):

\[ \dot{z}_{jl}^{1} = i\omega (\gamma_j) z_{jl}^{1} - \frac{3(\alpha d_1 + 2 \beta d_2 \cos \frac{2j\pi}{n})}{4[1 + \tau \gamma_j \exp(-i\omega (\gamma_j) \tau)]} \left( \left( z_{jl}^{1} e^{-i\omega (\gamma_j) \tau} + z_{jl}^{1} e^{i\omega (\gamma_j) \tau} \right)^3 + (z_{jl}^{1} e^{-i\omega (\gamma_j) \tau} + z_{jl}^{1} e^{i\omega (\gamma_j) \tau}) (z_{(n-j)l}^{1} e^{i\omega (\gamma_j) \tau} + z_{(n-j)l}^{1} e^{-i\omega (\gamma_j) \tau})^2 \right) + O(4). \]
\[ \dot{z}_{(n-j)l}^{1} = i\omega (\gamma_j) z_{(n-j)l}^{1} - \frac{3(\alpha d_1 + 2 \beta d_2 \cos \frac{2j\pi}{n})}{4[1 + \tau \gamma_j \exp(-i\omega (\gamma_j) \tau)]} \left( \left( z_{(n-j)l}^{1} e^{i\omega (\gamma_j) \tau} + z_{(n-j)l}^{1} e^{-i\omega (\gamma_j) \tau} \right)^3 + (z_{(n-j)l}^{1} e^{i\omega (\gamma_j) \tau} + z_{(n-j)l}^{1} e^{-i\omega (\gamma_j) \tau}) (z_{jl}^{1} e^{i\omega (\gamma_j) \tau} + z_{jl}^{1} e^{-i\omega (\gamma_j) \tau})^2 \right) + O(4). \]

Truncating the normal form equation to degree three we obtain

\[ \dot{z}_1 = \left( i\omega - \frac{3(\alpha d_1 + 2 \beta d_2 \cos \frac{2j\pi}{n})}{4[1 + \tau \gamma_j \exp(-i\omega \tau)]} \right) \left( 3|z_1|^2 + 2|z_2|^2 \right) z_1, \]
\[ \dot{z}_2 = \left( i\omega - \frac{3(\alpha d_1 + 2 \beta d_2 \cos \frac{2j\pi}{n})}{4[1 + \tau \gamma_j \exp(-i\omega \tau)]} \right) \left( 2|z_1|^2 + 3|z_2|^2 \right) z_2. \]
Note that \(-i\omega(\gamma_j)\) is a root of \(\Delta_j = \lambda + 1 - \gamma_j \exp(-\lambda \tau)\), we have

\[
\Re \left( \frac{\exp(-i\omega(\gamma_j)\tau)}{1 + \tau \gamma_j \exp(-i\omega(\gamma_j)\tau)} \right) = \frac{\gamma_j(1 + \tau \gamma_j^2)}{\omega(\gamma_j)^2 + (1 + \tau \gamma_j^2)^2}.
\]

Therefore there exist a series of supercritical \((\gamma_j(\alpha_d + 2\beta_d2 \cos \frac{2i\pi}{n}) > 0)\) or subcritical \((\gamma_j(\alpha_d + 2\beta_d2 \cos \frac{2i\pi}{n}) < 0)\) Hopf bifurcations and a sequence of stable invariant curves bifurcate from the trivial solution as \(\tau\) passes through its critical value \(\tau_l(\gamma_j)\).

### 4.2. Equivariant pitchfork bifurcations

As \(\gamma\) passes through the critical value \(\gamma^* = 1\), there exists a (repeated) real eigenvalue \(\lambda\) increasing though the critical value \(\lambda^* = 0\) and a positive real eigenvalue \(\lambda^+\) occurs. Since \(\frac{d\lambda}{d\gamma} |_{\gamma_j=1} = \frac{1}{1 + \tau} > 0\), an equivariant pitchfork bifurcation may be expected.

Choose \(q_{1+} = v_1^t (t = j, n - j)\) and its corresponding adjoint eigenvector \(q_{1+}^* = \frac{2}{n(1 + \tau)} v_1^t\).

Restricted to the corresponding center manifold with linear part \(\{y_j v_1^j + y_{n-j} v_1^{n-j}\}\), system (1.1) is locally (near the origin) topologically equivalent to one of the following complex normal forms

\[
\dot{y}_t(t) = \left\langle q_{1+}^* (\theta), \dot{x}_t \right\rangle = \left\langle q_{1+}^* (\theta), R(x_t) \right\rangle,
\]

\[
= -\frac{3(\alpha_d + 2\beta_d2 \cos \frac{2i\pi}{n})}{4(1 + \tau)} y_t(t) \left[ y_j^2 + y_{n-j}^2 \right] + O(4), \quad t = j \text{ or } n - j.
\]

Therefore a symmetrical breaking supercritical \((\alpha_d + 2\beta_d2 \cos \frac{2i\pi}{n} > 0)\) or subcritical \((\alpha_d + 2\beta_d2 \cos \frac{2i\pi}{n} < 0)\) pitchfork bifurcation occurs as \(\gamma_j\) gets across its critical value \(\gamma^* = 1\), which leads to the co-existence of multiple (asynchronous) nontrivial equilibrium points.

We summarize the bifurcating results which can be obtained by the methods and techniques proposed in this section:

**Theorem 4.1.** (Bifurcation by the trivial identity group \(\mathbb{I}\) action \((j = 0)\) or the \(\mathbb{Z}_2\) (mirror-reflecting) group action \((j = n/2)\)) Assume that (3.8) holds. Let \(\gamma = \alpha \pm 2\beta\) and \(d_\gamma = \alpha_d \pm 2\beta_d\). As \(\tau\) (respectively \(\gamma\)) passes through the critical value \(\tau_l\) (respectively \(\gamma^* = 1\)), Hopf (respectively pitchfork) bifurcations occur in system (1.1) and there exist synchronous/asynchronous closed limit cycles or nontrivial equilibrium points. If \(d_\gamma > 0\), it is supercritical for both pitchfork bifurcations and Hopf bifurcations \((\gamma > 1)\), and subcritical for Hopf bifurcations if \(\gamma < -1\); otherwise, if \(d_\gamma < 0\), the stability changes inversely.

**Theorem 4.2.** (\(\mathbb{D}_n\)-equivariant bifurcation \((j \neq 0, n/2)\)) Let \(d_{\gamma_j} = \alpha_d + 2\beta_d2 \cos \frac{2i\pi}{n}\). Assume that (3.8) holds. As \(\tau\) (respectively \(\gamma_j = \alpha \pm 2\beta \cos \frac{2i\pi}{n}\)) passes through the critical values \(\tau_l(\gamma_j)\) (respectively \(\alpha^* = 1\)), system (1.1) exhibits multiple bifurcations: equivariant Hopf (respectively pitchfork) bifurcations and there exist asynchronous (symmetry-breaking) closed limit cycles or nontrivial equilibrium points. If \(d_{\gamma_j} > 0\), it is supercritical for both pitchfork bifurcations and Hopf bifurcations \((\gamma_j > 1)\), and subcritical for Hopf bifurcations if \(\gamma_j < -1\); otherwise, if \(d_{\gamma_j} < 0\), the stability changes inversely. Furthermore,
(1) If $\gamma_j > 1$, for any time delay $\tau$, there exist two branches of symmetry-breaking asynchronous nontrivial equilibria:

(i) $x_i = x_{n+2-i}$ (with choice number $C^1_n = n$ mirror-reflecting points; invariant with the flip $\kappa$);

(ii) $x_i \neq x_{n+2-i}$ ($C^1_n$ standing points).

(2) If $|\gamma_j| > 1$ and $\tau > \tau_1(\gamma_j)$, there exist three branches of symmetry breaking oscillations, with the following patterns (invariant closed curves or limit cycles):

(i) $C^1_n$ mirror-reflecting waves: $x_i = x_{n+2-i}$ (invariant with the flip $\kappa$);

(ii) $C^1_n$ standing waves: $x_i \neq x_{n+2-i}$;

(iii) $C^1_2$ phase-locked waves: Each cell has the same waveform with phase shift (invariant with the cyclic group $\mathbb{Z}_n$).

Remark. The methods and techniques discussed as above, might offer some creative idea about filling the gap ($j \neq n/4$) in [2].

5. Conclusion

In this paper, we propose an efficient computation method in local symmetric bifurcation analysis of nonlinear delayed systems in the presence of the $\mathbb{D}_n$ symmetry. A new scheme is established for equivariant bifurcation analysis of delayed systems.

Mechanism, from the trivial equilibrium point to multiple nontrivial equilibria/periodic orbits and how equivariant bifurcations occur, is explored clearly. Physical properties, such as synchronization, asynchronization and spatio-temporal behavior by symmetrical breaking, are discussed.

Our case study may provide such an example that a ring of cells coupled with delayed nonlinear diffusion exhibits a series of symmetrical breaking bifurcations and spatio-temporal behavior even though the state is continuous by only a single variable. Nonlinear diffusion and pattern formations can be found in real natural world phenomena, for example, researchers are trying their best to develop theoretical models so that they can simulate the dissemination and diffusion in cancer surveillance, or the outbreak of SARS, Avian flu etc.

It is natural that there coexist multiple oscillation patterns, i.e., multistability in complex systems with multiple participants. Rich behavior of human being is a good example, such as the coexistence of different color, different languages, different culture and different living style etc.

Acknowledgments

Supported partially by NSFC, Tianyuan Foundation, China Education Scholarship Council, and Research Foundation of Beijing Jiao Tong University. This work was partially written and reported during the author’s visiting Department of Mathematics and Statistics, York University and Memorial University of Newfoundland. The author is greatly indebted to their great hospitality. He is also very grateful to Professor Dr. Wu Jianhong and Dr. Yuan Yuan for their helpful suggestions, interesting discussions and professional supports. The author is greatly indebted to the anonymous referee for valuable comments, which lead to an improvement of the original manuscript.
Appendix A. The computation of (4.1)

Proof. At $\tau = \tau_j(\gamma_j)$, it is known that there exist a pair of pure imaginary eigenvalues $\pm i\omega(\gamma_j)$. We choose $\lambda_j = i\omega(\gamma_j)$ (simply as $i\omega$), then $q_j^0(0) = q_{n-j}^0(0) = v_j = (1, \exp(i\frac{2j\pi}{n}), \ldots, \exp(i\frac{2(n-1)j\pi}{n}))^T$, $q_{n-j}^0(0) = \tilde{q}_j^0(0) = v_{n-j} = \tilde{v}_j$, and

$$q_j^0(-\tau) = \left(1, \exp\left(i\frac{2j\pi}{n}\right), \ldots, \exp\left(i\frac{2(n-1)j\pi}{n}\right)\right)^T e^{-i\omega \tau},$$

$$q_{n-j}^0(-\tau) = \left(1, \exp\left(-i\frac{2j\pi}{n}\right), \ldots, \exp\left(-i\frac{2(n-1)j\pi}{n}\right)\right)^T e^{-i\omega \tau},$$

$$\tilde{q}_j^0(-\tau) = \left(1, \exp\left(-i\frac{2j\pi}{n}\right), \ldots, \exp\left(-i\frac{2(n-1)j\pi}{n}\right)\right)^T e^{i\omega \tau}$$

and

$$\tilde{q}_{n-j}^0(-\tau) = \left(1, \exp\left(i\frac{2j\pi}{n}\right), \ldots, \exp\left(i\frac{2(n-1)j\pi}{n}\right)\right)^T e^{i\omega \tau}.$$

Related to $\{z_j^0 q_{j1}^0 + z_{(n-j)1}^0 \bar{q}_{(n-j)1}^0\} \Theta \{z_{j1}^0 \bar{q}_{j1}^0 + z_{(n-j)1}^0 q_{(n-j)1}^0\}$, if the system is restricted to the corresponding invariant manifold $C_0$, then,

$$x(t-\tau) = [z_{j1}^0 q_{j1}^0(-\tau) + z_{(n-j)1}^0 q_{(n-j)1}^0(-\tau) + z_{j1}^0 \bar{q}_{j1}^0(-\tau) + z_{(n-j)1}^0 q_{(n-j)1}^0(-\tau)] + O(r^2)$$

($l = j, n - j$). In view of the properties of $f$ and $g$ (3.8), it suffices to consider

$$x(t-\tau) = [z_{j1}^0 q_{j1}^0(-\tau) + z_{(n-j)1}^0 q_{(n-j)1}^0(-\tau) + z_{j1}^0 \bar{q}_{j1}^0(-\tau) + z_{(n-j)1}^0 q_{(n-j)1}^0(-\tau)],$$

i.e.,

$$x_1(t-\tau) = [z_{j1}^0 e^{-i\omega \tau} + z_{(n-j)1}^0 e^{i\omega \tau}] + [\bar{z}_{j1}^0 e^{i\omega \tau} + \bar{z}_{(n-j)1}^0 e^{-i\omega \tau}],$$

$$x_2(t-\tau) = \exp\left(i\frac{2j\pi}{n}\right)[z_{j1}^0 e^{-i\omega \tau} + z_{(n-j)1}^0 e^{i\omega \tau}]$$

$$\quad + \exp\left(-i\frac{2j\pi}{n}\right)[\bar{z}_{j1}^0 e^{i\omega \tau} + \bar{z}_{(n-j)1}^0 e^{-i\omega \tau}],$$

$$\vdots$$

$$x_n(t-\tau) = \exp\left(i\frac{2(n-1)j\pi}{n}\right)[z_{j1}^0 e^{-i\omega \tau} + z_{(n-j)1}^0 e^{i\omega \tau}]$$

$$\quad + \exp\left(-i\frac{2(n-1)j\pi}{n}\right)[\bar{z}_{j1}^0 e^{i\omega \tau} + \bar{z}_{(n-j)1}^0 e^{-i\omega \tau}],$$

which leads to
\[ x_1^3(t - \tau) = \left[ z_{j_1}^0 e^{-i\omega t} + z_{(n-j)}^0 e^{i\omega t} \right]^3 + \left[ z_{j_1}^0 e^{i\omega t} + z_{(n-j)}^0 e^{-i\omega t} \right]^3 + 3\left[z_{j_1}^0 e^{-i\omega t} + z_{(n-j)}^0 e^{i\omega t}\right] \left[z_{j_1}^0 e^{i\omega t} + z_{(n-j)}^0 e^{-i\omega t}\right] + 3\left[z_{j_1}^0 e^{i\omega t} + z_{(n-j)}^0 e^{-i\omega t}\right]^2, \]

\[ x_2^3(t - \tau) = \exp\left(i\frac{6j\pi}{n}\right)\left[z_{j_1}^0 e^{-i\omega t} + z_{(n-j)}^0 e^{i\omega t}\right]^3 + \exp\left(-i\frac{6j\pi}{n}\right)\left[z_{j_1}^0 e^{i\omega t} + z_{(n-j)}^0 e^{-i\omega t}\right]^3 + 3\exp\left(i\frac{2j\pi}{n}\right)\left[z_{j_1}^0 e^{-i\omega t} + z_{(n-j)}^0 e^{i\omega t}\right] \left[z_{j_1}^0 e^{i\omega t} + z_{(n-j)}^0 e^{-i\omega t}\right] + 3\exp\left(-i\frac{2j\pi}{n}\right)\left[z_{j_1}^0 e^{-i\omega t} + z_{(n-j)}^0 e^{i\omega t}\right] \left[z_{j_1}^0 e^{i\omega t} + z_{(n-j)}^0 e^{-i\omega t}\right]^2, \]

\[ x_3^3(t - \tau) = \exp\left(i\frac{6(n-1)j\pi}{n}\right)\left[z_{j_1}^0 e^{-i\omega t} + z_{(n-j)}^0 e^{i\omega t}\right]^3 + \exp\left(-i\frac{6(n-1)j\pi}{n}\right)\left[z_{j_1}^0 e^{i\omega t} + z_{(n-j)}^0 e^{-i\omega t}\right]^3 + 3\exp\left(i\frac{2(n-1)j\pi}{n}\right)\left[z_{j_1}^0 e^{-i\omega t} + z_{(n-j)}^0 e^{i\omega t}\right] \left[z_{j_1}^0 e^{i\omega t} + z_{(n-j)}^0 e^{-i\omega t}\right] + 3\exp\left(-i\frac{2(n-1)j\pi}{n}\right)\left[z_{j_1}^0 e^{-i\omega t} + z_{(n-j)}^0 e^{i\omega t}\right] \left[z_{j_1}^0 e^{i\omega t} + z_{(n-j)}^0 e^{-i\omega t}\right]^2. \]

Hence,

\[ x_2^3(t - \tau) + x_3^3(t - \tau) = \left[ \exp\left(i\frac{6j\pi}{n}\right) + \exp\left(-i\frac{6j\pi}{n}\right) \right]\left[z_{j_1}^0 e^{-i\omega t} + z_{(n-j)}^0 e^{i\omega t}\right]^3 + \left[ \exp\left(i\frac{2j\pi}{n}\right) + \exp\left(-i\frac{2(n-1)j\pi}{n}\right) \right]\left[z_{j_1}^0 e^{-i\omega t} + z_{(n-j)}^0 e^{i\omega t}\right]^3 + 3\left[z_{j_1}^0 e^{-i\omega t} + z_{(n-j)}^0 e^{i\omega t}\right] \left[z_{j_1}^0 e^{i\omega t} + z_{(n-j)}^0 e^{-i\omega t}\right] + 3\left[z_{j_1}^0 e^{i\omega t} + z_{(n-j)}^0 e^{-i\omega t}\right]^2, \]

\[ x_1^3(t - \tau) + x_3^3(t - \tau) = \left[ 1 + \exp\left(i\frac{6*2j\pi}{n}\right) \right]\left[z_{j_1}^0 e^{-i\omega t} + z_{(n-j)}^0 e^{i\omega t}\right]^3 + \left[ 1 + \exp\left(-i\frac{6*2j\pi}{n}\right) \right]\left[z_{j_1}^0 e^{i\omega t} + z_{(n-j)}^0 e^{-i\omega t}\right]^3. \]
\[
+ 3 \left[ 1 + \exp \left( i \frac{2 \ast 2 j \pi}{n} \right) \right] \left[ z_{0j}^0 e^{-i\omega t} + \bar{z}_{0(n-j)}^0 e^{i\omega t} \right]^2
\]
\times \left[ z_{0j}^0 e^{i\omega t} + z_{0(n-j)}^0 e^{-i\omega t} \right] + 3 \left[ 1 + \exp \left( -i \frac{2 \ast 2 j \pi}{n} \right) \right] \left[ z_{0j}^0 e^{-i\omega t} + \bar{z}_{0(n-j)}^0 e^{i\omega t} \right]
\times \left[ z_{0j}^0 e^{i\omega t} + z_{0(n-j)}^0 e^{-i\omega t} \right]^2,
\]

\[
x_{n-1}^3(t - \tau) + x_1^3(t - \tau) = \left[ 1 + \exp \left( i \frac{6(n-2)j \pi}{n} \right) \right] \left[ z_{0j}^0 e^{-i\omega t} + \bar{z}_{0(n-j)}^0 e^{i\omega t} \right]^3
\]
\times \left[ z_{0j}^0 e^{i\omega t} + z_{0(n-j)}^0 e^{-i\omega t} \right] + 3 \left[ 1 + \exp \left( -i \frac{6(n-2)j \pi}{n} \right) \right] \left[ z_{0j}^0 e^{i\omega t} + \bar{z}_{0(n-j)}^0 e^{-i\omega t} \right]^3
\times \left[ z_{0j}^0 e^{-i\omega t} + z_{0(n-j)}^0 e^{i\omega t} \right] + 3 \left[ 1 + \exp \left( i \frac{2(n-2)j \pi}{n} \right) \right] \left[ z_{0j}^0 e^{-i\omega t} + \bar{z}_{0(n-j)}^0 e^{i\omega t} \right]^2
\times \left[ z_{0j}^0 e^{i\omega t} + z_{0(n-j)}^0 e^{-i\omega t} \right] + 3 \left[ 1 + \exp \left( -i \frac{2(n-2)j \pi}{n} \right) \right] \left[ z_{0j}^0 e^{i\omega t} + \bar{z}_{0(n-j)}^0 e^{-i\omega t} \right]
\times \left[ z_{0j}^0 e^{-i\omega t} + z_{0(n-j)}^0 e^{i\omega t} \right]^2.
\]

The common scalar inner product in \( \mathbb{C}^n \) becomes

\[
\langle q_{j1}^0, 0 \rangle \langle N(x(t - \tau), 0) \rangle = \langle q_{j1}^0, N''(0)(x_1^3(t - \tau), \ldots, x_n^3(t - \tau)) \rangle + O(4)
\]
\[
= -\Upsilon \left( C_{30}^j \left[ z_{0j}^0 e^{-i\omega t} + \bar{z}_{0(n-j)}^0 e^{i\omega t} \right]^3 + C_{03}^j \left[ z_{0j}^0 e^{i\omega t} + \bar{z}_{0(n-j)}^0 e^{-i\omega t} \right]^3
\right.
\]
\[
+ C_{21}^j \left[ z_{0j}^0 e^{-i\omega t} + \bar{z}_{0(n-j)}^0 e^{i\omega t} \right]^2 \left[ z_{0j}^0 e^{i\omega t} + \bar{z}_{0(n-j)}^0 e^{-i\omega t} \right]
\]
\[
+ C_{12}^j \left[ z_{0j}^0 e^{-i\omega t} + \bar{z}_{0(n-j)}^0 e^{i\omega t} \right]^2 \left[ z_{0j}^0 e^{i\omega t} + \bar{z}_{0(n-j)}^0 e^{-i\omega t} \right]^2 \right) + O(4),
\]

where

\[
\Upsilon = \frac{1}{n[1 + \tau \gamma j \exp(-i\omega(\gamma_j)\tau)]},
\]

and

\[
C_{30}^j = \left( \alpha d_1 + 2\beta d_2 \cos \frac{6j \pi}{n} \right) \sum_{k=0}^{n-1} \exp \left( i \frac{4kj \pi}{n} \right),
\]
\[
C_{03}^j = \left( \alpha d_1 + 2\beta d_2 \cos \frac{6j \pi}{n} \right) \sum_{k=0}^{n-1} \exp \left( -i \frac{8kj \pi}{n} \right),
\]
\[ C_{21}^j = 3n \left( \alpha d_1 + 2\beta d_2 \cos \frac{2j\pi}{n} \right), \]
\[ C_{12}^j = 3 \left( \alpha d_1 + 2\beta d_2 \cos \frac{2j\pi}{n} \right) \sum_{k=0}^{n-1} \exp \left( -i \frac{4kj\pi}{n} \right). \]

Similarly, one can obtain
\[
\langle q_0^* (n-j) \mid N (x(t-\tau), 0) \rangle = \langle q_0^* (n-j) \mid G'' (0) (x_1^3 (t-\tau), \ldots, x_n^3 (t-\tau))^T \rangle + O(4)
\]
\[
= -\gamma \left( C_{30}^{n-j} \left[ z_{j1} e^{-i\omega \tau} + z_{n-j}^0 e^{i\omega \tau} \right]^3
+ C_{03}^{n-j} \left[ z_{j1} e^{i\omega \tau} + z_{n-j}^0 e^{-i\omega \tau} \right]^3
+ C_{21}^{n-j} \left[ z_{j1} e^{-i\omega \tau} + z_{n-j}^0 e^{i\omega \tau} \right] \left[ z_{j1} e^{i\omega \tau} + z_{n-j}^0 e^{-i\omega \tau} \right]^2
+ C_{12}^{n-j} \left[ z_{j1} e^{i\omega \tau} + z_{n-j}^0 e^{-i\omega \tau} \right] \left[ z_{j1} e^{-i\omega \tau} + z_{n-j}^0 e^{i\omega \tau} \right]^2 \right)
+ O(4),
\]

where
\[
C_{30}^{n-j} = \left( \alpha d_1 + 2\beta d_2 \cos \frac{6j\pi}{n} \right) \sum_{k=0}^{n-1} \exp \left( i \frac{8kj\pi}{n} \right),
\]
\[
C_{03}^{n-j} = \left( \alpha d_1 + 2\beta d_2 \cos \frac{6j\pi}{n} \right) \sum_{k=0}^{n-1} \exp \left( -i \frac{4kj\pi}{n} \right),
\]
\[
C_{21}^{n-j} = 3 \left( \alpha d_1 + 2\beta d_2 \cos \frac{2j\pi}{n} \right) \sum_{k=0}^{n-1} \exp \left( i \frac{4kj\pi}{n} \right),
\]
\[
C_{12}^{n-j} = 3n \left( \alpha d_1 + 2\beta d_2 \cos \frac{2j\pi}{n} \right).
\]

Since \( j \neq 0, n/2 \), we find that \( \exp(\pm i \frac{4j\pi}{n}) \neq 1 \). Therefore
\[
\sum_{k=0}^{n-1} \exp \left( \pm i \frac{4kj\pi}{n} \right) = 0,
\]
and \( C_{30}^{n-j} = C_{03}^{n-j} = 0 = C_{12}^{n-j} = C_{21}^{n-j} = 0 \). Furthermore, if \( j \neq n/4 \), i.e., \( n \neq 4m \) (\( m \) is one of the natural numbers), then \( \exp(i \frac{8j\pi}{n}) \neq 1 \). Therefore
\[
\sum_{k=0}^{n-1} \exp \left( i \frac{8kj\pi}{n} \right) = 0,
\]
and \( C_{03}^{n-j} = 0 \). In view of
\[
\dot{z}_{jl}(t) = i\omega z_{jl}^0 + \langle q_{jl}^0(0), N(x(t - \tau), 0) \rangle,
\]
\[
\dot{z}_{(n-j)l}(t) = i\omega z_{(n-j)l}^0 + \langle q_{(n-j)l}^0(0), N(x(t - \tau), 0) \rangle.
\]
we get (4.1).

If \( j = n/4 \), then \( \exp(i \frac{8j\pi}{n}) = 1 \), and
\[
\sum_{k=0}^{n-1} \exp \left( i \frac{8kj\pi}{n} \right) = n, \quad \cos \frac{2j\pi}{n} = \cos \frac{\pi}{2} = 0, \quad \cos \frac{6j\pi}{n} = \cos \frac{3\pi}{2} = 0.
\]
This implies
\[
C^j_{03} = C_{30}^{n-j} = n\alpha d_1, \quad C^j_{21} = C_{12}^{n-j} = 3n\alpha d_1,
\]
and similar results (also see [12]) can be obtained.

In conclusion, we find that
\[
C^j_{30} = 0 = C^j_{12},
\]
\[
C^j_{03} = \begin{cases} n\alpha d_1, & j = n/4, \\ 0, & j \neq n/4; \end{cases}
\]
\[
C^j_{21} = \begin{cases} 3n\alpha d_1, & j = n/4, \\ 3n(\alpha d_1 + 2\beta d_2 \cos \frac{2j\pi}{n}), & j \neq n/4; \end{cases}
\]
\[
C^{n-j}_{03} = 0 = C_{21}^{n-j},
\]
\[
C^{n-j}_{30} = \begin{cases} n\alpha d_1, & j = n/4, \\ 0, & j \neq n/4; \end{cases}
\]
\[
C^{n-j}_{12} = \begin{cases} n\alpha d_1, & j = n/4, \\ 3n(\alpha d_1 + 2\beta d_2 \cos \frac{2j\pi}{n}), & j \neq n/4. \end{cases}
\]

**Appendix B. The computation of (4.4)**

**Proof.** Choose
\[
q_j^1 = \left( 1, \cos \frac{2j\pi}{n}, \ldots, \cos \frac{2j(n-1)\pi}{n} \right)^T \exp(-i\omega\theta),
\]
\[
q_{n-j}^1 = \left( 0, \sin \frac{2j\pi}{n}, \ldots, \sin \frac{2j(n-1)\pi}{n} \right)^T \exp(-i\omega\theta).
\]

Then
\[
q_j^1(0) = \bar{q}_j^1(0) = \left( 1, \cos \frac{2j\pi}{n}, \ldots, \cos \frac{2j(n-1)\pi}{n} \right)^T,
\]
\[
q_{n-j}^1(0) = \bar{q}_j^1(0) = \left( 0, \sin \frac{2j\pi}{n}, \ldots, \sin \frac{2j(n-1)\pi}{n} \right)^T
\]
and

\[
q_j^1(-\tau) = \begin{pmatrix}
\frac{1}{\cos(\frac{2j\pi}{n})} \\
\vdots \\
\frac{1}{\cos(\frac{2(n-1)\pi}{n})}
\end{pmatrix} e^{-i\omega \tau}, \quad q_{n-j}^1(-\tau) = \begin{pmatrix}
0 \\
\ddots \\
0
\end{pmatrix} e^{-i\omega \tau},
\]

\[
\tilde{q}_j^1(-\tau) = \begin{pmatrix}
\frac{1}{\cos(\frac{2j\pi}{n})} \\
\vdots \\
\frac{1}{\cos(\frac{2(n-1)\pi}{n})}
\end{pmatrix} e^{i\omega \tau}, \quad \tilde{q}_{n-j}^1(-\tau) = \begin{pmatrix}
0 \\
\ddots \\
0
\end{pmatrix} e^{i\omega \tau}.
\]

Related to \([z_j^1 q_j^1 + \tilde{z}_j^1 \tilde{q}_j^1] \oplus \{z_{(n-j)}^1 q_{(n-j)}^1 + \tilde{z}_{(n-j)}^1 \tilde{q}_{(n-j)}^1\}\), if the system is restricted to the corresponding invariant manifold \(c_0\), then,

\[
x(t - \tau) = \left[z_j^1 q_j^1(-\tau) + \tilde{z}_j^1 \tilde{q}_j^1(-\tau) + z_{(n-j)}^1 q_{(n-j)}^1(-\tau) + \tilde{z}_{(n-j)}^1 \tilde{q}_{(n-j)}^1(-\tau)\right] + O(r^2),
\]

where \(t = j, n-j, r = \sqrt{\frac{r_j^2 + r_{n-j}^2}{r_j^2 + r_{n-j}^2}}, r_j^2 = z_j^1, r_{n-j}^2 = \tilde{z}_{(n-j)}^1\). Similarly as in the former case, we can obtain

\[
\{q_j^{1*}(0), N(x(t-\tau), 0)\} = \{q_j^{1*}(0), G''''(0)(x_1^3(t-\tau), \ldots, x_n^3(t-\tau))^T\} + O(4)
\]

\[
= -2\Upsilon \left(c_{30}^j \left[ z_j^1 e^{-i\omega \tau} + \tilde{z}_j^1 e^{i\omega \tau} \right]^3 + c_{03}^j \left[ z_{(n-j)}^1 e^{-i\omega \tau} + \tilde{z}_{(n-j)}^1 e^{i\omega \tau} \right]^3 \right. \\
\left. + c_{21}^j \left[ z_j^1 e^{-i\omega \tau} + \tilde{z}_j^1 e^{i\omega \tau} \right]^2 \left[ z_{(n-j)}^1 e^{-i\omega \tau} + \tilde{z}_{(n-j)}^1 e^{i\omega \tau} \right] + c_{12}^j \left[ z_j^1 e^{-i\omega \tau} + \tilde{z}_j^1 e^{i\omega \tau} \right] \left[ z_{(n-j)}^1 e^{-i\omega \tau} + \tilde{z}_{(n-j)}^1 e^{i\omega \tau} \right]^2 \right) + O(4),
\]

where

\[
c_{30}^j = \begin{cases} 
\frac{3n}{8} (\alpha d_1 + 2\beta d_2 \cos \frac{2j\pi}{n}), & j \neq n/4, \\
\frac{n}{8} \alpha d_1, & j = n/4;
\end{cases}
\]

\[
c_{12}^j = \begin{cases} 
\frac{3n}{8} (\alpha d_1 + 2\beta d_2 \cos \frac{2j\pi}{n}), & j \neq n/4, \\
0, & j = n/4;
\end{cases}
\]

\[
c_{21}^j = c_{03}^j = 0;
\]

and

\[
\{q_{(n-j)}^{1*}(0), N(x(t-\tau), 0)\} = \{q_{(n-j)}^{1*}(0), G''''(0)(x_1^3(t-\tau), \ldots, x_n^3(t-\tau))^T\} + O(4)
\]

\[
= -2\Upsilon \left(c_{30}^{n-j} \left[ z_j^{n-j} e^{-i\omega \tau} + \tilde{z}_j^{n-j} e^{i\omega \tau} \right]^3 \right. \\
\left. + c_{03}^{n-j} \left[ z_{(n-j)}^{n-j} e^{-i\omega \tau} + \tilde{z}_{(n-j)}^{n-j} e^{i\omega \tau} \right]^3 \right. \\
\left. + c_{21}^{n-j} \left[ z_j^{n-j} e^{-i\omega \tau} + \tilde{z}_j^{n-j} e^{i\omega \tau} \right]^2 \left[ z_{(n-j)}^{n-j} e^{-i\omega \tau} + \tilde{z}_{(n-j)}^{n-j} e^{i\omega \tau} \right] + c_{12}^{n-j} \left[ z_j^{n-j} e^{-i\omega \tau} + \tilde{z}_j^{n-j} e^{i\omega \tau} \right] \left[ z_{(n-j)}^{n-j} e^{-i\omega \tau} + \tilde{z}_{(n-j)}^{n-j} e^{i\omega \tau} \right]^2 \right) + O(4),
\]
\[ + c_{12} \left( z_{(n-j)l}^{1} e^{-i\omega\tau} + z_{(n-j)l}^{1} e^{i\omega\tau} \right) \]
\[ + c_{12} \left( z_{jl}^{1} e^{-i\omega\tau} + z_{jl}^{1} e^{i\omega\tau} \right) \left( z_{(n-j)l}^{1} e^{-i\omega\tau} + z_{(n-j)l}^{1} e^{i\omega\tau} \right)^2 \]
\[ + O(4), \]

where

\[ c_{n-j}^{n-j} = \begin{cases} \frac{3n}{8} (\alpha d_1 + 2 \beta d_2 \cos \frac{2j\pi}{n}), & j \neq n/4, \\ \frac{n}{2} \alpha d_1, & j = n/4; \end{cases} \]
\[ c_{n-j}^{n-j} = \begin{cases} \frac{3n}{8} (\alpha d_1 + 2 \beta d_2 \cos \frac{2j\pi}{n}), & j \neq n/4, \\ 0, & j = n/4; \end{cases} \]
\[ c_{12}^{n-j} = c_{30}^{n-j} = 0. \]  

(B.2)

For the detailed computation in (B.1) and (B.2), please see Appendices C and D.  

**Appendix C. The computation of (B.1)**

**Proof.** It is easy to find

\[ c_{30}^{j} = \alpha d_1 \sum_{k=0}^{n-1} \cos^4 \frac{2kj\pi}{n} \]
\[ + \beta d_2 \sum_{k=0}^{n-1} \left( \cos \frac{2kj\pi}{n} \cos \frac{2(k+1)j\pi}{n} + \cos \frac{2kj\pi}{n} \cos \frac{2(k-1)j\pi}{n} \right). \]

Since

\[ \cos^4 \frac{2kj\pi}{n} = \frac{3}{8} + \frac{1}{2} \cos \frac{4kj\pi}{n} + \frac{1}{8} \cos \frac{8kj\pi}{n}, \]
\[ \cos \frac{2kj\pi}{n} \cos \frac{2(k+1)j\pi}{n} = 1 + \cos \frac{4(k+1)j\pi}{n} n \left[ \cos \frac{4(k+1)j\pi}{n} \cos \frac{2j\pi}{n} \right. \]
\[ - \sin \frac{4(k+1)j\pi}{n} \sin \frac{2j\pi}{n} + \cos \frac{2j\pi}{n} \left] \right. \]
\[ = \frac{1}{4} \left[ 2 \cos \frac{4(k+1)j\pi}{n} \cos \frac{2j\pi}{n} - \sin \frac{4(k+1)j\pi}{n} \sin \frac{2j\pi}{n} \right. \]
\[ + \frac{3}{2} \cos \frac{2j\pi}{n} \left. \right] \]
\[ + \frac{1}{8} \left[ \cos \frac{8(k+1)j\pi}{n} \cos \frac{2j\pi}{n} - \sin \frac{8(k+1)j\pi}{n} \sin \frac{2j\pi}{n} \right], \]

and
\[
\cos \frac{2k\pi}{n} \cos^3 \frac{2(k-1)\pi}{n} = \frac{1 + \cos \frac{4(k-1)\pi}{n}}{4} \left[ \cos \frac{4(k-1)\pi}{n} \cos \frac{2\pi}{n} \right. \\
- \sin \frac{4(k-1)\pi}{n} \sin \frac{2\pi}{n} + \cos \frac{2\pi}{n} \left. \right]
\]
\[
= \frac{1}{4} \left[ 2\cos \frac{4(k-1)\pi}{n} \cos \frac{2\pi}{n} - \sin \frac{4(k-1)\pi}{n} \sin \frac{2\pi}{n} \\
+ \frac{3}{2} \cos \frac{2\pi}{n} \right]
+ \frac{1}{8} \left[ \cos \frac{8(k-1)\pi}{n} \cos \frac{2\pi}{n} - \sin \frac{8(k-1)\pi}{n} \sin \frac{2\pi}{n} \right].
\]

Combining with Proposition 3.3, we find that
\[
c_{30}^j = \begin{cases} 
\frac{3}{8} \left( \alpha d_1 + 2\beta d_2 \cos \frac{2j\pi}{n} \right), & \text{if } j \neq n/4, \\
\frac{9}{2} \alpha d_1, & \text{if } j = n/4.
\end{cases}
\]

It is easy to verify that
\[
c_{21}^j = \frac{3}{8} \alpha d_1 \sum_{k=0}^{n-1} \left( 1 - \cos \frac{8k\pi}{n} \right)
+ \frac{3}{2} \beta d_2 \sum_{k=0}^{n-1} \left( \cos \frac{2k\pi}{n} \sin \frac{2(k+1)\pi}{n} \sin \frac{4(k+1)\pi}{n} \right.
+ \cos \frac{2k\pi}{n} \sin \frac{2(k-1)\pi}{n} \sin \frac{4(k-1)\pi}{n} \right).
\]
\[
\cos \frac{2k\pi}{n} \sin \frac{2(k+1)\pi}{n} \sin \frac{4(k+1)\pi}{n} \cos \frac{2\pi}{n}
= \frac{1}{2} \sin \frac{4(k+1)\pi}{n} \left[ \sin \frac{4(k+1)\pi}{n} \cos \frac{2\pi}{n} \right.
- \cos \frac{4(k+1)\pi}{n} \sin \frac{2\pi}{n} + \sin \frac{2\pi}{n} \left. \right]
\]
\[
= \frac{1}{4} \left[ -\cos \frac{8(k+1)\pi}{n} \cos \frac{2\pi}{n} \\
- \sin \frac{8(k+1)\pi}{n} \sin \frac{2\pi}{n} + \cos \frac{2\pi}{n} + 2\sin \frac{2\pi}{n} \right].
\]

and
\[
\cos \frac{2k\pi}{n} \sin \frac{2(k-1)\pi}{n} \sin \frac{4(k-1)\pi}{n} \\
= \frac{1}{2} \sin \frac{4(k-1)\pi}{n} \left[ \sin \frac{2j\pi}{n} \cos \frac{2j\pi}{n} + \cos \frac{2(k-1)\pi}{n} \sin \frac{2j\pi}{n} - \sin \frac{2j\pi}{n} \right] \\
= \frac{1}{4} \left[ -\cos \frac{8(k-1)\pi}{n} \cos \frac{2j\pi}{n} + \sin \frac{8(k-1)\pi}{n} \sin \frac{2j\pi}{n} + \cos \frac{2j\pi}{n} - 2\sin \frac{2j\pi}{n} \right].
\]

Then
\[
c_j^{21} = \begin{cases} 
\frac{3n}{8} (\alpha d_1 + 2\beta d_2 \cos \frac{2j\pi}{n}), & j \neq n/4, \\
0, & \text{otherwise}.
\end{cases}
\]

Obviously, \(\sin \frac{2k\pi}{n} + \sin \frac{2(n-k)\pi}{n} = 0\) and \(\cos \frac{2k\pi}{n} = \cos \frac{2(n-k)\pi}{n}\). Therefore \(c_j^{30} = c_j^{12} = 0\). \(\square\)

**Appendix D. The computation of (B.2)**

**Proof.** As discussed above, we have
\[
c_{03}^{n-j} = \alpha d_1 \sum_{k=0}^{n-1} \sin^4 \frac{2k\pi}{n} \\
+ \beta d_2 \sum_{k=0}^{n-1} \left( \sin \frac{2k\pi}{n} \sin^3 \frac{2(k+1)\pi}{n} + \sin \frac{2k\pi}{n} \sin^3 \frac{2(k-1)\pi}{n} \right).
\]

By direct computation, we find
\[
\sin^4 \frac{2k\pi}{n} = \frac{3}{8} - \frac{1}{2} \cos \frac{4k\pi}{n} + \frac{1}{8} \cos \frac{8k\pi}{n},
\]
\[
\sin \frac{2k\pi}{n} \sin^3 \frac{2(k+1)\pi}{n} = \frac{1}{4} \left( 1 - \cos \frac{4(k+1)\pi}{n} \right) \left[ -\sin \frac{4(k+1)\pi}{n} \sin \frac{2j\pi}{n} - \cos \frac{4(k+1)\pi}{n} \cos \frac{2j\pi}{n} \right] \\
+ \frac{3}{2} \cos \frac{2j\pi}{n} \\
+ \frac{1}{8} \left[ \cos \frac{8(k+1)\pi}{n} \cos \frac{2j\pi}{n} + \sin \frac{8(k+1)\pi}{n} \sin \frac{2j\pi}{n} \right],
\]
and
\[
\sin \frac{2k\pi}{n} \sin^3 \frac{2(k-1)\pi}{n} = \frac{1}{4} \left( 1 - \cos \frac{4(k-1)\pi}{n} \right) \left[ \sin \frac{4(k-1)\pi}{n} \sin \frac{2\pi}{n} 
- \cos \frac{4(k-1)\pi}{n} \cos \frac{2\pi}{n} + \cos \frac{2\pi}{n} \right]
= \frac{1}{4} \left[ \sin \frac{4(k-1)\pi}{n} \sin \frac{2\pi}{n} + \frac{3}{2} \cos \frac{2\pi}{n} \right] + \frac{1}{8} \left[ \cos \frac{8(k-1)\pi}{n} \cos \frac{2\pi}{n} - \sin \frac{8(k-1)\pi}{n} \sin \frac{2\pi}{n} \right].
\]

Then
\[
c^n_{03} = \begin{cases} 
\frac{3n}{8} (\alpha d_1 + 2\beta d_2 \cos \frac{2j\pi}{n}) & , j \neq n/4, \\
\frac{n}{2} \alpha d_1 & , j = n/4.
\end{cases}
\]

Similarly, one can obtain
\[
c^n_{21} = \frac{3}{8} \alpha d_1 \sum_{k=0}^{n-1} \left( 1 - \cos \frac{8k\pi}{n} \right) + \frac{3}{2} \beta d_2 \sum_{k=0}^{n-1} \left( \sin \frac{2k\pi}{n} \cos \frac{2(k+1)\pi}{n} \sin \frac{4(k+1)\pi}{n} 
+ \sin \frac{2k\pi}{n} \cos \frac{2(k-1)\pi}{n} \sin \frac{4(k-1)\pi}{n} \right),
\]
\[
\sin \frac{2k\pi}{n} \cos \frac{2(k+1)\pi}{n} \sin \frac{4(k+1)\pi}{n} = \frac{1}{2} \sin \frac{4(k+1)\pi}{n} \left[ \sin \frac{4(k+1)\pi}{n} \cos \frac{2\pi}{n} 
- \cos \frac{4(k+1)\pi}{n} \sin \frac{2\pi}{n} + \sin \frac{2\pi}{n} \right]
= \frac{1}{4} \left[ -\cos \frac{8(k+1)\pi}{n} \cos \frac{2\pi}{n} 
- \sin \frac{8(k+1)\pi}{n} \sin \frac{2\pi}{n} + \cos \frac{2\pi}{n} + 2 \sin \frac{2\pi}{n} \right],
\]
and
\[
\sin \frac{2k\pi}{n} \cos \frac{2(k-1)\pi}{n} \sin \frac{4(k-1)\pi}{n} = \frac{1}{2} \sin \frac{4(k-1)\pi}{n} \left[ \sin \frac{4(k-1)\pi}{n} \cos \frac{2\pi}{n} 
- \cos \frac{4(k-1)\pi}{n} \sin \frac{2\pi}{n} + \sin \frac{2\pi}{n} \right].
\]
\[ + \cos \frac{4(k-1)j\pi}{n} \sin \frac{2j\pi}{n} - \sin \frac{2j\pi}{n} \]

\[ = \frac{1}{4} \left[ - \cos \frac{8(k-1)j\pi}{n} \cos \frac{2j\pi}{n} + \sin \frac{8(k-1)j\pi}{n} \sin \frac{2j\pi}{n} + \cos \frac{2j\pi}{n} - 2 \sin \frac{2j\pi}{n} \right]. \]

Therefore

\[ c_{21}^{n-j} = \begin{cases} \frac{3\pi}{8} (\alpha d_1 + 2\beta d_2 \cos \frac{2j\pi}{n}), & j \neq n/4, \\ 0, & j = n/4. \end{cases} \]

One can easily verify that \( c_{30}^{n-j} = c_{21}^{n-j} = 0. \]

References