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Two-Way Bidiagonalization Scheme for Downdating the Singular-Value Decomposition

Haesun Park*

Computer Science Department
University of Minnesota
Minneapolis, Minnesota 55455

and

Sabine Van Huffel†

ESAT Laboratory
Katholieke Universiteit Leuven
3001-Heverlee, Belgium

Submitted by Richard A. Brualdi

ABSTRACT

We present a method that transforms the problem of downdating the singular-value decomposition into a problem of diagonalizing a diagonal matrix bordered by one column. The first step in this diagonalization involves bidiagonalization of a diagonal matrix bordered by one column. For updating the singular-value decomposition, a two-way chasing scheme has been recently introduced, which reduces the total number of rotations by 50% compared to previously developed one-way chasing schemes. Here, a two-way chasing scheme is introduced for the bidiagonalization step in downdating the singular-value decomposition. We show how the matrix elements can be rearranged and how the nonzero elements can be chased away towards two corners of the matrix. The newly proposed scheme saves nearly 50% of the number of plane rotations required by one-way chasing schemes.

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†E-mail: `sabine.vanhuffel@esat.kuleuven.ac.be`. Senior Research Associate with the N.F.W.O. (Belgian National Fund for Scientific Research). Additional support was provided by the Belgian program on interuniversity attraction poles (IUAP-nr.50) initiated by the Belgian State—Prime Minister's Office—Science Policy Programming. The scientific responsibility is assumed by its authors.

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1. INTRODUCTION

Suppose we have a matrix $A \in \mathbf{C}^{m \times n}$ for which we also have a certain orthogonal decomposition, such as the QR decomposition or the singular-value decomposition (SVD). In many applications, we need to solve problems recursively, where the data matrix of one time step differs from that of the previous step only through addition of a new row (updating) or through deletion of an existing row (downdating). Obviously, we would like to obtain a new solution without recomputing the new decomposition all over again. In many of these updating and downdating problems, we need to restore the bandwidth of a banded matrix bordered by a row or a column.

For updating the SVD [3, 4], a two-way chasing scheme has been recently introduced [15]. We have generalized this scheme for the reduction of a more general banded matrix bordered by one or more rows and/or columns to banded form [11]. By splitting the matrix into two similarly structured submatrices and chasing nonzeros to the corners in two directions, the newly proposed patterns reduce the number of required rotations and hence the computational cost by 50% compared to the other existing one-way chasing algorithms [1, 4]. These methods can be used for updating the ordinary SVD and the partial SVD [13, 14], for example.

In this paper, we show that the problem of downdating a row in the SVD of a matrix can be transformed to a problem of bidiagonalizing a diagonal matrix bordered by a column and then finally diagonalizing this bidiagonal matrix. In Section 2, we briefly describe how to downdate the QR decomposition and show that downdating the QR decomposition is closely related to downdating the SVD. Section 3 then shows how the idea of downdating the QR decomposition can be adapted to downdating the SVD. In Section 4, we present the main characteristics of the one-way and two-way chasing schemes for the bidiagonalization step in downdating the SVD. For the two-way chasing scheme which requires only 50% of the computational cost of the one-way chasing scheme, we show how the matrix elements can be rearranged and how the nonzero elements can be chased away towards two corners of the matrix.

2. DOWNDATING THE QR DECOMPOSITION

We first briefly present an algorithm for downdating the QR decomposition, because this topic is well described in the literature [2, 6–8, 10] and also because downdating the SVD is closely related to downdating the QR decomposition.

Assume that a QR decomposition of a matrix $A \in \mathbf{C}^{m\times n}$, $m > n$, is given as

$$A = \begin{pmatrix} x^H \\ \widetilde{A} \end{pmatrix} = Q\begin{pmatrix} R \\ 0 \end{pmatrix} = \begin{pmatrix} q^H \\ Q^{(1)} \end{pmatrix}\begin{pmatrix} R \\ 0 \end{pmatrix}, \tag{2.1}$$

where $x^H \in \mathbf{C}^{1\times n}$ is the first row of A, $Q \in \mathbf{C}^{m\times m}$ is unitary, and q^H is the first row of Q. We can find a sequence of complex plane rotations J_i in the plane $(i, i+1)$, $1 \le i \le m-1$, which gives

$$q^H J_{m-1}J_{m-2}\cdots J_1 = e_1^T, \tag{2.2}$$

where $e_1 \in \mathbf{R}^{m\times 1}$ is the unit vector with 1 in the first position and 0 everywhere else. We define $P = J_{m-1}J_{m-2}\cdots J_1$ and apply P^H from the left to the matrix

$$\left(q \quad \begin{pmatrix} R \\ 0 \end{pmatrix}\right)$$

to obtain

$$P^H\left(q \quad \begin{pmatrix} R \\ 0 \end{pmatrix}\right) = P^H Q^H (e_1 \quad A) = \begin{pmatrix} 1 & y^H \\ 0 & \widetilde{R} \\ 0 & 0 \end{pmatrix}, \tag{2.3}$$

where $y^H \in \mathbf{C}^{1\times n}$ and $\widetilde{R} \in \mathbf{C}^{n\times n}$ is upper triangular. From (2.3), since $P^H Q^H e_1 = e_1$, $P^H Q^H$ has the form

$$P^H Q^H = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}^H \end{pmatrix} \tag{2.4}$$

for a unitary matrix $\widetilde{Q} \in \mathbf{C}^{(m-1)\times(m-1)}$. Thus from (2.3) and (2.4), we have

$$\begin{pmatrix} 1 & 0 \\ 0 & \widetilde{Q}^H \end{pmatrix}\begin{pmatrix} 1 & x^H \\ 0 & \widetilde{A} \end{pmatrix} = \begin{pmatrix} 1 & x^H \\ 0 & \widetilde{Q}^H\widetilde{A} \end{pmatrix} = \begin{pmatrix} 1 & y^H \\ 0 & \widetilde{R} \\ 0 & 0 \end{pmatrix},$$

which gives $y = x$ and the new QR decomposition

$$\widetilde{Q}^H\widetilde{A} = \begin{pmatrix} \widetilde{R} \\ 0 \end{pmatrix}$$

for $\widetilde{A}$. Equation (2.3) shows that the downdating transformation P is completely determined by the first row q^H of the unitary factor Q. However, in many cases, only the "skinny" Q factor which consists of the first n columns of Q is known, e.g., in the Gram-Schmidt or modified Gram-Schmidt algorithms, or the Q factor is not computed at all for the purpose of saving storage and computational costs. Downdating this skinny QR decomposition by means of a modified Gram-Schmidt-type algorithm is discussed in [2, 5]. For downdating the upper triangular factor in cases when the Q factor is not known, several algorithms, such as the LINPACK algorithm [8], and the CSNE and hybrid algorithms which produce more accurate solutions [2], have been developed.

Suppose we have the skinny QR decomposition

$$A = \begin{pmatrix} x^H \\ \widetilde{A} \end{pmatrix} = Q_n R = \begin{pmatrix} q_n^H \\ Q_n^{(1)} \end{pmatrix} R, \tag{2.5}$$

where $Q_n \in \mathbf{C}^{m \times n}$ consists of the first n columns of the unitary matrix Q, and $q_n^H \in \mathbf{C}^{1 \times n}$ is the first row of Q_n. Then since $x^H = q_n^H R$, we can obtain the first row q_n^H of Q_n by solving this triangular system. The complex rotations $J_{m-1}, \ldots, J_{n+1}$ in (2.2) do not affect the matrix R in the computation (2.3). Thus, we can simply compute $\gamma = \sqrt{1 - q_n^H q_n}$ and obtain the downdated triangular factor $\widetilde{R}$ from

$$K_1^H \cdots K_{n-1}^H K_n^H \begin{pmatrix} q_n & R \\ \gamma & 0 \end{pmatrix} = \begin{pmatrix} 1 & x^H \\ 0 & \widetilde{R} \end{pmatrix}, \tag{2.6}$$

where K_i^H is a complex plane rotation in the plane $(i, i+1)$ that annihilates the $(i+1)$th component of

$$K_{i+1}^H \cdots K_n^H \begin{pmatrix} q_n \\ \gamma \end{pmatrix}.$$

This is the LINPACK algorithm [8] for downdating the QR decomposition, where Q is assumed to be not available. The numerical properties of this method are discussed in [2, 10].

3. DOWNDATING THE SVD

We now show how the idea of downdating the QR decomposition can be adapted to downdating the SVD. In [7], the relation between downdating the QR decomposition and downdating any two-sided orthogonal transformations has been shown. Assume that the SVD of the matrix $A \in \mathbf{C}^{m\times n}$, $m > n$, is given by

$$A = \begin{pmatrix} x^H \\ \widetilde{A} \end{pmatrix} = U \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} V^H = \begin{pmatrix} u^H \\ U^{(1)} \end{pmatrix} \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} V^H, \qquad (3.7)$$

where $U \in \mathbf{C}^{m\times m}$ and $V \in \mathbf{C}^{n\times n}$ are unitary, $\Sigma \in \mathbf{R}^{n\times n}$ is diagonal, and $u^H \in \mathbf{C}^{1\times m}$ is the first row of U. We will show in the next section that there exist unitary matrices $F \in \mathbf{C}^{m\times m}$ and $G \in \mathbf{C}^{(n+1)\times(n+1)}$, a product of complex plane rotations each, such that

$$F^H \begin{pmatrix} u & \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} \end{pmatrix} G = \begin{pmatrix} 1 & w^H \\ 0 & B \\ 0 & 0 \end{pmatrix} \qquad (3.8)$$

where $U \in \mathbf{C}^{n\times n}$ is a bidiagonal matrix and $w^H \in \mathbf{C}^{1\times n}$ is a row vector. Moreover, it can be shown that there exists a matrix G of the form

$$G = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{G} \end{pmatrix} \qquad (3.9)$$

for some unitary matrix $\widetilde{G} \in \mathbf{C}^{n\times n}$. Consequently, G does not operate on the first column of

$$\begin{pmatrix} u & \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} \end{pmatrix}.$$

Then from (3.7) and (3.8), we have

$$F^H U^H (e_1 \quad AV) G = \begin{pmatrix} 1 & w^H \\ 0 & B \\ 0 & 0 \end{pmatrix}, \qquad (3.10)$$

and since $F^H U^H e_1 = e_1$, $F^H U^H$ has the form

$$F^H U^H = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{U}^H \end{pmatrix} \qquad (3.11)$$

for some unitary matrix $\widetilde{U} \in \mathbf{C}^{(m-1)\times(m-1)}$. Therefore, from (3.9), (3.10), and (3.11),

$$
\begin{aligned}
F^H U^H (e_1 \quad AV) G &= \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{U}^H \end{pmatrix} (e_1 \quad AV) \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{G} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{U}^H \end{pmatrix} \begin{pmatrix} 1 & x^H V \\ 0 & \widetilde{A} V \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{G} \end{pmatrix} \\
&= \begin{pmatrix} 1 & x^H V \widetilde{G} \\ 0 & \widetilde{U}^H \widetilde{A} V \widetilde{G} \end{pmatrix} = \begin{pmatrix} 1 & w^H \\ 0 & B \\ 0 & 0 \end{pmatrix},
\end{aligned}
$$

which gives the bidiagonal reduction of the downdated A:

$$
\widetilde{U}^H \widetilde{A} V \widetilde{G} = \begin{pmatrix} B \\ 0 \end{pmatrix}.
$$

Further reduction of the bidiagonal matrix B to a diagonal matrix can be performed by using QR iterations [6], resulting in the SVD of the matrix $\widetilde{A}$.

As in the QR decomposition, the matrix U is expensive to compute and store. However, many problems can be solved without the matrix U. We show how the singular values and the right-singular-vector matrix V can be downdated when the left singular vector matrix U is unknown. We assume rank $A = r \leq n$, and

$$
\Sigma = \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix},
$$

where $\Sigma_r \in \mathbf{R}^{r\times r}$ is a full-rank diagonal matrix. Then from (3.7), we have

$$
AV_r = \begin{pmatrix} x^H V_r \\ \widetilde{A} V_r \end{pmatrix} = U_r \Sigma_r = \begin{pmatrix} u_r^H \\ U_r^{(1)} \end{pmatrix} \Sigma_r, \tag{3.12}
$$

where $U_r \in \mathbf{C}^{m\times r}$ and $V_r \in \mathbf{C}^{n\times r}$ consist respectively of the first r columns of U and V, and $u_r^H \in \mathbf{C}^{1\times r}$ is the first row of U_r. Note that (3.12) has the same structure as (2.5) which was used for downdating the QR decomposition. Since

$$
x^H V_r = u_r^H \Sigma_r, \tag{3.13}
$$

we can obtain u_r^H. As in (2.6), we can then compute unitary matrices $F \in \mathbf{C}^{(r+1)\times(r+1)}$ and $G \in \mathbf{C}^{(r+1)\times(r+1)}$, each a product of complex plane rotations, such that

$$F^H \begin{pmatrix} u_r & \Sigma_r \\ \gamma & 0 \end{pmatrix} G = \begin{pmatrix} 1 & w_r^H \\ 0 & B_r \end{pmatrix} \tag{3.14}$$

where $B_r \in \mathbf{C}^{r\times r}$ is bidiagonal, $w_r^H \in \mathbf{C}^{1\times r}$ is a row vector and $\gamma = \sqrt{1 - u_r^H u_r}$. Thus, the problem is to transform a matrix of the form

$$\begin{pmatrix} \times & \times & 0 & 0 & \dots & 0 \\ \times & 0 & \times & 0 & \dots & 0 \\ \times & 0 & 0 & \times & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ \times & 0 & 0 & \dots & 0 & \times \\ \times & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad \text{into the form} \quad \begin{pmatrix} 1 & \times & \times & \times & \dots & \times \\ 0 & \times & \times & 0 & \dots & 0 \\ 0 & 0 & \times & \times & & \vdots \\ 0 & 0 & 0 & \times & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \times \\ 0 & 0 & 0 & \dots & 0 & \times \end{pmatrix} \tag{3.15}$$

by using complex rotations from the left and right sides of the first form in (3.15). In the following section, two schemes are presented that perform this bidiagonal reduction. We will see that although the first row, $(1 \; w_r^H)$, is allowed to be a full vector, the algorithms we present generate w_r^H which has zero components except for the first two components.

The complex plane rotations that constitute the downdating transformations F and G shown in (3.14) can be replaced by real plane rotations by using the fact that Σ_r is real even when A is complex [9]. Suppose we multiply two unitary diagonal matrices

$$P_1^H = \begin{pmatrix} P^H & 0 \\ 0 & e^{i\alpha_{r+1}} \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}, \tag{3.16}$$

where $P = \operatorname{diag}(e^{i\alpha_1}, \dots, e^{i\alpha_r})$ and $i = \sqrt{-1}$, from the left and the right, respectively, of the matrix

$$D_c \equiv \begin{pmatrix} u_r & \Sigma_r \\ \gamma & 0 \end{pmatrix}.$$

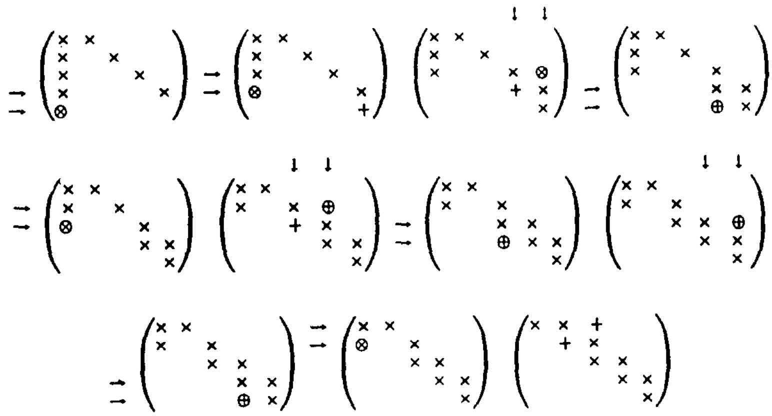

FIG. 1. Downdating a 4 × 4 diagonal submatrix bordered by one column to bidiagonal form using a one-way chasing scheme.

Then, since

$$D_r \equiv P_1^H D_c P_2 = \begin{pmatrix} P^H & 0 \\ 0 & e^{i\alpha_{r+1}} \end{pmatrix} \begin{pmatrix} u_r & \Sigma_r \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} = \begin{pmatrix} P^H u_r & P^H \Sigma_r P \\ e^{i\alpha_{r+1}}\gamma & 0 \end{pmatrix}, \tag{3.17}$$

we can choose α_j, $1 \le j \le r+1$, to make $P^H u_r$ and $e^{i\alpha_{r+1}}\gamma$ real, which makes the matrix D_r real, since $P^H \Sigma_r P = \Sigma_r$.

Thus, after D_c is made real via P_1 and P_2, we can use *real* plane rotations to transform D_r into the desired form. Note that in downdating the QR decomposition, the upper triangular factor and the row to be downdated cannot both be made real via simple transformations like diagonal unitary matrices.

4. BIDIAGONALIZATION SCHEMES FOR DOWNDATING THE SVD

4.1. One-Way Chasing Scheme

The bidiagonalization operation (3.15) on the $r \times r$ diagonal submatrix Σ_r bordered by one column can be performed by means of a *one-way* chasing scheme, as illustrated in Figure 1 for $r = 4$. As used conventionally, $\times$ represents possible nonzero entries of a matrix, and a blank space represents a zero. In the figures, the entry to be annihilated at the current reduction step is encircled, and a + represents a new fill-in entry. A pair of rows and a pair of columns involved in the plane rotation at each reduction step are marked by $\rightarrow$ and $\downarrow$, respectively. Denoting the rotation from the left in the plane $(i, i+1)$ as F_i and that from the right as G_i, the rotation sequence in the one-way chasing scheme is represented as

$$\begin{array}{lllllll} F_r & & & & & & \\ F_{r-1}, & G_r, & F_r, & & & & \\ F_{r-2}, & G_{r-1}, & F_{r-1}, & G_r, & F_r & & \\ \vdots & & & & & & \\ F_2, & G_3, & F_3, & G_4, & F_4, & \ldots, & G_r, \quad F_r \\ F_1 & & & & & & \end{array} \tag{4.18}$$

where the jth line represents the rotations that are required to annihilate the $(r+2-j)$th element of the current first column of D_c or D_r and to complete the $j \times j$ trailing bidiagonalization. The desired bidiagonal form is built from bottom to top while nonzeros are chased away to the lower-right corner. This proves that we can find unitary or orthogonal matrices F and G, products of the rotations F_i and G_i above, that reduce the matrix D_c or D_r to the form (3.15) from which the bidiagonal reduction of the downdated A follows. Since G_1 does not occur in the sequence (4.18), G does not operate on the first column of D_c or D_r implying that G is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \widetilde{G} \end{pmatrix},$$

as required. This one-way chasing scheme needs $1 + \sum_{i=2}^{r}(2i-3) = r^2 - 2r + 2$ plane rotations and $14r^2 - 28r + \mathcal{O}(1)$ flops (according to [11], using the definition of [6, p. 19]) to perform the required reduction on D_r.

4.2. Two-Way Chasing Scheme

A reduction of the number of plane rotations by up to 50% can be obtained by chasing nonzeros away in *two* directions, i.e., the upper left and lower right corners of the matrix. This is done as follows. First, we find unitary diagonal matrices P_1 and P_2 to make $D_r = P_1^H D_c P_2$ real as in (3.17). Suppose we find orthogonal matrices $F \in \mathbf{R}^{(r+1)\times(r+1)}$ and

$G \in \mathbf{R}^{(r+1)\times(r+1)}$, a product of plane rotations each, such that

$$F^T P_1^H \begin{pmatrix} u_r & \Sigma_r \\ \gamma & 0 \end{pmatrix} P_2 G = \begin{pmatrix} e_{k+1} & \begin{pmatrix} B_1 \\ w_r^T \\ B_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & B_1 \\ 1 & w_r^T \\ 0 & B_2 \end{pmatrix},$$

where

$$B_1 = \begin{pmatrix} B_{11} & 0_{k\times(r-k-1)} \end{pmatrix} \in \mathbf{R}^{k\times r}$$

and

$$B_{11} = \begin{pmatrix} \times & \times & 0 & \dots & 0 \\ 0 & \times & \times & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \times & \times \end{pmatrix} \in \mathbf{R}^{k\times(k+1)},$$

$$B_2 = \begin{pmatrix} 0_{(r-k)\times k} & B_{22} \end{pmatrix} \in \mathbf{R}^{(r-k)\times r}$$

and

$$B_{22} = \begin{pmatrix} \times & \times & 0 & \dots & 0 \\ 0 & \times & \times & & \vdots \\ 0 & & \ddots & \ddots & 0 \\ \vdots & & & \times & \times \\ 0 & \dots & 0 & 0 & \times \end{pmatrix} \in \mathbf{R}^{(r-k)\times(r-k)},$$

$k = r/2$ for even $r (k = (r+1)/2$ for odd $r)$, and $w_r^T \in \mathbf{R}^{1\times r}$. Then

$$\begin{aligned} &\begin{pmatrix} F & 0 \\ 0 & I_{m-r-1} \end{pmatrix}^T \begin{pmatrix} P_1 & 0 \\ 0 & I_{m-r-1} \end{pmatrix}^H \begin{pmatrix} u_r & \Sigma_r & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} P_2 & 0 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & I_{n-r} \end{pmatrix} \\ &= \begin{pmatrix} F & 0 \\ 0 & I_{m-r-1} \end{pmatrix}^T \begin{pmatrix} P_1 & 0 \\ 0 & I_{m-r-1} \end{pmatrix}^H W^H U^H (e_1 \ AV) \begin{pmatrix} P_2 & 0 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & I_{n-r} \end{pmatrix} \\ &= \begin{pmatrix} 0 & B_1 & 0 \\ 1 & w_r^T & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{4.19}$$

for some unitary matrix W^H that operates only on the rows $r+1,\ldots,m$. We can find G so that it does not operate on the first column of the matrix. Accordingly,

$$\begin{pmatrix} F & 0 \\ 0 & I_{m-r-1} \end{pmatrix}^T \begin{pmatrix} P_1 & 0 \\ 0 & I_{m-r-1} \end{pmatrix}^H W^H U^H e_1 = e_{k+1},$$

which means that

$$\begin{pmatrix} F & 0 \\ 0 & I_{m-r-1} \end{pmatrix}^T \begin{pmatrix} P_1 & 0 \\ 0 & I_{m-r-1} \end{pmatrix}^H W^H U^H = \begin{pmatrix} 0 & \widetilde{U}_1^H \\ 1 & 0 \\ 0 & \widetilde{U}_2^H \end{pmatrix} \tag{4.20}$$

for some $\widetilde{U}_1^H \in \mathbf{C}^{k\times(m-1)}$ and $\widetilde{U}_2^H \in \mathbf{C}^{(m-k-1)\times(m-1)}$, where

$$\begin{pmatrix} \widetilde{U}_1^H \\ \widetilde{U}_2^H \end{pmatrix} \in \mathbf{C}^{(m-1)\times(m-1)}$$

is unitary. Thus, from (4.19),

$$\begin{pmatrix} 0 & \widetilde{U}_1^H \\ 1 & 0 \\ 0 & \widetilde{U}_2^H \end{pmatrix} \begin{pmatrix} 1 & x^H V \\ 0 & \widetilde{A}V \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \widetilde{G} & 0 \\ 0 & 0 & I_{n-r} \end{pmatrix} = \begin{pmatrix} 0 & \widetilde{U}_1^H \widetilde{A} V_r \widetilde{G} & 0 \\ 1 & x^H V_r \widetilde{G} & 0 \\ 0 & \widetilde{U}_2^H \widetilde{A} V_r \widetilde{G} & 0 \end{pmatrix} \tag{4.21}$$

for a unitary $\widetilde{G} \in \mathbf{C}^{r\times r}$, and this completes the bidiagonal reduction of the downdated A:

$$\begin{pmatrix} \widetilde{U}_1^H \\ \widetilde{U}_2^H \end{pmatrix} \widetilde{A} V \begin{pmatrix} \widetilde{G} & 0 \\ 0 & I_{n-r} \end{pmatrix} = \begin{pmatrix} B_1 & 0 \\ B_2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.22}$$

The problem now is to transform the real matrix

$$D_r = P_1^H \begin{pmatrix} u_r & \Sigma_r \\ \gamma & 0 \end{pmatrix} P_2 = \begin{pmatrix} \widetilde{u}_r & \Sigma_r \\ \widetilde{\gamma} & 0 \end{pmatrix} \tag{4.23a}$$

into the form

$$\begin{pmatrix} 0 & B_1 \\ 1 & w_r^T \\ 0 & B_2 \end{pmatrix} \begin{matrix} k \\ 1 \\ r-k \end{matrix} \tag{4.23b}$$

For that, we first permute columns $(1,2),(2,3),\ldots,(k,k+1)$ by means of a permutation matrix $\Pi \in \mathbf{R}^{(r+1)\times(r+1)}$ such that the border column is moved to the middle:

$$\begin{pmatrix} \widetilde{u}_r & \Sigma_r \\ \widetilde{\gamma} & 0 \end{pmatrix} \Pi = \begin{pmatrix} \Sigma_{r1} & \widetilde{u}_{r1} & 0 \\ 0 & \widetilde{u}_{r2} & \Sigma_{r2} \\ 0 & \widetilde{\gamma} & 0 \end{pmatrix} \begin{matrix} k \\ r-k \\ 1 \end{matrix} \tag{4.24}$$

with

$$\widetilde{u}_r = \begin{pmatrix} \widetilde{u}_{r1} \\ \widetilde{u}_{r2} \end{pmatrix}, \quad \Sigma_{r1} = \mathrm{diag}\,(\sigma_1,\ldots,\sigma_k), \text{ and } \Sigma_{r2} = \mathrm{diag}(\sigma_{k+1},\ldots,\sigma_r).$$

The matrix now contains two similarly structured submatrices which can be reduced separately, as illustrated in Figure 2 for $r = 8$.

The upper left $k \times (k+1)$ submatrix $(\Sigma_{r1}\ \widetilde{u}_{r1})$ is reduced to bidiagonal form B_{11} by an ordinary one-way chasing scheme that applies to a diagonal matrix bordered by one column (see e.g. [1]). Denoting the rotation applied to the *entire* matrix from the left in the plane $(i, i+1)$ as F_i and that from the right as G_i, the rotation sequence in this one-way chasing scheme is represented as

$$\begin{matrix} F_1, & G_1 \\ F_2, & G_2, & F_1, & G_1 \\ F_3, & G_3, & F_2, & G_2, & F_1, & G_1 \\ \vdots \\ F_{k-1}, & G_{k-1}, & . & . & ., & F_2, & G_2, & F_1, & G_1 \end{matrix} \tag{4.25}$$

where the jth line represents the rotations that are required to annihilate the jth element of the current $\widetilde{u}_{r1}$ and to complete the $j \times j$ leading bidiagonalization. The reduction of this upper left submatrix is performed with $\sum_{i=1}^{k-1} 2i = k^2 - k$ plane rotations.

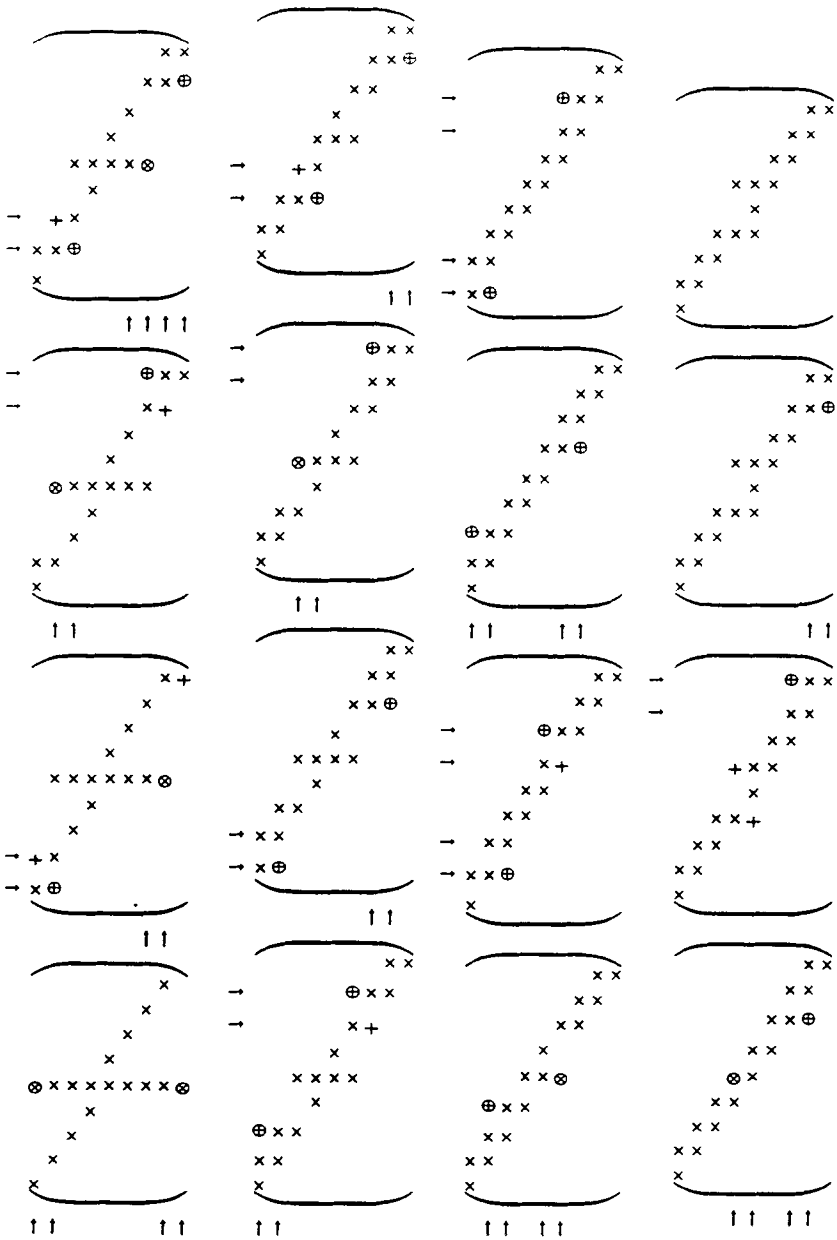

FIG. 2. Downdating an 8×8 diagonal bordered by one column to bidiagonal form using a two-way chasing scheme.

The lower right $(r-k+1)\times(r-k+1)$ submatrix

$$\begin{pmatrix} \widetilde{u}_{r2} & \Sigma_{r2} \\ \widetilde{\gamma} & 0 \end{pmatrix}$$

has the same structure as the left matrix in (3.15) and is reduced to bidiagonal form B_{22} using the same one-way chasing scheme, as illustrated in Figure 1, followed by the annihilation of the remaining nonzero at the $(1,3)$ entry, which is chased away to the lower-right corner in $2(r-k)-1$ rotations. Using the same notations as before, the rotation sequence is given by

$$\begin{array}{llllllll} F_r \\ F_{r-1}, & G_r, & F_r \\ F_{r-2}, & G_{r-1}, & F_{r-1}, & G_r, & F_r \\ \vdots \\ F_{k+1}, & G_{k+2}, & F_{k+2} & . & . & ., & G_r, & F_r \end{array} \tag{4.26}$$

where the jth line represents the rotations that are required to annihilate the $(r+2-j)$th element of the current $\binom{\tilde{u}_{r2}}{\tilde{\gamma}}$ and to complete the $j\times j$ trailing bidiagonalization. The reduction of this lower right submatrix is performed with $\sum_{i=1}^{r-k}(2i-1)=(r-k)^2$ plane rotations.

Finally, one extra plane rotation F_k is needed to reduce the matrix obtained thus far to the form

$$\begin{pmatrix} \times & \times & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \times & \times & & & \vdots & \vdots & & & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & & & \times & \times & 0 & \times & 0 & \cdots & \cdots & 0 \\ 0 & & \cdots & 0 & \times & \times & \times & 0 & \cdots & \cdots & 0 \\ 0 & & \cdots & \cdots & \cdots & 0 & \times & \times & 0 & \cdots & 0 \\ \vdots & & & & & & & \times & \times & \ddots & \vdots \\ \vdots & & & & & & & & \ddots & \ddots & 0 \\ \vdots & & & & & & & & & \times & \times \\ 0 & & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & 0 & \times \end{pmatrix}. \tag{4.27}$$

Hence, we can find F and G, products of the rotations F_i and G_i given above, such that

$$F\begin{pmatrix} \Sigma_{r1} & \widetilde{u}_{r1} & 0 \\ 0 & \widetilde{u}_{r2} & \Sigma_{r2} \\ 0 & \widetilde{\gamma} & 0 \end{pmatrix}G$$

is reduced to the form (4.27). By performing an inverse permutation Π^T, this final matrix can be brought into the form

$$\begin{pmatrix} 0 & B_1 \\ 1 & w_r^T \\ 0 & B_2 \end{pmatrix},$$

as required. Observe that no G_i operates on the permuted border column $\left(\begin{smallmatrix} \tilde{u}_r \\ \tilde{\gamma} \end{smallmatrix}\right)$, since G_k and G_{k+1} do not occur in the rotation sequences (4.25) and (4.26). This implies that G is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \widetilde{G} \end{pmatrix}$$

after back permutation, as required. In total, $r^2/2 - r/2 + 2$ real plane rotations and $7r^2 - 7r + \mathcal{O}(1)$ flops are needed to reduce the $r \times r$ bidiagonal submatrix in (4.24) bordered by one column by means of this two-way chasing scheme. Hence, we save $(r^2 - 3r + 2)/2$ rotations compared to the one-way chasing scheme, resulting in a reduction of the computation time by nearly 50% for large r.

5. CONCLUSIONS

In this paper, a new two-way chasing scheme has been proposed for downdating the singular-value decomposition (SVD) of a rank-r matrix. Suitable permutations split the matrix into two similarly structured submatrices that can be reduced separately during the bidiagonalization step. In this way, nonzeros are chased away simultaneously towards the two outer corners of the matrix, resulting in a reduction of the computation time of

$$\frac{1}{2}\left(\frac{r^2 - 3r + 2}{r^2 - 2r + 2}\right) \times 100\%$$

compared to one-way chasing schemes. These schemes can be easily parallelized analogously to the parallel schemes for reducing a diagonal matrix bordered by one row, as described in [12]. Following the same analysis as in [12], similar parallel architectures can be developed for the bidiagonalization of a diagonal matrix bordered by one column, to be used in downdating the SVD.

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