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The geometric structure of complex fluids

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ABSTRACT

This paper develops the theory of affine Euler–Poincaré and affine Lie–Poisson reductions and applies these processes to various examples of complex fluids, including Yang–Mills and Hall magnetohydrodynamics for fluids and superfluids, spin glasses, microfluids, and liquid crystals. As a consequence of the Lagrangian approach, the variational formulation of the equations is determined. On the Hamiltonian side, the associated Poisson brackets are obtained by reduction of a canonical cotangent bundle. A Kelvin–Noether circulation theorem is presented and is applied to these examples.

Contents

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1. Introduction

The equations of motion of an *adiabatic compressible fluid* are given by

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \nabla_{\mathbf{u}}\mathbf{u} = \dfrac{1}{\rho}\operatorname{grad} p, \\ \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \quad \dfrac{\partial S}{\partial t} + \operatorname{div}(S\mathbf{u}) = 0, \end{cases} \tag{1.1}$$

where ρ is the mass density, S is the entropy density, and p is the pressure. It was shown in [25] that this system, as well as its magnetohydrodynamic extension, admit a non-canonical Poisson formulation, that is, Eq. (1.1) can be written as

$$\dot{f} = \{f, h\},$$

relative to a Hamiltonian function h. It is of great (mathematical and physical) interest to obtain these Poisson brackets by a reduction procedure from a canonical Hamiltonian formulation on a cotangent bundle. In [24], the non-canonical Poisson bracket associated to (1.1) is obtained via Lie–Poisson reduction for a semidirect product group involving the diffeomorphisms group of the fluid container and the space of the advected quantities ρ and S. The Lagrangian formulation of these equations is given in [17].

In the same spirit, the non-canonical Hamiltonian structure for *adiabatic Yang–Mills charged fluids* discovered in [10] is obtained by reduction from a canonical formulation in [8], by using a Kaluza–Klein point of view involving the automorphism group of the principal bundle of the theory. This paper gives also the Euler–Poincaré formulation of these equations.

Non-canonical Hamiltonian structures for a wide class of non-dissipative fluid models were derived in [11,14–16], and [12]. These examples include *Yang–Mills magnetohydrodynamics*, *spin glasses*, and various models of *superfluids*, and involve Lie–Poisson brackets *with cocycles*. Remarkably, from a mathematical point of view, the Hamiltonian structures of many of these models are identical. This Hamiltonian structure together with the corresponding variational principles are studied in more detail, with an application to liquid crystals, in [13]. We will refer to all these models as *complex fluids*.

In this paper we show the remarkable property that these Lie–Poisson brackets with cocycles can also be obtained by Poisson reduction from a canonical Hamiltonian structure. The cocycle in the Hamiltonian structure appears only after reduction and it is due to the presence of an affine term added to the cotangent lifted action. The associated reduction process is naturally called *affine Lie–Poisson reduction*.

An important example of such an affine action is given by the usual action of the automorphism group of a principal bundle on the connection forms. As a result, we obtain, in a natural way, covariant differentials and covariant divergences in the expression of the Poisson brackets and of the reduced equations. These gauge theory aspects in the case of complex fluids are mathematically and physically interesting since they represent a bridge to other possible gauge theories in physics.

The new affine Lie–Poisson reduction principle presented in this paper unifies all previous approaches in the problem of complex fluid dynamics. Eringen and Holm have extensively studied the roles of auxiliary variables that were Lie algebra valued 1-forms frozen into the fluid flow. In particular, Holm observed that these variables contribute generalized 2-cocycles in the Lie–Poisson structures of ideal complex fluids. We implement Holm's observation as a fundamental principle which, for physical interactions in complex fluids, is similar to the principle of spontaneous symmetry breaking in the coupling of charges and fields in electromagnetism. This new geometrically formulated principle applies also to Lie algebra valued charges (e.g., spins, Yang–Mills charges, etc.). The present paper reformulates the theory of complex fluids in the framework of the fundamental principles of condensed-matter physics. Some results concerning the affine Lie–Poisson reduction and its application to Yang–Mills magnetohydrodynamics and superfluids were announced in [9]. It is clear that other geometric applications in condensed-matter physics are forthcoming, such as micropolar elasticity, for example.

The paper also formulates a parallel theory on the Lagrangian side by extending the Euler–Poincaré framework for semidirect products developed in [17] to the case of an affine representation of the configuration space Lie group on the vector space whose dual are the convected variables. The resulting *affine Euler–Poincaré reduction* is natural in two senses. Firstly, in the case of complex fluids and at the reduced level, it coincides with that given in [13]. Secondly, in the hyperregular case and through the Legendre transformation, it is compatible with the affine Lie–Poisson reduction for semidirect products discussed previously. In addition, the Euler–Poincaré formulation immediately leads to analogs of the Kelvin circulation theorem for complex fluids.

Organization of the paper. The paper is organized as follows. The first third, comprising Sections 2–7, presents only theoretical results. These are developed and applied to a wide range of examples in the last Section 8 forming the bulk of the paper.

The theoretical part begins by recalling at the end of this introduction some needed facts about the Lagrangian and Hamiltonian reductions for semidirect products. It is well known that these processes form the basic framework for the geometric formulation of various models of simple fluids. In Section 2 we present the theory of affine Euler–Poincaré reduction for a general Lie group G acting by affine representation on a dual vector space V^*. This theory is specialized, in Section 3, to the case of general complex fluids. More precisely, we describe concretely the group G and the cocycle needed in the affine representation, in order to obtain by reduction the general equations for complex fluids. In order to carry out the Hamiltonian side of the theory, we need to recall and state some results concerning the reduction of a canonical symplectic structure with a magnetic term. This is the subject of Section 4, whose principal result states that, under some conditions, reducing a canonical symplectic form relative to a cotangent lift with an affine term is equivalent to reducing a magnetic symplectic form relative to the cotangent lift. This observation is used in Section 5 to obtain the theory of affine Lie–Poisson reduction for a general Lie group G acting by affine representation on a dual vector space V^*. In particular, we compute the associated momentum map and Poisson bracket as well as the symplectic reduced spaces. This theory is also shown to be a particular case of the process of reduction by stages for nonequivariant momentum maps. In Section 6 we specialize these results to the case of the group and the cocycle involved in the description of complex fluid dynamics. In particular, we compute the associated Poisson bracket and momentum map. This part constitutes the Hamiltonian side of the theory developed in Section 3. The Kelvin–Noether theorem is a version of the classical Noether

conservation law that holds for solutions of the Euler–Poincaré equations. For example, an application of this theorem to the compressible fluid gives the Kelvin circulation theorem. The generalization of this result to the case of affine Euler–Poincaré equations is the subject of Section 7.

The rest of the paper is devoted to applications dealing with spin systems and complex fluids, all of them contained in Section 8. In Section 8.1 we start with the example of spin systems, since it illustrates the applicability of our theory in a very simple situation that exhibits, nevertheless, some of the key difficulties of more complicated fluid models. We then treat, in Sections 8.2–8.7, the examples of Yang–Mills and Hall magnetohydrodynamics for fluids and superfluids as well as the HVBK dynamics for ^{4}He with vortices. In each case, we formulate in detail the Lagrangian and Hamiltonian reduction processes as well as the associated Poisson bracket and circulation theorems. In order to compare fluids and superfluids from a Hamiltonian point of view, we present in Section 8.8 a summary of the models treated so far. In Section 8.9 we study the example of the Volovik–Dotsenko spin glasses and try to understand, from a Hamiltonian point of view, the passage from a given spin system to its hydrodynamic analogue. This process allows us to understand mathematically the link between the two approaches to spin glasses appearing in the current literature. All the Hamiltonian structures obtained so far by reduction of the canonical structure, coincide with the ones obtained previously and by various different methods in a series of papers by Holm and Kupershmidt.

In the case of microfluids, treated in Section 8.10, we determine the Lagrangian and Hamiltonian structure of three models proposed by Eringen, namely, the micropolar, the microstretch, and the micromorphic models, corresponding to three groups associated to the internal structure of the fluid particles. Since our theory applies to any group, we can obtain new models of microfluids such as the anisotropic micropolar or anisotropic microstretch models. In the case of microstretch and micropolar fluids, we show that the internal degree of freedom can be modeled by the group of invertible or unit quaternions, respectively. In Section 8.11, we determine and compare the Lagrangian and Hamiltonian structures of three models of liquid crystals dynamics. We also treat the case of Eringen's polymeric liquid crystals and give some information concerning the new model of anisotropic micropolar liquid crystals.

We close this introduction by recalling some needed facts about Euler–Poincaré and Lie–Poisson reduction for semidirect products.

Notations for semidirect products. In the Euler–Poincaré reduction for semidirect products (see [17]) one is given a Lie group G and a *right* representation $\rho : G \to \mathrm{Aut}(V)$ of G on the vector space V. As a set, the semidirect product $S = G \circledS V$ is the Cartesian product $S = G \times V$ whose group multiplication is given by

$$(g_1, v_1)(g_2, v_2) = \big(g_1 g_2, v_2 + \rho_{g_2}(v_1)\big).$$

The Lie algebra of S is the semidirect product Lie algebra, $\mathfrak{s} = \mathfrak{g} \circledS V$, whose bracket has the expression

$$\mathrm{ad}_{(\xi_1, v_1)}(\xi_2, v_2) = \big[(\xi_1, v_1), (\xi_2, v_2)\big] = \big([\xi_1, \xi_2], v_1\xi_2 - v_2\xi_1\big),$$

where $v\xi$ denotes the induced action of $\mathfrak{g}$ on V, that is,

$$v\xi := \frac{d}{dt}\bigg|_{t=0} \rho_{\exp(t\xi)}(v) \in V.$$

From the expression for the Lie bracket, it follows that for $(\xi, v) \in \mathfrak{s}$ and $(\mu, a) \in \mathfrak{s}^*$ we have

$$\mathrm{ad}^*_{(\xi, v)}(\mu, a) = \big(\mathrm{ad}^*_\xi \mu + v \diamond a, a\xi\big),$$

where $a\xi \in V^*$ and $v \diamond a \in \mathfrak{g}^*$ are given by

$$a\xi := \frac{d}{dt}\bigg|_{t=0} \rho^*_{\exp(-t\xi)}(a) \quad \text{and} \quad \langle v \diamond a, \xi\rangle_{\mathfrak{g}} := -\langle a\xi, v\rangle_V,$$

and where $\langle\cdot,\cdot\rangle_{\mathfrak{g}}:\mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$ and $\langle\cdot,\cdot\rangle_V:V^*\times V\to\mathbb{R}$ are the duality pairings.

Lagrangian semidirect product theory.

- Assume that we have a function $L:TG\times V^*\to\mathbb{R}$ which is right G-invariant.
- In particular, if $a_0\in V^*$, define the Lagrangian $L_{a_0}:TG\to\mathbb{R}$ by $L_{a_0}(v_g):=L(v_g,a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of G_{a_0} on G, where G_{a_0} is the isotropy group of a_0.
- Right G-invariance of L permits us to define $l:\mathfrak{g}\times V^*\to\mathbb{R}$ by $l:=L|_{\mathfrak{g}\times V^*}$ and hence

$$l\big(T_gR_{g^{-1}}(v_g),\rho_g^*(a_0)\big)=L(v_g,a_0).$$

- For a curve $g(t)\in G$, let $\xi(t):=TR_{g(t)^{-1}}(\dot g(t))$ and define the curve $a(t)$ as the unique solution of the following linear differential equation with time-dependent coefficients

$$\dot a(t)=-a(t)\xi(t),$$

with initial condition $a(0)=a_0$. The solution can be written as $a(t)=\rho^*_{g(t)}(a_0)$.

Theorem 1.1. *With the preceding notations, the following are equivalent*:

(i) *With a_0 held fixed, Hamilton's variational principle*

$$\delta\int_{t_1}^{t_2}L_{a_0}\big(g(t),\dot g(t)\big)\,dt=0$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

(ii) *$g(t)$ satisfies the Euler–Lagrange equations for L_{a_0} on G.*

(iii) *The constrained variational principle*

$$\delta\int_{t_1}^{t_2}l\big(\xi(t),a(t)\big)\,dt=0$$

holds on $\mathfrak{g}\times V^*$, upon using variations of the form

$$\delta\xi=\frac{\partial\eta}{\partial t}-[\xi,\eta],\qquad \delta a=-a\eta,$$

where $\eta(t)\in\mathfrak{g}$ vanishes at the endpoints.

(iv) *The Euler–Poincaré equations hold on $\mathfrak{g}\times V^*$*:

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta\xi}=-\operatorname{ad}^*_\xi\frac{\delta l}{\delta\xi}+\frac{\delta l}{\delta a}\diamond a. \tag{1.2}$$

It is worth noting that there is a remarkable symmetry breaking from $L:TG\times V^*\to\mathbb{R}$ to $L_{a_0}:TG\to\mathbb{R}$. Whereas the function L is right G-invariant, that is, $L(v_hg,\rho^*_{g^{-1}}(a))=L(v_h,a)$ for all $v_h\in TG$, $a\in V^*$, $g\in G$, the Lagrangian L_{a_0} is only G_{a_0}-invariant, that is, $L_{a_0}(v_hk)=L_{a_0}(v_h)$ for all $v_h\in TG$, $k\in G_{a_0}$. Note, however, that L_{a_0} is a Lagrangian function on TG, whereas L is not since it is not defined on a tangent bundle. The relationship between these two different invariance properties are dealt with in the previous theorem and the evolution $a(t)=\rho^*_{g(t)}a_0$ or, equivalently, the equation $\dot a(t)=-a(t)\xi(t)$, $a(0)=a_0$, that needs to be added to (1.2) in order to get a complete system of equations for all the unknowns, is exactly due to this symmetry breaking from G to G_{a_0}. We shall observe

the same phenomenon throughout the paper when working out the affine reduction theorem on both the Lagrangian and Hamiltonian sides.

Hamiltonian semidirect product theory. Let $S := G \circledS V$ be the semidirect product defined at the beginning of this section. The lift of right translation of S on T^*S induces a right action on $T^*G \times V^*$. Consider a Hamiltonian function $H : T^*G \times V^* \to \mathbb{R}$ right invariant under the S-action on $T^*G \times V^*$. In particular, the function $H_{a_0} := H|_{T^*G\times\{a_0\}} : T^*G \to \mathbb{R}$ is invariant under the induced action of the isotropy subgroup $G_{a_0} := \{g \in G \mid \rho_g^* a_0 = a_0\}$ for any $a_0 \in V^*$. The following theorem is an easy consequence of the semidirect product reduction theorem (see [24]) and the reduction by stages method (see [22]).

Theorem 1.2. *For $\alpha(t) \in T^*_{g(t)}G$ and $\mu(t) := T^*R_{g(t)}(\alpha(t)) \in \mathfrak{g}^*$, the following are equivalent*:

(i) *$\alpha(t)$ satisfies Hamilton's equations for H_{a_0} on T^*G.*
(ii) *The Lie–Poisson equation holds on $\mathfrak{s}^*$:*

$$\frac{\partial}{\partial t}(\mu, a) = -\operatorname{ad}^*_{\left(\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a}\right)}(\mu, a) = -\left(\operatorname{ad}^*_{\frac{\delta h}{\delta \mu}} \mu + \frac{\delta h}{\delta a} \diamond a, a\frac{\delta h}{\delta \mu}\right), \qquad a(0) = a_0,$$

where $\mathfrak{s}$ is the semidirect product Lie algebra $\mathfrak{s} = \mathfrak{g} \circledS V$. The associated Poisson bracket is the Lie–Poisson bracket on the semidirect product Lie algebra $\mathfrak{s}^*$, that is,

$$\{f, g\}(\mu, a) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\right\rangle + \left\langle a, \frac{\delta f}{\delta a}\frac{\delta g}{\delta \mu} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mu}\right\rangle.$$

As on the Lagrangian side, the evolution of the advected quantities is given by $a(t) = \rho^*_{g(t)}(a_0)$.

For example, one can start with a Lagrangian L_{a_0} as at the beginning of this section, suppose that the Legendre transformation $\mathbb{F}L_{a_0}$ is invertible and form the corresponding Hamiltonian $H_{a_0} = E_{a_0} \circ \mathbb{F}L_{a_0}^{-1}$, where E_{a_0} is the energy of L_{a_0}. Then the function $H : T^*G \times V^* \to \mathbb{R}$ so defined is S-invariant and one can apply this theorem. At the level of the reduced space, to a reduced Lagrangian $l : \mathfrak{g} \times V^* \to \mathbb{R}$ we associate the reduced Hamiltonian $h : \mathfrak{g}^* \circledS V^* \to \mathbb{R}$ given by

$$h(\mu, a) := \langle \mu, \xi\rangle - l(\xi, a), \quad \mu = \frac{\delta l}{\delta \xi}.$$

Since

$$\frac{\delta h}{\delta \mu} = \xi \quad \text{and} \quad \frac{\delta h}{\delta a} = -\frac{\delta l}{\delta a},$$

we see that the Lie–Poisson equations for h on $\mathfrak{s}^*$ are equivalent to the Euler–Poincaré equations (1.2) for l together with the advection equation $\dot{a} + a\xi = 0$.

Links with reduction by stages. Consider the semidirect product Lie group $S = G \circledS V$ acting by right translation on its cotangent bundle T^*S. An equivariant momentum map relative to the canonical symplectic form is given by

$$\mathbf{J}_R\big(\alpha_f, (u, a)\big) = T^*L_{(f,u)}\big(\alpha_f, (u, a)\big) = \big(T^*_e L_f(\alpha_f) + u \diamond a, a\big).$$

Since V is a closed normal subgroup of S, it also acts on T^*S and has a momentum map $\mathbf{J}_V : T^*S \to V^*$ given by

$$\mathbf{J}_V\big(\alpha_f, (u, a)\big) = a.$$

Reducing T^*S by V at the value a we get the first reduced space $(T^*S)_a = \mathbf{J}_V^{-1}(a)/V$. The isotropy subgroup G_a, consisting of elements of G that leave the point a fixed, acts freely and properly on $(T^*S)_a$ and has an induced equivariant momentum map $\mathbf{J}_a : (T^*S)_a \to \mathfrak{g}_a^*$, where $\mathfrak{g}_a$ is the Lie algebra of G_a. Reducing $(T^*S)_a$ at the point $\mu_a := \mu|\mathfrak{g}_a$, we get the second reduced space $((T^*S)_a)_{\mu_a} = \mathbf{J}_a^{-1}(\mu_a)/(G_a)_{\mu_a}$.

Using the Semidirect Product Reduction [24] or the Reduction by Stages Theorem [22], the two-stage reduced space $((T^*S)_a)_{\mu_a}$ is symplectically diffeomorphic to the reduced space $(T^*S)_{(\mu,a)} = \mathbf{J}_R^{-1}(\mu, a)/G_{(\mu,a)}$ obtained by reducing T^*S by the whole group S at the point $(\mu, a) \in \mathfrak{s}^*$.

The first symplectic reduced space $((T^*S)_a, \Omega_a)$ is symplectically diffeomorphic to the canonical symplectic manifold (T^*G, Ω) and the second reduced space $(((T^*S)_a)_{\mu_a}, (\Omega_a)_{\mu_a})$ is symplectically diffeomorphic to the coadjoint orbit $(\mathcal{O}_{(\mu,a)}, \omega_{(\mu,a)})$ together with its orbit symplectic form. Note also that we can consider the right G-invariant Hamiltonian $H : T^*G \times V^* \to \mathbb{R}$ as being the Poisson reduction of an S-invariant Hamiltonian $\overline{H} : T^*S \to \mathbb{R}$ by the normal subgroup $\{e\} \times V$ since $(T^*S)/(\{e\} \times V) \cong T^*G \times V^*$. Here the G-action on $T^*G \times V^*$ is induced by the lift of right translation of S. Theorem 1.2 is then a trivial consequence of these observations.

2. Affine Lagrangian semidirect product theory

Consider the *right* contragredient representation $\rho_{g^{-1}}^*$ of G on the vector space V^*. We can form an affine *right* representation $\theta_g(a) = \rho_{g^{-1}}^*(a) + c(g)$, where $c \in \mathcal{F}(G, V^*)$ is a contragredient representation valued right group one-cocycle, that is, it verifies the property $c(fg) = \rho_{g^{-1}}^*(c(f)) + c(g)$ for all $f, g \in G$. This implies that $c(e) = 0$ and $c(g^{-1}) = -\rho_g^*(c(g))$. Note that

$$\left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(t\xi)}(a) = a\xi + \mathbf{d}c(\xi)$$

and

$$\big\langle a\xi + \mathbf{d}c(\xi), v\big\rangle_V = \big\langle \mathbf{d}c^T(v) - v \diamond a, \xi\big\rangle_{\mathfrak{g}},$$

where $\mathbf{d}c : \mathfrak{g} \to V^*$ is defined by $\mathbf{d}c(\xi) := T_e c(\xi)$, and $\mathbf{d}c^T : V \to \mathfrak{g}^*$ is defined by

$$\big\langle \mathbf{d}c^T(v), \xi\big\rangle_{\mathfrak{g}} := \big\langle \mathbf{d}c(\xi), v\big\rangle_V.$$

- Assume that we have a function $L : TG \times V^* \to \mathbb{R}$ which is right G-invariant under the affine action $(v_h, a) \mapsto (T_h R_g(v_h), \theta_g(a)) = (T_h R_g(v_h), \rho_{g^{-1}}^*(a) + c(g))$.
- In particular, if $a_0 \in V^*$, define the Lagrangian $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) := L(v_g, a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of $G_{a_0}^c$ on G, where $G_{a_0}^c$ is the isotropy group of a_0 with respect to the affine action θ.
- Right G-invariance of L permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l\big(T_g R_{g^{-1}}(v_g), \theta_{g^{-1}}(a_0)\big) = L(v_g, a_0).$$

- For a curve $g(t) \in G$, let $\xi(t) := TR_{g(t)^{-1}}(\dot{g}(t))$ and define the curve $a(t)$ as the unique solution of the following affine differential equation with time-dependent coefficients

$$\dot{a}(t) = -a(t)\xi(t) - \mathbf{d}c\big(\xi(t)\big),$$

with initial condition $a(0) = a_0$. The solution can be written as $a(t) = \theta_{g(t)^{-1}}(a_0)$.

Theorem 2.1. *With the preceding notations, the following are equivalent*:

(i) *With a_0 held fixed, Hamilton's variational principle*

$$\delta \int_{t_1}^{t_2} L_{a_0}\big(g(t), \dot{g}(t)\big)\, dt = 0 \tag{2.1}$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.
(ii) *$g(t)$ satisfies the Euler–Lagrange equations for L_{a_0} on G.*
(iii) *The constrained variational principle*

$$\delta \int_{t_1}^{t_2} l\big(\xi(t), a(t)\big)\, dt = 0 \tag{2.2}$$

holds on $\mathfrak{g} \times V^*$, upon using variations of the form

$$\delta\xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \qquad \delta a = -a\eta - \mathbf{d}c(\eta),$$

where $\eta(t) \in \mathfrak{g}$ vanishes at the endpoints.
(iv) *The affine Euler–Poincaré equations hold on $\mathfrak{g} \times V^*$:*

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} = -\operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a - \mathbf{d}c^T\left(\frac{\delta l}{\delta a}\right). \tag{2.3}$$

Proof. The equivalence of (i) and (ii) is true in general.

Next we show the equivalence of (iii) and (iv). Indeed, using the definitions, integrating by parts, and taking into account that $\eta(t_1) = \eta(t_2) = 0$, we compute the variation of the integral to be

$$\begin{aligned}
\delta \int_{t_1}^{t_2} l\big(\xi(t), a(t)\big)\, dt &= \int_{t_1}^{t_2} \left(\left\langle \frac{\delta l}{\delta \xi}, \delta\xi \right\rangle + \left\langle \delta a, \frac{\delta l}{\delta a} \right\rangle \right) dt \\
&= \int_{t_1}^{t_2} \left(\left\langle \frac{\delta l}{\delta \xi}, \dot{\eta} - \operatorname{ad}_\xi \eta \right\rangle - \left\langle a\eta + \mathbf{d}c(\eta), \frac{\delta l}{\delta a} \right\rangle \right) dt \\
&= \int_{t_1}^{t_2} \left(\left\langle -\frac{d}{dt}\frac{\delta l}{\delta \xi} - \operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi}, \eta \right\rangle - \left\langle -\frac{\delta l}{\delta a} \diamond a + \mathbf{d}c^T\left(\frac{\delta l}{\delta a}\right), \eta \right\rangle \right) dt \\
&= \int_{t_1}^{t_2} \left(\left\langle -\frac{d}{dt}\frac{\delta l}{\delta \xi} - \operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a - \mathbf{d}c^T\left(\frac{\delta l}{\delta a}\right), \eta \right\rangle \right) dt
\end{aligned}$$

and so the result follows.

Finally we show that (i) and (iii) are equivalent. First note that the G-invariance of $L : TG \times V^* \to \mathbb{R}$ and the definition of $a(t) = \theta_{g(t)^{-1}}(a_0)$ imply that the integrands in (2.1) and (2.2) are equal. It is known that all variations $\delta g(t) \in TG$ of $g(t)$ with fixed endpoints induce and are induced by variations $\delta\xi(t) \in \mathfrak{g}$ of $\xi(t)$ of the form $\delta\xi = \dot{\eta} - [\xi, \eta]$ with $\eta(t)$ vanishing at the endpoints; the relation between $\delta g(t)$ and $\eta(t)$ is given by $\eta(t) = TR_{g(t)^{-1}}(\delta g(t))$. See [2] for details.

Thus, if (i) holds, we define $\eta(t) = TR_{g(t)^{-1}}(\delta g(t))$ for a variation $\delta g(t)$ with fixed endpoints. Then if we let $\delta\xi(t) = TR_{g(t)^{-1}}(\dot{g}(t))$, we have $\delta\dot{\xi} = \dot{\eta} - [\xi, \eta]$. In addition, the variation of $a(t) = \theta_{g(t)^{-1}}(a_0)$

is $\delta a(t) = -a(t)\eta(t) - \mathbf{d}c(\eta(t))$. Conversely, if $\delta\xi = \dot{\eta} - [\xi, \eta]$ with $\eta(t)$ vanishing at the endpoints, we define $\delta g(t) = TR_{g(t)}(\eta(t))$. This $\delta g(t)$ is the general variation of $g(t)$ vanishing at the endpoints. From $\delta a(t) = -a(t)\eta(t) - \mathbf{d}c(\eta(t))$ it follows that the variation of $\theta_{g(t)}(a(t)) = a_0$ vanishes, which is consistent with the dependence of L_{a_0} only on $g(t), \dot{g}(t)$. □

3. Lagrangian approach to continuum theories of perfect complex fluids

Recall that in the case of the motion of a fluid on an orientable manifold $\mathcal{D}$, the configuration space is the group $G = \mathrm{Diff}(\mathcal{D})$ of all diffeomorphisms of $\mathcal{D}$. In the case of incompressible fluids, one chooses the subgroup $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})$ of all volume preserving diffeomorphisms, with respect to a fixed volume form on $\mathcal{D}$. Besides the diffeomorphism group, the other basic object is the vector space V^* of advected quantities on which G acts by representations. Typical advected quantities are, for example, the *mass density*, the *specific entropy*, or the *magnetic field*. One can obtain the fluid equations by choosing the appropriate Lagrangian and Hamiltonian functions and by applying the semidirect Euler–Poincaré or Lie–Poisson reduction processes (Theorems 1.1 and 1.2), see [24] and [17].

The goal of this section is to extend these formulations to the case of complex fluids. At the reduced level, the Euler–Poincaré equations for complex fluids are given in [13] (Eqs. (3.23), (3.24), (3.32) and (3.33)). The two key observations we make regarding these equations are the following. First, the two equations (3.23) and (3.24) suggest that the configuration manifold $\mathrm{Diff}(\mathcal{D})$ has to be enlarged to a bigger group G in order to contain variables involving the Lie group $\mathcal{O}$ of order parameters. Second, the two advection equations (3.32) and (3.33) suggest that there is a new advected quantity on which the group G acts by affine representation. Making use of these two observations, we construct below the appropriate configuration space and the appropriate affine action for the dynamics of complex fluids. By using the general process of affine Euler–Poincaré reduction developed before (Theorem 2.1), we get (a generalization of) the equations given in [13].

Here and in all examples that follow, there are fields different from the velocity field for which we shall never specify the boundary conditions. We make the general assumption, valid throughout the paper, that all integrations by parts have vanishing boundary terms, or that the problem has periodic boundary conditions (in which case $\mathcal{D}$ is a boundaryless three-dimensional manifold). Of course, if one would try to get an analytically rigorous result, the boundary conditions for all fields need to be carefully specified.

The configuration manifold. Consider a finite-dimensional Lie group $\mathcal{O}$. In applications $\mathcal{O}$ will be called the *order parameter Lie group*. Recall that in the case of the motion of a fluid on an orientable manifold $\mathcal{D}$, the configuration space is the group $G = \mathrm{Diff}(\mathcal{D})$ of all diffeomorphisms of $\mathcal{D}$. In the case of complex fluids, the basic idea is to enlarge this group to the semidirect product of groups $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$. Here $\mathcal{F}(\mathcal{D}, \mathcal{O})$ denotes the group of all mappings χ defined on $\mathcal{D}$ with values in the Lie group $\mathcal{O}$ of order parameters. The diffeomorphism group acts on $\mathcal{F}(\mathcal{D}, \mathcal{O})$ via the *right* action

$$(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathcal{O}) \mapsto \chi \circ \eta \in \mathcal{F}(\mathcal{D}, \mathcal{O}).$$

Therefore, the group multiplication is given by

$$(\eta, \chi)(\varphi, \psi) = \big(\eta \circ \varphi, (\chi \circ \varphi)\psi\big).$$

Recall that the tangent space to $\mathrm{Diff}(\mathcal{D})$ at η is

$$T_\eta \mathrm{Diff}(\mathcal{D}) = \big\{\mathbf{u}_\eta : \mathcal{D} \to T\mathcal{D} \;\big|\; \mathbf{u}_\eta(x) \in T_{\eta(x)}\mathcal{D}\big\},$$

the tangent space to $\mathcal{F}(\mathcal{D}, \mathcal{O})$ at χ is

$$T_\chi \mathcal{F}(\mathcal{D}, \mathcal{O}) = \big\{v_\chi : \mathcal{D} \to T\mathcal{O} \;\big|\; v_\chi(x) \in T_{\chi(x)}\mathcal{O}\big\}.$$

A direct computation shows that the tangent map of right translation is

$$TR_{(\varphi,\psi)}(\mathbf{u}_\eta, v_\chi) = \big(\mathbf{u}_\eta \circ \varphi, TR_\psi(v_\chi \circ \varphi)\big).$$

For simplicity we fix a volume form μ on $\mathcal{D}$. Therefore we can identify the cotangent space $T^*_\eta \mathrm{Diff}(\mathcal{D})$ with a space of one-forms over η, that is,

$$T^*_\eta \mathrm{Diff}(\mathcal{D}) = \big\{\mathbf{m}_\eta : \mathcal{D} \to T^*\mathcal{D} \;\big|\; \mathbf{m}_\eta(x) \in T^*_{\eta(x)}\mathcal{D}\big\}.$$

The cotangent space of $\mathcal{F}(\mathcal{D}, \mathcal{O})$ at χ is naturally given by

$$T^*_\chi \mathcal{F}(\mathcal{D}, \mathcal{O}) = \big\{\kappa_\chi : \mathcal{D} \to T^*\mathcal{O} \;\big|\; \kappa_\chi(x) \in T^*_{\chi(x)}\mathcal{O}\big\}.$$

Using these identifications, the cotangent map of right translation is computed to be

$$T^*R_{(\varphi,\psi)}(\mathbf{m}_\eta, \kappa_\chi) = J\big(\varphi^{-1}\big)\big(\mathbf{m}_\eta \circ \varphi^{-1}, T^*R_{\psi\circ\varphi^{-1}}\big(\kappa_\chi \circ \varphi^{-1}\big)\big),$$

where $J(\varphi^{-1})$ is the Jacobian determinant of the diffeomorphism φ^{-1}. The corresponding cotangent lift, defined by $R^{T^*}_{(\varphi,\psi)} := T^*R_{(\varphi,\psi)^{-1}}$, is given by

$$R^{T^*}_{(\varphi,\psi)}(\mathbf{m}_\eta, \kappa_\chi) = J(\varphi)\big(\mathbf{m}_\eta \circ \varphi, T^*R_{\psi^{-1}}(\kappa_\chi \circ \varphi)\big).$$

The Lie algebra $\mathfrak{g}$ of the semidirect product group is

$$\mathfrak{g} = \mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o}),$$

and the Lie bracket is computed to be

$$\mathrm{ad}_{(\mathbf{u},v)}(\mathbf{v}, \zeta) = (\mathrm{ad}_{\mathbf{u}}\mathbf{v}, \mathrm{ad}_v\zeta + \mathbf{d}v\cdot\mathbf{v} - \mathbf{d}\zeta\cdot\mathbf{u}),$$

where $\mathrm{ad}_{\mathbf{u}}\mathbf{v} = -[\mathbf{u}, \mathbf{v}]$, $\mathrm{ad}_v\zeta \in \mathcal{F}(\mathcal{D}, \mathfrak{o})$ is given by $\mathrm{ad}_v\zeta(x) := \mathrm{ad}_{v(x)}\zeta(x)$, and $\mathbf{d}v\cdot\mathbf{v} \in \mathcal{F}(\mathcal{D}, \mathfrak{o})$ is given by $\mathbf{d}v\cdot\mathbf{v}(x) := \mathbf{d}v(x)(\mathbf{v}(x))$.

Using the previous identification of cotangent spaces, the dual Lie algebra $\mathfrak{g}^*$ can be identified with

$$\Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$$

through the pairing

$$\big\langle(\mathbf{m}, \kappa), (\mathbf{u}, v)\big\rangle = \int_{\mathcal{D}} (\mathbf{m}\cdot\mathbf{u} + \kappa\cdot v)\mu.$$

The dual map to $\mathrm{ad}_{(\mathbf{u},v)}$ is

$$\mathrm{ad}^*_{(\mathbf{u},v)}(\mathbf{m}, \kappa) = \big(\pounds_{\mathbf{u}}\mathbf{m} + (\mathrm{div}\,\mathbf{u})\mathbf{m} + \kappa\cdot\mathbf{d}v, \mathrm{ad}^*_v\kappa + \mathrm{div}(\mathbf{u}\kappa)\big). \tag{3.1}$$

This formula needs some explanation. The symbol $\kappa\cdot\mathbf{d}v \in \Omega^1(\mathcal{D})$ denotes the one-form defined by

$$\kappa\cdot\mathbf{d}v(v_x) := \kappa(x)\big(\mathbf{d}v(v_x)\big)$$

and $\operatorname{ad}^*_\nu \kappa \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$ denotes the $\mathfrak{o}^*$-valued mapping defined by

$$\operatorname{ad}^*_\nu \kappa(x) := \operatorname{ad}^*_{\nu(x)}\big(\kappa(x)\big).$$

The expression $\mathbf{u}\kappa$ denotes the 1-contravariant tensor field with values in $\mathfrak{o}^*$ defined by

$$\mathbf{u}\kappa(\alpha_x) := \alpha_x\big(\mathbf{u}(x)\big)\kappa(x) \in \mathfrak{o}^*.$$

Since $\mathbf{u}\kappa$ is a generalization of the notion of a vector field, we denote by $\mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ the space of all 1-contravariant tensor fields with values in $\mathfrak{o}^*$. In (3.1), $\operatorname{div}(\mathbf{u})$ denotes the divergence of the vector field $\mathbf{u}$ with respect to the fixed volume form μ. Recall that it is defined by the condition

$$(\operatorname{div}\mathbf{u})\mu = \pounds_{\mathbf{u}}\mu.$$

This operator can be naturally extended to the space $\mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ as follows. For $w \in \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ we write $w = w_a\varepsilon^a$ where (ε^a) is a basis of $\mathfrak{o}^*$ and $w_a \in \mathfrak{X}(\mathcal{D})$. We define $\operatorname{div}: \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*) \to \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$ by the equality

$$\operatorname{div} w := (\operatorname{div} w_a)\varepsilon^a.$$

Note that if $w = \mathbf{u}\kappa$ then

$$\operatorname{div}(\mathbf{u}\kappa) = \mathbf{d}\kappa \cdot \mathbf{u} + (\operatorname{div}\mathbf{u})\kappa.$$

The space of advected quantities. In physical applications, the affine representation space V^* of $G = \operatorname{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$ is a direct product $V_1^* \oplus V_2^*$, where V_i^* are subspaces of the space of all tensor fields on $\mathcal{D}$ (possibly with values in a vector space). Moreover:

- V_1^* is only acted upon by the component $\operatorname{Diff}(\mathcal{D})$ of G.
- The action of G on V_2^* is affine, with the restriction that the affine term only depends on the second component $\mathcal{F}(\mathcal{D}, \mathcal{O})$ of G.

In this way, we obtain the affine representation

$$(a, \gamma) \in V_1^* \oplus V_2^* \mapsto \big(a\eta, \gamma(\eta, \chi) + C(\chi)\big) \in V_1^* \oplus V_2^*, \tag{3.2}$$

where $\gamma(\eta, \chi)$ denotes the representation of $(\eta, \chi) \in G$ on $\gamma \in V_2^*$, and $C \in \mathcal{F}(\mathcal{F}(\mathcal{D}, \mathcal{O}), V_2^*)$ satisfies the identity

$$C\big((\chi \circ \varphi)\psi\big) = C(\chi)(\varphi, \psi) + C(\psi) \tag{3.3}$$

for all $\chi, \psi \in \mathcal{F}(\mathcal{D}, \mathcal{O})$ and $\varphi \in \operatorname{Diff}(\mathcal{D})$. Note that this is equivalent to say that the representation ρ and the affine term c of the previous section have the particular form

$$\rho^*_{(\eta,\chi)^{-1}}(a, \gamma) = \big(a\eta, \gamma(\eta, \chi)\big) \quad \text{and} \quad c(\eta, \chi) = \big(0, C(\chi)\big).$$

The infinitesimal action of $(\mathbf{u}, \nu) \in \mathfrak{g}$ on $\gamma \in V_2^*$ induced by the representation of G on V_2^* is

$$\begin{aligned}\gamma(\mathbf{u}, \nu) &:= \frac{d}{dt}\bigg|_{t=0} \gamma\big(\exp(t\mathbf{u}), \exp(t\nu)\big) = \frac{d}{dt}\bigg|_{t=0} \gamma\big(\exp(t\mathbf{u}), e\big)\big(e, \exp(t\nu)\big)\\ &= \frac{d}{dt}\bigg|_{t=0} \gamma\big(\exp(t\mathbf{u}), e\big) + \frac{d}{dt}\bigg|_{t=0} \gamma\big(e, \exp(t\nu)\big) =: \gamma\mathbf{u} + \gamma\nu.\end{aligned}$$

Therefore, for $(v, w) \in V_1 \oplus V_2$ and $(a, \gamma) \in V_1^* \oplus V_2^*$ we have

$$(v, w) \diamond (a, \gamma) = (v \diamond a + w \diamond_1 \gamma, w \diamond_2 \gamma),$$

where $\diamond_1$ and $\diamond_2$ are associated to the induced representations of the first and second component of G on V_2^*. On the right-hand side, the diamond operation $\diamond$ is associated to the representation of $\mathrm{Diff}(\mathcal{D})$ on V_1^*. The space V_1^* is naturally the dual of some space V_1 of tensor fields on $\mathcal{D}$. For example the (p, q) tensor fields are naturally in duality with the (q, p) tensor fields. For $a \in V_1^*$ and $v \in V_1$, the duality pairing is given by

$$\langle a, v \rangle = \int_{\mathcal{D}} (a \cdot v)\mu,$$

where $\cdot$ denotes the contraction of tensor fields.

Since the affine cocycle has the particular form $c(\eta, \chi) = (0, C(\chi))$, we obtain that

$$\mathbf{d}c^T(v, w) = \big(0, \mathbf{d}C^T(w)\big).$$

For a Lagrangian $l = l(\mathbf{u}, v, a, \gamma) : [\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o})] \times [V_1^* \oplus V_2^*] \to \mathbb{R}$, the affine Euler–Poincaré equations (2.3) become

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \mathbf{u}} = -\pounds_{\mathbf{u}}\dfrac{\delta l}{\delta \mathbf{u}} - (\operatorname{div}\mathbf{u})\dfrac{\delta l}{\delta \mathbf{u}} - \dfrac{\delta l}{\delta v}\cdot \mathbf{d}v + \dfrac{\delta l}{\delta a}\diamond a + \dfrac{\delta l}{\delta \gamma}\diamond_1 \gamma, \\ \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta v} = -\operatorname{ad}_v^*\dfrac{\delta l}{\delta v} - \operatorname{div}\left(\mathbf{u}\dfrac{\delta l}{\delta v}\right) + \dfrac{\delta l}{\delta \gamma}\diamond_2 \gamma - \mathbf{d}C^T\left(\dfrac{\delta l}{\delta \gamma}\right), \end{cases} \tag{3.4}$$

and the advection equations are

$$\begin{cases} \dot{a} + a\mathbf{u} = 0, \\ \dot{\gamma} + \gamma\mathbf{u} + \gamma v + \mathbf{d}C(v) = 0. \end{cases} \tag{3.5}$$

Remark on the duality pairing. We remind the reader that, for simplicity, we have assumed that $\mathcal{D}$ is orientable and we have also fixed a volume form $\mu \in \Omega^n(\mathcal{D})$. Thus, using the L^2 pairing, the dual space of the Lie algebra $\mathfrak{X}(\mathcal{D})$ of vector fields is identified with the one-forms $\Omega^1(\mathcal{D})$. Therefore, the functional derivative $\delta l/\delta \mathbf{u}$ is a one-form on $\mathcal{D}$.

The natural dual of $\mathfrak{X}(\mathcal{D})$ is the space $\Omega^1(\mathcal{D}) \otimes \mathcal{D}en(\mathcal{D})$ of one-form densities on $\mathcal{D}$. In this general approach, valid also for non-orientable manifolds, the duality pairing of $\alpha \otimes \omega \in \Omega^1(\mathcal{D}) \otimes \mathcal{D}en(\mathcal{D})$ with $\mathbf{u}$ reads

$$\int_{\mathcal{D}} (\alpha \cdot \mathbf{u})\omega.$$

Then the functional derivative $\delta l/\delta \mathbf{u}$ is interpreted as a one-form density on $\mathcal{D}$. The same remark applies for the dual of vector valued tensors on $\mathcal{D}$, in particular for the Lie algebra $\mathcal{F}(\mathcal{D}, \mathfrak{o})$.

We chose to work on oriented manifolds and to avoid the use of $\Omega^1(\mathcal{D}) \otimes \mathcal{D}en(\mathcal{D})$ because this general case would considerably complicate the writing of various formulas and equations, without adding any essential new information. We emphasize that all results of this paper are also valid for non-orientable manifolds by appealing to the usual exterior differential calculus and the Stokes theorem for twisted k-forms on $\mathcal{D}$ (see [1, Supplement 7.2A] for details).

Basic example. Take $V_2^* := \Omega^1(\mathcal{D}, \mathfrak{o})$, the space of all one-forms on $\mathcal{D}$ with values in $\mathfrak{o}$. This space is naturally the dual of the space $V_2 = \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ of contravariant tensor fields with values in $\mathfrak{o}^*$, the duality pairing being given, for $\gamma \in \Omega^1(\mathcal{D}, \mathfrak{o})$ and $w \in \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$, by

$$\langle \gamma, w \rangle := \int_{\mathcal{D}} (\gamma \cdot w)\mu,$$

where $\gamma \cdot w$ denotes the contraction of tensors. In what follows, we still let V_1 be an arbitrary vector space. In concrete examples, its dual V_1^* is formed by classical advected quantities, such as mass density, entropy density, or magnetic field, for example.

We consider for (3.2) the affine representation of $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$ on $V_1^* \times \Omega^1(\mathcal{D}, \mathfrak{o})$ defined by

$$(a, \gamma) \mapsto \big(a\eta, \mathrm{Ad}_{\chi^{-1}} \eta^* \gamma + \chi^{-1} T\chi\big), \tag{3.6}$$

where $(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$, $(a, \gamma) \in V_1^* \times \Omega^1(\mathcal{D}, \mathfrak{o})$, and $\mathrm{Ad}_{\chi^{-1}} \eta^* \gamma + \chi^{-1} T\chi$ is the $\mathfrak{o}$-valued one-form given by

$$\big(\mathrm{Ad}_{\chi^{-1}} \eta^* \gamma + \chi^{-1} T\chi\big)(v_x) := \mathrm{Ad}_{\chi(x)^{-1}}\big(\eta^* \gamma(v_x)\big) + \chi(x)^{-1} T_x \chi(v_x)$$

for $v_x \in T_x\mathcal{D}$. One can check that $\gamma(\eta, \chi) := \mathrm{Ad}_{\chi^{-1}} \eta^* \gamma$ is a right representation of G on V_2^* and that $C(\chi) = \chi^{-1} T\chi$ verifies the condition (3.3). In fact, (3.6) corresponds to the action of the automorphism group of the trivial principal bundle $\mathcal{O} \times \mathcal{D}$ on the space of connections.

For this example we have

$$\gamma \mathbf{u} = \pounds_{\mathbf{u}} \gamma, \quad \gamma \nu = -\operatorname{ad}_\nu \gamma \quad \text{and} \quad \mathbf{d}C(\nu) = \mathbf{d}\nu,$$

where $\operatorname{ad}_\nu \gamma \in \Omega^1(\mathcal{D}, \mathfrak{o})$ denotes the one-form given by

$$(\operatorname{ad}_\nu \gamma)(v_x) := \operatorname{ad}_{\nu(x)}\big(\gamma(v_x)\big) = \big[\nu(x), \gamma(v_x)\big],$$

and $\mathbf{d}\nu \in \Omega^1(\mathcal{D}, \mathfrak{o})$ is given by $\mathbf{d}\nu(v_x) := T_x \nu(v_x) \in \mathfrak{o}$ for all $v_x \in T_x\mathcal{D}$.

A direct computation shows that

$$\begin{aligned} w \diamond_1 \gamma &= (\operatorname{div} w) \cdot \gamma - w \cdot \mathbf{i}_{\_}\, \mathbf{d}\gamma \in \Omega^1(\mathcal{D}), \\ w \diamond_2 \gamma &= -\operatorname{Tr}\big(\operatorname{ad}^*_\gamma w\big) \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*), \\ \mathbf{d}C^T(w) &= -\operatorname{div} w \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*), \end{aligned}$$

where Tr denotes the trace of the $\mathfrak{o}^*$-valued $(1,1)$ tensor

$$\operatorname{ad}^*_\gamma w : T^*\mathcal{D} \times T\mathcal{D} \to \mathfrak{o}^*, \qquad (\alpha_x, v_x) \mapsto \operatorname{ad}^*_{\gamma(v_x)}\big(w(\alpha_x)\big).$$

In coordinates we have $\operatorname{Tr}(\operatorname{ad}^*_\gamma w) = \operatorname{ad}^*_{\gamma_i} w^i$. Making use of these computations, the affine Euler–Poincaré equations (3.4) become

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \mathbf{u}} = -\pounds_{\mathbf{u}} \dfrac{\delta l}{\delta \mathbf{u}} - \Big(\operatorname{div} \mathbf{u}\Big)\dfrac{\delta l}{\delta \mathbf{u}} - \dfrac{\delta l}{\delta \nu} \cdot \mathbf{d}\nu + \dfrac{\delta l}{\delta a} \diamond a + \Big(\operatorname{div} \dfrac{\delta l}{\delta \gamma}\Big) \cdot \gamma - \dfrac{\delta l}{\delta \gamma} \cdot \mathbf{i}_{\_}\, \mathbf{d}\gamma, \\ \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \nu} = -\operatorname{ad}^*_\nu \dfrac{\delta l}{\delta \nu} + \operatorname{div}\Big(\dfrac{\delta l}{\delta \gamma} - \mathbf{u}\dfrac{\delta l}{\delta \nu}\Big) - \operatorname{Tr}\Big(\operatorname{ad}^*_\gamma \dfrac{\delta l}{\delta \gamma}\Big), \end{cases} \tag{3.7}$$

and the advection equations are

$$\begin{cases} \dot{a} + a\mathbf{u} = 0, \\ \dot{\gamma} + \mathbf{\pounds}_{\mathbf{u}}\gamma - \mathrm{ad}_{\nu}\,\gamma + \mathbf{d}\nu = 0. \end{cases}$$

These are, up to sign conventions, the equations for complex fluids as given in [13].

We now rewrite these equations using the covariant differentiation associated to a connection. A one-form $\gamma \in \Omega^1(\mathcal{D}, \mathfrak{o})$ can be considered as a connection one-form on the trivial principal $\mathcal{O}$-bundle $\mathcal{O} \times \mathcal{D} \to \mathcal{D}$, namely,

$$(v_x, \xi_{\mathsf{h}}) \in T_x\mathcal{D} \times T_{\mathsf{h}}\mathcal{O} \mapsto \mathrm{Ad}_{\mathsf{h}^{-1}}\big(\gamma(x)(v_x) + TR_{\mathsf{h}^{-1}}(\xi_{\mathsf{h}})\big) \in \mathfrak{o}. \tag{3.8}$$

The covariant differential associated to this principal connection will be denoted by $\mathbf{d}^{\gamma}$. Therefore, for a function $\nu \in \mathcal{F}(\mathcal{D}, \mathfrak{o})$, we have

$$\mathbf{d}^{\gamma}\nu(\mathbf{v}) := \mathbf{d}\nu(\mathbf{v}) + \big[\gamma(\mathbf{v}), \nu\big]. \tag{3.9}$$

The covariant divergence of $w \in \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ is the function

$$\mathrm{div}^{\gamma}\, w := \mathrm{div}\, w - \mathrm{Tr}\big(\mathrm{ad}^*_{\gamma}\, w\big) \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*), \tag{3.10}$$

defined as minus the adjoint of the covariant differential, that is,

$$\int_{\mathcal{D}} \big(\mathbf{d}^{\gamma}\nu \cdot w\big)\mu = -\int_{\mathcal{D}} \big(\nu \cdot \mathrm{div}^{\gamma}\, w\big)\mu \tag{3.11}$$

for all $\nu \in \mathcal{F}(\mathcal{D}, \mathfrak{o})$. Note that the Lie derivative of $\gamma \in \Omega^1(\mathcal{D}, \mathfrak{o})$ can be written as

$$\begin{aligned} \mathbf{\pounds}_{\mathbf{u}}\gamma(\mathbf{v}) &= \mathbf{d}\big(\gamma(\mathbf{u})\big)(\mathbf{v}) + \mathbf{i}_{\mathbf{u}}\mathbf{d}\gamma(\mathbf{v}) \\ &= \mathbf{d}^{\gamma}\big(\gamma(\mathbf{u})\big)(\mathbf{v}) - \big[\gamma(\mathbf{v}), \gamma(\mathbf{u})\big] + \mathbf{d}\gamma^{\gamma}(\mathbf{u}, \mathbf{v}) - \big[\gamma(\mathbf{u}), \gamma(\mathbf{v})\big] \\ &= \mathbf{d}^{\gamma}\big(\gamma(\mathbf{u})\big)(\mathbf{v}) + \mathbf{i}_{\mathbf{u}}B(\mathbf{v}), \end{aligned} \tag{3.12}$$

where

$$B := \mathbf{d}^{\gamma}\gamma = \mathbf{d}\gamma + [\gamma, \gamma]$$

is the curvature of the connection induced by γ.

Note also that, using covariant differentiation, we have

$$w \diamond_1 \gamma = (\mathrm{div}\, w) \cdot \gamma - w \cdot \mathbf{i}_{\_}\mathbf{d}\gamma = \big(\mathrm{div}^{\gamma}\, w\big) \cdot \gamma - w \cdot \mathbf{i}_{\_}B.$$

Therefore, in terms of $\mathbf{d}^{\gamma}$, div^{γ}, and $B = \mathbf{d}^{\gamma}\gamma$, the equations read

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \mathbf{u}} = -\mathbf{\pounds}_{\mathbf{u}}\dfrac{\delta l}{\delta \mathbf{u}} - (\mathrm{div}\,\mathbf{u})\dfrac{\delta l}{\delta \mathbf{u}} - \dfrac{\delta l}{\delta \nu} \cdot \mathbf{d}\nu + \dfrac{\delta l}{\delta a} \diamond a + \left(\mathrm{div}^{\gamma}\,\dfrac{\delta l}{\delta \gamma}\right) \cdot \gamma - \dfrac{\delta l}{\delta \gamma} \cdot \mathbf{i}_{\_}B, \\ \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \nu} = -\mathrm{ad}^*_{\nu}\,\dfrac{\delta l}{\delta \nu} - \mathrm{div}\left(\mathbf{u}\dfrac{\delta l}{\delta \nu}\right) + \mathrm{div}^{\gamma}\,\dfrac{\delta l}{\delta \gamma}, \end{cases} \tag{3.13}$$

and

$$\begin{cases} \dot{a} + a\mathbf{u} = 0, \\ \dot{\gamma} + \mathbf{d}^{\gamma}\big(\gamma(\mathbf{u})\big) + \mathbf{i}_{\mathbf{u}} B + \mathbf{d}^{\gamma}\nu = 0. \end{cases}$$

The *B*-representation. In this example we want to reformulate the reduction process as well as the equations of motion (3.13) in terms of another set of variables, namely, $(\mathbf{u}, \nu, a, B)$, where $B = \mathbf{d}^{\gamma}\gamma = \mathbf{d}\gamma + [\gamma, \gamma] \in \Omega^2(\mathcal{D}, \mathfrak{o})$ is the *curvature* of γ, instead of $(\mathbf{u}, \nu, a, \gamma)$. As we shall see below, with this choice of variables, the action (3.6) becomes linear instead of affine. We shall also assume that the Lagrangian L, and hence also l, depend on γ only through B. To do this we shall use the standard Euler–Poincaré reduction for semidirect products (Theorem 1.1).

Indeed, if $\gamma' = \mathrm{Ad}_{\chi^{-1}}\eta^*\gamma + \chi^{-1}T\chi$ then we have $\mathbf{d}^{\gamma'}\gamma' = \mathrm{Ad}_{\chi^{-1}}\eta^*\mathbf{d}^{\gamma}\gamma$. Thus the representation of $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$ on $V_1^* \oplus \Omega^2(\mathcal{D}, \mathfrak{o})$ is given by

$$(a, B) \mapsto (a\eta, \mathrm{Ad}_{\chi^{-1}}\eta^* B).$$

The associated infinitesimal action of $(\mathbf{u}, \nu) \in \mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o})$ is

$$(a, B)(\mathbf{u}, \nu) = \big(a\mathbf{u}, B(\mathbf{u}, \nu)\big) = (a\mathbf{u}, \pounds_{\mathbf{u}} B - \mathrm{ad}_{\nu} B).$$

The space $\Omega^2(\mathcal{D}, \mathfrak{o})$ is, in a natural way, dual to the space $\Omega_2(\mathcal{D}, \mathfrak{o}^*)$ of 2-contravariant skew-symmetric tensor fields with values in $\mathfrak{o}^*$. The duality pairing is given by contraction and integration with respect to the fixed volume form μ. More generally, we can consider the space $\Omega_k(\mathcal{D}, \mathfrak{o}^*)$ of k-contravariant skew-symmetric tensor fields with values in $\mathfrak{o}^*$ and we can define the divergence operators, $\mathrm{div}, \mathrm{div}^{\gamma} : \Omega_k(\mathcal{D}, \mathfrak{o}^*) \to \Omega_{k-1}(\mathcal{D}, \mathfrak{o}^*)$, to be minus the adjoint of the exterior derivatives $\mathbf{d}$ and $\mathbf{d}^{\gamma}$, respectively. For example, div^{γ} is defined on $\Omega_k(\mathcal{D}, \mathfrak{o}^*)$ by

$$\int_{\mathcal{D}} \big(\mathbf{d}^{\gamma}\alpha \cdot \omega\big)\mu = -\int_{\mathcal{D}} \big(\alpha \cdot \mathrm{div}^{\gamma}\,\omega\big)\mu, \tag{3.14}$$

where $\alpha \in \Omega^{k-1}(\mathcal{D}, \mathfrak{o})$ and $\omega \in \Omega_k(\mathcal{D}, \mathfrak{o}^*)$. Note that we have used the notations $\Omega_1(\mathcal{D}, \mathfrak{o}^*) = \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ and $\Omega_0(\mathcal{D}, \mathfrak{o}^*) = \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$.

Using the duality pairing defined before, for $(v, b) \in V_1 \oplus \Omega_2(\mathcal{D}, \mathfrak{o}^*)$ and $(a, B) \in V_1^* \oplus \Omega^2(\mathcal{D}, \mathfrak{o})$, the diamond operation is given by

$$(v, b) \diamond (a, B) = (v \diamond a + b \diamond_1 B, b \diamond_2 B),$$

where

$$b \diamond_1 B = (\mathrm{div}\, b) \cdot \mathbf{i}_{\_} B - b \cdot \mathbf{i}_{\_}\mathbf{d}B \in \Omega^1(\mathcal{D})$$

and

$$b \diamond_2 B = -\mathrm{Tr}\big(\mathrm{ad}_B^* b\big) = -\mathrm{ad}_{B_{ij}}^* b^{ij} \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*).$$

The Euler–Poincaré equations (1.2) are in this case

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \mathbf{u}} = -\pounds_{\mathbf{u}}\dfrac{\delta l}{\delta \mathbf{u}} - (\mathrm{div}\,\mathbf{u})\dfrac{\delta l}{\delta \mathbf{u}} - \dfrac{\delta l}{\delta \nu} \cdot \mathbf{d}\nu + \dfrac{\delta l}{\delta a} \diamond a + \left(\mathrm{div}\dfrac{\delta l}{\delta B}\right) \cdot \mathbf{i}_{\_} B - \dfrac{\delta l}{\delta B} \cdot \mathbf{i}_{\_}\mathbf{d}B, \\ \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \nu} = -\mathrm{ad}_{\nu}^*\dfrac{\delta l}{\delta \nu} + \dfrac{\delta l}{\delta B} \diamond_2 B - \mathrm{Tr}\left(\mathrm{ad}_B^*\dfrac{\delta l}{\delta B}\right), \end{cases} \tag{3.15}$$

and the advection equations are

$$\begin{cases} \dot{a} + a\mathbf{u} = 0, \\ \dot{B} + \pounds_{\mathbf{u}} B - \mathrm{ad}_{\nu} B = 0. \end{cases}$$

We now show that when B is the curvature of the connection γ, that is, $B = \mathbf{d}^{\gamma}\gamma$, then the affine Euler–Poincaré equations (3.13) imply the standard Euler–Poincaré equations (3.15). Define the map

$$\Phi : \Omega^1(\mathcal{D}, \mathfrak{o}) \to \Omega^2(\mathcal{D}, \mathfrak{o}) \quad \text{by} \quad \Phi(\gamma) = \mathbf{d}^{\gamma}\gamma$$

and suppose that the Lagrangians $l^1(\mathbf{u}, \nu, a, \gamma)$ and $l^2(\mathbf{u}, \nu, a, B)$ are related by

$$l^2\big(\mathbf{u}, \nu, a, \Phi(\gamma)\big) = l^1(\mathbf{u}, \nu, a, \gamma).$$

We have $T_{\gamma}\Phi(\alpha) = \mathbf{d}\alpha + [\gamma, \alpha] + [\alpha, \gamma] = \mathbf{d}^{\gamma}\alpha$ and $\Phi(\mathrm{Ad}_{\chi^{-1}}\eta^*\gamma + \chi^{-1}T\chi) = \mathrm{Ad}_{\chi^{-1}}\eta^*\Phi(\gamma)$. Therefore, by taking the derivative, the infinitesimal action verifies

$$B\mathbf{u} + B\nu = \mathbf{d}^{\gamma}(\gamma\mathbf{u} + \gamma\nu + \mathbf{d}\nu),$$

or, explicitly,

$$\pounds_{\mathbf{u}} B - \mathrm{ad}_{\nu} B = \mathbf{d}^{\gamma}(\pounds_{\mathbf{u}}\gamma - \mathrm{ad}_{\nu}\gamma + \mathbf{d}\nu). \tag{3.16}$$

This proves that the advection equations for γ imply the one for B. Using the preceding formula and the definition of the diamond operation we find that

$$\big(\mathrm{div}^{\gamma} b\big) \diamond_1 \gamma = -b \diamond_1 B \quad \text{and} \quad \big(\mathrm{div}^{\gamma} b\big) \diamond_2 \gamma + \mathrm{div}\big(\mathrm{div}^{\gamma} b\big) = -b \diamond_2 B,$$

or, explicitly,

$$\mathrm{div}^{\gamma}\big(\big(\mathrm{div}^{\gamma} b\big)\gamma\big) - \mathrm{div}^{\gamma} b \cdot \mathbf{i}_{\_} B = -\mathrm{div}\, b \cdot \mathbf{i}_{\_} B + b \cdot \mathbf{i}_{\_}\mathbf{d}B \quad \text{and} \quad \mathrm{div}^{\gamma}\big(\mathrm{div}^{\gamma} b\big) = \mathrm{Tr}\big(\mathrm{ad}^*_B b\big).$$

Using the equality

$$-\mathrm{div}^{\gamma}\left(\frac{\delta l^2}{\delta B}\right) \circ \Phi = \frac{\delta l^1}{\delta \gamma},$$

we obtain

$$\frac{\delta l^1}{\delta \gamma} \diamond_1 \gamma = \frac{\delta l^2}{\delta B} \diamond_1 B \quad \text{and} \quad \frac{\delta l^1}{\delta \gamma} \diamond_2 \gamma + \mathrm{div}\left(\frac{\delta l^1}{\delta \gamma}\right) = \frac{\delta l^2}{\delta B} \diamond_2 B. \tag{3.17}$$

This proves that the affine Euler–Poincaré equations (3.13) imply the standard Euler–Poincaré equations (3.15).

4. Affine and magnetic Lie–Poisson reduction

The goal of this section is to carry out a generalization of the standard process of Lie–Poisson reduction for Lie groups, which is motivated by the example of complex fluids. The only modification lies in the fact that the Lie group G acts on its cotangent bundle by a cotangent lift *plus an affine term*. The principal result of this section states that, under some conditions, reducing a canonical symplectic form relative to a cotangent lift with an affine term is equivalent to reducing a magnetic symplectic form relative to the right cotangent lift. At the reduced level, we obtain affine Lie–Poisson brackets and affine coadjoint orbits, whose affine terms depend on the affine term in the action.

Consider the cotangent lift $R_g^{T^*}$ of the right translation R_g on a Lie group G. Recall that $R_g^{T^*}$ is the right action of G on T^*G given by

$$R_g^{T^*}(\alpha_f) = T^*R_{g^{-1}}(\alpha_f).$$

Consider the map $\Psi_g : T^*G \to T^*G$ defined by

$$\Psi_g(\alpha_f) := R_g^{T^*}(\alpha_f) + C_g(f), \tag{4.1}$$

where $C : G \times G \to T^*G$ is a smooth map such that $C_g(f) \in T^*_{fg}G$, for all $f, g \in G$. The map Ψ_g is seen here as a modification of the cotangent lift by an affine term C. The following lemma gives the conditions guaranteeing that the map Ψ_g is a right action.

Lemma 4.1. *Consider the map Ψ_g defined in* (4.1)*. The following are equivalent.*

(i) *Ψ_g is a right action.*
(ii) *For all $f, g, h \in G$, the affine term C verifies the property*

$$C_{gh}(f) = C_h(fg) + R_h^{T^*}\big(C_g(f)\big). \tag{4.2}$$

(iii) *There exists a one-form $\alpha \in \Omega^1(G)$ such that $C_g(f) = \alpha(fg) - R_g^{T^*}(\alpha(f))$.*

Proof. The equivalence between (i) and (ii) is a direct computation. Suppose that (ii) holds. By setting $f = e$ in formula (4.2), we obtain that $C_{gh}(e) = C_h(g) + R_h^{T^*}(C_g(e))$. Therefore we can define the one-form α by $\alpha(g) := C_g(e)$, and we have $C_h(g) = \alpha(gh) - R_h^{T^*}(\alpha(g))$. Conversely, suppose that (iii) holds. Then a direct computation shows that (4.2) holds. □

We denote by $\mathcal{C}(G)$ the space of all maps $C : G \times G \to T^*G$, $(g, f) \mapsto C_g(f) \in T^*_{fg}G$ verifying the property (4.2). Note that given an affine term $C \in \mathcal{C}(G)$, the one-form α in item (iii), is only determined up to a right-invariant one-form. Denoting by $\Omega^1_R(G)$ the space of all right-invariant one-forms on G, we have an isomorphism between $\mathcal{C}(G)$ and $\Omega^1(G)/\Omega^1_R(G)$. This space is clearly isomorphic to the space $\Omega^1_0(G)$ of all one-forms α on G such that $\alpha(e) = 0$. From now on, when we say that *the* one-form α is associated to C, we shall always assume that $\alpha(e) = 0$ which then guarantees the uniqueness of this one-form.

In order to carry out the symplectic reduction associated to an affine action Ψ_g of the form (4.1), we make two crucial observations (see Theorem 4.6).

- Let $\alpha \in \Omega^1_0(G)$ be the one-form associated to Ψ_g and consider the associated fiber translation t_α on T^*G defined by

$$t_\alpha(\beta_f) := \beta_f - \alpha(f).$$

Then t_α, viewed as a map from the canonical cotangent bundle $(T^*G, \Omega_{\text{can}})$ to the magnetic cotangent bundle $(T^*G, \Omega_{\text{can}} - \pi_G^* \mathbf{d}\alpha)$, is a *symplectic map*. Moreover, t_α is equivariant with respect to the action Ψ_g on $(T^*G, \Omega_{\text{can}})$ and the cotangent lift $R_g^{T^*}$ on $(T^*G, \Omega - \pi_G^* \mathbf{d}\alpha)$.

- Suppose that $\mathbf{d}\alpha$ is G-invariant. Then the action Ψ_g is *symplectic* relative to the canonical symplectic form Ω_{can}.

From these observations we conclude that, under some conditions to be specified later on, reducing the canonical cotangent bundle $(T^*G, \Omega_{\text{can}})$ relative to an affine action of the form (4.1) is equivalent to reducing the magnetic cotangent bundle $(T^*G, \Omega_{\text{can}} - \pi_G^* \mathbf{d}\alpha)$ relative to the cotangent lift of right translations. It is therefore useful to recall below some facts about the reduction of magnetic cotangent bundles.

Some facts about magnetic cotangent bundle reduction. We first recall the following result (Theorem 7.1.1 in [22]) about the existence of momentum maps associated to cotangent bundles with magnetic terms.

Theorem 4.2. *Let $\mathcal{B}$ be a closed two-form on a connected configuration manifold Q. Let $\Phi : G \times Q \to Q$ be a free and proper right action which leaves the form $\mathcal{B}$ invariant. Consider the cotangent lift $\Phi_g^{T^*}$ of the G-action Φ_g to the symplectic manifold $(T^*Q, \Omega_{\text{can}} - \pi_Q^* \mathcal{B})$, where Ω_{can} is the canonical symplectic form and $\pi_Q : T^*Q \to Q$ is the cotangent bundle projection. Suppose that there is a smooth map $\phi : Q \to \mathfrak{g}^*$ that satisfies*

$$\mathbf{i}_{\xi_Q} \mathcal{B} = \mathbf{d}\langle \phi, \xi \rangle$$

for all $\xi \in \mathfrak{g}$. Then the following hold:

(i) *The map $\mathbf{J} = \mathbf{J}_{\text{can}} - \phi \circ \pi_Q$, where $\mathbf{J}_{\text{can}}$ is the standard momentum map for the G-action relative to the canonical symplectic form, is a momentum map for the cotangent lifted action of G on T^*Q with symplectic form $\Omega_{\text{can}} - \pi_Q^* \mathcal{B}$.*

(ii) *The momentum map is, in general, not equivariant. Its nonequivariance $\mathfrak{g}^*$-valued group one-cocycle $\sigma : G \to \mathfrak{g}^*$ is given by*

$$\sigma(g) = -\phi\big(\Phi_g(q)\big) + \mathrm{Ad}_g^*\big(\phi(q)\big), \tag{4.3}$$

with the right-hand side independent of $q \in Q$. The cocycle identity is

$$\sigma(gh) = \mathrm{Ad}_h^*\big(\sigma(g)\big) + \sigma(h).$$

Note that the G-invariance of $\mathcal{B}$ ensures that the action Φ^{T^*} is symplectic relative to the symplectic form $\Omega_{\text{can}} - \pi_Q^* \mathcal{B}$. The momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is equivariant relative to the right affine action of G on $\mathfrak{g}^*$ given by

$$\theta_g(\lambda) := \mathrm{Ad}_g^* \lambda + \sigma(g).$$

We turn now to the particular case when the configuration manifold is a Lie group G and the action is the cotangent lift of right translation, that is, $\Phi_g^{T^*} = R_g^{T^*}$. We first recall from [22, Theorem 7.2.1] the magnetic Lie–Poisson reduction theorem.

Theorem 4.3. *Consider a closed G-invariant two-form $\mathcal{B}$ on G. The Poisson reduced space for the right cotangent lifted action of G on $(T^*G, \Omega_{\text{can}} - \pi_G^* \mathcal{B})$ is $\mathfrak{g}^*$ with Poisson bracket given by*

$$\{f,g\}_{\mathcal{B}}(\mu)=\left\langle \mu,\left[\frac{\delta f}{\delta \mu},\frac{\delta g}{\delta \mu}\right]\right\rangle+\mathcal{B}(e)\left(\frac{\delta f}{\delta \mu},\frac{\delta g}{\delta \mu}\right)$$

for $f,g\in\mathcal{F}(\mathfrak{g}^*)$.

Note that for a Hamiltonian $h:\mathfrak{g}^*\to\mathbb{R}$, the corresponding Hamiltonian vector field is given by

$$X_h(\mu)=-\operatorname{ad}^*_{\frac{\delta h}{\delta \mu}}\mu-\mathcal{B}(e)\left(\frac{\delta h}{\delta \mu},\cdot\right).$$

When the magnetic term $\mathcal{B}$ is absent, we recover the standard Lie–Poisson bracket on $\mathfrak{g}^*$. In this particular case, the symplectic reduced spaces $\mathbf{J}_R^{-1}(\mu)/G_\mu$ are symplectically diffeomorphic to the coadjoint orbits $\mathcal{O}_\mu=\{\operatorname{Ad}^*_g(\mu)\mid g\in G\}$, where $\mathbf{J}_R$ is the standard right momentum map for the G-action relative to the canonical symplectic form and G_μ is the coadjoint isotropy group of $\mu\in\mathfrak{g}^*$. The symplectic diffeomorphism is induced by the map

$$\varphi:\mathbf{J}_R^{-1}(\mu)\to\mathcal{O}_\mu,\quad \varphi(\alpha_g)=T^*R_g(\alpha_g)=\operatorname{Ad}^*_{g^{-1}}\mu.$$

When a magnetic term $\mathcal{B}$ is present, the existence of a momentum map is not guaranteed, and we shall use the result of Theorem 4.2. Recall that the infinitesimal generator of ξ for the right action is the left-invariant extension of ξ; that is, $\xi_G=\xi^L$. Suppose that there is a smooth map $\phi:G\to\mathfrak{g}^*$ that satisfies

$$\mathbf{i}_{\xi^L}\mathcal{B}=\mathbf{d}\langle\phi,\xi\rangle$$

for all $\xi\in\mathfrak{g}$. Since ϕ is determined by this equation only up to a constant, we can always impose the condition $\phi(e)=0$. From Theorem 4.2, the map $\mathbf{J}=\mathbf{J}_R-\phi\circ\pi_G$ is a momentum map for the cotangent lifted action of G on T^*G with symplectic form $\Omega_{\text{can}}-\pi_G^*\mathcal{B}$. In this case, using the relation (4.3), the nonequivariance cocycle is simply given by $\sigma=-\phi$. We denote by G_μ^σ the isotropy group of μ relative to the affine action $\theta_g(\mu)=\operatorname{Ad}^*_g(\mu)+\sigma(g)$. The following result (see [22, Theorem 7.2.2]) shows that the reduction of a magnetic cotangent bundle of a Lie group at a given point $\mu\in\mathfrak{g}^*$ is symplectically diffeomorphic with the affine coadjoint orbit $\mathcal{O}_\mu^\sigma$ passing through μ.

Theorem 4.4. *Consider a closed G-invariant two-form $\mathcal{B}$ on G, suppose that there is a smooth map $\phi:G\to\mathfrak{g}^*$ that satisfies*

$$\mathbf{i}_{\xi^L}\mathcal{B}=\mathbf{d}\langle\phi,\xi\rangle$$

for all $\xi\in\mathfrak{g}$, and consider the momentum map $\mathbf{J}=\mathbf{J}_{\text{can}}-\phi\circ\pi_Q$. Then for each $\mu\in\mathfrak{g}^*$, the symplectic reduced space $\mathbf{J}^{-1}(\mu)/G_\mu^\sigma$ is symplectically diffeomorphic to

$$\mathcal{O}_\mu^\sigma=\left\{\theta_g(\mu)=\operatorname{Ad}^*_g\mu+\sigma(g)\;\middle|\; g\in G\right\},$$

the affine orbit through μ. The tangent space at $\lambda=\theta_g^\sigma(\mu)\in\mathcal{O}_\mu^\sigma$ to $\mathcal{O}_\mu^\sigma$ is given by

$$T_\lambda\mathcal{O}_\mu^\sigma=\left\{\operatorname{ad}^*_\xi\lambda-\Sigma(\xi,\cdot)\;\middle|\;\xi\in\mathfrak{g}\right\},$$

where $\Sigma(\xi,\cdot):=-T_e\sigma(\xi)=-\mathcal{B}(e)(\xi,\cdot)$. The symplectic structure on $\mathcal{O}_\mu^\sigma$ has the expression

$$\omega^+_{\mathcal{B}}(\lambda)\left(\operatorname{ad}^*_\xi\lambda-\Sigma(\xi,\cdot),\operatorname{ad}^*_\eta\lambda-\Sigma(\eta,\cdot)\right)=\langle\lambda,[\xi,\eta]\rangle-\Sigma(\xi,\eta),$$

which we call the magnetic orbit symplectic form.

The symplectic diffeomorphism between the symplectic reduced spaces and the affine coadjoint orbits is constructed as follows. Consider the smooth map $\varphi : \mathbf{J}^{-1}(\mu) \to \mathcal{O}_\mu^\sigma$ defined for $\alpha_g \in \mathbf{J}^{-1}(\mu)$ by

$$\varphi(\alpha_g) := \theta_{g^{-1}}(\mu) = \mathrm{Ad}^*_{g^{-1}}\,\mu + \sigma\big(g^{-1}\big).$$

Then φ is G_μ^σ-invariant and induces a symplectic diffeomorphism

$$\overline{\varphi} : \big(\mathbf{J}^{-1}(\mu)/G_\mu^\sigma, \Omega_\mu\big) \to \big(\mathcal{O}_\mu^\sigma, \omega_{\mathcal{B}}^+\big).$$

Note that we have $\varphi(\alpha_g) = T^*R_g(\alpha_g)$. Indeed, since $\alpha_g \in \mathbf{J}^{-1}(\mu)$, we have $\mu = T^*L_g(\alpha_g) - \phi(g)$, therefore we obtain

$$\begin{aligned}\varphi(\alpha_g) &= \mathrm{Ad}^*_{g^{-1}}\,\mu + \sigma\big(g^{-1}\big)\\ &= T^*R_g(\alpha_g) + \mathrm{Ad}^*_{g^{-1}}\big(\sigma(g)\big) + \sigma\big(g^{-1}\big)\\ &= T^*R_g(\alpha_g).\end{aligned}$$

The general theory of symplectic reduction implies that the affine coadjoint orbits $(\mathcal{O}_\mu^\sigma, \omega_{\mathcal{B}}^+)$ are the symplectic leaves of the Poisson manifold $(\mathfrak{g}^*, \{\,,\,\}_{\mathcal{B}})$, where $\{\,,\,\}_{\mathcal{B}}$ denotes the Poisson bracket in Theorem 4.3.

The following proposition shows that when the magnetic term $\mathcal{B}$ is an exact two-form, then the magnetic cotangent bundle is symplectomorphic to the canonical symplectic cotangent bundle. Through this symplectomorphism, the cotangent lift is transformed into an affine action. In the particular case of a Lie group G acting on its cotangent bundle by right cotangent lift, this affine action is of the form (4.1).

Proposition 4.5. *Assume that all the hypotheses of Theorem* 4.2 *are satisfied and suppose that* $\mathcal{B} = \mathbf{d}\alpha$. *Let* $t_\alpha : (T^*Q, \Omega_{\mathrm{can}}) \to (T^*Q, \Omega_{\mathrm{can}} - \pi_Q^*\mathcal{B})$ *be the fiber translation defined by*

$$t_\alpha(\beta_q) := \beta_q - \alpha(q).$$

Then the following hold:

(i) t_α *is a symplectic fiber translation.*
(ii) *The symplectic action* Ψ *on* $(T^*Q, \Omega_{\mathrm{can}})$ *induced by* Φ^{T^*} *through the map* t_α, *that is,* $\Psi_g := t_\alpha^{-1} \circ \Phi \circ t_\alpha$ *for any* $g \in G$, *is the affine action given by*

$$\Psi_g(\beta_q) = \Phi_g^{T^*}(\beta_q) + C_g(q), \quad \text{where} \quad C_g(q) := \alpha\big(\Phi_g(q)\big) - \Phi^{T^*}\big(\alpha(q)\big).$$

(iii) *A momentum map relative to the* G*-action* Ψ *on* $(T^*Q, \Omega_{\mathrm{can}})$ *is given by*

$$\mathbf{J}_\alpha = \mathbf{J} \circ t_\alpha = \mathbf{J}_{\mathrm{can}} \circ t_\alpha - \phi \circ \pi_Q;$$

its nonequivariance cocycle equals the nonequivariance cocycle of $\mathbf{J}$.

Proof. (i) See Proposition 6.6.2 in [23].

(ii) We have

$$\begin{aligned}\Psi_g(\beta_q) &= t_\alpha^{-1}\big(\Phi_g^{T^*}\big(t_\alpha(\beta_q)\big)\big) = t_\alpha^{-1}\big(\Phi_g^{T^*}\big(\beta_q - \alpha(q)\big)\big)\\ &= \Phi_g^{T^*}\big(\beta_q - \alpha(q)\big) + \alpha\big(\Phi_g(q)\big)\\ &= \Phi_g^{T^*}(\beta_q) + \alpha\big(\Phi_g(q)\big) - \Phi_g^{T^*}\big(\alpha(q)\big)\\ &= \Phi_g^{T^*}(\beta_q) + C_g(q).\end{aligned}$$

(iii) This is a consequence of the fact that $\mathbf{J}_\alpha = \mathbf{J}\circ t_\alpha$. The nonequivariance one-cocycle of $\mathbf{J}_\alpha$ is

$$\begin{aligned}\sigma_\alpha(g) &= \mathbf{J}_\alpha\big(\Psi_g(\beta_q)\big) - \mathrm{Ad}_g^*\big(\mathbf{J}_\alpha(\beta_q)\big)\\ &= \mathbf{J}\big(\Phi_g^{T^*}\big(t_\alpha(\beta_q)\big)\big) - \mathrm{Ad}_g^*\big(\mathbf{J}\big(t_\alpha(\beta_q)\big)\big)\\ &= \sigma(g). \qquad \square\end{aligned}$$

Note that $\mathbf{d}\alpha$ is assumed to be G-invariant. When the one-form α is also G-invariant, then the affine term C vanishes. Therefore, the most interesting case happens when α is not G-invariant but $\mathbf{d}\alpha$ is.

Affine Lie–Poisson reduction. We apply now the previous results concerning the reduction with magnetic terms to our initial problem, that is, the Lie–Poisson reduction of the canonical cotangent bundle $(T^*G, \Omega_{\mathrm{can}})$ with respect to an affine action of the form (4.1). We obtain below, as an easy consequence of the previous theorems, the main result of this section.

Theorem 4.6. *Consider the symplectic manifold $(T^*G, \Omega_{\mathrm{can}})$, and the affine action*

$$\Psi_g(\beta_f) := R_g^{T^*}(\beta_f) + C_g(f),$$

where $C \in \mathcal{C}(G)$. Let $\alpha \in \Omega_0^1(G)$ be the one-form associated to Ψ_g. Then the following hold:

(i) *The fiber translation $t_\alpha : (T^*G, \Omega_{\mathrm{can}}) \to (T^*G, \Omega_{\mathrm{can}} - \pi_G^*\mathbf{d}\alpha)$ is a symplectic map. The action induced by Ψ_g on $(T^*G, \Omega_{\mathrm{can}} - \pi_G^*\mathbf{d}\alpha)$ through t_α is simply the cotangent lift $R_g^{T^*}$.*
(ii) *Suppose that $\mathbf{d}\alpha$ is G-invariant. Then the action Ψ_g is symplectic relative to the canonical symplectic form Ω_{can}.*
(iii) *Suppose that there is a smooth map $\phi : G \to \mathfrak{g}^*$ that satisfies*

$$\mathbf{i}_{\xi_L}\mathbf{d}\alpha = \mathbf{d}\langle\phi, \xi\rangle$$

for all $\xi \in \mathfrak{g}$. Then the map $\mathbf{J}_\alpha = \mathbf{J}_R \circ t_\alpha - \phi \circ \pi_G$ is a momentum map for the action Ψ_g relative to the canonical symplectic form. We can always choose ϕ such that $\phi(e) = 0$. In this case, the nonequivariance one-cocycle of $\mathbf{J}_\alpha$ is $\sigma = -\phi$.
(iv) *The symplectic reduced space $(\mathbf{J}_\alpha^{-1}(\mu)/G_\mu^\sigma, \Omega_\mu)$ is symplectically diffeomorphic to the affine coadjoint orbit $(\mathcal{O}_\mu^\sigma, \omega_{\mathcal{B}}^+)$, the symplectic diffeomorphism being induced by the G_μ^σ-invariant smooth map*

$$\psi : \mathbf{J}_\alpha^{-1}(\mu) \to \mathcal{O}_\mu^\sigma, \quad \psi(\alpha_g) := \Psi_{g^{-1}}(\alpha_g).$$

Proof. (i) That t_α is a symplectic map follows from item (i) in Proposition 4.5. From item (ii) in Proposition 4.5, we know that the action induced on $(T^*G, \Omega_{\mathrm{can}})$ by the right cotangent lift on $(T^*G, \Omega_{\mathrm{can}} - \pi_Q^*\mathbf{d}\alpha)$ through the map t_α is the affine action whose affine term is given by α. This

is precisely the action Ψ_g. Thus we conclude that Ψ_g induces the right cotangent lifted action on $(T^*G, \Omega_{\text{can}} - \pi_Q^* \mathbf{d}\alpha)$.

(ii) When $\mathbf{d}\alpha$ is G-invariant, we know that the right cotangent lift is symplectic relative to $\Omega - \pi_Q^* \mathbf{d}\alpha$. Since t_α is symplectic, we conclude the result.

(iii) This follows from items (i) and (ii) in Theorem 4.2. Since ϕ can be chosen modulo a constant term, we can impose the condition $\phi(e) = 0$. From the relation (4.3) we obtain the equality $\phi = -\sigma$.

(iv) The symplectic diffeomorphism t_α induces a symplectic diffeomorphism between the reduced spaces. Therefore, by Theorem 4.4, we obtain that $(\mathbf{J}_\alpha^{-1}(\mu)/G_\mu^\sigma, \Omega_\mu)$ is symplectically diffeomorphic to the affine coadjoint orbit $(\mathcal{O}_\mu^\sigma, \omega_\mu^\sigma)$, the symplectic diffeomorphism being induced by the map $\psi := \varphi \circ t_\alpha : \mathbf{J}_\alpha^{-1}(\mu) \to \mathcal{O}_\mu^\sigma$. We have

$$\begin{aligned}\psi(\beta_g) &= \varphi\big(t_\alpha(\beta_g)\big) = T^*R_g\big(\beta_g - \alpha(g)\big) = R_{g^{-1}}^{T^*}(\beta_g) - R_{g^{-1}}^{T^*}\big(\alpha(g)\big)\\ &= R_{g^{-1}}^{T^*}(\beta_g) + C_{g^{-1}}(g) = \Psi_g(\beta_g),\end{aligned}$$

where in the fourth equality we used the relation $C_g(f) = \alpha(fg) - R_g^{T^*}(\alpha(f))$. □

The affine coadjoint orbits $(\mathcal{O}_\mu^\sigma, \omega_\sigma^+)$ are symplectic leaves in the affine Lie–Poisson space $(\mathfrak{g}^*, \{,\}_\sigma^+)$, where

$$\{f, g\}_\sigma^+(\mu) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\right\rangle - \Sigma\left(\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right).$$

Note that this bracket is the Lie–Poisson bracket on the Poisson submanifold $\mathfrak{g}^* \times \{1\} \subset \hat{\mathfrak{g}}^* := \mathfrak{g}^* \times \mathbb{R}$, where $\hat{\mathfrak{g}}$ is the one-dimensional central extension of $\mathfrak{g}$ defined by the cocycle $-\Sigma$.

Assume that $-\Sigma$ integrates to a group two-cocycle $B : G \times G \to \mathbb{R}$, that is,

$$-\Sigma(\xi, \eta) = \frac{d^2}{dt\, ds}\bigg|_{t=s=0} \big(B\big(g(t), h(s)\big) - B\big(h(s), g(t)\big)\big),$$

where $t \mapsto g(t)$ and $s \mapsto h(s)$ are smooth curves through $e \in G$ with tangent vectors $\xi = \frac{dg(t)}{dt}|_{t=0}$ and $\eta = \frac{dh(s)}{ds}|_{s=0}$. Let $\widehat{G}$ be the central extension of G defined by the two-cocycle B and recall that the Lie algebra of $\widehat{G}$ equals $\hat{\mathfrak{g}}$. Then the affine coadjoint orbit $\mathcal{O}_\mu^\sigma$ is obtained by usual Lie–Poisson reduction of $T^*\widehat{G}$ relative to the lift of right translation at $(\mu, 1)$. See [22, §6.2] for more details.

5. Affine Hamiltonian semidirect product theory

This is the Hamiltonian version of Section 2. More precisely, we carry out the Poisson and symplectic reductions of a canonical cotangent bundle $(T^*S, \Omega_{\text{can}})$, where $S = G \circledS V$ is the semidirect product of a Lie group G and a vector space V and where S acts on its cotangent bundle by cotangent lift *plus an affine term*. We will see that this process is a particular case of the theory developed in the previous section.

Consider the semidirect product Lie group $S := G \circledS V$ associated to a right representation $\rho : G \to \text{Aut}(V)$. The cotangent lift of the right translation is given by

$$\begin{aligned}R_{(g,v)}^{T^*}\big(\alpha_f, (u, a)\big) &:= T^*_{(f,u)(g,v)}R_{(g,v)^{-1}}\big(\alpha_f, (u, a)\big)\\ &= \big(T^*_{fg}R_{g^{-1}}(\alpha_f), v + \rho_g(u), \rho^*_{g^{-1}}(a)\big)\\ &= \big(R_g^{T^*}(\alpha_f), v + \rho_g(u), \rho^*_{g^{-1}}(a)\big) \in T^*_{(f,u)(g,v)}S.\end{aligned}$$

We modify this cotangent lifted action by an affine term of the form

$$C_{(g,v)}(f,u) := \big(0_{fg}, v + \rho_g(u), c(g)\big) \tag{5.1}$$

for a group one-cocycle $c \in \mathcal{F}(G, V^*)$, that is, verifying the property $c(fg) = \rho^*_{g^{-1}}(c(f)) + c(g)$, as in Section 2. The resulting affine right action on T^*S is therefore given by

$$\begin{aligned}\Psi_{(g,v)}\big(\alpha_f, (u,a)\big) &:= R^{T^*}_{(g,v)}\big(\alpha_f, (u,a)\big) + C_{(g,v)}(f,u)\\ &= \big(R^{T^*}_g(\alpha_f), v + \rho_g(u), \rho^*_{g^{-1}}(a) + c(g)\big).\end{aligned} \tag{5.2}$$

This action is clearly of the form (4.1). We now check that property (4.2) holds. This will prove that $\Psi_{(g,v)}$ is a right action. Indeed,

$$\begin{aligned}&C_{(h,w)}\big((f,u)(g,v)\big) + R^{T^*}_{(h,w)}\big(C_{(g,v)}(f,u)\big)\\ &\quad= C_{(h,w)}\big(fg, v + \rho_g(u)\big) + R^{T^*}_{(h,w)}\big(0_{fg}, v + \rho_g(u), c(g)\big)\\ &\quad= \big(0_{fgh}, w + \rho_h\big(v + \rho_g(u)\big), c(h)\big) + \big(R^{T^*}_h(0_{fg}), w + \rho_h\big(v + \rho_g(u)\big), \rho^*_{h^{-1}}\big(c(g)\big)\big)\\ &\quad= \big(0_{fgh}, w + \rho_h\big(v + \rho_g(u)\big), c(h) + \rho^*_{h^{-1}}\big(c(g)\big)\big)\\ &\quad= \big(0_{fgh}, w + \rho_h(v) + \rho_{gh}(u), c(gh)\big)\\ &\quad= C_{(gh, w+\rho_h(v))}(f,u) = C_{(g,v)(h,w)}(f,u).\end{aligned}$$

In the following lemmas, we compute the one-form $\alpha \in \Omega^1_0(S)$ associated to C and we show that it verifies the hypotheses of Theorem 4.6. Recall that α is defined by $\alpha(g,v) := C_{(g,v)}(e,0)$.

Lemma 5.1. *The one-form $\alpha \in \Omega^1_0(S)$ associated to the affine term* (5.1) *is given by*

$$\alpha(g,v)\big(\xi_g, (v,u)\big) = \big\langle c(g), u\big\rangle \tag{5.3}$$

for $(\xi_g, (v,u)) \in T_{(g,v)}S$. Moreover $\mathcal{B} := \mathbf{d}\alpha$ is S-invariant and its value at the identity is given by

$$\mathcal{B}(e,0)\big((\xi,u),(\eta,w)\big) = \big\langle \mathbf{d}c(\xi), w\big\rangle - \big\langle \mathbf{d}c(\eta), u\big\rangle. \tag{5.4}$$

Proof. For $(\xi_g, (v,u)) \in T_{(g,v)}S$ we have $\alpha(g,v) := C_{(g,v)}(e,0) = (0_g, v, c(g))$. Therefore we obtain the equality $\alpha(g,v)(\xi_g,(v,u)) = \langle c(g), u\rangle$. We now prove the right-invariance of $\mathbf{d}\alpha$. For $(\xi,u) \in \mathfrak{s}$ define the associated right-invariant vector field $(\xi,u)^R \in \mathfrak{X}(S)$ by

$$(\xi,u)^R(g,v) := TR_{(g,v)}(\xi,u) = \big(\xi^R(g), v, \rho_g(u)\big),$$

where $\xi^R(g) := T_eR_g(\xi)$. Note that we have $\alpha((\xi,u)^R)(g,v) = \langle c(g), \rho_g(u)\rangle = -\langle c(g^{-1}), u\rangle$. Therefore

$$\begin{aligned}&R^*_{(g,v)}\mathbf{d}\alpha(e,0)\big((\xi,u),(\eta,w)\big)\\ &\quad= \mathbf{d}\alpha(g,v)\big(TR_{(g,v)}(\xi,u), TR_{(g,v)}(\eta,w)\big)\\ &\quad= \mathbf{d}\alpha(g,v)\big((\xi,u)^R(g,v), (\eta,w)^R(g,v)\big)\\ &\quad= \mathbf{d}\big(\alpha\big((\eta,w)^R\big)\big)(g,v)\big((\eta,w)^R(g,v)\big) - \mathbf{d}\big(\alpha\big((\eta,w)^R\big)\big)(g,v)\big((\xi,u)^R(g,v)\big)\\ &\qquad- \alpha\big(\big[(\xi,u)^R, (\eta,w)^R\big]\big)(g,v)\end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{dt}\Big|_{t=0} \alpha\big((\eta, w)^R\big)\big(\exp(t\xi)g, v + \rho_g(ut)\big) - \frac{d}{dt}\Big|_{t=0} \alpha\big((\xi, u)^R\big)\big(\exp(t\eta)g, v + \rho_g(wt)\big) \\
&\quad + \alpha\big(\big([\xi, \eta], u\eta - w\xi\big)^R\big)(g, v) \\
&= -\frac{d}{dt}\Big|_{t=0} \big\langle c\big(g^{-1}\exp(t\xi)^{-1}\big), w\big\rangle + \frac{d}{dt}\Big|_{t=0} \big\langle c\big(g^{-1}\exp(t\eta)^{-1}\big), u\big\rangle \\
&\quad + \alpha(g, v)\big([\xi, \eta]^R(g), \rho_g(u\eta - w\xi)\big) \\
&= \big\langle c\big(g^{-1}\big)\xi + \mathbf{d}c(\xi), w\big\rangle - \big\langle c\big(g^{-1}\big)\eta + \mathbf{d}c(\eta), u\big\rangle - \big\langle c\big(g^{-1}\big), u\eta - w\xi\big\rangle \\
&= \big\langle \mathbf{d}c(\xi), w\big\rangle - \big\langle \mathbf{d}c(\eta), u\big\rangle = \mathbf{d}\alpha(e, 0)\big((\xi, u), (\eta, w)\big),
\end{aligned}
$$

where in the fourth equality we used the identity

$$
\big[(\xi, u)^R, (\eta, w)^R\big] = -\big[(\xi, u), (\eta, w)\big]^R = -\big([\xi, \eta], u\eta - w\xi\big)^R,
$$

and for the sixth equality the identity

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \big\langle c\big(g^{-1}\exp(t\xi)^{-1}\big), w\big\rangle &= \frac{d}{dt}\Big|_{t=0} \big\langle \rho^*_{\exp(t\xi)}\big(c\big(g^{-1}\big)\big) + c\big(\exp(t\xi)^{-1}\big), w\big\rangle \\
&= -\big\langle c\big(g^{-1}\big)\xi + \mathbf{d}c(\xi), w\big\rangle. \qquad \square
\end{aligned}
$$

This lemma shows that hypothesis (ii) of Theorem 4.6 is verified. This implies that the action $\Psi_{(g,v)}$ is symplectic with respect to the canonical symplectic form Ω_{can} on T^*S. We now check hypothesis (iii) of Theorem 4.6. This implies that the action $\Psi_{(g,v)}$ admits a momentum map relative to the canonical symplectic form.

Lemma 5.2. *The map $\phi : S \to \mathfrak{s}^*$ defined by*

$$
\phi(g, v) = \big(\mathbf{d}c^T(v) - v \diamond c(g), -c(g)\big),
$$

verifies the property

$$
\mathbf{i}_{(\xi,u)^L}\mathbf{d}\alpha = \mathbf{d}\big\langle \phi, (\xi, u)\big\rangle,
$$

where $(\xi, u)^L \in \mathfrak{X}(S)$ is the left-invariant vector field induced by $(\xi, u) \in \mathfrak{s}$.

Proof. We will use the following formulas:

$$
\begin{aligned}
\big\langle \phi(g, v), (\xi, u)\big\rangle &= -\big\langle c(g), u\big\rangle + \big\langle v, c(g)\xi + \mathbf{d}c(\xi)\big\rangle, \\
(\xi, u)^L(g, v) &= TL_{(g,v)}(\xi, u) = \big(TL_g(\xi), v, u + v\xi\big), \\
TR_{(g,v)^{-1}}\big((\xi, u)^L(g, v)\big) &= \big(\mathrm{Ad}_g\,\xi, \rho_{g^{-1}}(u + v\xi)\big).
\end{aligned}
$$

By the right-invariance of $\mathbf{d}\alpha$, we have

$$
\begin{aligned}
&\mathbf{i}_{(\xi,u)^L}\mathbf{d}\alpha(g,v)\big((\eta,w)^L(g,v)\big)\\
&\quad=\mathbf{d}\alpha(g,v)\big((\xi,u)^L(g,v),(\eta,w)^L(g,v)\big)\\
&\quad=\mathbf{d}\alpha(e,0)\big(\big(\mathrm{Ad}_g\,\xi,\rho_{g^{-1}}(u+v\xi)\big),\big(\mathrm{Ad}_g\,\eta,\rho_{g^{-1}}(w+v\eta)\big)\big)\\
&\quad=\big\langle\mathbf{d}c(\mathrm{Ad}_g\,\xi),\rho_{g^{-1}}(w+v\eta)\big\rangle-\big\langle\mathbf{d}c(\mathrm{Ad}_g\,\eta),\rho_{g^{-1}}(u+v\xi)\big\rangle\\
&\quad=\big\langle Tgc\big(TL_g(\xi)\big),w+v\eta\big\rangle-\big\langle Tgc\big(TL_g(\eta)\big),u+v\xi\big\rangle.
\end{aligned}
$$

For the last equality, we use that

$$
\begin{aligned}
\rho^*_{g^{-1}}\big(\mathbf{d}c(\mathrm{Ad}_g\,\xi)\big)&=\rho^*_{g^{-1}}\frac{d}{dt}\bigg|_{t=0}c\big(g\exp(t\xi)g^{-1}\big)=\frac{d}{dt}\bigg|_{t=0}\big(c\big(g\exp(t\xi)\big)-c(g)\big)\\
&=Tgc\big(TL_g(\xi)\big).
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
&\mathbf{d}\big\langle\phi,(\xi,u)\big\rangle(g,v)\big((\eta,w)^L(g,v)\big)\\
&\quad=\frac{d}{dt}\bigg|_{t=0}\big\langle\phi\big(g\exp(t\eta),wt+\rho_{\exp(t\eta)}(v)\big),(\xi,u)\big\rangle\\
&\quad=\frac{d}{dt}\bigg|_{t=0}-\big\langle c\big(g\exp(t\eta)\big),u\big\rangle+\big\langle wt+\rho_{\exp(t\eta)}(v),c\big(g\exp(t\eta)\big)\xi+\mathbf{d}c(\xi)\big\rangle\\
&\quad=-\big\langle Tgc\big(TL_g(\eta)\big),u\big\rangle+\big\langle w+v\eta,c(g)\xi+\mathbf{d}c(\xi)\big\rangle+\frac{d}{dt}\bigg|_{t=0}\big\langle v,c\big(g\exp(t\eta)\big)\xi\big\rangle\\
&\quad=-\big\langle Tgc\big(TL_g(\eta)\big),u\big\rangle+\big\langle w+v\eta,Tgc\big(TL_g(\xi)\big)\big\rangle-\big\langle v\xi,Tgc\big(TL_g(\eta)\big)\big\rangle\\
&\quad=\big\langle Tgc\big(TL_g(\xi)\big),w+v\eta\big\rangle-\big\langle Tgc\big(TL_g(\eta)\big),u+v\xi\big\rangle.
\end{aligned}
$$

Thus we obtain that

$$
\mathbf{i}_{(\xi,u)^L}\mathbf{d}\alpha=\mathbf{d}\big\langle\phi,(\xi,u)\big\rangle.\qquad\square
$$

The momentum map. By item (iii) of Theorem 4.6 and using $\alpha\in\Omega^1_0(S)$ given by (5.3), we obtain that a momentum map for the right action $\Psi_{(g,v)}(\alpha_f,(u,a))=(R^{T^*}_g(\alpha_f),v+\rho_g(u),\rho^*_{g^{-1}}(a)+c(g))$ is given by

$$
\begin{aligned}
\mathbf{J}_\alpha\big(\beta_f,(u,a)\big)&=\mathbf{J}_R\big(t_\alpha\big(\beta_f,(u,a)\big)\big)-\phi(f,u)\\
&=\mathbf{J}_R\big(\beta_f,\big(u,a-c(f)\big)\big)-\phi(f,u)\\
&=T^*L_{(f,u)}\big(\beta_f,\big(u,a-c(f)\big)\big)-\phi(f,u)\\
&=\big(T^*L_f(\beta_f)+u\diamond\big(a-c(f)\big),a-c(f)\big)-\big(\mathbf{d}c^T(u)-u\diamond c(f),-c(f)\big)\\
&=\big(T^*L_f(\beta_f)+u\diamond a-\mathbf{d}c^T(u),a\big),
\end{aligned}
\tag{5.5}
$$

with nonequivariance one-cocycle

$$
\sigma(f,u)=-\phi(f,u)=\big(u\diamond c(f)-\mathbf{d}c^T(u),c(f)\big)\in\mathfrak{s}^*.
\tag{5.6}
$$

Poisson bracket and Hamiltonian vector fields. Using Theorem 4.3 and the expression of $\mathcal{B}$ given in Lemma 5.1, we obtain that the reduced Poisson bracket on $\mathfrak{s}^*$ is given by

$$\begin{aligned}\{f,g\}_{\mathcal{B}}(\mu,a) &= \left\langle (\mu,a), \left[\left(\frac{\delta f}{\delta \mu}, \frac{\delta f}{\delta a}\right), \left(\frac{\delta g}{\delta \mu}, \frac{\delta g}{\delta a}\right)\right]\right\rangle + \mathcal{B}(e,0)\left(\left(\frac{\delta f}{\delta \mu}, \frac{\delta f}{\delta a}\right), \left(\frac{\delta g}{\delta \mu}, \frac{\delta g}{\delta a}\right)\right)\\ &= \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\right\rangle + \left\langle a, \frac{\delta f}{\delta a}\frac{\delta g}{\delta \mu} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mu}\right\rangle + \left\langle \mathbf{d}c\left(\frac{\delta f}{\delta \mu}\right), \frac{\delta g}{\delta a}\right\rangle - \left\langle \mathbf{d}c\left(\frac{\delta g}{\delta \mu}\right), \frac{\delta f}{\delta a}\right\rangle.\end{aligned}$$

Given a Hamiltonian function $h:\mathfrak{s}^*\to\mathbb{R}$, the corresponding Hamiltonian vector field with respect to the bracket $\{\,,\}_{\mathcal{B}}$, is given by

$$\begin{aligned}X_h(\mu,a) &= -\operatorname{ad}^*_{\left(\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a}\right)}(\mu,a) - \mathcal{B}(e,0)\left(\left(\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a}\right), \cdot\right)\\ &= \left(-\operatorname{ad}^*_{\frac{\delta h}{\delta \mu}}\mu - \frac{\delta h}{\delta a}\diamond a + \mathbf{d}c^T\left(\frac{\delta h}{\delta a}\right), -a\frac{\delta h}{\delta \mu} - \mathbf{d}c\left(\frac{\delta h}{\delta \mu}\right)\right).\end{aligned}$$

The symplectic reduced spaces. By item (iv) of Theorem 4.6, the reduced space

$$\left(\mathbf{J}_\alpha^{-1}(\mu,a)/S^\sigma_{(\mu,a)}, \Omega_{(\mu,a)}\right)$$

is symplectically diffeomorphic to the affine coadjoint orbit $(\mathcal{O}^\sigma_{(\mu,a)}, \omega^+_{\mathcal{B}})$. More precisely, we have

$$\begin{aligned}\mathcal{O}^\sigma_{(\mu,a)} &= \left\{\operatorname{Ad}^*_{(g,u)}(\mu,a) + \sigma(g,u) \;\middle|\; (g,u)\in S\right\}\\ &= \left\{\left(\operatorname{Ad}^*_g\mu + u\diamond\left(\rho^*_{g^{-1}}(a) + c(g)\right) - \mathbf{d}c^T(u), \rho^*_{g^{-1}}(a) + c(g)\right) \;\middle|\; (g,u)\in S\right\}. \qquad (5.7)\end{aligned}$$

Note also that the bilinear form Σ appearing in the formula of the affine orbit symplectic form (see Theorem 4.4), is given by

$$\begin{aligned}\Sigma\big((\xi,u),\cdot\big) &= -T_{(e,0)}\sigma(\xi,u) = -\frac{d}{dt}\Big|_{t=0}\left(tu\diamond c\big(\exp(t\xi)\big) - \mathbf{d}c^T(tu), c\big(\exp(t\xi)\big)\right)\\ &= \big(\mathbf{d}c^T(u), -\mathbf{d}c(\xi)\big),\end{aligned}$$

where $(\xi,u)\in\mathfrak{s}$. The tangent space to the affine coadjoint orbit $\mathcal{O}^\sigma_{(\mu,a)}$ at (λ,b) is equal to

$$\begin{aligned}T_{(\lambda,b)}\mathcal{O}^\sigma_{(\mu,a)} &= \left\{\operatorname{ad}^*_{(\xi,u)}(\lambda,b) - \Sigma\big((\xi,u),\cdot\big) \;\middle|\; (\xi,u)\in\mathfrak{s}\right\}\\ &= \left\{\left(\operatorname{ad}^*_\xi\lambda + u\diamond b - \mathbf{d}c^T(u), b\xi + \mathbf{d}c(\xi)\right) \;\middle|\; (\xi,u)\in\mathfrak{s}\right\}.\end{aligned}$$

The symplectic structure on $\mathcal{O}^\sigma_{(\mu,a)}$ is given by

$$\begin{aligned}&\omega^+_{\mathcal{B}}(\lambda,b)\big(\left(\operatorname{ad}^*_\xi\lambda + u\diamond b - \mathbf{d}c^T(u), b\xi + \mathbf{d}c(\xi)\right), \left(\operatorname{ad}^*_\eta\lambda + w\diamond b - \mathbf{d}c^T(w), b\eta + \mathbf{d}c(\eta)\right)\big)\\ &= \big\langle (\lambda,b), \big[(\xi,u),(\eta,w)\big]\big\rangle - \Sigma\big((\xi,u),(\eta,w)\big)\\ &= \big\langle \lambda, [\xi,\eta]\big\rangle + \langle b, u\eta - w\xi\rangle + \big\langle \mathbf{d}c(\eta), u\big\rangle - \big\langle \mathbf{d}c(\xi), w\big\rangle. \qquad (5.8)\end{aligned}$$

Recall that the affine coadjoint orbits are the symplectic leaves of the Poisson manifold $(\mathfrak{s}^*, \{\,,\}_{\mathcal{B}})$.

We now apply these results to the main result of this section, that is, the Hamiltonian counterpart of Theorem 2.1

Consider a Hamiltonian function $H : T^*G \times V^* \to \mathbb{R}$ right invariant under the G-action

$$(\alpha_h, a) \mapsto \big(R_g^{T^*}(\alpha_h), \theta_g(a)\big) := \big(R_g^{T^*}(\alpha_h), \rho^*_{g^{-1}}(a) + c(g)\big). \tag{5.9}$$

This G-action on $T^*G \times V^*$ is induced by the S-action (5.2) on T^*S given by

$$\begin{aligned}\Psi_{(g,v)}\big(\alpha_h, (u,a)\big) &= R^{T^*}_{(g,v)}\big(\alpha_h, (u,a)\big) + C_{(g,v)}(h,u)\\ &= \big(R_g^{T^*}(\alpha_h), v + \rho_g(u), \rho^*_{g^{-1}}(a) + c(g)\big).\end{aligned}$$

Note also that we can think of this Hamiltonian $H : T^*G \times V^* \to \mathbb{R}$ as being the Poisson reduction of an S-invariant Hamiltonian $\overline{H} : T^*S \to \mathbb{R}$ by the normal subgroup $\{e\} \times V$ since $(T^*S)/(\{e\} \times V) \cong T^*G \times V^*$.

In particular, the function $H_{a_0} := H|_{T^*G \times \{a_0\}} : T^*G \to \mathbb{R}$ is invariant under the induced action of the isotropy subgroup $G^c_{a_0}$ of a_0 relative to the affine action θ, for any $a_0 \in V^*$. Recall that $\theta_g(a) := \rho^*_{g^{-1}}(a) + c(g)$ for any $g \in G$ and $a \in V^*$. The following theorem is a generalization of Theorem 1.2 and is also a consequence of the reduction by stages method for nonequivariant momentum maps, together with the results obtained in Section 4 and at the beginning of the present section.

Theorem 5.3. *For $\alpha(t) \in T^*_{g(t)}G$ and $\mu(t) := T^*_e R_{g(t)}(\alpha(t)) \in \mathfrak{g}^*$, the following are equivalent*:

(i) *$\alpha(t)$ satisfies Hamilton's equations for H_{a_0} on T^*G.*
(ii) *The following affine Lie–Poisson equation holds on $\mathfrak{s}^*$:*

$$\frac{\partial}{\partial t}(\mu, a) = \left(-\operatorname{ad}^*_{\frac{\delta h}{\delta \mu}} \mu - \frac{\delta h}{\delta a} \diamond a + \mathbf{d}c^T\left(\frac{\delta h}{\delta a}\right), -a\frac{\delta h}{\delta \mu} - \mathbf{d}c\left(\frac{\delta h}{\delta \mu}\right)\right), \qquad a(0) = a_0,$$

where $\mathfrak{s}$ is the semidirect product Lie algebra $\mathfrak{s} = \mathfrak{g} \circledS V$. The associated Poisson bracket is the following affine Lie–Poisson bracket on the dual $\mathfrak{s}^*$:

$$\begin{aligned}\{f, g\}_{\mathcal{B}}(\mu, a) = &\left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\right\rangle + \left\langle a, \frac{\delta f}{\delta a}\frac{\delta g}{\delta \mu} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mu}\right\rangle\\ &+ \left\langle \mathbf{d}c\left(\frac{\delta f}{\delta \mu}\right), \frac{\delta g}{\delta a}\right\rangle - \left\langle \mathbf{d}c\left(\frac{\delta g}{\delta \mu}\right), \frac{\delta f}{\delta a}\right\rangle.\end{aligned}$$

As on the Lagrangian side, the evolution of the advected quantities is given by $a(t) = \theta_{g(t)^{-1}}(a_0)$.

Proof. Recall that the momentum map relative to the canonical symplectic form on T^*S and to the action

$$\Psi_{(g,v)}\big(\beta_f, (u,a)\big) = \big(R_g^{T^*}(\beta_f), v + \rho_g(u), \rho^*_{g^{-1}}(a) + c(g)\big)$$

is given by

$$\mathbf{J}_\alpha\big(\beta_f, (u,a)\big) = \big(T^*L_f(\beta_f) + u \diamond a - \mathbf{d}c^T(u), a\big).$$

The action $\Psi_{(g,v)}$ of S induces an action of V given by

$$\big(\beta_f, (u,a)\big) \mapsto \big(\beta_f, (v+u, a)\big).$$

Since V is a closed normal subgroup of S, this action admits a momentum map given by

$$\mathbf{J}_V\big(\beta_f,(u,a)\big)=a.$$

Since V is an Abelian group, the coadjoint isotropy group of $a\in V^*$ is $V_a=V$ and the first reduced space $(T^*S)_{a_a}=\mathbf{J}^{-1}(a)/V$ is symplectically diffeomorphic to the canonical symplectic manifold $(T^*G,\Omega_{\mathrm{can}})$. The action Ψ of S on T^*S restricts to an action Ψ^a of $G_a^c\circledS V$ on $\mathbf{J}_V^{-1}(a)$, where $G_a^c:=\{g\in G\mid \rho_{g^{-1}}^*(a)+c(g)=a\}$. Passing to the quotient spaces, this action induces an action of G_a^c on $(T^*S)_a$, which is readily seen to be the cotangent lifted action of G_a^c on T^*G. We denote by $\mathbf{J}_a:(T^*S)_a\to(\mathfrak{g}_a^c)^*$ the associated equivariant momentum map, where $\mathfrak{g}_a^c$ is the Lie algebra of G_a^c. Reducing $(T^*S)_a$ at the point $\mu_a:=\mu|\mathfrak{g}_a^c$, we get the second reduced space $((T^*S)_a)_{\mu_a}=\mathbf{J}_a^{-1}(\mu_a)/(G_a^c)_{\mu_a}$, with symplectic form denoted by $(\Omega_a)_{\mu_a}$.

By the Reduction by Stages Theorem for nonequivariant momentum maps, the second reduced space is symplectically diffeomorphic to the reduced space

$$\big(\mathbf{J}_\alpha^{-1}(\mu,a)/S^\sigma_{(\mu,a)},\Omega_{(\mu,a)}\big)$$

obtained by reducing T^*S by the whole group S at the point $(\mu,a)\in\mathfrak{s}^*$. Here $S^\sigma_{(\mu,a)}$ denotes the isotropy group of the affine coadjoint action with cocycle σ given in (5.6).

As shown at the beginning of this section, this reduced space is symplectically diffeomorphic to the affine coadjoint orbit

$$\big(\mathcal{O}^\sigma_{(\mu,a)},\omega^+_{\mathcal{B}}\big)$$

endowed with the affine orbit symplectic form described in (5.8). These affine coadjoint orbits are the symplectic leaves of the Poisson manifold $(\mathfrak{s}^*,\{\,,\,\}_{\mathcal{B}})$.

Note finally that we can consider the right-invariant Hamiltonian $H:T^*G\times V^*\to\mathbb{R}$, as coming from an S-invariant Hamiltonian $\overline{H}:T^*S\to\mathbb{R}$.

The theorem is then a consequence of all these observations. □

Reconstruction of dynamics. We give now some details concerning the passage from the reduced formulation (ii) to the canonical formulation (i). Let $(\mu(t),a(t))\in\mathfrak{g}^*\times V^*$ be the solution of the affine Lie–Poisson equations, with initial condition (μ_0,a_0). Then the curve $\alpha(t):=T^*R_{g(t)^{-1}}(\mu(t))$, where $g(t)\in G$ satisfies the linear ordinary differential equation with time-dependent coefficients

$$\dot g(t)=TR_{g(t)}\left(\frac{\delta h}{\delta\mu}\right),\qquad g(0)=e,$$

is the solution of Hamilton's equations associated to H_{a_0} on T^*G and with initial condition $\alpha(0)=\mu_0$. Note that for the curve $g(t)$ defined above, we have $a(t)=\theta_{g(t)^{-1}}(a_0)$.

The preceding theorem is compatible with Theorem 2.3. Indeed, we can start with a Lagrangian $L_{a_0}:TG\to\mathbb{R}$ as in Section 2, that is, we have a function $L:TG\times V^*\to\mathbb{R}$ which is right G-invariant under the affine action $(v_h,a)\mapsto(T_hR_g(v_h),\theta_g(a))=(T_hR_g(v_h),\rho_{g^{-1}}^*(a)+c(g))$, such that $L_{a_0}(v_g)=L(v_g,a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of $G_{a_0}^c$ on G. Suppose that the Legendre transformation $\mathbb{F}L_{a_0}$ is invertible and form the corresponding Hamiltonian $H_{a_0}=E_{a_0}\circ\mathbb{F}L_{a_0}^{-1}$, where E_{a_0} is the energy of L_{a_0}. Then the function $H:T^*G\times V^*\to\mathbb{R}$ so defined is S-invariant and one can apply this theorem. At the level of the reduced space, to a reduced Lagrangian $l:\mathfrak{g}\times V^*\to\mathbb{R}$ we associate the reduced Hamiltonian $h:\mathfrak{g}^*\circledS V^*\to\mathbb{R}$ given by

$$h(\mu,a):=\langle\mu,\xi\rangle-l(\xi,a),\quad \mu=\frac{\delta l}{\delta\xi}.$$

Since

$$\frac{\delta h}{\delta \mu} = \xi \quad \text{and} \quad \frac{\delta h}{\delta a} = -\frac{\delta l}{\delta a},$$

we see that the affine Lie–Poisson equations for h on $\mathfrak{s}^*$ are equivalent to the affine Euler–Poincaré equations (2.3) for l together with the affine advection equation

$$\dot{a} + a\xi + \mathbf{d}c(\xi) = 0.$$

6. Hamiltonian approach to continuum theories of perfect complex fluids

This section is the Hamiltonian version of Section 3. Recall that in Section 3 we have applied Theorem 2.1 to the case of complex fluids. Here we apply the Hamiltonian analogue of this theorem, namely Theorem 5.3.

Recall that for complex fluids we apply the abstract theory for $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$ and $V^* = V_1^* \oplus V_2^*$. The representations of G on V and V^* are of the form

$$\rho_{(\eta,\chi)}(v, w) = \big(v\eta, w(\eta, \chi)\big) \quad \text{and} \quad \rho^*_{(\eta,\chi)^{-1}}(a, \gamma) = \big(a\eta, \gamma(\eta, \chi)\big).$$

This implies that the infinitesimal actions of $\mathfrak{g}$ on V and V^* are of the form

$$(v, w)(\mathbf{u}, \nu) = (v\mathbf{u}, w\mathbf{u} + w\nu) \quad \text{and} \quad (a, \gamma)(\mathbf{u}, \nu) = (a\mathbf{u}, \gamma\mathbf{u} + \gamma\nu).$$

We therefore obtain the diamond operation

$$(v, w) \diamond (a, \gamma) = (v \diamond a + w \diamond_1 \gamma, w \diamond_2 \gamma).$$

Since the affine term has the particular form $c(\eta, \chi) = (0, C(\chi))$, we obtain the equalities

$$\mathbf{d}c(\mathbf{u}, \nu) = \big(0, \mathbf{d}C(\nu)\big) \quad \text{and} \quad \mathbf{d}c^T(v, w) = \big(0, \mathbf{d}C^T(w)\big).$$

We now compute some useful expressions in the particular case of complex fluids by using the general formulas of Section 5.

By Theorem 5.3, we obtain that the affine Lie–Poisson bracket for complex fluids is

$$\begin{aligned}
\{f, g\}(\mathbf{m}, \kappa, a, \gamma) = &\int_{\mathcal{D}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}}\right]\mu \\
&+ \int_{\mathcal{D}} \kappa \cdot \left(\mathrm{ad}_{\frac{\delta f}{\delta \kappa}} \frac{\delta g}{\delta \kappa} + \mathbf{d}\frac{\delta f}{\delta \kappa} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \kappa} \cdot \frac{\delta f}{\delta \mathbf{m}}\right)\mu \\
&+ \int_{\mathcal{D}} a \cdot \left(\frac{\delta f}{\delta a}\frac{\delta g}{\delta \mathbf{m}} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mathbf{m}}\right)\mu \\
&+ \int_{\mathcal{D}} \gamma \cdot \left(\frac{\delta f}{\delta \gamma}\frac{\delta g}{\delta \mathbf{m}} + \frac{\delta f}{\delta \gamma}\frac{\delta g}{\delta \kappa} - \frac{\delta g}{\delta \gamma}\frac{\delta f}{\delta \mathbf{m}} - \frac{\delta g}{\delta \gamma}\frac{\delta f}{\delta \kappa}\right)\mu \\
&+ \int_{\mathcal{D}} \left(\mathbf{d}C\left(\frac{\delta f}{\delta \kappa}\right) \cdot \frac{\delta g}{\delta \gamma} - \mathbf{d}C\left(\frac{\delta g}{\delta \kappa}\right) \cdot \frac{\delta f}{\delta \gamma}\right)\mu.
\end{aligned}$$

The first four terms give the Lie–Poisson bracket on the dual Lie algebra

$$\big([\mathfrak{X}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathfrak{o})]\circledS[V_1\oplus V_2]\big)^*\cong\Omega^1(\mathcal{D})\times\mathcal{F}(\mathcal{D},\mathfrak{o}^*)\times V_1^*\times V_2^*.$$

The last term is due to the presence of the affine term C in the representation. Since C depends only on the group $\mathcal{F}(\mathcal{D},\mathcal{O})$, this term does not involve the functional derivatives with respect to $\mathbf{m}$.

The symplectic leaves of this bracket are the affine coadjoint orbits in the dual Lie algebra $([\mathfrak{X}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathfrak{o})]\circledS[V_1\oplus V_2])^*$. The expression of the tangent spaces and of the affine orbit symplectic forms involves the bilinear form Σ which is defined in this case on $[\mathfrak{X}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathfrak{o})]\circledS[V_1\oplus V_2]$ by

$$\Sigma\big((\mathbf{u}_1,v_1,v_1,w_1),(\mathbf{u}_2,v_2,v_2,w_2)\big)=\mathbf{d}C(v_2)\cdot w_1-\mathbf{d}C(v_1)\cdot w_2.$$

For a Hamiltonian $h=h(\mathbf{m},\kappa,a,\gamma):\Omega^1(\mathcal{D})\times\mathcal{F}(\mathcal{D},\mathfrak{o}^*)\times V_1^*\times V_2^*\to\mathbb{R}$, the affine Lie–Poisson equations of Theorem 5.3 become

$$\begin{cases}\dfrac{\partial}{\partial t}\mathbf{m}=-\pounds_{\frac{\delta h}{\delta\mathbf{m}}}\mathbf{m}-\operatorname{div}\left(\dfrac{\delta h}{\delta\mathbf{m}}\right)\mathbf{m}-\kappa\cdot\mathbf{d}\dfrac{\delta h}{\delta\kappa}-\dfrac{\delta h}{\delta a}\diamond a-\dfrac{\delta h}{\delta\gamma}\diamond_1\gamma,\\[2ex]\dfrac{\partial}{\partial t}\kappa=-\operatorname{ad}^*_{\frac{\delta h}{\delta\kappa}}\kappa-\operatorname{div}\left(\dfrac{\delta h}{\delta\mathbf{m}}\kappa\right)-\dfrac{\delta h}{\delta\gamma}\diamond_2\gamma+\mathbf{d}C^T\left(\dfrac{\delta h}{\delta\gamma}\right),\\[2ex]\dfrac{\partial}{\partial t}a=-a\dfrac{\delta h}{\delta\mathbf{m}},\\[2ex]\dfrac{\partial}{\partial t}\gamma=-\gamma\dfrac{\delta h}{\delta\mathbf{m}}-\gamma\dfrac{\delta h}{\delta\kappa}-\mathbf{d}C\left(\dfrac{\delta h}{\delta\kappa}\right).\end{cases}\tag{6.1}$$

As explained in the previous section, when the reduced Hamiltonian h is defined by a Lagrangian l through the Legendre transformation

$$h(\mathbf{m},\kappa,a,\gamma)=\big\langle(\mathbf{m},\kappa),(\mathbf{u},v)\big\rangle-l(\mathbf{u},v,a,\gamma),\quad(\mathbf{m},\kappa)=\left(\frac{\delta l}{\delta\mathbf{u}},\frac{\delta l}{\delta v}\right),$$

we have

$$\frac{\delta h}{\delta\mathbf{m}}=\mathbf{u},\qquad\frac{\delta h}{\delta\kappa}=v,\qquad\frac{\delta h}{\delta a}=-\frac{\delta l}{\delta a},\qquad\frac{\delta h}{\delta\gamma}=-\frac{\delta l}{\delta\gamma}.$$

Using these equalities, we see directly that the affine Lie–Poisson equations (6.1) are equivalent to the affine Euler–Poincaré equations (3.4), together with the advection equations (3.5).

Using formula (5.5), the momentum map of the affine right action of the semidirect product $[\mathrm{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathcal{O})]\circledS[V_1\oplus V_2]$ on its cotangent bundle is computed to be

$$\begin{aligned}&\mathbf{J}_\alpha\big(\mathbf{m}_\eta,\kappa_\chi,(v,w),(a,\gamma)\big)\\&\quad=\big(T^*\eta\circ\mathbf{m}_\eta+T^*\chi\circ\kappa_\chi+v\diamond a+w\diamond_1\gamma,T^*L_\chi\circ\kappa_\chi+w\diamond_2\gamma-\mathbf{d}C^T(w),(a,\gamma)\big).\end{aligned}$$

In this formula we need to elaborate on the meaning of the expression $T^*\chi\circ\kappa_\chi\in\Omega^1(\mathcal{D})$. Thus, by definition, for any $u_x\in T_x\mathcal{D}$ we set $\langle(T^*\chi\circ\kappa_\chi)(x),u_x\rangle:=\langle T^*_x\chi(\kappa_\chi(x)),u_x\rangle=\langle\kappa_\chi(x),T_x\chi(u_x)\rangle$. In this last expression, recall that $\kappa_\chi:\mathcal{D}\to T^*\mathcal{O}$ covers $\chi:\mathcal{D}\to\mathcal{O}$.

By the general theory, the nonequivariance cocycle of $\mathbf{J}_\alpha$ is given by $\sigma=-\phi$, where ϕ is computed (using Lemma 5.2) to be

$$\phi : \big[\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})\big] \circledS [V_1 \oplus V_2] \to \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) \times V_1^* \times V_2^*,$$
$$\phi(\eta, \chi, v, w) = \big(\mathbf{d}c^T(v, w) - (v, w) \diamond c(\eta, \chi), -c(\eta, \chi)\big)$$
$$= \big(-w \diamond_1 C(\chi), \mathbf{d}C^T(w) - w \diamond_2 C(\chi), 0, -C(\chi)\big).$$

One can also compute the momentum map $\mathbf{J}_{(a_0,\gamma_0)}$ appearing in the proof of Theorem 5.3 at the second stage of reduction. It is associated to the cotangent lifted action of the isotropy group

$$\big(\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})\big)_{(a_0,\gamma_0)} = \big\{(\eta, \chi) \;\big|\; \big(a_0\eta, \gamma_0(\eta, \chi) + C(\chi)\big) = (a_0, \gamma_0)\big\}$$

on the canonical cotangent bundle $T^*(\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O}))$. It is given by

$$\mathbf{J}_{(a_0,\gamma_0)}(\mathbf{m}_\eta, \kappa_\chi) = (T^*\eta \circ \mathbf{m}_\eta + T^*\chi \circ \kappa_\chi, T^*L_\chi \circ \kappa_\chi). \tag{6.2}$$

Basic example. As in the example given in Section 3, we consider the particular case when $V_2^* = \Omega^1(\mathcal{D}, \mathfrak{o})$ and the affine representation is

$$(a, \gamma) \mapsto \big(a\eta, \mathrm{Ad}_{\chi^{-1}} \eta^*\gamma + \chi^{-1}T\chi\big). \tag{6.3}$$

Recall that in this particular case we have

$$\gamma\mathbf{u} = \pounds_{\mathbf{u}}\gamma = \mathbf{d}^\gamma\big(\gamma(\mathbf{u})\big) + \mathbf{i}_{\mathbf{u}}\mathbf{d}^\gamma\gamma,$$

$$\gamma v = -\mathrm{ad}_v\,\gamma, \quad \mathbf{d}C(v) = \mathbf{d}v, \quad \text{and} \quad \mathbf{d}C^T(w) = -\mathrm{div}(w).$$

The diamond operations are given by

$$w \diamond_1 \gamma = \big(\mathrm{div}^\gamma\, w\big) \cdot \gamma - w \cdot \mathbf{i}\_\,\mathbf{d}^\gamma\gamma \quad \text{and} \quad w \diamond_2 \gamma = -\mathrm{Tr}\big(\mathrm{ad}^*_\gamma\, w\big).$$

The affine Lie–Poisson equations (6.1) become

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{m} = -\pounds_{\frac{\delta h}{\delta \mathbf{m}}}\mathbf{m} - \mathrm{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\right)\mathbf{m} - \kappa \cdot \mathbf{d}\dfrac{\delta h}{\delta \kappa} - \dfrac{\delta h}{\delta a} \diamond a \\ \qquad\qquad - \left(\mathrm{div}^\gamma \dfrac{\delta h}{\delta \gamma}\right)\gamma + \dfrac{\delta h}{\delta \gamma} \cdot \mathbf{i}\_\,\mathbf{d}^\gamma\gamma, \\ \dfrac{\partial}{\partial t}\kappa = -\mathrm{ad}^*_{\frac{\delta h}{\delta \kappa}}\,\kappa - \mathrm{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\kappa\right) - \mathrm{div}^\gamma \dfrac{\delta h}{\delta \gamma}, \\ \dfrac{\partial}{\partial t}a = -a\dfrac{\delta h}{\delta \mathbf{m}}, \\ \dfrac{\partial}{\partial t}\gamma = -\mathbf{d}^\gamma\left(\gamma\left(\dfrac{\delta h}{\delta \mathbf{m}}\right)\right) - \mathbf{i}_{\frac{\delta h}{\delta \mathbf{m}}}\mathbf{d}^\gamma\gamma - \mathbf{d}^\gamma\dfrac{\delta h}{\delta \kappa}. \end{cases} \tag{6.4}$$

So we recover, by a reduction from a canonical situation, Eqs. (3.44) of [13], up to sign conventions, as well as their Hamiltonian structure. In matrix notation and with respect to local coordinates we have

$$\begin{bmatrix} \dot{m}_i \\ \dot{\kappa}_a \\ \dot{a} \\ \dot{\gamma}_i^a \end{bmatrix} = -\begin{bmatrix} m_k\partial_i + \partial_k m_i & \kappa_b\partial_i & (\Box \diamond a)_i & \partial_j\gamma_i^b - \gamma_{j,i}^b \\ \partial_k\kappa_a & \kappa_c C_{ba}^c & 0 & \delta_a^b\partial_j - C_{ca}^b\gamma_j^c \\ a\Box\partial_k & 0 & 0 & 0 \\ \gamma_k^a\partial_i + \gamma_{i,k}^a & \delta_b^a\partial_i + C_{cb}^a\gamma_i^c & 0 & 0 \end{bmatrix} \begin{bmatrix} (\delta h/\delta m)^k \\ (\delta h/\delta \kappa)^b \\ \delta h/\delta a \\ (\delta h/\delta \gamma)_b^j \end{bmatrix}. \tag{6.5}$$

This matrix appears in [16, formula (2.26a)] and in [13, formula (3.46)] as the common Hamiltonian structure for various hydrodynamical systems. For another derivation of this Hamiltonian structure based on reduction see [3].

The associated affine Lie–Poisson bracket is

$$\begin{aligned}\{f,g\}(\mathbf{m},\kappa,a,\gamma) = &\int_{\mathcal{D}} \mathbf{m}\cdot\left[\frac{\delta f}{\delta \mathbf{m}},\frac{\delta g}{\delta \mathbf{m}}\right]\mu\\ &+\int_{\mathcal{D}} \kappa\cdot\left(\mathrm{ad}_{\frac{\delta f}{\delta \kappa}}\frac{\delta g}{\delta \kappa}+\mathbf{d}\frac{\delta f}{\delta \kappa}\cdot\frac{\delta g}{\delta \mathbf{m}}-\mathbf{d}\frac{\delta g}{\delta \kappa}\cdot\frac{\delta f}{\delta \mathbf{m}}\right)\mu\\ &+\int_{\mathcal{D}} a\cdot\left(\frac{\delta f}{\delta a}\frac{\delta g}{\delta \mathbf{m}}-\frac{\delta g}{\delta a}\frac{\delta f}{\delta \mathbf{m}}\right)\mu\\ &+\int_{\mathcal{D}}\left[\left(\mathbf{d}^{\gamma}\frac{\delta f}{\delta \kappa}+\pounds_{\frac{\delta f}{\delta \mathbf{m}}}\gamma\right)\cdot\frac{\delta g}{\delta \gamma}-\left(\mathbf{d}^{\gamma}\frac{\delta g}{\delta \kappa}+\pounds_{\frac{\delta g}{\delta \mathbf{m}}}\gamma\right)\cdot\frac{\delta f}{\delta \gamma}\right]\mu. \end{aligned}\tag{6.6}$$

The momentum map is computed to be

$$\begin{aligned}&\mathbf{J}_{\alpha}\big(\mathbf{m}_{\eta},\kappa_{\chi},(v,w),(a,\gamma)\big)\\ &\quad=\big(T^{*}\eta\circ\mathbf{m}_{\eta}+T^{*}\chi\circ\kappa_{\chi}+v\diamond a+\big(\mathrm{div}^{\gamma}\,w\big)\cdot\gamma-w\cdot\mathbf{i}_{\_}\,\mathbf{d}^{\gamma}\gamma,T^{*}L_{\chi}\circ\kappa_{\chi}+\mathrm{div}^{\gamma}\,w,(a,\gamma)\big).\end{aligned}$$

The *B*-representation. As on the Lagrangian side, we consider the case where the Hamiltonian is given in terms of the curvature $B=\mathbf{d}^{\gamma}\gamma$. Recall that the *affine* representation of $\mathrm{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathcal{O})$ on a connection one-form on the trivial principal $\mathcal{O}$-bundle $\mathcal{O}\times\mathcal{D}\to\mathcal{D}$ induced by $\gamma\in\Omega^{1}(\mathcal{D},\mathfrak{o})$ is

$$\gamma(\eta,\chi):=\mathrm{Ad}_{\chi^{-1}}\,\eta^{*}\gamma+\chi^{-1}T\chi.$$

It induces the *linear* representation

$$B(\eta,\chi):=\mathrm{Ad}^{*}_{\chi^{-1}}\,\eta^{*}B$$

on the curvature $B=\mathbf{d}^{\gamma}\gamma$. The Lie–Poisson reduction for semidirect products (see Theorem 1.2) gives the equations

$$\begin{cases}\dfrac{\partial}{\partial t}\mathbf{m}=-\pounds_{\frac{\delta h}{\delta \mathbf{m}}}\mathbf{m}-\mathrm{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\right)\mathbf{m}-\kappa\cdot\mathbf{d}\dfrac{\delta h}{\delta \kappa}-\dfrac{\delta h}{\delta a}\diamond a-\mathrm{div}\,\dfrac{\delta h}{\delta B}\cdot\mathbf{i}_{\_}\,B+\dfrac{\delta h}{\delta B}\cdot\mathbf{i}_{\_}\,\mathbf{d}B,\\ \dfrac{\partial}{\partial t}\kappa=-\mathrm{ad}^{*}_{\frac{\delta h}{\delta \kappa}}\,\kappa-\mathrm{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\kappa\right)+\mathrm{Tr}\left(\mathrm{ad}^{*}_{B}\dfrac{\delta h}{\delta B}\right),\\ \dfrac{\partial}{\partial t}a=-a\dfrac{\delta h}{\delta \mathbf{m}},\\ \dfrac{\partial}{\partial t}B=-\pounds_{\frac{\delta h}{\delta \mathbf{m}}}B+\mathrm{ad}_{\frac{\delta h}{\delta \kappa}}\,B.\end{cases}\tag{6.7}$$

Using formulas (3.17), we can prove, as on the Lagrangian side, that the Lie–Poisson equations (6.7) are compatible with the affine Lie–Poisson equations (6.4). In matrix notation and with respect to local coordinates, the Lie–Poisson equations read

$$\begin{bmatrix} \dot{m}_i \\ \dot{\kappa}_a \\ \dot{a} \\ \dot{B}^a_{ij} \end{bmatrix} = -\begin{bmatrix} m_k\partial_i + \partial_k m_i & \kappa_b\partial_i & (\Box \diamond a)_i & M^b_{ikl} \\ \partial_k\kappa_a & \kappa_c C^c_{ba} & 0 & -C^b_{ca}B^c_{kl} \\ a\Box\partial_k & 0 & 0 & 0 \\ N^a_{ijk} & C^a_{cb}B^c_{ij} & 0 & 0 \end{bmatrix} \begin{bmatrix} (\delta h/\delta m)^k \\ (\delta h/\delta\kappa)^b \\ \delta h/\delta a \\ (\delta h/\delta B)^{kl}_b \end{bmatrix},$$

where

$$M^b_{ikl} = -B^b_{kl,i} + \partial_k B^b_{il} - \partial_l B^b_{ik} \quad \text{and} \quad N^a_{ijk} = B^a_{ij,k} + B^a_{kj}\partial_i - B^a_{ki}\partial_j.$$

As before, we recover by a reduction from a canonical cotangent bundle, the Hamiltonian structures appearing in [16, formula (2.28)] and [13, p. 152], and we have explained in which sense this matrix is Lie–Poisson, as already noted in these papers. More precisely, we have found the Lie group which corresponds to the Lie algebra underlying this Hamiltonian structure. This group is given by

$$\big[\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})\big] \circledS \big[V_1 \oplus \Omega_2(\mathcal{D}, \mathfrak{o}^*)\big],$$

where $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$ acts on $\Omega^2(\mathcal{D}, \mathfrak{o}) = [\Omega_2(\mathcal{D}, \mathfrak{o}^*)]^*$ by the representation

$$B \mapsto \mathrm{Ad}_{\chi^{-1}}\, \eta^* B,$$

and where the space V_1^* is only acted upon by the subgroup $\mathrm{Diff}(\mathcal{D})$.

The Lie–Poisson bracket is computed to be

$$\begin{aligned} \{f, g\}(\mathbf{m}, \kappa, a, B) = &\int_{\mathcal{D}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}}\right]\mu \\ &+ \int_{\mathcal{D}} \kappa \cdot \left(\mathrm{ad}_{\frac{\delta f}{\delta \kappa}} \frac{\delta g}{\delta \kappa} + \mathbf{d}\frac{\delta f}{\delta \kappa} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \kappa} \cdot \frac{\delta f}{\delta \mathbf{m}}\right)\mu \\ &+ \int_{\mathcal{D}} a \cdot \left(\frac{\delta f}{\delta a}\frac{\delta g}{\delta \mathbf{m}} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mathbf{m}}\right)\mu \\ &+ \int_{\mathcal{D}} \left[(\pounds_{\frac{\delta f}{\delta \mathbf{m}}} B - \mathrm{ad}_{\frac{\delta f}{\delta \kappa}} B) \cdot \frac{\delta g}{\delta B} - (\pounds_{\frac{\delta g}{\delta \mathbf{m}}} B - \mathrm{ad}_{\frac{\delta g}{\delta \kappa}} B) \cdot \frac{\delta f}{\delta B}\right]\mu. \end{aligned}$$

Using formulas (3.16) and

$$\frac{\delta f}{\delta \gamma} = -\,\mathrm{div}^{\gamma}\, \frac{\delta f}{\delta B},$$

we obtain that the map

$$(\mathbf{m}, \nu, a, \gamma) \mapsto (\mathbf{m}, \nu, a, \mathbf{d}^{\gamma}\gamma)$$

is a Poisson map relative to the affine Lie–Poisson bracket $\{,\}$ given in (6.6) and the Lie–Poisson bracket associated to the B-representation.

7. The circulation theorems

The Kelvin–Noether theorem is a version of the Noether theorem that holds for solutions of the Euler–Poincaré equations. For example, an application of this theorem to the compressible adiabatic fluid gives the Kelvin circulation theorem

$$\frac{d}{dt}\oint_{\gamma_t}\mathbf{u}^\flat=\oint_{\gamma_t}T\,\mathbf{d}s,$$

where $\gamma_t\subset\mathcal{D}$ is a closed curve which moves with the fluid velocity $\mathbf{u}$, $T=\partial e/\partial s$ is the temperature, and e,s denote respectively the specific internal energy and the specific entropy. The Kelvin–Noether theorem associated to Euler–Poincaré reduction for semidirect products is presented in [17]. We now adapt this result to the case of affine Euler–Poincaré reduction.

Kelvin–Noether theorem. We work under the hypotheses and the notations of Section 2. Let $\mathcal{C}$ be a manifold on which G acts on the left and suppose we have an equivariant map $\mathcal{K}:\mathcal{C}\times V^*\to\mathfrak{g}^{**}$, that is, for all $g\in G$, $a\in V^*$, $c\in\mathcal{C}$, we have

$$\big\langle\mathcal{K}\big(gc,\theta_g(a)\big),\mu\big\rangle=\big\langle\mathcal{K}(c,a),\mathrm{Ad}^*_g\,\mu\big\rangle,$$

where gc denotes the action of G on $\mathcal{C}$, and θ_g is the affine action of G on V^*.

Define the Kelvin–Noether quantity $I:\mathcal{C}\times\mathfrak{g}\times V^*\to\mathbb{R}$ by

$$I(c,\xi,a):=\left\langle\mathcal{K}(c,a),\frac{\delta l}{\delta\xi}(\xi,a)\right\rangle.$$

The same proof as in [17] yields the following result.

Theorem 7.1. *Fixing $c_0\in\mathcal{C}$, let $\xi(t),a(t)$ satisfy the affine Euler–Poincaré equations* (2.3) *and define $g(t)$ to be the solution of $\dot g(t)=TR_{g(t)}\xi(t)$ and, say, $g(0)=e$. Let $c(t)=g(t)c_0$ and $I(t):=I(c(t),\xi(t),a(t))$. Then*

$$\frac{d}{dt}I(t)=\left\langle\mathcal{K}\big(c(t),a(t)\big),\frac{\delta l}{\delta a}\diamond a-\mathbf{d}c^T\left(\frac{\delta l}{\delta a}\right)\right\rangle.$$

As we will see in the applications, some examples do not admit a Lagrangian formulation. Nevertheless, a Kelvin–Noether theorem is still valid for the Hamiltonian formulation. Keeping the same notations as before, the Kelvin–Noether quantity is now the mapping $J:\mathcal{C}\times\mathfrak{g}^*\times V^*\to\mathbb{R}$ defined by

$$J(c,\mu,a):=\big\langle\mathcal{K}(c,a),\mu\big\rangle,$$

and we have the following result.

Theorem 7.2. *Fixing $c_0\in\mathcal{C}$, let $\mu(t),a(t)$ satisfy the affine Lie–Poisson equations of Theorem* 5.3 *and define $g(t)$ to be the solution of*

$$\dot g(t)=TR_{g(t)}\left(\frac{\delta h}{\delta\mu}\right),\qquad g(0)=e.$$

Let $c(t)=g(t)c_0$ and $J(t):=J(c(t),\mu(t),a(t))$. Then

$$\frac{d}{dt}J(t)=\left\langle\mathcal{K}\big(c(t),a(t)\big),-\frac{\delta h}{\delta a}\diamond a+\mathbf{d}c^T\left(\frac{\delta h}{\delta a}\right)\right\rangle.$$

This result follows from the reconstruction of dynamics as described in Section 5. Of course, when the Hamiltonian h comes from a Lagrangian by Legendre transformation, then this theorem is a corollary of Theorem 7.1.

In the case of dynamics on the group $G = \mathrm{Diff}(\mathcal{D})$, the standard choice for the equivariant map $\mathcal{K}$ is

$$\big\langle \mathcal{K}(c,a), \mathbf{m} \big\rangle := \oint_c \frac{1}{\rho}\mathbf{m}, \tag{7.1}$$

where $c \in \mathcal{C} = \mathrm{Emb}(S^1, \mathcal{D})$, the manifold of all embeddings of the circle S^1 in $\mathcal{D}$, $\mathbf{m} \in \Omega^1(\mathcal{D})$, and ρ is advected as $(J\eta)(\rho \circ \eta)$. There is a generalization of this map in the case of the group $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$, see §7 in [8]. Therefore, Theorems 7.1 and 7.2 can be applied in the case of the affine Euler–Poincaré and Lie–Poisson equations (3.4) and (6.1). Nevertheless we shall not use this point of view here and we apply the Kelvin–Noether theorem only to the first component of the group G, namely the group $\mathrm{Diff}(\mathcal{D})$, and we obtain the following result.

Theorem 7.3. *Consider the affine Euler–Poincaré equations for complex fluids* (3.4)*. Suppose that one of the linear advected variables, say ρ, is advected as $(J\eta)(\rho \circ \eta)$. Then, using the map* (7.1)*, we have*

$$\frac{d}{dt}\oint_{c_t} \frac{1}{\rho}\frac{\delta l}{\delta \mathbf{u}} = \oint_{c_t} \frac{1}{\rho}\left(-\frac{\delta l}{\delta \nu}\cdot \mathbf{d}\nu + \frac{\delta l}{\delta a}\diamond a + \frac{\delta l}{\delta \gamma}\diamond_1 \gamma \right),$$

where c_t is a loop in $\mathcal{D}$ which moves with the fluid velocity $\mathbf{u}$.

Similarly, consider the affine Lie–Poisson equations for complex fluids (6.1)*. Suppose that one of the linear advected variables, say ρ, is advected as $(J\eta)(\rho \circ \eta)$. Then, using the map* (7.1)*, we have*

$$\frac{d}{dt}\oint_{c_t} \frac{1}{\rho}\mathbf{m} = \oint_{c_t} \frac{1}{\rho}\left(-\kappa\cdot \mathbf{d}\frac{\delta h}{\delta \kappa} - \frac{\delta h}{\delta a}\diamond a - \frac{\delta h}{\delta \gamma}\diamond_1 \gamma \right),$$

where c_t is a loop in $\mathcal{D}$ which moves with the fluid velocity $\mathbf{u}$, defined by the equality

$$\mathbf{u} := \frac{\delta h}{\delta \mathbf{m}}.$$

***γ*-circulation.** The γ-circulation is associated to the equation

$$\frac{\partial}{\partial t}\gamma + \pounds_{\mathbf{u}}\gamma = -\mathbf{d}\nu + \mathrm{ad}_\nu \gamma.$$

Let η_t be the flow of the vector field $\mathbf{u}$, let c_0 be a loop in $\mathcal{D}$ and let $c_t := \eta_t \circ c_0$. Then, by change of variables, we have

$$\frac{d}{dt}\oint_{c_t}\gamma = \frac{d}{dt}\oint_{c_0}\eta_t^*\gamma = \oint_{c_0}\eta_t^*(\dot\gamma + \pounds_{\mathbf{u}}\gamma) = \oint_{c_0}\eta_t^*(-\mathbf{d}\nu + \mathrm{ad}_\nu\gamma) = \oint_{c_t}\mathrm{ad}_\nu\gamma \in \mathfrak{o}.$$

8. Applications

8.1. Spin systems

We thank D. Holm for challenging us with this example as a simple model for many applications. It illustrates the applicability of Theorems 2.1 and 5.3 in a very simple situation that exhibits, nevertheless, some of the key difficulties of more complicated fluid models. Let $\mathcal{D}$ be a manifold and $\mathcal{O}$ a Lie group thought of as the "order parameters" of some fluid model. Take $G = \mathcal{F}(\mathcal{D}, \mathcal{O}) \ni \chi$ and $V^* = \Omega^1(\mathcal{D}, \mathfrak{o}) \ni \gamma$. Consider the affine G-representation on V^* given by

$$\theta_\chi(\gamma) := \mathrm{Ad}_{\chi^{-1}}\gamma + \chi^{-1}T\chi.$$

This is simply the gauge transformation of the connection on the trivial principal $\mathcal{O}$-bundle $\mathcal{O}\times\mathcal{D}\to\mathcal{D}$ induced by γ (see (3.8)). The associated diamond operation is given by

$$w\diamond\gamma = -\mathrm{Tr}\big(\mathrm{ad}^*_\gamma w\big).$$

Since $c(\chi) = \chi^{-1}T\chi$ we obtain, as in the case of complex fluids,

$$\mathbf{d}c(v) = \mathbf{d}v \quad\text{and}\quad \mathbf{d}c^T(w) = -\operatorname{div} w.$$

As before, we use the notations $v\in\mathcal{F}(\mathcal{D},\mathfrak{o})$ and $w\in\mathfrak{X}(\mathcal{D},\mathfrak{o}^*)$. The affine Euler–Poincaré equations (2.3) become

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta v} = -\mathrm{ad}^*_v\frac{\delta l}{\delta v} + \operatorname{div}^\gamma\frac{\delta l}{\delta\gamma}, \tag{8.1}$$

where $(\mathrm{ad}^*_v\kappa)(x) = \mathrm{ad}^*_{v(x)}(\kappa(x))$. The evolution equation for γ is

$$\frac{\partial}{\partial t}\gamma + \mathbf{d}^\gamma v = 0.$$

Similarly, the affine Lie–Poisson equations of Theorem 5.3 become

$$\begin{cases} \dfrac{\partial}{\partial t}\kappa = -\mathrm{ad}^*_{\frac{\delta h}{\delta\kappa}}\kappa - \operatorname{div}^\gamma\dfrac{\delta h}{\delta\gamma}, \\ \dfrac{\partial}{\partial t}\gamma + \mathbf{d}^\gamma\dfrac{\delta h}{\delta\kappa} = 0. \end{cases} \tag{8.2}$$

In the particular case $\mathcal{D} = \mathbb{R}^3$ and $\mathcal{O} = SO(3)$, this system of equations appears in the context of the *macroscopic description of spin glasses*, see Eqs. (28) and (29) in [6] and references therein. See also Eqs. (3.9), (3.10) in [18], system (1) in [19] and references therein for an application to *magnetic media*. In this context, the variable κ is interpreted as the *spin density*, v is the *infinitesimal spin rotation*, and the curvature $\mathbf{d}^\gamma\gamma$ is the *disclination density*.

Interesting choices for the Lagrangian are

$$l_\perp(v,\gamma) = \frac{1}{2}\int_{\mathcal{D}}\big\|[v,\gamma]\big\|^2\mu, \qquad l_\parallel(v,\gamma) = \frac{1}{2}\int_{\mathcal{D}}\big\|k(v,\gamma)\big\|^2\mu,$$

and

$$l_{SG}(v,\gamma) = \frac{\epsilon}{2}\int_{\mathcal{D}}\|v\|^2\mu - \frac{\rho}{2}\int_{\mathcal{D}}\|\gamma\|^2\mu. \tag{8.3}$$

In the expression of the Lagrangian $l_\perp$, $[\nu,\gamma]$ denotes the $\mathfrak{o}$-valued one-form on $\mathcal{D}$ given by $v_x \mapsto [\nu(x),\gamma(x)(v_x)] \in \mathfrak{o}$. The norm is associated to the metric (gk), on the vector bundle of $\mathfrak{o}$-valued one-forms on $\mathcal{D}$, induced by a Riemannian metric g on $\mathcal{D}$ and by an Ad-invariant inner product k on $\mathfrak{o}$. More precisely, the metric (gk) on the vector bundle of $\mathfrak{o}$-valued k-forms on $\mathcal{D}$ is given in the following way. The Riemannian metric g induces a Riemannian metric $\bar{g}$ on the vector bundles $\Lambda^k\mathcal{D}\to\mathcal{D}$ of exterior k-forms on $\mathcal{D}$. For $\alpha_x,\beta_x\in\Lambda^k(\mathcal{D},\mathfrak{o})_x$, we can write $\alpha_x=\alpha^a f_a$ and $\beta_x=\beta^a f_a$, where $\{f_a\}$ is a basis of $\mathfrak{o}$ and $\alpha^a,\beta^a\in(\Lambda^k M)_x$. So we define

$$(gk)_x(\alpha_x,\beta_x) := k_{ab}\bar{g}\big(\alpha^a,\beta^b\big),$$

where $k_{ab}:=k(f_a,f_b)$. This construction is independent of the basis. If the manifold is taken to be $\mathbb{R}$ and if $\mathcal{O}=SO(3)$, then the Lagrangian $l_\perp$ reads

$$l_\perp(\nu,\gamma)=\frac{1}{2}\int_{-\infty}^{\infty}\|\nu\times\gamma\|^2\mu.$$

This choice is reminiscent of the *Skyrme model*, a nonlinear topological model of pions in nuclear physics (see [26]).

In the expression of the Lagrangian $l_\parallel$, $k(\nu,\gamma)$ denotes the one-form on $\mathcal{D}$ given by $v_x\mapsto k(\nu(x),\gamma(x)(v_x))$. The norm is taken relative to the Riemannian metric g on $\mathcal{D}$.

Interestingly, when the Lagrangians $l_\perp$ or $l_\parallel$ are used, the affine Euler–Poincaré equations (8.1) simplify to

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \nu}=\operatorname{div}\frac{\delta l}{\delta\gamma}, \tag{8.4}$$

where on the right-hand side we have the usual divergence operator in $\mathbb{R}^3$.

In the expression of the Lagrangian l_{SG}, we used the symbols ϵ and ρ for the *constants of susceptibility* and the *rigidity*. The norms are respectively associated to the inner product k and to the metric (gk). The associated Hamiltonian reads

$$h_{SG}(\kappa,\gamma)=\frac{1}{2\epsilon}\int_{\mathcal{D}}\|\kappa\|^2\mu+\frac{\rho}{2}\int_{\mathcal{D}}\|\gamma\|^2\mu,$$

and is used, with $\mathcal{O}=SO(3)$, in the context of the *macroscopic description of spin glasses*, see expression (26) in [6]. In this case the affine Lie–Poisson equation (8.2) reads

$$\begin{cases}\dfrac{\partial}{\partial t}\kappa=-\rho\operatorname{div}\gamma^\sharp,\\[2mm] \dfrac{\partial}{\partial t}\gamma+\dfrac{1}{\epsilon}\mathbf{d}^\gamma\kappa^\sharp=0,\end{cases} \tag{8.5}$$

where $\gamma^\sharp\in\mathfrak{X}(\mathcal{D},\mathfrak{o}^*)$ and $\kappa^\sharp\in\mathcal{F}(\mathcal{D},\mathfrak{o})$ are associated to γ and κ via the metrics.

More general expressions, such as

$$h(\kappa,\gamma)=\frac{1}{2\epsilon}\int_{\mathcal{D}}\|\kappa\|^2\mu+\frac{1}{2\epsilon_1}\int_{\mathcal{D}}\|\kappa\|^4\mu+\frac{\rho}{2}\int_{\mathcal{D}}\|\gamma\|^2\mu+\frac{\rho_1}{2}\int_{\mathcal{D}}\|\gamma\|^4\mu+\frac{q}{2}\int_{\mathcal{D}}\|\kappa\cdot\gamma\|^2\mu,$$

are used in the theory of magnetic media, see e.g. Eq. (3) in [19].

Lagrangian reduction. Consider a Lagrangian

$$L : T\mathcal{F}(\mathcal{D},\mathcal{O}) \times \Omega^1(\mathcal{D},\mathfrak{o}) \to \mathbb{R}, \quad L = L(v_\chi, \gamma)$$

such that, for all $\psi \in \mathcal{F}(\mathcal{D},\mathcal{O})$ we have

$$L\big(TR_\psi \circ v_\chi, \mathrm{Ad}_{\psi^{-1}}\gamma + \psi^{-1}T\psi\big) = L(v_\chi,\gamma).$$

Let χ be a curve in $\mathcal{F}(\mathcal{D},\mathcal{O})$ and consider the curve $v := TR_{\chi^{-1}} \circ \dot{\chi}$. For $\gamma_0 \in \Omega^1(\mathcal{D},\mathfrak{o})$ consider the solution γ of the equation

$$\frac{\partial}{\partial t}\gamma + \mathbf{d}^\gamma v = 0,$$

with initial condition γ_0. This solution is given by $\gamma = \mathrm{Ad}_\chi \gamma_0 + \chi T\chi^{-1}$. Then, by Theorem 2.1, χ is a solution of the Euler–Lagrange equations associated to L_{γ_0} if and only if v is solution of (8.1). Note that in the special case $\gamma_0 = 0$, the evolution of γ is given by the important relation

$$\gamma = \chi T\chi^{-1}$$

and the disclination density vanishes, that is, $\mathbf{d}\gamma + [\gamma,\gamma] = 0$. This hypothesis is usually assumed in the examples treated in [6] and [19], and is referred to as the *Maurer–Cartan constraint*. Recall that the vanishing of the curvature is preserved by the flow of (8.2). In our approach the variable χ can be interpreted as the *Lagrangian evolution of the spin*.

Hamiltonian reduction. Consider a Hamiltonian

$$H : T^*\mathcal{F}(\mathcal{D},\mathcal{O}) \times \Omega^1(\mathcal{D},\mathfrak{o}) \to \mathbb{R}, \quad H = H(\kappa_\chi,\gamma)$$

such that, for all $\psi \in \mathcal{F}(\mathcal{D},\mathcal{O})$ we have

$$H\big(T^*R_{\psi^{-1}} \circ \kappa_\chi, \mathrm{Ad}_{\psi^{-1}}\gamma + \psi^{-1}T\psi\big) = H(\kappa_\chi,\gamma).$$

Then, by Theorem 5.3, a curve $\kappa_\chi \in T^*\mathcal{F}(\mathcal{D},\mathcal{O})$ is a solution of the Hamilton equations associated to H_{γ_0} if and only if the curve

$$\kappa := T^*R_\chi \circ \kappa_\chi$$

is a solution of the affine Lie–Poisson equation (8.2).

The associated affine Lie–Poisson bracket is given by

$$\{f,g\}(\kappa,\gamma) = \int_{\mathcal{D}} \kappa \cdot \left(\mathrm{ad}_{\frac{\delta f}{\delta \kappa}}\frac{\delta g}{\delta \kappa}\right)\mu + \int_{\mathcal{D}} \left(\mathbf{d}^\gamma \frac{\delta f}{\delta \kappa}\cdot\frac{\delta g}{\delta \gamma} - \mathbf{d}^\gamma \frac{\delta g}{\delta \kappa}\cdot\frac{\delta f}{\delta \gamma}\right)\mu.$$

Thus, we have recovered the Poisson bracket of [6], by reduction of the canonical structure. The momentum map is

$$\mathbf{J}_\alpha\big(\kappa_\chi,(w,\gamma)\big) = \big(T^*L_\chi \circ \kappa_\chi + \mathrm{div}^\gamma w, \gamma\big)$$

and the reduced symplectic spaces are affine coadjoint orbits in $(\mathcal{F}(\mathcal{D},\mathfrak{o}) \circledS \Omega^1(\mathcal{D},\mathfrak{o}))^*$. Using formula (5.7), we obtain that they are given by

$$\mathcal{O}^\sigma_{(\kappa,\gamma)} = \big\{\big(\mathrm{Ad}^*_\chi \kappa + \mathrm{div}^{\theta_\chi(\gamma)} w, \theta_\chi(\gamma)\big) \,\big|\, (\chi, w) \in \mathcal{F}(\mathcal{D},\mathcal{O}) \circledS \mathfrak{X}(\mathcal{D},\mathfrak{o}^*)\big\}.$$

8.2. Yang–Mills magnetohydrodynamics

Magnetohydrodynamics models the motion of an electrically charged and perfectly conducting fluid. In the balance of momentum law, one must add the Lorentz force of the magnetic field created by the fluid in motion. In addition, the hypothesis of infinite conductivity leads one to the conclusion that magnetic lines are frozen in the fluid, i.e. that they are transported along the particle paths. This hypothesis leads to the equation

$$\frac{\partial}{\partial t}B + \pounds_{\mathbf{u}} B = 0.$$

This model can be extend to incorporate non-Abelian Yang–Mills interactions and is known under the name of Yang–Mills magnetohydrodynamics; see [14] for a derivation of this model. Recall that for Yang–Mills theory, the field B is seen as the curvature of a connection A on a principal bundle. Clearly the connection A represents the variable γ in the general theory developed previously, on which the automorphism group acts by affine transformations. This shows that the abstract formalism developed previously is very natural in the context of Yang–Mills theory. Note that there is a more general model of fluid motion with Yang–Mills charged particles, namely the Euler–Yang–Mills equations. The Hamiltonian structure of these equations is given in [10], see also [8] for the associated Lagrangian and Hamiltonian reductions.

As remarked in [16], at the reduced level, the Hamiltonian structure of Yang–Mills magnetohydrodynamics is given by the matrix (6.5). In this paragraph we carry out the corresponding affine Lie–Poisson reduction.

The group G is chosen to be the semidirect product of the diffeomorphism group with the group of $\mathcal{O}$-valued function on $\mathcal{D}$, that is, $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$. The order parameter Lie group $\mathcal{O}$ represents here the *symmetry group of particle interaction*. For example, $\mathcal{O} = S^1$ corresponds to electromagnetism, $\mathcal{O} = SU(2)$ and $\mathcal{O} = SU(3)$ correspond to weak and strong interactions, respectively. The advected quantities are the *mass density* ρ, the *specific entropy* s, and the *potential of the Yang–Mills field* A. Therefore, we set

$$a = (\rho, s) \in V_1^* = \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}) \quad \text{and} \quad A = \gamma \in \Omega^1(\mathcal{D}, \mathfrak{o}).$$

The action of $(\eta, \chi) \in G$ on $(\rho, s) \in V_1^*$ is the usual right representation of the fluid relabeling group on the mass density and entropy. It is given by

$$(\rho, s)(\eta, \chi) = \big(J\eta(\rho \circ \eta), s \circ \eta\big).$$

The right affine action of $(\eta, \chi) \in G$ on $A \in \Omega^1(M, \mathfrak{o})$ is given, as in the example (6.3), by

$$A \mapsto \mathrm{Ad}_{\chi^{-1}}\, \eta^* A + \chi^{-1} T\chi\,.$$

Since the variable $\kappa \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$ is interpreted as the *gauge-charge density*, we use the notations $Q \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) = T_0^*\mathcal{F}(\mathcal{D}, \mathcal{O})$ and $Q_\chi \in T_\chi^*\mathcal{F}(\mathcal{D}, \mathcal{O})$.

The Hamiltonian $H_{(\rho,s,A)} : T^*(\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})) \to \mathbb{R}$ is given by

$$H_{(\rho,s,A)}(\mathbf{m}_\eta, Q_\chi) = \frac{1}{2}\int_{\mathcal{D}} \frac{1}{\rho}\|\mathbf{m}_\eta\|^2 \mu + \int_{\mathcal{D}} \rho e\big(\rho\big(J\eta^{-1}\big), s\big)\mu + \frac{1}{2}\int_{\mathcal{D}} \big\|\mathbf{d}^A A \cdot T\eta^{-1}\big\|^2 (J\eta)\mu, \tag{8.6}$$

where e denotes the *specific internal energy*, the norm in the first term is associated to a Riemannian metric g on $\mathcal{D}$, and the norm in the third term is associated to the metric (gk), on the vector bundle

of $\mathfrak{o}$-valued k-forms on $\mathcal{D}$, induced by the metric g and by an Ad-invariant inner product k on $\mathfrak{o}$. For details see Section 8.1.

The metric (gk) can be used to identify $\Omega_k(\mathcal{D},\mathfrak{o}^*)$ and its dual $\Omega^k(\mathcal{D},\mathfrak{o})$, by raising and lowering indices. Through this identification, the operators div and div^A act also on $\Omega^k(\mathcal{D},\mathfrak{o})$. The Hamiltonian $H(\mathbf{m}_\eta, Q_\chi, \rho, s, A)$ is invariant under the right action of (φ,ψ) given by

$$\begin{aligned}&(\mathbf{m}_\eta, Q_\chi, \rho, s, A)\\ &\quad\mapsto \big(J\varphi(\mathbf{m}_\eta\circ\varphi),\, J\varphi(T^*R_{\psi^{-1}}\circ Q_\chi\circ\varphi),\, J\varphi(\rho\circ\varphi),\, s\circ\varphi,\, \operatorname{Ad}_{\psi^{-1}}\varphi^*A+\psi^{-1}T\psi\big).\end{aligned}$$

Therefore, the hypotheses of Theorem 5.3 are satisfied and the reduced Hamiltonian $h:\Omega^1(\mathcal{D})\times\mathcal{F}(\mathcal{D},\mathfrak{o}^*)\times V_1^*\times V_2^*\to\mathbb{R}$ is given by

$$h(\mathbf{m}, Q, \rho, s, A)=\frac{1}{2}\int_{\mathcal{D}}\frac{1}{\rho}\|\mathbf{m}\|^2\mu+\int_{\mathcal{D}}\rho e(\rho,s)\mu+\frac{1}{2}\int_{\mathcal{D}}\big\|\mathbf{d}^A A\big\|^2\mu,$$

where the norms are respectively associated to the metrics g and (gk). We now compute the affine Lie–Poisson equations (6.4) associated to this Hamiltonian. The functional derivatives are

$$\mathbf{u}:=\frac{\delta h}{\delta\mathbf{m}}=\frac{1}{\rho}\mathbf{m}^\sharp,\qquad \nu:=\frac{\delta h}{\delta Q}=0$$

and

$$\frac{\delta h}{\delta\rho}=-\frac{1}{2}\|\mathbf{u}\|^2+e+\rho\frac{\partial e}{\partial\rho},\qquad \frac{\delta h}{\delta s}=\rho\frac{\partial e}{\partial s},\qquad \frac{\delta h}{\delta A}=-\operatorname{div}^A\mathbf{d}^A A,$$

where div^A is defined by (3.14). The advection equations are

$$\frac{\partial}{\partial t}\rho+\operatorname{div}(\rho\mathbf{u})=0,\qquad \frac{\partial}{\partial t}s+\mathbf{d}s(\mathbf{u})=0,\qquad \frac{\partial}{\partial t}A+\mathbf{d}^A\big(A(\mathbf{u})\big)+\mathbf{i}_{\mathbf{u}}B=0.$$

The equation for the gauge charge is

$$\frac{\partial}{\partial t}Q=-\operatorname{div}(Q\,\mathbf{u})+\operatorname{div}^A\operatorname{div}^A\mathbf{d}^A A=-\operatorname{div}(Q\,\mathbf{u}).$$

Indeed, for all $f\in\mathcal{F}(\mathcal{D},\mathfrak{o})$ we have

$$\int_{\mathcal{D}}k\big(\operatorname{div}^A\operatorname{div}^A\mathbf{d}^A A, f\big)\mu=\int_{\mathcal{D}}(gk)\big(\mathbf{d}^A A,\mathbf{d}^A\mathbf{d}^A f\big)\mu=\int_{\mathcal{D}}(gk)\big(B,[B,f]\big)\mu=0,$$

where we used the equality $\mathbf{d}^A\mathbf{d}^A f=[B,f]$ for $B=\mathbf{d}^A A$ and the fact that, in an orthonormal frame with respect to g, we have

$$\begin{aligned}(gk)\big(B,[B,f]\big)&=B^a_{ij}[B,f]^b_{ij}k_{ab}=B^a_{ij}[B_{ij},f]^b k_{ab}\\&=k\big(B_{ij},[B_{ij},f]\big)=-k\big([B_{ij},B_{ij}],f\big)=0.\end{aligned}$$

This proves that $\operatorname{div}^A\operatorname{div}^A\mathbf{d}^A A=0$.

Using the advection equation for ρ we obtain the equality

$$\left(\frac{\partial}{\partial t}\mathbf{m} + \pounds_{\mathbf{u}}\mathbf{m} + (\operatorname{div}\mathbf{u})\mathbf{m}\right)^{\sharp} = \rho\left(\frac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} + \nabla\mathbf{u}^{T}\cdot\mathbf{u}\right).$$

We also have

$$\begin{aligned}
-\left(\frac{\delta h}{\delta\rho}\diamond\rho + \frac{\delta h}{\delta s}\diamond s\right)^{\sharp} &= \frac{\rho}{2}\operatorname{grad}\|\mathbf{u}\|^{2} - \rho\operatorname{grad}e - \rho\operatorname{grad}\left(\rho\frac{\partial e}{\partial\rho}\right) + \rho\frac{\partial e}{\partial s}\operatorname{grad}s\\
&= \rho\left(\nabla\mathbf{u}^{T}\cdot\mathbf{u}\right) - \rho\operatorname{grad}\left(\rho\frac{\partial e}{\partial\rho}\right) - \rho\frac{\partial e}{\partial\rho}\operatorname{grad}\rho\\
&= \rho\left(\nabla\mathbf{u}^{T}\cdot\mathbf{u}\right) - \operatorname{grad}\left(\rho^{2}\frac{\partial e}{\partial\rho}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\delta h}{\delta A}\diamond_{1} A &= \left(\operatorname{div}^{A}\frac{\delta h}{\delta A}\right)A - \frac{\delta h}{\delta A}\cdot\mathbf{i}_{_}\,\mathbf{d}^{A}A\\
&= 0 + (gk)\left(\operatorname{div}^{A}\mathbf{d}^{A}A, \mathbf{i}_{_}\,\mathbf{d}^{A}A\right).
\end{aligned}$$

Therefore we obtain that the first line of (6.4) becomes

$$\frac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\frac{1}{\rho}\left(\operatorname{grad}p + (gk)\left(\mathbf{i}_{\_}\,B, \operatorname{div}^{A}B\right)^{\sharp}\right),$$

where $B = \mathbf{d}^{A}A$ and $p = \rho^{2}\frac{\partial e}{\partial\rho}$. In summary, we have obtained the equations of Yang–Mills magnetohydrodynamics

$$\begin{cases}
\dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\dfrac{1}{\rho}\left(\operatorname{grad}p + (gk)\left(\mathbf{i}_{_}\,B, \operatorname{div}^{A}B\right)^{\sharp}\right), \quad B = \mathbf{d}^{A}A,\\
\dfrac{\partial}{\partial t}Q + \operatorname{div}(Q\,\mathbf{u}) = 0, \quad \dfrac{\partial}{\partial t}A + \mathbf{d}^{A}\left(A(\mathbf{u})\right) + \mathbf{i}_{\mathbf{u}}B = 0,\\
\dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \quad \dfrac{\partial}{\partial t}s + \mathbf{d}s(\mathbf{u}) = 0.
\end{cases} \tag{8.7}$$

We now treat the particular case of magnetohydrodynamics, that is, the case $\mathcal{O} = S^{1}$. In order to recover the standard equations we suppose that $\mathcal{D}$ is three-dimensional. In this case we can define the *magnetic potential* $\mathbf{A} := A^{\sharp} \in \mathfrak{X}(\mathcal{D})$ and the *magnetic field* $\mathbf{B} := (\star B)^{\sharp} \in \mathfrak{X}(\mathcal{D})$. Since the group is Abelian, covariant differentiation coincides with usual differentiation and the equality $\mathbf{d}^{A}A = B$ reads $\operatorname{curl}\mathbf{A} = \mathbf{B}$. Using the identities $(\operatorname{div}B)^{\sharp} = -\operatorname{curl}\mathbf{B}$ and $(\mathbf{i}_{\mathbf{u}}B)^{\sharp} = \mathbf{B}\times\mathbf{u}$ we obtain

$$g(\mathbf{i}_{\_}\,B, \operatorname{div}B)^{\sharp} = -(\mathbf{i}_{(\operatorname{div}B)^{\sharp}}B)^{\sharp} = \mathbf{B}\times\operatorname{curl}\mathbf{B}.$$

Suppose that all particles have mass m. The electric charge q is such that $Q = \rho\frac{q}{m}$, therefore the equation for Q in (8.7) becomes

$$\frac{\partial}{\partial t}q + \mathbf{d}q(\mathbf{u}) = 0.$$

If we suppose that at time $t = 0$ all the particles have the same charge, then this charge remains constant for all time. By making use af these remarks and hypotheses, Eqs. (8.7) become

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\dfrac{1}{\rho}(\operatorname{grad} p + \mathbf{B} \times \operatorname{curl}\mathbf{B}), \quad \mathbf{B} = \operatorname{curl}\mathbf{A}, \\ \dfrac{\partial}{\partial t}\mathbf{A} + \operatorname{grad}\big[g(\mathbf{A},\mathbf{u})\big] + \mathbf{B} \times \mathbf{u} = 0, \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \quad \dfrac{\partial}{\partial t}s + \mathbf{d}s(\mathbf{u}) = 0. \end{cases} \tag{8.8}$$

Thus, we have recovered the equations for magnetohydrodynamics.

Turning back to the general case and using Theorem 5.3, we obtain the following result.

Hamiltonian reduction for Yang–Mills magnetohydrodynamics. A smooth path $(\mathbf{m}_\eta, Q_\chi) \in T^*[\operatorname{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})]$ is a solution of Hamilton's equations associated to the Hamiltonian $H_{(\rho_0, s_0, A_0)}$ given in (8.6) if and only if the curve

$$\big(\rho\mathbf{u}^\flat, Q\big) =: (\mathbf{m}, Q) := J\big(\eta^{-1}\big)\big(\mathbf{m}_\eta \circ \eta^{-1}, T^*R_{\chi\circ\eta^{-1}}\big(Q_\chi \circ \eta^{-1}\big)\big)$$

is a solution of the system (8.7) with initial conditions (ρ_0, s_0, A_0).

The evolution of the advected quantities is given by

$$\rho = J\big(\eta^{-1}\big)\big(\rho_0 \circ \eta^{-1}\big), \qquad s = s_0 \circ \eta^{-1},$$
$$A = \operatorname{Ad}_{\chi\circ\eta^{-1}}\eta_* A_0 + \big(\chi \circ \eta^{-1}\big)T\big(\chi^{-1} \circ \eta^{-1}\big) = \eta_*\big(\operatorname{Ad}_\chi A_0 + \chi T\chi^{-1}\big).$$

This theorem is interesting from two points of view. Firstly, it allows us to recover the non-canonical Hamiltonian structure given in [16] by a reduction from a canonical cotangent bundle. Secondly, it generalizes to the non-Abelian case the Hamiltonian reduction for magnetohydrodynamics given in [24].

The associated affine Lie–Poisson bracket is that given in (6.6), where the third term takes the explicit form

$$\int_{\mathcal{D}} \rho\left(\mathbf{d}\left(\frac{\delta f}{\delta\rho}\right)\frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\left(\frac{\delta g}{\delta\rho}\right)\frac{\delta f}{\delta \mathbf{m}}\right)\mu + \int_{\mathcal{D}} s\left(\operatorname{div}\left(\frac{\delta f}{\delta s}\frac{\delta g}{\delta \mathbf{m}}\right) - \operatorname{div}\left(\frac{\delta g}{\delta s}\frac{\delta f}{\delta \mathbf{m}}\right)\right)\mu.$$

Since the Hamiltonian depends on A only through its curvature, the equations can be formulated using the B-representation. One simply replaces the equation for A by those for its curvature B, namely,

$$\frac{\partial}{\partial t}B + \pounds_{\mathbf{u}}B = 0.$$

In particular, we obtain that the force term $(gk)(\mathbf{i}_{\_}B, \operatorname{div}^A B)^\sharp$ depends only on the curvature B and not on the connection one-form A.

In the particular case of magnetohydrodynamics, the evolution equation of $\mathbf{B}$ reads

$$\frac{\partial}{\partial t}\mathbf{B} + \operatorname{curl}(\mathbf{B} \times \mathbf{u}) = 0.$$

In the general case of Yang–Mills magnetohydrodynamics, the Kelvin–Noether theorem gives

$$\frac{d}{dt}\oint_{c_t}\mathbf{u}^\flat = \oint_{c_t} T\mathbf{d}s - \oint_{c_t}\frac{1}{\rho}(gk)\big(\mathbf{i}_{\_}B, \operatorname{div}^A B\big),$$

and the γ-circulation gives

$$\frac{d}{dt}\oint_{c_t} A = 0,$$

where c_t is a loop which moves with the fluid velocity $\mathbf{u}$, that is, $c_t = \eta_t \circ c_0$.

8.3. Hall magnetohydrodynamics

As we will see, Hall magnetohydrodynamics does not require the use of the affine Lie–Poisson reduction developed in this paper. However, in view of the next paragraph about superfluids, we quickly recall here from [11] the Hamiltonian formulation of these equations. We will obtain this Hamiltonian structure by a Lie–Poisson reduction for semidirect products, associated to the direct product group $G := \mathrm{Diff}(\mathcal{D}) \times \mathrm{Diff}(\mathcal{D})$. The advected quantities are

$$(\rho, s; n) \in \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}).$$

The variables ρ and s are, as before, the *mass density* and the *specific entropy*, on which only the first diffeomorphism group acts as

$$(\rho, s) \mapsto \big((J\eta)(\rho \circ \eta), s \circ \eta\big).$$

The variable n is the *electron charge density*, on which only the second diffeomorphism group acts as

$$n \mapsto (J\xi)(n \circ \xi).$$

By Lie–Poisson reduction, for a Hamiltonian $h = h(\mathbf{m}, \rho, s; \mathbf{n}, n)$ defined on the dual Lie algebra

$$\big([\mathfrak{X}(\mathcal{D}) \circledS \big(\mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D})\big)] \times [\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D})]\big)^* \cong [\Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D})] \times [\Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D})],$$

we obtain the coupled Lie–Poisson equations

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{m} = -\pounds_{\frac{\delta h}{\delta \mathbf{m}}}\mathbf{m} - \operatorname{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\right)\mathbf{m} - \dfrac{\delta h}{\delta \rho} \diamond \rho - \dfrac{\delta h}{\delta s} \diamond s, \\ \dfrac{\partial}{\partial t}\rho = -\operatorname{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\rho\right), \\ \dfrac{\partial}{\partial t}s = -\mathbf{d}s\left(\dfrac{\delta h}{\delta \mathbf{m}}\right) \end{cases} \tag{8.9}$$

and

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{n} = -\pounds_{\frac{\delta h}{\delta \mathbf{n}}}\mathbf{n} - \operatorname{div}\left(\dfrac{\delta h}{\delta \mathbf{n}}\right)\mathbf{n} - \dfrac{\delta h}{\delta n} \diamond n, \\ \dfrac{\partial}{\partial t}n = -\operatorname{div}\left(\dfrac{\delta h}{\delta \mathbf{n}}n\right). \end{cases} \tag{8.10}$$

Here $\mathbf{m}$ is the total momentum density of the fluid and, as will be discussed below, $\mathbf{n}$ is interpreted as the momentum density associated to the *electron fluid velocity* $\mathbf{v} := \delta h/\delta \mathbf{n}$.

From the second Lie–Poisson system we obtain that the evolution of $\mathbf{n}/n$ is given by

$$\frac{\partial}{\partial t}(\mathbf{n}/n) = -\pounds_{\frac{\delta h}{\delta \mathbf{n}}}(\mathbf{n}/n) - \operatorname{grad}\frac{\delta h}{\delta n}. \tag{8.11}$$

The Hamiltonian for Hall magnetohydrodynamics is

$$h(\mathbf{m},\rho,s;\mathbf{n},n) := \frac{1}{2}\int_{\mathcal{D}} \frac{1}{\rho}\left\|\mathbf{m}-\frac{a\rho}{R}A\right\|^2\mu + \int_{\mathcal{D}} \rho e(\rho,s)\mu + \frac{1}{2}\int_{\mathcal{D}} \|\mathbf{d}A\|^2\mu, \tag{8.12}$$

where the one-form A, defined by

$$A := R\frac{\mathbf{n}}{n} \in \Omega^1(\mathcal{D}),$$

is the *magnetic vector potential*, the constants a, R are respectively the *ion charge-to-mass ratio* and the *Hall scaling parameter*, and the norms are taken with respect to a fixed Riemannian metric g on $\mathcal{D}$. The functional derivatives are computed to be

$$\mathbf{u} := \frac{\delta h}{\delta \mathbf{m}} = \frac{1}{\rho}\mathbf{m}^\sharp - \frac{a}{R}A^\sharp, \qquad \frac{\delta h}{\delta \rho} = -\frac{1}{2}\|\mathbf{u}\|^2 - \frac{a}{R}A\cdot\mathbf{u} + e + \rho\frac{\partial e}{\partial \rho}, \qquad \frac{\delta h}{\delta s} = \rho\frac{\partial e}{\partial s},$$

and

$$\mathbf{v} := \frac{\delta h}{\delta \mathbf{n}} = -\frac{a\rho}{n}\mathbf{u} - \frac{R}{n}(\operatorname{div} B)^\sharp, \qquad \frac{\delta h}{\delta n} = \frac{a\rho}{n^2}\mathbf{n}\cdot\mathbf{u} + \frac{R}{n^2}\mathbf{n}\cdot(\operatorname{div} B)^\sharp = -\frac{1}{R}A\cdot\mathbf{v}, \qquad B := \mathbf{d}A.$$

Recall from (3.14) that div is defined on $\Omega_k(\mathcal{D})$ as the negative of the adjoint of the exterior differential $\mathbf{d}$ on $\Omega^k(\mathcal{D})$. However, the metric g on $\mathcal{D}$ gives an identification of $\Omega_k(\mathcal{D})$ with $\Omega^k(\mathcal{D})$ and hence we can regard div as defined also on $\Omega^k(\mathcal{D})$. It follows that on $\Omega^k(\mathcal{D})$ we have $\operatorname{div} = -\delta$, where δ is the usual codifferential induced by g and $\mathbf{d}$.

The variable $\mathbf{v}$ is interpreted as the *electron fluid velocity*. The advection equations for ρ, s, and n are given by

$$\frac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \quad \frac{\partial}{\partial t}s + \mathbf{d}s(\mathbf{u}) = 0, \quad \text{and} \quad \frac{\partial}{\partial t}n + \operatorname{div}(n\mathbf{v}) = 0.$$

Using the expression of $\mathbf{v}$ in terms of $\mathbf{u}$ we obtain that $\operatorname{div}(n\mathbf{v}) = -a\operatorname{div}(\rho\mathbf{u})$ (since $B = \mathbf{d}A$ implies that $\operatorname{div}((\operatorname{div} B)^\sharp) = 0$) which proves that

$$\frac{\partial}{\partial t}(a\rho + n) = 0.$$

Thus, if we assume that $a\rho_0 + n_0 = 0$ for the initial conditions, we have $a\rho + n = 0$ for all time.

Using the definition of A and (8.11), we obtain that the equation for A is given by

$$\frac{\partial}{\partial t}A = -\mathbf{i}_{\mathbf{v}}B.$$

Using the equations for A and ρ, we obtain the following equation for $\mathbf{u}$:

$$\frac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\frac{1}{\rho}\left(\operatorname{grad} p - \frac{a\rho}{R}(\mathbf{i}_{\mathbf{v}-\mathbf{u}}B)^\sharp\right), \quad p = \rho^2\frac{\partial e}{\partial \rho}.$$

Suppose that the initial conditions ρ_0 and n_0 verify the equality $a\rho_0 + n_0 = 0$. As we have seen above, the equality remains valid for all times, and we obtain $\mathbf{v} = \mathbf{u} + \frac{R}{a\rho}(\operatorname{div} B)^\sharp$. Using this, the equations above simplify and we obtain the system

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\dfrac{1}{\rho}\big(\operatorname{grad} p - (\mathbf{i}_{(\operatorname{div} B)^\sharp} B)^\sharp\big), \\ \dfrac{\partial}{\partial t}\rho = -\operatorname{div}(\rho\mathbf{u}), \quad \dfrac{\partial}{\partial t}s = -\mathbf{d}s(\mathbf{u}), \\ \dfrac{\partial}{\partial t}A = -\mathbf{i}_{\mathbf{u}}B - \dfrac{R}{a\rho}\mathbf{i}_{(\operatorname{div} B)^\sharp} B. \end{cases} \tag{8.13}$$

When $\mathcal{D}$ is three-dimensional, we can define the *magnetic potential* $\mathbf{A} := A^\sharp$ and the *magnetic field* $\mathbf{B} := (\star B)^\sharp$. In this case the previous equations read

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\dfrac{1}{\rho}(\operatorname{grad} p + \mathbf{B} \times \operatorname{curl}\mathbf{B}), \\ \dfrac{\partial}{\partial t}\rho = -\operatorname{div}(\rho\mathbf{u}), \quad \dfrac{\partial}{\partial t}s = -\mathbf{d}s(\mathbf{u}), \\ \dfrac{\partial}{\partial t}\mathbf{A} = \mathbf{u} \times \mathbf{B} + \dfrac{R}{a\rho}\mathbf{B} \times \operatorname{curl}\mathbf{B}. \end{cases} \tag{8.14}$$

These are the classical equations of Hall magnetohydrodynamics. Note that we can pass from the equations for magnetohydrodynamics to those for Hall magnetohydrodynamics by simply replacing the advection law for A by *Ohm's law*. In terms of the magnetic field B, one simply replaces the advection law

$$\frac{\partial}{\partial t}B + \pounds_{\mathbf{u}}B = 0,$$

where $\mathbf{u}$ is the fluid velocity, by the equation

$$\frac{\partial}{\partial t}B + \pounds_{\mathbf{v}}B = 0,$$

where $\mathbf{v}$ is the electron fluid velocity.

In fact, the Hamiltonian h given in (8.12), is the value at the identity of the right-invariant Hamiltonian $H(\mathbf{m}_\eta, \rho, s; \mathbf{n}_\xi, n) = H_{(\rho,s;n)}(\mathbf{m}_\eta; \mathbf{n}_\xi)$, where

$$H_{(\rho,s;n)} : T^*\big(\operatorname{Diff}(\mathcal{D}) \times \operatorname{Diff}(\mathcal{D})\big) \to \mathbb{R}$$

is given by

$$H_{(\rho,s;n)}(\mathbf{m}_\eta; \mathbf{n}_\xi) = \frac{1}{2}\int_{\mathcal{D}} \left\| \mathbf{m}_\eta - a\rho \frac{\mathbf{n}_\xi \circ \xi^{-1} \circ \eta}{n \circ \xi^{-1} \circ \eta} \right\|^2 \mu + \int_{\mathcal{D}} \rho e\big(\rho J\eta^{-1}, s\big)\mu$$
$$+ \frac{R^2}{2}\int_{\mathcal{D}} \left\| \mathbf{d}\left(\frac{\mathbf{n}_\xi \circ \xi^{-1}}{n \circ \xi^{-1}} \right) \right\|^2 \mu.$$

Hamiltonian reduction for Hall magnetohydrodynamics. Suppose that $a\rho_0 + n_0 = 0$. A curve $(\mathbf{m}_\eta, \mathbf{n}_\xi) \in T^*(\mathrm{Diff}(\mathcal{D}) \times \mathrm{Diff}(\mathcal{D}))$ is a solution of Hamilton's equations associated to $H_{(\rho_0, s_0; n_0)}$ if and only if the curve

$$(\mathbf{m}, \mathbf{n}) := \big(J\big(\eta^{-1}\big)\big(\mathbf{m}_\eta \circ \eta^{-1}\big), J\big(\xi^{-1}\big)\big(\mathbf{n}_\xi \circ \xi^{-1}\big)\big)$$

is a solution of Eqs. (8.13) where $A := \frac{R}{n}\mathbf{n} = -\frac{R}{a\rho}\mathbf{n}$, since $a\rho + n = 0$. Moreover the evolution of the advected quantities is given by

$$\rho = J\big(\eta^{-1}\big)\big(\rho_0 \circ \eta^{-1}\big), \qquad s = s_0 \circ \eta^{-1}, \qquad n = J\big(\xi^{-1}\big)\big(n_0 \circ \xi^{-1}\big).$$

Let us assume from now on that the initial conditions ρ_0 and n_0 are related by $a\rho_0 + n_0 = 0$. We have seen that this implies that $a\rho + n = 0$. From the relations above we conclude the interesting result that the action of $\xi^{-1} \circ \eta$ fixes n_0, that is, $J(\xi^{-1} \circ \eta)(n_0 \circ \xi^{-1} \circ \eta) = n_0$. Conversely, given this relation and the condition $a\rho_0 + n_0 = 0$, it is easily seen that $a\rho + n = 0$.

The Lie–Poisson bracket associated to these equations is clearly the sum of two Lie–Poisson brackets associated to the semidirect products $\mathrm{Diff}(\mathcal{D}) \circledS [\mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D})]$ and $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D})$.

The Kelvin–Noether theorem associated to the variable $\mathbf{m}$ gives

$$\frac{d}{dt} \oint_{c_t} \left(\mathbf{u}^\flat + \frac{a}{R} A \right) = \oint_{c_t} T \, \mathbf{d}s,$$

which can be rewritten as

$$\frac{d}{dt} \oint_{c_t} \mathbf{u}^\flat = \oint_{c_t} T \, \mathbf{d}s + \oint_{c_t} \frac{1}{\rho} \mathbf{i}_{(\mathrm{div}\, B)^\sharp} B,$$

where c_t is a loop which moves with the *fluid velocity* $\mathbf{u}$, that is, $c_t = \eta_t \circ c_0$. The Kelvin–Noether theorem associated to the variable $\mathbf{n}$ gives

$$\frac{d}{dt} \oint_{d_t} A = 0,$$

where d_t is a loop which moves with the *electron fluid velocity* $\mathbf{v}$, that is, $d_t = \xi_t \circ d_0$.

8.4. Multivelocity superfluids

Superfluidity is a rare state of matter encountered in few fluids at extremely low temperatures. Such materials exhibit strange behavior such as the lack of viscosity, the ability to flow through very small channels that are impermeable to ordinary fluids, and the fact that it can form a layer whose thickness is that of one atom on the walls of the container in which it is placed. In addition, the rotational speed of a superfluid is quantized, that is, the fluid can rotate only at certain values of the speed. Superfluidity is considered to be a manifestation of quantum mechanical effects at macroscopic level. Typical examples of superfluids are ^{3}He, whose atoms are fermions and the superfluid transition occurs by Cooper pairing between atoms rather than electrons, and ^{4}He, whose atoms are bosons and the superfluidity is a consequence of Bose–Einstein condensation in an interacting system.

For example, at temperatures close to absolute zero, a solution of ^{3}He and ^{4}He has its hydrodynamics described by three velocities: two superfluid velocities $\mathbf{v}_s^1, \mathbf{v}_s^2$ and one normal fluid velocity $\mathbf{v}_n$. If other kinds of superfluids are present, one needs to introduce additional superfluid velocities. For a history of the equations considered below and the Hamiltonian structure for multivelocity superfluids see [15].

The two-fluid model. We first treat the two-fluid model, that is, the case of one superfluid velocity $\mathbf{v}_s$ and one normal-fluid velocity $\mathbf{v}_n$. Remarkably, this Hamiltonian structure can be obtained by affine Lie–Poisson reduction, with order parameter Lie group $\mathcal{O} = S^1$. In this paragraph we also carry out the corresponding Lagrangian formulation, by applying the general theory of affine Euler–Poincaré reduction to the semidirect product group $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, S^1)$.

The linear advected quantity is the *entropy density* S on which a diffeomorphism η acts as

$$S \mapsto (J\eta)(S \circ \eta).$$

The affine advected quantity is the *superfluid velocity* $\mathbf{v}_s$, on which the element $(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, S^1)$ acts as

$$\mathbf{v}_s \mapsto \left(\eta^* \mathbf{v}_s^\flat + \mathbf{d}\chi\right)^\sharp.$$

This action is simply the affine representation (3.6) for the Lie group $\mathcal{O} = S^1$. Here, the advected quantity $\mathbf{v}_s$ is a vector field and not a one-form, since it represents a velocity and hence formula (3.6) was changed accordingly. As will be seen, in this formalism, the *mass density* does not appear as an advected quantity in the representation space V^*; it is a momentum, that is, one of the variables in the dual Lie algebra $\mathfrak{g}^* = \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D})$.

The reduced Lagrangian for superfluids is $l : [\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D})] \circledS [\mathcal{F}(\mathcal{D}) \oplus \mathfrak{X}(\mathcal{D})] \to \mathbb{R}$ given by

$$l(\mathbf{v}_n, \nu, S, \mathbf{v}_s) := \frac{1}{2} \int_{\mathcal{D}} \rho \|\mathbf{v}_n\|^2 \mu + \int_{\mathcal{D}} (\rho\nu)\mu - \int_{\mathcal{D}} \varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n)\mu, \tag{8.15}$$

where $\mathbf{v}_n$ is the *velocity of the normal flow*. The internal energy density ε is seen here as a function of three variables $\varepsilon = \varepsilon(\rho, S, \mathbf{r}) : \mathbb{R} \times \mathbb{R} \times T\mathcal{D} \to \mathbb{R}$. The norm in the first term is taken relative to a fixed Riemannian metric g on $\mathcal{D}$. Note that for superfluids it is more convenient to work with the internal energy and entropy *per unit volume* and not *per unit mass* as in the preceding examples. We make the following definitions:

$$\mu_{\mathrm{chem}} := \frac{\partial \varepsilon}{\partial \rho}(\rho, S, \mathbf{v}_s - \mathbf{v}_n) \in \mathcal{F}(\mathcal{D}), \qquad T := \frac{\partial \varepsilon}{\partial S}(\rho, S, \mathbf{v}_s - \mathbf{v}_n) \in \mathcal{F}(\mathcal{D}),$$

$$\mathbf{p} := \frac{\partial \varepsilon}{\partial \mathbf{r}}(\rho, S, \mathbf{v}_s - \mathbf{v}_n) \in \Omega^1(\mathcal{D}).$$

The interpretation of the quantities μ_{chem}, T, and $\mathbf{p}$ is obtained from the following thermodynamic derivative identity for the internal energy (superfluid first law):

$$\mathbf{d}\big(\varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n)\big) = \mu_{\mathrm{chem}} \mathbf{d}\rho + T\, \mathbf{d}S + \mathbf{p} \cdot \nabla_{-}(\mathbf{v}_s - \mathbf{v}_n) \in \Omega^1(\mathcal{D}),$$

where ∇ denotes the Levi-Civita covariant derivative associated to the metric g. The function μ_{chem} is the *chemical potential*, T is the *temperature*, and $\mathbf{p}$ is the *relative momentum density*. The mass density ρ is the function $\rho = \rho(\mathbf{v}_n, \nu, S, \mathbf{v}_s)$ defined implicitly by the condition

$$\mu_{\mathrm{chem}} := \frac{\partial \varepsilon}{\partial \rho}(\rho, S, \mathbf{v}_s - \mathbf{v}_n) = \frac{1}{2}\|\mathbf{v}_n\|^2 + \nu. \tag{8.16}$$

Therefore, ρ is not a variable in this approach. By the implicit function theorem, the relation above defines a unique function ρ, provided the function ε verifies the condition

$$\frac{\partial^2 \varepsilon}{\partial \rho^2}(r, s, v_x) \neq 0, \quad \text{for all } (r, s, v_x) \in \mathbb{R} \times \mathbb{R} \times T\mathcal{D}. \tag{8.17}$$

The affine Euler–Poincaré equations for the two-fluid model. Using the definition (8.16) of the function ρ, we compute below the functional derivatives of l. We have

$$\begin{aligned}
\frac{\delta l}{\delta \mathbf{v}_n} &= \rho \mathbf{v}_n^\flat + \frac{1}{2}\|\mathbf{v_n}\|^2 \frac{\partial \rho}{\partial \mathbf{v}_n} + \nu \frac{\partial \rho}{\partial \mathbf{v}_n} - \mu_{\text{chem}} \frac{\partial \rho}{\partial \mathbf{v}_n} + \mathbf{p} = \rho \mathbf{v}_n^\flat + \mathbf{p} =: \mathbf{m},\\
\frac{\delta l}{\delta \nu} &= \frac{1}{2}\|\mathbf{v_n}\|^2 \frac{\partial \rho}{\partial \nu} + \rho + \nu \frac{\partial \rho}{\partial \nu} - \mu_{\text{chem}} \frac{\partial \rho}{\partial \nu} = \rho,\\
\frac{\delta l}{\delta S} &= \frac{1}{2}\|\mathbf{v_n}\|^2 \frac{\partial \rho}{\partial S} + \nu \frac{\partial \rho}{\partial S} - \mu_{\text{chem}} \frac{\partial \rho}{\partial S} - T = -T,\\
\frac{\delta l}{\delta \mathbf{v}_s} &= \frac{1}{2}\|\mathbf{v_n}\|^2 \frac{\partial \rho}{\partial \mathbf{v}_s} + \nu \frac{\partial \rho}{\partial \mathbf{v}_s} - \mu_{\text{chem}} \frac{\partial \rho}{\partial \mathbf{v}_s} - \mathbf{p} = -\mathbf{p}.
\end{aligned}$$

Using the affine Euler–Poincaré equations (3.13), we obtain the following equations for ρ, S, and $\mathbf{v}_s$:

$$\begin{gathered}
\frac{\partial}{\partial t}\rho + \operatorname{div}\big(\rho \mathbf{v}_n + \mathbf{p}^\sharp\big) = 0, \qquad \frac{\partial}{\partial t} S + \operatorname{div}(S\mathbf{v}_n) = 0,\\
\frac{\partial}{\partial t}\mathbf{v}_s + \operatorname{grad}\left(g(\mathbf{v}_s, \mathbf{v}_n) + \mu_{\text{chem}} - \frac{1}{2}\|\mathbf{v}_n\|^2 \right) + \big(\mathbf{i}_{\mathbf{v}_n} \mathbf{d}\mathbf{v}_s^\flat\big)^\sharp = 0.
\end{gathered}$$

The last equation can be rewritten as

$$\frac{\partial}{\partial t}\mathbf{v}_s + \nabla_{\mathbf{v}_s}\mathbf{v}_s = -\operatorname{grad}\left(\mu_{\text{chem}} - \frac{1}{2}\|\mathbf{v}_s - \mathbf{v}_n\|^2 \right) + \big(\mathbf{i}_{\mathbf{v}_s - \mathbf{v}_n} \mathbf{d}\mathbf{v}_s^\flat\big)^\sharp. \tag{8.18}$$

When $\mathcal{D}$ is three-dimensional, the last term reads

$$\big(\mathbf{i}_{\mathbf{v}_s - \mathbf{v}_n} \mathbf{d}\mathbf{v}_s^\flat\big)^\sharp = \big(\star \mathbf{d}\mathbf{v}_s^\flat\big)^\sharp \times (\mathbf{v}_s - \mathbf{v}_n) = \operatorname{curl}\mathbf{v}_s \times (\mathbf{v}_s - \mathbf{v}_n) = (\mathbf{v}_n - \mathbf{v}_s) \times \operatorname{curl}\mathbf{v}_s.$$

Using the equality

$$-\rho \mathbf{d}\left(\mu_{\text{chem}} - \frac{1}{2}\|\mathbf{v}_n\|^2 \right) - S\mathbf{d}T = -\mathbf{d}p - \mathbf{p} \cdot \nabla_{\_}\mathbf{v}_s + \mathbf{m} \cdot \nabla_{\_}\mathbf{v}_n,$$

where $p := -\varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n) + \mu_{\text{chem}}\rho + ST$ is the *Euler pressure law*, the equation for $\mathbf{m}$ is computed as follows:

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{m} &= -\pounds_{\mathbf{v}_n}\mathbf{m} - \operatorname{div}(\mathbf{v}_n)\mathbf{m} - \rho \mathbf{d}\left(\mu_{\text{chem}} - \frac{1}{2}\|\mathbf{v}_n\|^2 \right) - S\mathbf{d}T - \operatorname{div}\big(\mathbf{p}^\sharp\big)\mathbf{v}_s^\flat + \mathbf{d}\mathbf{v}_s^\flat\big(_, \mathbf{p}^\sharp\big)\\
&= -\pounds_{\mathbf{v}_n}\mathbf{m} - \operatorname{div}(\mathbf{v}_n)\mathbf{m} + \mathbf{m} \cdot \nabla_{_}\mathbf{v}_n - \mathbf{d}p - \mathbf{p} \cdot \nabla_{_}\mathbf{v}_s - \operatorname{div}\big(\mathbf{p}^\sharp\big)\mathbf{v}_s^\flat + \mathbf{d}\mathbf{v}_s^\flat\big(_, \mathbf{p}^\sharp\big)\\
&= -\nabla_{\mathbf{v}_n}\mathbf{m} - \operatorname{div}(\mathbf{v}_n)\mathbf{m} - \mathbf{d}p - \nabla_{\mathbf{p}^\sharp}\mathbf{v}_n^\flat - \operatorname{div}\big(\mathbf{p}^\sharp\big)\mathbf{v}_n^\flat\\
&= -\operatorname{Div}\mathbf{T},
\end{aligned}$$

where $\mathbf{T}$ is the $(1,1)$ *superfluid stress tensor* defined by

$$\mathbf{T} := \mathbf{v}_n \otimes \mathbf{m} + \mathbf{p}^\sharp \otimes \mathbf{v}_s^\flat + p\delta,$$

δ is the Kronecker $(1,1)$ tensor, and Div is the divergence of a $(1,1)$ tensor, defined as the trace of the bilinear map

$$(\alpha, v) \mapsto \nabla_v \mathbf{T}(\alpha, \_) \in \Omega^1(\mathcal{D}).$$

In coordinates, we have

$$\mathbf{T}^i_j = \mathbf{v}^i_n \mathbf{m}_j + \mathbf{p}^i \mathbf{v}_{sj} + p\delta^i_j$$

and

$$(\operatorname{Div}\mathbf{T})_j = (\nabla \mathbf{T})^i_{ji} = \partial_i \mathbf{T}^i_j + \mathbf{T}^l_j \Gamma^i_{il} - \mathbf{T}^i_l \Gamma^l_{ij}.$$

The equations for superfluid dynamics are therefore given by

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{m} = -\operatorname{Div}\mathbf{T}, \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}\big(\rho \mathbf{v}_n + \mathbf{p}^\sharp\big) = 0, \quad \dfrac{\partial}{\partial t} S + \operatorname{div}(S\mathbf{v}_n) = 0, \\ \dfrac{\partial}{\partial t}\mathbf{v}_s + \operatorname{grad}\left(g(\mathbf{v}_s, \mathbf{v}_n) + \mu_{\text{chem}} - \dfrac{1}{2}\|\mathbf{v}_n\|^2 \right) + \big(\mathbf{i}_{\mathbf{v}_n} \mathbf{d}\mathbf{v}^\flat_s\big)^\sharp = 0, \end{cases} \tag{8.19}$$

where

$$\mathbf{T} := \mathbf{v}_n \otimes \mathbf{m} + \mathbf{p}^\sharp \otimes \mathbf{v}^\flat_s + p\delta, \quad \text{and} \quad p := -\varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n) + \mu_{\text{chem}}\rho + ST.$$

Thus we have recovered Eqs. (1a)–(1d) in [15], in the particular case of the two-fluid model. By Legendre transformation of the reduced Lagrangian (8.15), we obtain the reduced Hamiltonian

$$h(\mathbf{m}, \rho, S, \mathbf{v}_s) = -\frac{1}{2}\int_{\mathcal{D}} \rho \|\mathbf{v}_n\|^2 \mu + \int_{\mathcal{D}} (\mathbf{m}\cdot\mathbf{v}_n)\mu + \int_{\mathcal{D}} \varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n)\mu, \tag{8.20}$$

where $\mathbf{v}_n = \mathbf{v}_n(\mathbf{m}, \rho, S, \mathbf{v}_s)$ is the vector field defined by the implicit condition

$$\mathbf{m} - \rho \mathbf{v}^\flat_n = \frac{\partial \varepsilon}{\partial \mathbf{r}}(\rho, S, \mathbf{v}_s - \mathbf{v}_n) =: \mathbf{p}. \tag{8.21}$$

By the implicit function theorem, the above relation defines a unique vector field $\mathbf{v}_n$, provided the function ε verifies the condition that

$$u_x \mapsto \frac{\partial^2 \varepsilon}{\partial \mathbf{r}^2}(r, s, v_x)\cdot u_x - r u_x$$

is a bijective linear map. If (8.17) holds, then this condition is equivalent to saying that the Legendre transformation is invertible. Of course, the functions l and h are the values at the identity of the corresponding unreduced right-invariant Lagrangian and Hamiltonian $L(\mathbf{v}_\eta, \nu_\chi, S, \mathbf{v}_s) = L_{(S,\mathbf{v}_s)}(\mathbf{v}_\eta, \nu_\chi)$ and $H(\mathbf{m}_\eta, \rho_\chi, S, \mathbf{v}_s) = H_{(S,\mathbf{v}_s)}(\mathbf{m}_\eta, \rho_\chi)$, where

$$L_{(S,\mathbf{v}_s)} : T\big(\operatorname{Diff}(\mathcal{D}) \circledS \mathcal{F}\big(\mathcal{D}, S^1\big)\big) \to \mathbb{R} \quad \text{and} \quad H_{(S,\mathbf{v}_s)} : T^*\big(\operatorname{Diff}(\mathcal{D}) \circledS \mathcal{F}\big(\mathcal{D}, S^1\big)\big) \to \mathbb{R}.$$

Lagrangian reduction for superfluids. A curve $(\eta, \chi) \in \operatorname{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, S^1)$ is a solution of the Euler–Lagrange equations associated to the Lagrangian $L_{(S_0,\mathbf{v}_{s0})}$ if and only if the curve

$$(\mathbf{v}_n, \nu) := \big(\dot\eta \circ \eta^{-1}, (TR_{\chi^{-1}}\dot\chi)\circ\eta^{-1}\big)$$

is a solution of the superfluids equations (8.19) with initial conditions $(S_0, \mathbf{v}_{s0})$.

The evolution of the advected quantities is given by

$$S = J(\eta^{-1})\big(S_0 \circ \eta^{-1}\big) \quad \text{and} \quad \mathbf{v}_s = \big(\eta_*\big(\mathbf{v}_{s0}^\flat + \mathbf{d}\chi^{-1}\big)\big)^\sharp.$$

Note that the evolution of the *superfluid vorticity* is given by

$$\mathbf{d}\mathbf{v}_s^\flat = \eta_* \mathbf{d}\mathbf{v}_{s0}^\flat,$$

therefore, the irrotationality condition $\mathbf{d}\mathbf{v}_s^\flat = 0$ ($\operatorname{curl}\mathbf{v}_s = 0$ for the three-dimensional case) is preserved.

Hamiltonian reduction for superfluids. A curve $(\mathbf{m}_\eta, \rho_\chi) \in T^*[\operatorname{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, S^1)]$ is a solution of Hamilton's equations associated to the superfluid Hamiltonian $H_{(S_0,\mathbf{v}_{s0})}$ if and only if the curve

$$(\mathbf{m}, \rho) := J\big(\eta^{-1}\big)\big(\mathbf{m}_\eta \circ \eta^{-1}, \rho_\chi \circ \eta^{-1}\big)$$

is a solution of the system (8.19) with initial conditions $(S_0, \mathbf{v}_{s0})$.

The associated Poisson bracket for superfluids is

$$\begin{aligned}\{f, g\}(\mathbf{m}, \rho, S, \mathbf{v}_s) = {} & \int_{\mathcal{D}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}}\right]\mu + \int_{\mathcal{D}} \rho \cdot \left(\mathbf{d}\frac{\delta f}{\delta \rho} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \rho} \cdot \frac{\delta f}{\delta \mathbf{m}}\right)\mu \\ & + \int_{\mathcal{D}} S \cdot \left(\mathbf{d}\frac{\delta f}{\delta S} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta S} \cdot \frac{\delta f}{\delta \mathbf{m}}\right)\mu \\ & + \int_{\mathcal{D}} \left[\left(\mathbf{d}\frac{\delta f}{\delta \rho} + \pounds_{\frac{\delta f}{\delta \mathbf{m}}}\mathbf{v}_s^\flat\right) \cdot \frac{\delta g}{\delta \mathbf{v}_s}^\sharp - \left(\mathbf{d}\frac{\delta g}{\delta \rho} + \pounds_{\frac{\delta g}{\delta \mathbf{m}}}\mathbf{v}_s^\flat\right) \cdot \frac{\delta f}{\delta \mathbf{v}_s}^\sharp\right]\mu. \end{aligned} \tag{8.22}$$

Multivelocity superfluids. We now quickly explain how to generalize the preceding approach to the case of superfluids with m velocities. Consider the semidirect product

$$G := \operatorname{Diff}(\mathcal{D}) \circledS \underbrace{\big[\mathcal{F}\big(\mathcal{D}, S^1\big) \times \cdots \times \mathcal{F}\big(\mathcal{D}, S^1\big)\big]}_{m \text{ times}},$$

where the group on the right is the direct product of the groups $\mathcal{F}(\mathcal{D}, S^1)$. The semidirect product is associated to the right action of $\operatorname{Diff}(\mathcal{D})$ given by

$$\big(\chi^1, \ldots, \chi^m\big) \mapsto \big(\chi^1 \circ \eta, \ldots, \chi^m \circ \eta\big).$$

The affine advected quantities are the m superfluid velocities $(\mathbf{v}_s^1, \ldots, \mathbf{v}_s^m) \in \mathfrak{X}(\mathcal{D})^m$ on which an element $(\eta, \chi^1, \ldots, \chi^m)$ acts as

$$\big(\mathbf{v}_s^1, \ldots, \mathbf{v}_s^m\big) \mapsto \big(\eta^*\big(\mathbf{v}_s^1\big)^\flat + \mathbf{d}\chi^1, \ldots, \eta^*\big(\mathbf{v}_s^m\big)^\flat + \mathbf{d}\chi^m\big)^\sharp.$$

The reduced Lagrangian is defined on $[\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D})^m] \circledS [\mathcal{F}(\mathcal{D}) \oplus \mathfrak{X}(\mathcal{D})^m]$ and is given by

$$l\big(\mathbf{v}_n, \big(\nu^\alpha\big), S, \big(\mathbf{v}_s^\alpha\big)\big) := \frac{1}{2}\int_{\mathcal{D}} \rho \|\mathbf{v}_n\| \mu + \sum_{\alpha=1}^m \int_{\mathcal{D}} \rho^\alpha \nu^\alpha - \int_{\mathcal{D}} \varepsilon\big(\big(\rho^\alpha\big), S, \big(\mathbf{v}_s^\alpha - \mathbf{v}_n\big)\big)\mu,$$

where ρ^α, the *mass density* of the condensate particles with flow $\mathbf{v}_s^\alpha$, is the function $\rho^\alpha = \rho^\alpha(\mathbf{v}_n, (\nu^\alpha), S, (\mathbf{v}_s^\alpha))$ defined by the condition

$$\mu_{\text{chem}}^\alpha := \frac{\partial \varepsilon}{\partial \rho^\alpha}\big((\rho^\alpha), S, (\mathbf{v}_s^\alpha - \mathbf{v}_n)\big) = \frac{1}{2}\|\mathbf{v}_n\|^2 + \nu^\alpha.$$

By the implicit function theorem, these conditions uniquely determine ρ^α, provided the matrix

$$\left(\frac{\partial^2 \varepsilon}{\partial \rho^\alpha \partial \rho^\beta}\big((r^i), s, (v_x^j)\big)\right)_{\alpha\beta}$$

is invertible for all $r^i, s, v_x^j \in \mathbb{R}\times\mathbb{R}\times T\mathcal{D}$. The variable ρ is defined by $\rho := \rho^1 + \cdots + \rho^m$ and denotes the *total mass density*. Using the notations

$$\mu_{\text{chem}}^\alpha := \frac{\partial \varepsilon}{\partial \rho^\alpha}\big((\rho^\alpha), S, (\mathbf{v}_s^\alpha - \mathbf{v}_n)\big) \in \mathcal{F}(\mathcal{D}), \qquad T := \frac{\partial \varepsilon}{\partial S}\big((\rho^\alpha), S, (\mathbf{v}_s^\alpha - \mathbf{v}_n)\big) \in \mathcal{F}(\mathcal{D}),$$

$$\mathbf{p}^\alpha := \frac{\partial \varepsilon}{\partial \mathbf{r}^\alpha}\big((\rho^\alpha), S, (\mathbf{v}_s^\alpha - \mathbf{v}_n)\big) \in \Omega^1(\mathcal{D}),$$

we obtain the functional derivatives

$$\frac{\delta l}{\delta \mathbf{v}_n} = \rho \mathbf{v}_n^\flat + \sum_{\alpha=1}^m \mathbf{p}^\alpha, \qquad \frac{\delta l}{\delta \nu^\alpha} = \rho^\alpha, \qquad \frac{\delta l}{\delta S} = -T, \qquad \frac{\delta l}{\delta \mathbf{v}_s^\alpha} = -\mathbf{p}^\alpha$$

and the thermodynamic derivative identity for the internal energy

$$\mathbf{d}\big(\varepsilon\big((\rho^\alpha), S, (\mathbf{v}_s^\alpha - \mathbf{v}_n)\big)\big) = \sum_{\alpha=1}^m \mu_{\text{chem}}^\alpha \mathbf{d}\rho^\alpha + T\mathbf{d}S + \sum_{\alpha=1}^m \mathbf{p}^\alpha \cdot \nabla_-\big(\mathbf{v}_s^\alpha - \mathbf{v}_n\big) \in \Omega^1(\mathcal{D}).$$

Using the affine Euler–Poincaré equations associated to the group $\text{Diff}(\mathcal{D}) \circledS [\mathcal{F}(\mathcal{D}, S^1) \times \cdots \times \mathcal{F}(\mathcal{D}, S^1)]$, we obtain the equations for multivelocity superfluids

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{m} = -\operatorname{Div}\mathbf{T}, \\ \dfrac{\partial}{\partial t}\rho^\alpha + \operatorname{div}\big(\rho^\alpha \mathbf{v}_n + (\mathbf{p}^\alpha)^\sharp\big) = 0, \quad \dfrac{\partial}{\partial t}S + \operatorname{div}(S\mathbf{v}_n) = 0, \\ \dfrac{\partial}{\partial t}\mathbf{v}_s^\alpha + \operatorname{grad}\left(g\big(\mathbf{v}_s^\alpha, \mathbf{v}_n\big) + \mu_{\text{chem}}^\alpha - \dfrac{1}{2}\|\mathbf{v}_n\|^2\right) + \big(\mathbf{i}_{\mathbf{v}_n}\mathbf{d}\big(\mathbf{v}_s^\alpha\big)^\flat\big)^\sharp = 0, \end{cases} \tag{8.23}$$

where $\alpha = 1, \ldots, m$. The stress tensor $\mathbf{T}$ and the pressure p are given by

$$\mathbf{T} := \mathbf{v}_n \otimes \mathbf{m} + \sum_{\alpha=1}^m (\mathbf{p}^\alpha)^\sharp \otimes (\mathbf{v}_s^\alpha)^\flat + p\delta,$$

$$p := -\varepsilon\big((\rho^\alpha), S, (\mathbf{v}_s^\alpha - \mathbf{v}_n)\big) + \sum_{\alpha=1}^m \mu_{\text{chem}}^\alpha \rho^\alpha + ST.$$

By Legendre transformation, we obtain the Hamiltonian

$$h\big(\mathbf{m}, (\rho^\alpha), S, (\mathbf{v}_s^\alpha)\big) = -\frac{1}{2}\int_{\mathcal{D}} \rho\|\mathbf{v}_n\|^2\mu + \int_{\mathcal{D}} (\mathbf{m}\cdot\mathbf{v}_n)\mu + \int_{\mathcal{D}} \varepsilon\big((\rho^\alpha), S, (\mathbf{v}_s^\alpha - \mathbf{v}_n)\big)\mu,$$

where $\mathbf{v}_n = \mathbf{v}_n(\mathbf{m}, (\rho^\alpha), S, (\mathbf{v}_s^\alpha))$ is the vector field defined by the implicit condition

$$\mathbf{m} - \rho\mathbf{v}_n^\flat = \sum_{\alpha=1}^m \frac{\partial\varepsilon}{\partial\mathbf{r}^\alpha}\big((\rho^\alpha), S, (\mathbf{v}_s^\alpha - \mathbf{v}_n)\big).$$

By the implicit function theorem, the above relation defines a unique function $\mathbf{v}_n$, provided the function ε verifies the condition that the linear map

$$u_x \mapsto \sum_{\alpha,\beta=1}^m \frac{\partial^2\varepsilon}{\partial\mathbf{r}^\alpha\partial\mathbf{r}^\beta}(r, s, v_x)\cdot u_x - r u_x$$

is bijective.

Lagrangian and Hamiltonian reductions hold as in the two-fluid model. The evolutions of S and $\mathbf{v}_s^\alpha$ are given by

$$S = J\big(\eta^{-1}\big)\big(S_0\circ\eta^{-1}\big) \quad \text{and} \quad \mathbf{v}_s^\alpha = \eta_*\big(\big(\mathbf{v}_{s0}^\alpha\big)^\flat + \mathbf{d}\big(\chi^\alpha\big)^{-1}\big)^\sharp,$$

and the irrotationality condition $\mathbf{d}(\mathbf{v}_s^\alpha)^\flat = 0$ is preserved.

The associated Poisson bracket is given by

$$\begin{aligned}\{f, g\}(\mathbf{m}, \rho, S, \mathbf{v}_s) = {}& \int_{\mathcal{D}} \mathbf{m}\cdot\left[\frac{\delta f}{\delta\mathbf{m}}, \frac{\delta g}{\delta\mathbf{m}}\right]\mu + \sum_{\alpha=1}^m\int_{\mathcal{D}} \rho^\alpha\cdot\left(\mathbf{d}\frac{\delta f}{\delta\rho^\alpha}\cdot\frac{\delta g}{\delta\mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta\rho^\alpha}\cdot\frac{\delta f}{\delta\mathbf{m}}\right)\mu \\ & + \int_{\mathcal{D}} S\cdot\left(\mathbf{d}\frac{\delta f}{\delta S}\cdot\frac{\delta g}{\delta\mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta S}\cdot\frac{\delta f}{\delta\mathbf{m}}\right)\mu \\ & + \sum_{\alpha=1}^m\int_{\mathcal{D}}\left[\left(\mathbf{d}\frac{\delta f}{\delta\rho^\alpha} + \pounds_{\frac{\delta f}{\delta\mathbf{m}}}\big(\mathbf{v}_s^\alpha\big)^\flat\right)\cdot\frac{\delta g}{\delta\mathbf{v}_s^\alpha}^\sharp - \left(\mathbf{d}\frac{\delta g}{\delta\rho^\alpha} + \pounds_{\frac{\delta g}{\delta\mathbf{m}}}\big(\mathbf{v}_s^\alpha\big)^\flat\right)\cdot\frac{\delta f}{\delta\mathbf{v}_s^\alpha}^\sharp\right]\mu.\end{aligned}$$

The γ-circulation gives

$$\frac{d}{dt}\oint_{c_t}\big(\mathbf{v}_s^\alpha\big)^\flat = 0, \quad \text{for all } \alpha = 1, \dots, m,$$

where c_t is a loop which moves with the *normal fluid velocity* $\mathbf{v}_n$.

8.5. Superfluid Yang–Mills magnetohydrodynamics

In this paragraph we combine the Hamiltonian structures of Yang–Mills magnetohydrodynamics and superfluid dynamics, to obtain a new physical model for the theory of superfluid Yang–Mills magnetohydrodynamics as well as the corresponding Hamiltonian structure. In the Abelian case we recover the theory and the Hamiltonian structure derived in [15]. We need a slight generalization of

the geometric framework developed in Sections 3 and 6, namely we consider the group semidirect product

$$\mathrm{Diff}(\mathcal{D}) \circledS \big(\mathcal{F}(\mathcal{D}, \mathcal{O}) \times \mathcal{F}\big(\mathcal{D}, S^1\big)\big),$$

where $\mathcal{F}(\mathcal{D}, \mathcal{O}) \times \mathcal{F}(\mathcal{D}, S^1)$ is a direct product of groups on which $\mathrm{Diff}(\mathcal{D})$ acts as

$$(\chi_1, \chi_2) \mapsto (\chi_1 \circ \eta, \chi_2 \circ \eta).$$

The affine advected quantities are the *potential of the Yang–Mills fluid* A and the *superfluid velocity* $\mathbf{v}_s$, on which (η, χ_1, χ_2) acts as

$$A \mapsto \mathrm{Ad}_{\chi_1^{-1}} \eta^* A + \chi_1^{-1} T \chi_1 \quad \text{and} \quad \mathbf{v}_s \mapsto \big(\eta^* \mathbf{v}_s^\flat + \mathbf{d}\chi_2\big)^\sharp.$$

The reduced Hamiltonian is defined on the dual of the Lie algebra

$$\big[\mathfrak{X}(\mathcal{D}) \circledS \big(\mathcal{F}(\mathcal{D}, \mathfrak{o}) \times \mathcal{F}(\mathcal{D})\big)\big] \circledS \big(\mathcal{F}(\mathcal{D}) \times \Omega^1(\mathcal{D}, \mathfrak{o}) \times \mathfrak{X}(\mathcal{D})\big)$$

and is given by

$$h(\mathbf{m}, Q, \rho, S, A, \mathbf{v}_s) = -\frac{1}{2} \int_{\mathcal{D}} \rho \|\mathbf{v}_n\|^2 \mu + \int_{\mathcal{D}} (\mathbf{m} \cdot \mathbf{v}_n) \mu + \int_{\mathcal{D}} \varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n) \mu + \frac{1}{2} \int_{\mathcal{D}} \big\|\mathbf{d}^A A\big\|^2 \mu,$$

where $\mathbf{v}_n$ is the *normal fluid velocity* defined as in (8.21). This is simply the Hamiltonian (8.20) plus the energy of the Yang–Mills field. The norms are respectively associated to the metrics g and (gk), where g is a Riemannian metric on $\mathcal{D}$ and k is an Ad-invariant inner product on $\mathfrak{o}$. The affine Lie–Poisson equations associated to this Hamiltonian are computed to be

$$\begin{cases} \dfrac{\partial}{\partial t} \mathbf{m} = -\operatorname{Div} \mathbf{T}, \quad \dfrac{\partial}{\partial t} \rho + \operatorname{div}\big(\rho \mathbf{v}_n + \mathbf{p}^\sharp\big) = 0, \\ \dfrac{\partial}{\partial t} Q + \operatorname{div}(Q \mathbf{v}_n) = 0, \quad \dfrac{\partial}{\partial t} S + \operatorname{div}(S \mathbf{v}_n) = 0, \\ \dfrac{\partial}{\partial t} \mathbf{v}_s + \operatorname{grad}\left(g(\mathbf{v}_s, \mathbf{v}_n) + \mu_{\mathrm{chem}} - \dfrac{1}{2} \|\mathbf{v}_n\|^2 \right) + \big(\mathbf{i}_{\mathbf{v}_n} \mathbf{d} \mathbf{v}_s^\flat\big)^\sharp = 0, \\ \dfrac{\partial}{\partial t} A + \mathbf{d}^A\big(A(\mathbf{v}_n)\big) + \mathbf{i}_{\mathbf{v}_n} B = 0, \quad B := \mathbf{d}^A A. \end{cases} \tag{8.24}$$

For superfluid Yang–Mills magnetohydrodynamics the stress tensor is given by

$$\mathbf{T} := \mathbf{v}_n \otimes \mathbf{m} + \mathbf{p}^\sharp \otimes \mathbf{v}_s^\flat + B \cdot B + p\delta,$$

where $B \cdot B$ is the $(1,1)$ tensor field defined by

$$(B \cdot B)^i_j := B^b_{ij} B^{li}_b,$$

and where the *pressure* is given by $p := -\varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n) + \mu_{\mathrm{chem}} \rho + ST - \frac{1}{2}\|B\|^2$.

The corresponding Hamiltonian reduction and affine Lie–Poisson bracket can be found as before and the evolutions of the advected quantities are given by

$$S = J\big(\eta^{-1}\big)\big(S_0 \circ \eta^{-1}\big), \quad A = \eta_*\big(\mathrm{Ad}_\chi A_0 + \chi_1 T \chi_1^{-1}\big) \quad \text{and} \quad \mathbf{v}_s = \big(\eta_*\big(\mathbf{v}_{s0}^\flat + \mathbf{d}\chi_2^{-1}\big)\big)^\sharp.$$

As in the preceding example, it is possible to generalize this approach to multivelocity superfluids. In this case, the γ-circulation gives

$$\frac{d}{dt}\oint_{c_t}\left(\mathbf{v}_s^\alpha\right)^\flat = 0, \quad \text{for all } \alpha = 1,\dots,m, \quad \text{and} \quad \frac{d}{dt}\oint_{c_t} A = 0,$$

where c_t is a loop which moves with the *normal fluid velocity* $\mathbf{v}_n$.

8.6. Superfluid Hall magnetohydrodynamics

The Hamiltonian formulation of superfluid Hall magnetohydrodynamics is given in [15]. As one can guess, the Hamiltonian structure of these equations combines the Hamiltonian structures of Hall magnetohydrodynamics and of superfluids. This is still true at the group level and we will obtain the equations by affine Lie–Poisson reduction associated to the group

$$G := \left[\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}\left(\mathcal{D}, S^1\right)\right] \times \mathrm{Diff}(\mathcal{D}).$$

In this expression, the symbol $\times$ denotes the *direct product* of the two groups. The advected quantities are

$$(S, \mathbf{u}; n) \in \mathcal{F}(\mathcal{D}) \times \mathfrak{X}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}).$$

The variable S is the *entropy density* of the normal flow, the other variables will be interpreted later. The action of $(\eta, \chi; \xi) \in G$ is given by

$$(S, \mathbf{u}; n) \mapsto \left(J\eta(S \circ \eta), \left(\eta^* \mathbf{u}^\flat + \mathbf{d}\chi\right)^\sharp; J\xi(n \circ \xi)\right).$$

The resulting affine Lie–Poisson equations consist of two systems, the affine Lie–Poisson equations associated to the variables $(\mathbf{m}, \rho, S, \mathbf{u})$ and the Lie–Poisson equations associated to the variables $(\mathbf{n}, n)$.

The Hamiltonian of superfluid Hall magnetohydrodynamics is defined on the dual Lie algebra

$$\begin{aligned}&\left(\left[\left(\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D})\right) \circledS \left(\mathcal{F}(\mathcal{D}) \oplus \mathfrak{X}(\mathcal{D})\right)\right] \times \left[\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D})\right]\right)^* \\ &\quad \cong \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}) \times \Omega^1(\mathcal{D}) \times \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D})\end{aligned}$$

and is given by

$$\begin{aligned}h(\mathbf{m}, \rho, S, \mathbf{u}; \mathbf{n}, n) := &-\frac{1}{2}\int_{\mathcal{D}} \rho \|\mathbf{v}_n\|^2 \mu + \int_{\mathcal{D}} \left(\left(\mathbf{m} - \frac{a\rho}{R} A\right) \cdot \mathbf{v}_n\right)\mu \\ &+ \int_{\mathcal{D}} \varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n)\mu + \frac{1}{2}\int_{\mathcal{D}} \|\mathbf{d}A\|^2 \mu,\end{aligned}$$

where $\mathbf{m}$ is the total momentum density of the fluid, $\mathbf{n}$ is interpreted below as the momentum density associated to the *electron fluid velocity* $\mathbf{v} := \delta h/\delta \mathbf{n}$, $\mathbf{v}_n$ is the *velocity of the normal flow*, $\mathbf{v}_s := \mathbf{u} - \frac{a}{R} A^\sharp$ is the *superfluid velocity*, and ε is the *internal energy density*. The one-form A is defined by

$$A := R\frac{\mathbf{n}}{n}.$$

The norm in the first term is taken with respect to a fixed Riemannian metric g on $\mathcal{D}$. The velocity $\mathbf{v}_n$ is the function $\mathbf{v}_n = \mathbf{v}_n(\mathbf{m}, \rho, S, \mathbf{u}; \mathbf{n}, n)$ defined by the implicit condition

$$\mathbf{m} - \rho \mathbf{v}_n^\flat - \frac{a\rho}{R} A = \frac{\partial \varepsilon}{\partial \mathbf{r}}(\rho, S, \mathbf{v}_s - \mathbf{v}_n) =: \mathbf{p}.$$

By the implicit function theorem, the above relation defines a unique function $\mathbf{v}_n$, provided the function ε verifies the condition that the linear map

$$u_x \mapsto \frac{\partial^2 \varepsilon}{\partial \mathbf{r}^2}(r, s, v_x, w_x) \cdot u_x - r u_x$$

is bijective for all $(r, s, v_x, w_x) \in \mathbb{R} \times \mathbb{R} \times T\mathcal{D} \times T\mathcal{D}$.

Using the notations

$$\mu_{\text{chem}} := \frac{\partial \varepsilon}{\partial \rho}(\rho, S, \mathbf{v}_s - \mathbf{v}_n) \in \mathcal{F}(\mathcal{D}), \qquad T := \frac{\partial \varepsilon}{\partial S}(\rho, S, \mathbf{v}_s - \mathbf{v}_n) \in \mathcal{F}(\mathcal{D}),$$

$$\mathbf{p} := \frac{\partial \varepsilon}{\partial \mathbf{r}}(\rho, S, \mathbf{v}_s - \mathbf{v}_n) \in \Omega^1(\mathcal{D}),$$

the functional derivatives of h are computed to be

$$\frac{\delta h}{\delta \mathbf{m}} = \mathbf{v}_n, \qquad \frac{\delta h}{\delta \rho} = -\frac{1}{2}\|\mathbf{v}_n\|^2 - \frac{a}{R} A \cdot \mathbf{v}_n + \mu_{\text{chem}}, \qquad \frac{\delta h}{\delta S} = T, \qquad \frac{\delta h}{\delta \mathbf{u}} = \mathbf{p},$$

$$\mathbf{v} := \frac{\delta h}{\delta \mathbf{n}} = -\frac{1}{n}\big(a\rho \mathbf{v}_n + a\mathbf{p}^\sharp + R(\operatorname{div} B)^\sharp\big), \qquad \frac{\delta h}{\delta n} = -\frac{1}{R} A \cdot \mathbf{v}.$$

The vector field $\mathbf{v}$ is interpreted as the *electron fluid velocity*. The equations for ρ, S, and n are given by

$$\frac{\partial}{\partial t}\rho + \operatorname{div}\big(\rho \mathbf{v}_n + \mathbf{p}^\sharp\big) = 0, \quad \frac{\partial}{\partial t} S + \operatorname{div}(S\mathbf{v}_n) = 0, \quad \text{and} \quad \frac{\partial}{\partial t} n + \operatorname{div}(n\mathbf{v}) = 0.$$

Using the expression of $\mathbf{v}$ in terms of $\mathbf{v}_n$ we obtain that $\operatorname{div}(n\mathbf{v}) = -a \operatorname{div}(\rho \mathbf{v}_n + \mathbf{p}^\sharp)$ which proves that

$$\frac{\partial}{\partial t}(a\rho + n) = 0.$$

Thus, if we assume that the initial conditions verify $a\rho_0 + n_0 = 0$, then we have $a\rho + n = 0$ for all time. The equations for A and $\mathbf{u}$ are computed to be

$$\frac{\partial}{\partial t} A = -\mathbf{i}_{\mathbf{v}_n} B - \frac{1}{\rho} \mathbf{i}_{\mathbf{p}^\sharp} B - \frac{R}{a\rho} \mathbf{i}_{(\operatorname{div} B)^\sharp} B,$$

$$\frac{\partial}{\partial t}\mathbf{u} + \operatorname{grad}\left(g(\mathbf{v}_s, \mathbf{v}_n) + \mu_{\text{chem}} - \frac{1}{2}\|\mathbf{v}_n\|^2 \right) + \big(\mathbf{i}_{\mathbf{v}_n} \mathbf{d}\mathbf{u}^\flat\big)^\sharp = 0.$$

From these two equations we obtain the evolution of the superfluid velocity $\mathbf{v}_s = \mathbf{u} - \frac{a}{R} A^\sharp$ as

$$\frac{\partial}{\partial t}\mathbf{v}_s = -\operatorname{grad}\left(g(\mathbf{v}_s, \mathbf{v}_n) + \mu_{\text{chem}} - \frac{1}{2}\|\mathbf{v}_n\|^2 \right) + \left(\frac{a}{R\rho} \mathbf{i}_{\mathbf{p}^\sharp} B + \frac{1}{\rho} \mathbf{i}_{(\operatorname{div} B)^\sharp} B - \mathbf{i}_{\mathbf{v}_n} \mathbf{d}\mathbf{v}_s^\flat \right)^\sharp.$$

Doing computations similar to those for superfluids we obtain that the equation for $\mathbf{m}+\mathbf{n}$ is given by

$$\frac{\partial}{\partial t}(\mathbf{m}+\mathbf{n}) = -\operatorname{Div}\mathbf{T},$$

where the stress tensor $\mathbf{T}$ is given by

$$\mathbf{T} = \mathbf{v}_n \otimes \left(\rho \mathbf{v}_n^\flat + \mathbf{p}\right) + \mathbf{p}^\sharp \otimes \mathbf{v}_s^\flat + B\cdot B + p\delta, \quad p = \rho\mu_{\text{chem}} + ST - \varepsilon - \frac{1}{2}\|B\|^2.$$

Thus, we have obtained the following equations

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{m}+\mathbf{n}) = -\operatorname{Div}\mathbf{T}, \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}\left(\rho\mathbf{v}_n + \mathbf{p}^\sharp\right) = 0, \quad \dfrac{\partial}{\partial t}S + \operatorname{div}(S\mathbf{v}_n) = 0, \\ \dfrac{\partial}{\partial t}A = -\mathbf{i}_{\mathbf{v}_n}B - \dfrac{1}{\rho}\mathbf{i}_{\mathbf{p}^\sharp}B - \dfrac{R}{a\rho}\mathbf{i}_{(\operatorname{div}B)^\sharp}B, \\ \dfrac{\partial}{\partial t}\mathbf{v}_s = -\operatorname{grad}\left(g(\mathbf{v}_s,\mathbf{v}_n) + \mu_{\text{chem}} - \dfrac{1}{2}\|\mathbf{v}_n\|^2\right) \\ \qquad\qquad + \left(\dfrac{a}{R\rho}\mathbf{i}_{\mathbf{p}^\sharp}B + \dfrac{1}{\rho}\mathbf{i}_{(\operatorname{div}B)^\sharp}B - \mathbf{i}_{\mathbf{v}_n}\mathbf{d}\mathbf{v}_s^\flat\right)^\sharp. \end{cases} \tag{8.25}$$

These are the equations for superfluid Hall magnetohydrodynamics as given in [15, Eqs. (35a)–(35e)]. When $\mathcal{D}$ is three-dimensional, the two last equations read

$$\frac{\partial}{\partial t}A^\sharp = \left(\mathbf{v}_n + \frac{1}{\rho}\mathbf{p}^\sharp - \frac{R}{a\rho}\operatorname{curl}\mathbf{B}\right)\times\mathbf{B} \quad \text{and}$$

$$\frac{\partial}{\partial t}\mathbf{v}_s = -\operatorname{grad}\left(g(\mathbf{v}_s,\mathbf{v}_n) + \mu_{\text{chem}} - \frac{1}{2}\|\mathbf{v}_n\|^2\right) + \mathbf{v}_n\times\operatorname{curl}\mathbf{v}_s + \frac{1}{\rho}\left(\operatorname{curl}\mathbf{B} - \frac{a}{R}\mathbf{p}^\sharp\right)\times\mathbf{B}.$$

Hamiltonian reduction for superfluid Hall magnetohydrodynamics. Consider the right-invariant Hamiltonian function $H(\mathbf{m}_\eta, \rho_\chi, S, \mathbf{u}; \mathbf{n}_\xi, n) = H_{(S,\mathbf{u};n)}(\mathbf{m}_\eta, \rho_\chi; \mathbf{n}_\xi)$ induced by h and suppose that we have $a\rho_0 + n_0 = 0$. A smooth curve

$$(\mathbf{m}_\eta, \rho_\chi; \mathbf{n}_\xi) \in T^*\left[\left(\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}\left(\mathcal{D}, S^1\right)\right)\times\operatorname{Diff}(\mathcal{D})\right]$$

is a solution of Hamilton's equations associated to $H_{(S_0,\mathbf{u}_0;n_0)}$ and with the initial condition ρ_0 if and only if the curve

$$(\mathbf{m}, \rho; \mathbf{n}) := \left(J\left(\eta^{-1}\right)\left(\mathbf{m}_\eta\circ\eta^{-1}\right), J\left(\eta^{-1}\right)\left(\rho_\chi\circ\eta^{-1}\right); J\left(\xi^{-1}\right)\left(\mathbf{n}\circ\xi^{-1}\right)\right)$$

is a solution of Eqs. (8.25), where $\mathbf{v}_s = \mathbf{u} - aA^\sharp/R = \mathbf{u} - a\mathbf{n}/n$.

The Poisson bracket for superfluid Hall magnetohydrodynamics is the sum of the affine Lie–Poisson bracket associated to the variables $(\mathbf{m}, \rho, S, \mathbf{u})$ and the Lie–Poisson bracket associated to the variables $(\mathbf{n}, n)$.

The γ-circulation gives

$$\frac{d}{dt}\oint_{c_t}\mathbf{u}^\flat = 0.$$

Using the definition $\mathbf{v}_s := \mathbf{u} - \frac{a}{R}A^\sharp$, we obtain

$$\frac{d}{dt}\oint_{c_t}\left(\mathbf{v}_s^\flat + \frac{a}{R}A^\sharp\right) = 0,$$

where c_t is a loop which moves with the *normal fluid velocity* $\mathbf{v}_n$. The Kelvin–Noether theorem associated to the variable $\mathbf{n}$ gives

$$\frac{d}{dt}\oint_{d_t} A = 0,$$

where d_t is a loop which moves with the *electron fluid velocity* $\mathbf{v}$.

8.7. HVBK dynamics for superfluid ^{4}He with vortices

The Hall–Vinen–Bekarevich–Khalatnikov (HVBK) equations describe superfluid Helium turbulence. We consider the version of HVBK equations, together with its Hamiltonian structure, as given in [12]. It turns out that this Hamiltonian structure is the same as that of superfluid Hall magnetohydrodynamics, that is, it is obtained by affine Lie–Poisson reduction associated to the group

$$G := \left[\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}\left(\mathcal{D}, S^1\right)\right] \times \mathrm{Diff}(\mathcal{D}).$$

As before, the advected quantities are

$$(S, \mathbf{u}; n) \in \mathcal{F}(\mathcal{D}) \times \mathfrak{X}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}),$$

where S is the *entropy density of the normal flow*. The action of $(\eta, \chi; \xi) \in G$ is given by

$$(S, \mathbf{u}; n) \mapsto \left(J\eta(S \circ \eta), \left(\eta^*\mathbf{u}^\flat + \mathbf{d}\chi\right)^\sharp; J\xi(n \circ \xi)\right).$$

The resulting affine Lie–Poisson equations consist of two systems, the affine Lie–Poisson equations associated to the variables $(\mathbf{m}, \rho, S, \mathbf{u})$ and the Lie–Poisson equations associated to the variables $(\mathbf{n}, n)$.

For simplicity we assume that the manifold $\mathcal{D}$ is three-dimensional. The Hamiltonian of HVBK dynamics is defined on the dual Lie algebra

$$\begin{aligned}&\left(\left[\left(\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D})\right) \circledS \left(\mathcal{F}(\mathcal{D}) \oplus \mathfrak{X}(\mathcal{D})\right)\right] \times \left[\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D})\right]\right)^* \\ &\quad \cong \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}) \times \Omega^1(\mathcal{D}) \times \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D})\end{aligned}$$

and is given by

$$h(\mathbf{m}, \rho, S, \mathbf{u}; \mathbf{n}, n) := -\frac{1}{2}\int_{\mathcal{D}} \rho\|\mathbf{v}_n\|^2\mu + \int_{\mathcal{D}}\left((\mathbf{m} - \rho A)\cdot\mathbf{v}_n\right)\mu + \int_{\mathcal{D}} \varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n, \boldsymbol{\omega})\mu,$$

where $\mathbf{m}$ is the total momentum density of the fluid, $\mathbf{n}$ is interpreted below as the momentum density associated to the *vortex fluid velocity* $\mathbf{v}_l := \delta h/\delta\mathbf{n}$, $\mathbf{v}_n$ is the *velocity of the normal flow*, $\mathbf{v}_s := \mathbf{u} - A^\sharp$ is the *superfluid velocity*, $\boldsymbol{\omega} := \mathrm{curl}\,\mathbf{v}_s$ is the *superfluid vorticity*, and ε is the *internal energy density*. The one-form A is defined by

$$A := -\frac{\mathbf{n}}{n}.$$

Note that the Hamiltonian of HVBK dynamics is similar to that of Hall magnetohydrodynamics, for $R = a = -1$. The norm in the first term and the operator curl are taken with respect to a fixed Riemannian metric g on $\mathcal{D}$. The velocity $\mathbf{v}_n$ is the function $\mathbf{v}_n = \mathbf{v}_n(\mathbf{m}, \rho, S, \mathbf{u}; \mathbf{n}, n)$ defined by the implicit condition

$$\mathbf{m} - \rho \mathbf{v}_n^\flat - \rho A = \frac{\partial \varepsilon}{\partial \mathbf{r}}(\rho, S, \mathbf{v}_s - \mathbf{v}_n, \boldsymbol{\omega}) =: \mathbf{p}.$$

By the implicit function theorem, the above relation defines a unique function $\mathbf{v}_n$, provided the function ε verifies the condition that the linear map

$$u_x \mapsto \frac{\partial^2 \varepsilon}{\partial \mathbf{r}^2}(r, s, v_x, w_x) \cdot u_x - r u_x$$

is bijective, for all $(r, s, v_x, w_x) \in \mathbb{R} \times \mathbb{R} \times T\mathcal{D} \times T\mathcal{D}$.

We make the following definitions

$$\mu_{\text{chem}} := \frac{\partial \varepsilon}{\partial \rho}(\rho, S, \mathbf{v}_s - \mathbf{v}_n, \boldsymbol{\omega}) \in \mathcal{F}(\mathcal{D}), \qquad T := \frac{\partial \varepsilon}{\partial S}(\rho, S, \mathbf{v}_s - \mathbf{v}_n, \boldsymbol{\omega}) \in \mathcal{F}(\mathcal{D}),$$

$$\mathbf{p} := \frac{\partial \varepsilon}{\partial \mathbf{r}}(\rho, S, \mathbf{v}_s - \mathbf{v}_n, \boldsymbol{\omega}) \in \Omega^1(\mathcal{D}), \qquad \lambda := \frac{\partial \varepsilon}{\partial \boldsymbol{\omega}}(\rho, S, \mathbf{v}_s - \mathbf{v}_n, \boldsymbol{\omega}) \in \Omega^1(\mathcal{D}).$$

The interpretation of the quantities $\mu_{\text{chem}}, T, \mathbf{p}$, and λ is obtained from the following thermodynamic derivative identity for the internal energy:

$$\mathbf{d}\big(\varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n, \boldsymbol{\omega})\big) = \mu_{\text{chem}} \mathbf{d}\rho + T\mathbf{d}S + \mathbf{p} \cdot \nabla_{(\mathbf{v}_s - \mathbf{v}_n)} + \lambda \cdot \nabla_{\boldsymbol{\omega}} \in \Omega^1(\mathcal{D}),$$

where ∇ denotes the Levi-Civita covariant derivative associated to the metric g. The functional derivatives of h are computed to be

$$\frac{\delta h}{\delta \mathbf{m}} = \mathbf{v}_n, \qquad \frac{\delta h}{\delta \rho} = -\frac{1}{2}\|\mathbf{v}_n\|^2 - A \cdot \mathbf{v}_n + \mu_{\text{chem}}, \qquad \frac{\delta h}{\delta S} = T, \qquad \frac{\delta h}{\delta \mathbf{u}} = \mathbf{p} + \operatorname{curl} \lambda,$$

$$\mathbf{v}_l := \frac{\delta h}{\delta \mathbf{n}} = \frac{1}{n}\big(\rho \mathbf{v}_n + \mathbf{p}^\sharp + \operatorname{curl} \lambda\big), \qquad \frac{\delta h}{\delta n} = A \cdot \mathbf{v}_l.$$

The vector field $\mathbf{v}_l$ will be interpreted as the *vortex line velocity*. The equations for ρ, S, and n are given by

$$\frac{\partial}{\partial t}\rho + \operatorname{div}\big(\rho \mathbf{v}_n + \mathbf{p}^\sharp + \operatorname{curl} \lambda\big) = 0, \quad \frac{\partial}{\partial t} S + \operatorname{div}(S \mathbf{v}_n) = 0, \quad \text{and} \quad \frac{\partial}{\partial t} n + \operatorname{div}(n \mathbf{v}_l) = 0.$$

Using the expression of $\mathbf{v}_l$ in terms of $\mathbf{v}_n$ we obtain that $\operatorname{div}(n\mathbf{v}_l) = \operatorname{div}(\rho \mathbf{v}_n + \mathbf{p}^\sharp + \operatorname{curl} \lambda)$ which proves that

$$\frac{\partial}{\partial t}(\rho - n) = 0.$$

Thus, if we assume that the initial conditions verify $\rho_0 = n_0$, then we have $\rho = n$ for all time. The equations for A and $\mathbf{u}$ are computed to be

$$\frac{\partial}{\partial t} A^\sharp = \frac{1}{\rho}\big(\rho \mathbf{v}_n + \mathbf{p}^\sharp + \operatorname{curl} \lambda\big) \times \operatorname{curl} A^\sharp = \mathbf{v}_l \times \operatorname{curl} A^\sharp,$$

$$\frac{\partial}{\partial t}\mathbf{u} + \operatorname{grad}\left(g(\mathbf{v}_s, \mathbf{v}_n) + \mu_{\text{chem}} - \frac{1}{2}\|\mathbf{v}_n\|^2\right) + \operatorname{curl} \mathbf{u} \times \mathbf{v}_n = 0.$$

As in superfluid dynamics, this equation preserves the condition $\operatorname{curl}\mathbf{u}=0$, and we will suppose that it holds initially: $\operatorname{curl}\mathbf{u}_0=0$. In this case we have $\boldsymbol{\omega}=-\operatorname{curl}A^\sharp$ and the equations above read

$$\frac{\partial}{\partial t}A^\sharp+\mathbf{v}_l\times\boldsymbol{\omega}=0\quad\text{and}\quad\frac{\partial}{\partial t}\mathbf{u}+\operatorname{grad}\left(g(\mathbf{v}_s,\mathbf{v}_n)+\mu_{\text{chem}}-\frac{1}{2}\|\mathbf{v}_n\|^2\right)=0. \tag{8.26}$$

From these two equations we obtain that the evolution of the superfluid velocity $\mathbf{v}_s=\mathbf{u}-A^\sharp$ is given by

$$\frac{\partial}{\partial t}\mathbf{v}_s+\boldsymbol{\omega}\times\mathbf{v}_l=-\operatorname{grad}\left(g(\mathbf{v}_s,\mathbf{v}_n)+\mu_{\text{chem}}-\frac{1}{2}\|\mathbf{v}_n\|^2\right), \tag{8.27}$$

which can be rewritten as

$$\frac{\partial}{\partial t}\mathbf{v}_s+\nabla_{\mathbf{v}_s}\mathbf{v}_s=-\operatorname{grad}\left(\mu_{\text{chem}}-\frac{1}{2}\|\mathbf{v}_s-\mathbf{v}_n\|^2\right)+\mathbf{f},\quad\mathbf{f}=(\mathbf{v}_l-\mathbf{v}_s)\times\boldsymbol{\omega},$$

(compare to (8.18)). Doing computations similar to those for superfluids we obtain that the equation for $\mathbf{m}+\mathbf{n}$ is given by

$$\frac{\partial}{\partial t}(\mathbf{m}+\mathbf{n})=-\operatorname{Div}\mathbf{T},$$

where $\mathbf{T}$ is the *HVBK stress tensor* given by

$$\mathbf{T}=\mathbf{v}_n\otimes\left(\rho\mathbf{v}_n^\flat+\mathbf{p}\right)+\mathbf{p}^\sharp\otimes\mathbf{v}_s^\flat-\boldsymbol{\omega}\otimes\lambda^\flat+p\delta,\quad p=\rho\mu_{\text{chem}}+ST-\varepsilon+\boldsymbol{\omega}^\flat\cdot\lambda.$$

Thus, we have obtained the following equations

$$\begin{cases}\dfrac{\partial}{\partial t}\mathbf{J}=-\operatorname{Div}\mathbf{T},\\[2mm]\dfrac{\partial}{\partial t}\rho+\operatorname{div}\mathbf{J}=0,\quad\dfrac{\partial}{\partial t}S+\operatorname{div}(S\mathbf{v}_n)=0,\\[2mm]\dfrac{\partial}{\partial t}\mathbf{v}_s+\nabla_{\mathbf{v}_s}\mathbf{v}_s=-\operatorname{grad}\left(\mu_{\text{chem}}-\dfrac{1}{2}\|\mathbf{v}_s-\mathbf{v}_n\|^2\right)+\mathbf{f},\quad\mathbf{f}=(\mathbf{v}_l-\mathbf{v}_s)\times\boldsymbol{\omega},\end{cases} \tag{8.28}$$

where we have used the notation

$$\mathbf{J}:=\rho\mathbf{v}_n^\flat+\mathbf{p}=\mathbf{m}+\mathbf{n}$$

for the *total momentum density*. These are the equations of HBVK dynamics as given in Eq. (1) of [12] with $R=0$.

Hamiltonian reduction for HVBK dynamics. Consider the right-invariant Hamiltonian $H(\mathbf{m}_\eta,\rho_\chi,S,\mathbf{u};\mathbf{n}_\xi,n)=H_{(S,\mathbf{u};n)}(\mathbf{m}_\eta,\rho_\chi;\mathbf{n}_\xi)$ induced by h. Suppose that $\rho_0=n_0$ and $\operatorname{curl}\mathbf{u}_0=0$ and let $(\mathbf{m}_\eta,\rho_\chi;\mathbf{n}_\xi)$ be a solution curve in $T^*[(\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},S^1))\times\operatorname{Diff}(\mathcal{D})]$ of Hamilton's equations associated to $H_{(S_0,\mathbf{u}_0;n_0)}$ and with the initial condition ρ_0. Then the curve

$$(\mathbf{m},\rho;\mathbf{n}):=\left(J\left(\eta^{-1}\right)\left(\mathbf{m}_\eta\circ\eta^{-1}\right),J\left(\eta^{-1}\right)\left(\rho_\chi\circ\eta^{-1}\right);J\left(\xi^{-1}\right)\left(\mathbf{n}\circ\xi^{-1}\right)\right)$$

is a solution of the HVBK equations (8.28), where $\mathbf{v}_s=\mathbf{u}-A^\sharp=\mathbf{u}+\mathbf{n}/n$. To obtain the converse of this assertion, it is not enough to assume that $(\mathbf{m},\rho,S,\mathbf{v}_s,\mathbf{n})$ verifies (8.28). It is also required that A or $\mathbf{u}$ verify the corresponding equation in (8.26).

Note that from the equalities $\mathbf{v}_s = \mathbf{u} - A^\sharp$ and $\operatorname{curl}\mathbf{u} = 0$ we obtain that the variables $\mathbf{u}$ and $A^\sharp$ are interpreted respectively as the *potential* and the *rotational* components of the superfluid velocity $\mathbf{v}_s$.

As in Section 8.6, we have

$$\frac{d}{dt}\oint_{c_t}\left(\mathbf{v}_s^\flat + A\right) = 0 \quad \text{and} \quad \frac{d}{dt}\oint_{d_t} A = 0,$$

where c_t and d_t are loops which move with the *normal fluid velocity* $\mathbf{v}_n$ and the *vortex line velocity* $\mathbf{v}_l$, respectively. Using Eq. (8.27) for the evolution of the superfluid velocity $\mathbf{v}_s$, we obtain the *vortex Kelvin theorem*

$$\frac{d}{dt}\oint_{d_t}\mathbf{v}_s^\flat = 0.$$

By the Stokes theorem, this can be rewritten as

$$\frac{d}{dt}\iint_{S_t}\left(\boldsymbol{\omega}^\flat\cdot\mathbf{n}\right)dS = 0,$$

where $\mathbf{n}$ is the unit vector normal to the surface S_t whose boundary ∂S_t is a loop which moves with the vortex line velocity $\mathbf{v}_l$. This is the conservation of the flux of superfluid vorticity through any surface whose boundary moves with the velocity $\mathbf{v}_l$.

In [12] it is also supposed that $\frac{\partial\varepsilon}{\partial\mathbf{r}}$ is collinear to $\mathbf{r}$, more precisely, that there exists a positive function ρ_s such that

$$\mathbf{p} = \rho_s(\mathbf{v}_s - \mathbf{v}_n).$$

The function ρ_s is interpreted as the *superfluid mass density*, and the *density of the normal fluid* is given by $\rho_n := \rho - \rho_s$. Using these notations, the total momentum $\mathbf{J}$ and the stress tensor $\mathbf{T}$ can be rewritten as

$$\mathbf{J} = \rho_n\mathbf{v}_n + \rho_s\mathbf{v}_s$$

and

$$\mathbf{T} = \mathbf{v}_n\otimes\rho_n\mathbf{v}_n^\flat + \mathbf{v}_s\otimes\rho_s\mathbf{v}_s^\flat - \boldsymbol{\omega}\otimes\lambda^\flat + p\delta.$$

8.8. Classical fluids versus superfluids

We summarize below the examples that have been studied in the previous sections.

Classical fluids. In order to compare the two theories, we use the notation $\mathbf{v}_n$ for the fluid velocity. We also express the dynamics in terms of the entropy density S and internal energy density ε (and not in terms of the specific entropy s and internal energy e). We have

$$\varepsilon(\rho, S) = \rho e(\rho, S/\rho),$$

and the first law of thermodynamics

$$\mathbf{d}e = \frac{p}{\rho^2}\mathbf{d}\rho + T\mathbf{d}s, \quad p = \rho^2\frac{\partial e}{\partial\rho},$$

reads

$$\mathbf{d}\varepsilon = \mu_{\text{chem}}\mathbf{d}\rho + T\mathbf{d}S, \quad p = \rho\mu_{\text{chem}} + ST - \varepsilon.$$

(i) *Basic hydrodynamics*
Symmetry group $\text{Diff}(\mathcal{D})$, momentum $\mathbf{m} \in \Omega^1(\mathcal{D})$.
Advected quantities $(\rho, S) \in \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D})$.
Hamiltonian

$$h(\mathbf{m}, \rho, S) = \frac{1}{2}\int_{\mathcal{D}} \frac{1}{\rho}\|\mathbf{m}\|^2\mu + \int_{\mathcal{D}} \varepsilon(\rho, S)\mu,$$

$$\mathbf{m} = \rho\mathbf{v}_n^\flat.$$

Stress tensor formulation $\dot{\mathbf{m}} = -\operatorname{Div}\mathbf{T}$, where

$$\mathbf{T} = \mathbf{v}_\mathbf{n} \otimes \rho\mathbf{v}_n^\flat + p\delta, \quad p = \rho\mu_{\text{chem}} + ST - \varepsilon.$$

(ii) *Yang–Mills magnetohydrodynamics*
Symmetry group $\text{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$, momenta $(\mathbf{m}, Q) \in \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$.
Advected quantities $(\rho, S, A) \in \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}) \times \Omega^1(\mathcal{D}, \mathfrak{o})$.
Hamiltonian

$$h(\mathbf{m}, Q, \rho, S, A) = \frac{1}{2}\int_{\mathcal{D}} \frac{1}{\rho}\|\mathbf{m}\|^2\mu + \int_{\mathcal{D}} \varepsilon(\rho, S)\mu + \frac{1}{2}\int_{\mathcal{D}} \|\mathbf{d}^A A\|^2\mu,$$

$$\mathbf{m} = \rho\mathbf{v}_n^\flat.$$

Stress tensor formulation $\dot{\mathbf{m}} = -\operatorname{Div}\mathbf{T}$, where

$$\mathbf{T} = \mathbf{v}_\mathbf{n} \otimes \rho\mathbf{v}_n^\flat + B \cdot B + p\delta, \quad p = \rho\mu_{\text{chem}} + ST - \varepsilon - \frac{1}{2}\|B\|^2.$$

(iii) *Hall magnetohydrodynamics*
Symmetry group $\text{Diff}(\mathcal{D}) \times \text{Diff}(\mathcal{D})$, momenta $(\mathbf{m}, \mathbf{n}) \in \Omega^1(\mathcal{D}) \times \Omega^1(\mathcal{D})$.
Advected quantities $(\rho, S; n) \in \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D})$.
Hamiltonian

$$h(\mathbf{m}, \rho, S; \mathbf{n}, n) = \frac{1}{2}\int_{\mathcal{D}} \frac{1}{\rho}\left\|\mathbf{m} - \frac{a\rho}{R}A\right\|^2\mu + \int_{\mathcal{D}} \varepsilon(\rho, S)\mu + \frac{1}{2}\int_{\mathcal{D}} \|\mathbf{d}A\|^2\mu, \quad A := R\frac{\mathbf{n}}{n},$$

$$\mathbf{m} = \rho\mathbf{v}_n^\flat + \frac{a\rho}{R}A.$$

Initial conditions

$$a\rho_0 + n_0 = 0 \implies \{a\rho + n = 0 \text{ and } \mathbf{m} + \mathbf{n} = \rho\mathbf{v}_n^\flat\}.$$

Stress tensor formulation $\dot{\mathbf{m}} + \dot{\mathbf{n}} = -\operatorname{Div}\mathbf{T}$, where

$$\mathbf{T} = \mathbf{v}_\mathbf{n} \otimes \rho\mathbf{v}_n^\flat + B \cdot B + p\delta, \quad p = \rho\mu_{\text{chem}} + ST - \varepsilon - \frac{1}{2}\|B\|^2.$$

Superfluids. As before, we denote by $\mathbf{v}_n$ the velocity of the normal flow and by $\mathbf{v}_s$ the superfluid velocity. For simplicity we treat the two fluid model in this summary. Generalization to multifluid models follows as above in the examples. The superfluid first law reads

$$\mathbf{d}\varepsilon = \mu_{\text{chem}}\mathbf{d}\rho + T\mathbf{d}S + \mathbf{p}\cdot\nabla_{\_}(\mathbf{v}_s - \mathbf{v}_n),$$

where ε is the internal energy density. For HBVK dynamics the term $\lambda\cdot\nabla_{\_}\boldsymbol{\omega}$ has to be added.

(i) *Basic superfluid hydrodynamics*
Symmetry group $\text{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},S^1)$, momenta $(\mathbf{m},\rho)\in\Omega^1(\mathcal{D})\times\mathcal{F}(\mathcal{D})$.
Advected quantities $(S,\mathbf{v}_s)\in\mathcal{F}(\mathcal{D})\times\mathfrak{X}(\mathcal{D})$.
Hamiltonian

$$h(\mathbf{m},\rho,S,\mathbf{v}_s) = -\frac{1}{2}\int_{\mathcal{D}}\rho\|\mathbf{v}_n\|^2\mu + \int_{\mathcal{D}}(\mathbf{m}\cdot\mathbf{v}_n)\mu + \int_{\mathcal{D}}\varepsilon(\rho,S,\mathbf{v}_s-\mathbf{v}_n)\mu,$$

$$\mathbf{m} = \rho\mathbf{v}_n^\flat + \mathbf{p}.$$

Stress tensor formulation $\dot{\mathbf{m}} = -\operatorname{Div}\mathbf{T}$, where

$$\mathbf{T} = \mathbf{v}_n\otimes\left(\rho\mathbf{v}_n^\flat + \mathbf{p}\right) + \mathbf{p}^\sharp\otimes\mathbf{v}_s^\flat + p\delta,\quad p = \rho\mu_{\text{chem}} + ST - \varepsilon.$$

(ii) *Superfluid Yang–Mills magnetohydrodynamics*
Symmetry group $\text{Diff}(\mathcal{D})\circledS(\mathcal{F}(\mathcal{D},\mathcal{O})\times\mathcal{F}(\mathcal{D},S^1))$, momenta $(\mathbf{m},Q,\rho)\in\Omega^1(\mathcal{D})\times\mathcal{F}(\mathcal{D},\mathfrak{o}^*)\times\mathcal{F}(\mathcal{D})$.
Advected quantities $(S,A,\mathbf{v}_s)\in\mathcal{F}(\mathcal{D})\times\Omega^1(\mathcal{D},\mathfrak{o})\times\mathfrak{X}(\mathcal{D})$.
Hamiltonian

$$\begin{aligned}h(\mathbf{m},Q,\rho,S,A,\mathbf{v}_s)&\\ = -\frac{1}{2}\int_{\mathcal{D}}\rho\|\mathbf{v}_n\|^2\mu + \int_{\mathcal{D}}(\mathbf{m}\cdot\mathbf{v}_n)\mu + \int_{\mathcal{D}}\varepsilon(\rho,S,\mathbf{v}_s-\mathbf{v}_n)\mu + \frac{1}{2}\int_{\mathcal{D}}\left\|\mathbf{d}^A A\right\|^2\mu,&\end{aligned}$$

$$\mathbf{m} = \rho\mathbf{v}_n^\flat + \mathbf{p}.$$

Stress tensor formulation $\dot{\mathbf{m}} = -\operatorname{Div}\mathbf{T}$, where

$$\mathbf{T} = \mathbf{v}_\mathbf{n}\otimes\left(\rho\mathbf{v}_n^\flat + \mathbf{p}\right) + \mathbf{p}^\sharp\otimes\mathbf{v}_s^\flat + B\cdot B + p\delta,\quad p = \rho\mu_{\text{chem}} + ST - \varepsilon - \frac{1}{2}\|B\|^2.$$

(iii) *Superfluid Hall magnetohydrodynamics*
Symmetry group $[\text{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},S^1)]\times\text{Diff}(\mathcal{D})$, momenta $(\mathbf{m},\rho,\mathbf{n})\in\Omega^1(\mathcal{D})\times\Omega^1(\mathcal{D})\times\mathcal{F}(\mathcal{D})$.
Advected quantities $(S,\mathbf{u};n)\in\mathcal{F}(\mathcal{D})\times\mathfrak{X}(\mathcal{D})\times\mathcal{F}(\mathcal{D})$.
Hamiltonian

$$\begin{aligned}h(\mathbf{m},\rho,S,\mathbf{u};\mathbf{n},n) = -\frac{1}{2}\int_{\mathcal{D}}\rho\|\mathbf{v}_n\|^2\mu + \int_{\mathcal{D}}\left(\left(\mathbf{m}-\frac{a\rho}{R}A\right)\cdot\mathbf{v}_n\right)\mu&\\ + \int_{\mathcal{D}}\varepsilon(\rho,S,\mathbf{v}_s-\mathbf{v}_n)\mu + \frac{1}{2}\int_{\mathcal{D}}\|\mathbf{d}A\|^2\mu,&\end{aligned}$$

where

$$A := R\frac{\mathbf{n}}{n}, \qquad \mathbf{v}_s := \mathbf{u} - \frac{a}{R}A^\sharp,$$
$$\mathbf{m} = \rho\mathbf{v}_n^\flat + \frac{a\rho}{R}A + \mathbf{p}.$$

Initial conditions

$$a\rho_0 + n_0 = 0 \quad\Longrightarrow\quad \{a\rho + n = 0 \text{ and } \mathbf{m} + \mathbf{n} = \rho\mathbf{v}_n^\flat + \mathbf{p}\},$$
$$\operatorname{curl}\mathbf{u}_0 = 0 \quad\Longrightarrow\quad \operatorname{curl}\mathbf{u} = 0.$$

Stress tensor formulation $\dot{\mathbf{m}} + \dot{\mathbf{n}} = -\operatorname{Div}\mathbf{T}$, where

$$\mathbf{T} = \mathbf{v}_n \otimes (\rho\mathbf{v}_n^\flat + \mathbf{p}) + \mathbf{p}^\sharp \otimes \mathbf{v}_s^\flat + B\cdot B + p\delta, \quad p = \rho\mu_{\text{chem}} + ST - \varepsilon - \frac{1}{2}\|B\|^2.$$

(iv) *HBVK hydrodynamics*
Symmetry group $[\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D}, S^1)]\times\operatorname{Diff}(\mathcal{D})$, momenta $(\mathbf{m}, \rho, \mathbf{n}) \in \Omega^1(\mathcal{D})\times\Omega^1(\mathcal{D})\times\mathcal{F}(\mathcal{D})$.
Advected quantities $(S, \mathbf{u}; n) \in \mathcal{F}(\mathcal{D})\times\mathfrak{X}(\mathcal{D})\times\mathcal{F}(\mathcal{D})$.
Hamiltonian

$$h(\mathbf{m}, \rho, S, \mathbf{u}; \mathbf{n}, n) = -\frac{1}{2}\int_{\mathcal{D}} \rho\|\mathbf{v}_n\|^2\mu + \int_{\mathcal{D}}((\mathbf{m} - \rho A)\cdot\mathbf{v}_n)\mu$$
$$+ \int_{\mathcal{D}} \varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n, \boldsymbol{\omega})\mu,$$
$$A := -\frac{\mathbf{n}}{n}, \qquad \mathbf{v}_s := \mathbf{u} - A^\sharp, \qquad \boldsymbol{\omega} := \operatorname{curl}\mathbf{v}_s,$$
$$\mathbf{m} = \rho\mathbf{v}_n^\flat + \rho A + \mathbf{p}.$$

Initial conditions

$$\rho_0 = n_0 \quad\Longrightarrow\quad \{\rho = n \text{ and } \mathbf{m} + \mathbf{n} = \rho\mathbf{v}_n^\flat + \mathbf{p} =: \mathbf{J}\}, \qquad \operatorname{curl}\mathbf{u}_0 = 0 \quad\Longrightarrow\quad \operatorname{curl}\mathbf{u} = 0.$$

Stress tensor formulation $\dot{\mathbf{m}} + \dot{\mathbf{n}} = -\operatorname{Div}\mathbf{T}$, where

$$\mathbf{T} = \mathbf{v}_n \otimes (\rho\mathbf{v}_n^\flat + \mathbf{p}) + \mathbf{p}^\sharp \otimes \mathbf{v}_s^\flat - \boldsymbol{\omega}\otimes\lambda^\flat + p\delta, \quad p = \rho\mu_{\text{chem}} + ST - \varepsilon + \boldsymbol{\omega}^\flat\cdot\lambda.$$

The hypothesis $\mathbf{p} = \rho_s(\mathbf{v}_s - \mathbf{v}_n)$ implies the equalities $\mathbf{J} = \rho_n\mathbf{v}_n + \rho_s\mathbf{v}_s$ and

$$\mathbf{T} = \mathbf{v}_n \otimes \rho_n\mathbf{v}_n^\flat + \mathbf{v}_s \otimes \rho_s\mathbf{v}_s^\flat + \boldsymbol{\omega}\otimes\lambda^\flat + p\delta.$$

8.9. Volovik–Dotsenko theory of spin glasses

In [27], the authors use a Poisson bracket approach to derive the equations of nonplanar magnet with disclinations and of spin glass. These models are referred to as the *Volovik–Dotsenko spin glasses*. In [16], the Hamiltonian structure of the Volovik–Dotsenko spin glasses is shown to be isomorphic to that of Yang–Mills magnetohydrodynamics. Thus, it can be obtained using the affine Lie–Poisson reduction developed in the present paper. In this section we also carry out the Lagrangian version of the approach given in [16], to which we refer for additional comments on the physics of spin glasses.

The advected variables are

$$\rho \in V_1^* = \mathcal{F}(\mathcal{D}) \quad\text{and}\quad \gamma \in V_2^* = \Omega^1(\mathcal{D}, \mathfrak{o}).$$

The variable ρ is the *defect inertial-mass density* and the curvature $B = \mathbf{d}^\gamma \gamma$ is interpreted as the *disclination density*.

The reduced Lagrangian $l : [\mathfrak{X}(D) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o})] \circledS [V_1^* \oplus V_2^*] \to \mathbb{R}$ is given by

$$l(\mathbf{u}, \nu, \rho, \gamma) = \frac{1}{2} \int_{\mathcal{D}} \rho \|\mathbf{u}\|^2 \mu + \frac{\epsilon}{2} \int_{\mathcal{D}} \|\gamma(\mathbf{u}) + \nu\|^2 \mu - \frac{1}{2} \int_{\mathcal{D}} \rho \|\gamma\|^2 \mu, \tag{8.29}$$

where the norms are associated to the metrics g, k, and (gk), respectively, and ϵ is the *constant of susceptibility*. The unreduced Lagrangian L is the right-invariant function induced by l on the cotangent bundle. Since it has a complicated expression, we do not give the formula for L. We will justify the choice of this Lagrangian by showing that its Legendre transformation yields the Hamiltonian of the Volovik–Dotsenko spin glasses.

This Lagrangian appears as a generalization of the Lagrangian

$$l_{SG}(\nu, \gamma) = \frac{\epsilon}{2} \int_{\mathcal{D}} \|\nu\|^2 \mu - \frac{1}{2} \int_{\mathcal{D}} \rho \|\gamma\|^2 \mu, \tag{8.30}$$

associated to the macroscopic description of spin glasses in [6], where ρ is the *constant of rigidity*, see Section 8.1. In order to understand mathematically the passage form the Lagrangian (8.30) to the Lagrangian (8.29), we consider the following general situation.

General case. Consider a Lagrangian $l_\rho : \mathcal{F}(\mathcal{D}, \mathfrak{o}) \oplus \Omega^1(\mathcal{D}, \mathfrak{o}) \to \mathbb{R}$, $l_\rho = l_\rho(\nu, \gamma)$ associated to a spin system and depending on a parameter ρ interpreted as the *spin rigidity*. Recall that the affine Euler–Poincaré and advection equations are (see Section 8.1)

$$\begin{cases} \dfrac{\partial}{\partial t} \dfrac{\delta l_\rho}{\delta \nu} = -\operatorname{ad}^*_\nu \dfrac{\delta l_\rho}{\delta \nu} + \operatorname{div}^\gamma \dfrac{\delta l_\rho}{\delta \gamma}, \\ \dfrac{\partial}{\partial t} \gamma + \mathbf{d}^\gamma \nu = 0. \end{cases} \tag{8.31}$$

To l_ρ we associate the Lagrangian $l : [\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o})] \circledS [\mathcal{F}(\mathcal{D}) \oplus \Omega^1(\mathcal{D}, \mathfrak{o})] \to \mathbb{R}$ given by

$$l(\mathbf{u}, \nu, \rho, \gamma) := \frac{1}{2} \int_{\mathcal{D}} \rho \|\mathbf{u}\|^2 \mu + l_\rho(\overline{\nu}, \gamma),$$

where $\overline{\nu} := \gamma(\mathbf{u}) + \nu$ and ρ is now a variable. Remark the analogy with the process of minimal coupling. In the case where l_ρ is given by (8.30), we recover the Volovik–Dotsenko Lagrangian (8.29).

The functional derivatives of l are computed to be

$$\kappa := \frac{\delta l}{\delta \nu} = \frac{\delta l_\rho}{\delta \overline{\nu}} \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*), \qquad \mathbf{m} := \frac{\delta l}{\delta \mathbf{u}} = \rho \mathbf{u}^\flat + \frac{\delta l_\rho}{\delta \overline{\nu}} \cdot \gamma \in \Omega^1(\mathcal{D}),$$

and

$$\frac{\delta l}{\delta \rho} = \frac{1}{2} \|\mathbf{u}\|^2 + \frac{\delta l_\rho}{\delta \rho} \in \mathcal{F}(\mathcal{D}), \qquad \frac{\delta l}{\delta \gamma} = \frac{\delta l_\rho}{\delta \overline{\nu}} \mathbf{u} + \frac{\delta l_\rho}{\delta \gamma} \in \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*).$$

The advection equations are

$$\frac{\partial}{\partial t} \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} \gamma + \mathbf{i}_\mathbf{u} B + \mathbf{d}^\gamma \overline{\nu} = 0.$$

We now compute the affine Euler–Poincaré equations. We have

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\delta l_\rho}{\delta \overline{\nu}} &= -\operatorname{ad}^*_{\nu}\frac{\delta l_\rho}{\delta \overline{\nu}} - \operatorname{div}\left(\frac{\delta l_\rho}{\delta \overline{\nu}}\mathbf{u}\right) + \operatorname{div}^\gamma\left(\frac{\delta l_\rho}{\delta \overline{\nu}}\mathbf{u} + \frac{\delta l_\rho}{\delta \gamma}\right)\\ &= -\operatorname{ad}^*_{\overline{\nu}-\gamma(\mathbf{u})}\frac{\delta l_\rho}{\delta \overline{\nu}} - \operatorname{div}\left(\frac{\delta l_\rho}{\delta \overline{\nu}}\mathbf{u}\right) + \operatorname{div}\left(\frac{\delta l_\rho}{\delta \overline{\nu}}\mathbf{u}\right) - \operatorname{Tr}\left(\operatorname{ad}^*_{\gamma}\frac{\delta l_\rho}{\delta \overline{\nu}}\mathbf{u}\right) + \operatorname{div}^\gamma\left(\frac{\delta l_\rho}{\delta \gamma}\right)\\ &= -\operatorname{ad}^*_{\overline{\nu}}\frac{\delta l_s}{\delta \overline{\nu}} + \operatorname{div}^\gamma\left(\frac{\delta l_s}{\delta \gamma}\right).\end{aligned}$$

Using the equations for $\rho, \gamma, \frac{\delta l_\rho}{\delta \overline{\nu}}$, the relation $\mathbf{m} = \rho\mathbf{u}^\flat + \frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\gamma$, and the identity

$$\pounds_{\mathbf{u}}\left(\frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\gamma\right) = \mathbf{d}\frac{\delta l_\rho}{\delta \overline{\nu}}(\mathbf{u})\cdot\gamma + \frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\pounds_{\mathbf{u}}\gamma$$

we obtain

$$\begin{aligned}&\left(\frac{\partial}{\partial t}\mathbf{m} + \pounds_{\mathbf{u}}\mathbf{m} + (\operatorname{div}\mathbf{u})\mathbf{m}\right)^\sharp\\ &= \rho\left(\frac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} + \nabla\mathbf{u}^T\cdot\mathbf{u}\right) + \frac{\partial}{\partial t}\left(\frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\gamma\right) + \pounds_{\mathbf{u}}\left(\frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\gamma\right) + (\operatorname{div}\mathbf{u})\frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\gamma\\ &= \rho\left(\frac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} + \nabla\mathbf{u}^T\cdot\mathbf{u}\right) - \operatorname{ad}^*_{\overline{\nu}}\frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\gamma + \operatorname{div}^\gamma\left(\frac{\delta l_\rho}{\delta \gamma}\right)\cdot\gamma - \frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\mathbf{i}_{\mathbf{u}}B - \frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\mathbf{d}^\gamma\overline{\nu}\\ &\quad + \mathbf{d}\frac{\delta l_\rho}{\delta \overline{\nu}}(\mathbf{u})\cdot\gamma + \frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\pounds_{\mathbf{u}}\gamma + (\operatorname{div}\mathbf{u})\frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\gamma\\ &= \rho\left(\frac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} + \nabla\mathbf{u}^T\cdot\mathbf{u}\right) - \operatorname{ad}^*_{\overline{\nu}}\frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\gamma + \operatorname{div}^\gamma\left(\frac{\delta l_\rho}{\delta \gamma}\right)\cdot\gamma\\ &\quad + \frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\mathbf{d}^\gamma\left(\gamma(\mathbf{u}) - \overline{\nu}\right) + \operatorname{div}\left(\frac{\delta l_\rho}{\delta \overline{\nu}}\mathbf{u}\right)\cdot\gamma,\end{aligned}$$

where in the last equality we used the identity (3.12) and the definition of the curvature B. We also have

$$\left(\frac{\delta l}{\delta \rho}\diamond\rho\right)^\sharp = \rho\left(\nabla\mathbf{u}^T\cdot\mathbf{u} + \operatorname{grad}\frac{\delta l_\rho}{\delta \rho}\right)$$

and

$$\frac{\delta l}{\delta \gamma}\diamond_1\gamma = \operatorname{div}^\gamma\left(\frac{\delta l_\rho}{\delta \overline{\nu}}\mathbf{u} + \frac{\delta l_\rho}{\delta \gamma}\right)\cdot\gamma - \left(\frac{\delta l_\rho}{\delta \overline{\nu}}\mathbf{u} + \frac{\delta l_\rho}{\delta \gamma}\right)\cdot\mathbf{i}_{\_}B.$$

Thus the affine Euler–Poincaré equation for the variable $\mathbf{u}$ is

$$\begin{aligned}&\rho\left(\frac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} + \nabla\mathbf{u}^T\cdot\mathbf{u}\right) - \operatorname{ad}^*_{\overline{\nu}}\frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\gamma + \operatorname{div}^\gamma\left(\frac{\delta l_\rho}{\delta \gamma}\right)\cdot\gamma + \frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\mathbf{d}^\gamma\left(\gamma(\mathbf{u}) - \overline{\nu}\right) + \operatorname{div}\left(\frac{\delta l_\rho}{\delta \overline{\nu}}\mathbf{u}\right)\cdot\gamma\\ &= -\frac{\delta l_\rho}{\delta \overline{\nu}}\cdot\mathbf{d}\nu + \rho\left(\nabla\mathbf{u}^T\cdot\mathbf{u} + \operatorname{grad}\frac{\delta l_\rho}{\delta \rho}\right) + \operatorname{div}^\gamma\left(\frac{\delta l_\rho}{\delta \overline{\nu}}\mathbf{u} + \frac{\delta l_\rho}{\delta \gamma}\right)\cdot\gamma - \left(\frac{\delta l_\rho}{\delta \overline{\nu}}\mathbf{u} + \frac{\delta l_\rho}{\delta \gamma}\right)\cdot\mathbf{i}_{\_}B,\end{aligned}$$

which, after remarkable cancellations, reads

$$\frac{\partial}{\partial t}\mathbf{u}+\nabla_{\mathbf{u}}\mathbf{u}=\operatorname{grad}\frac{\delta l_\rho}{\delta\rho}-\frac{1}{\rho}\left(\frac{\delta l_\rho}{\delta\overline{\nu}}\mathbf{u}+\frac{\delta l_\rho}{\delta\gamma}\right)\cdot\mathbf{i}_{\_}B.$$

Thus the affine Euler–Poincaré equations associated to l are given by

$$\begin{cases}\dfrac{\partial}{\partial t}\mathbf{u}+\nabla_{\mathbf{u}}\mathbf{u}=\operatorname{grad}\dfrac{\delta l_\rho}{\delta\rho}-\dfrac{1}{\rho}\left[\left(\dfrac{\delta l_\rho}{\delta\overline{\nu}}\mathbf{u}+\dfrac{\delta l_\rho}{\delta\gamma}\right)\cdot\mathbf{i}_{\_}B\right]^\sharp,\\ \dfrac{\partial}{\partial t}\dfrac{\delta l_\rho}{\delta\overline{\nu}}=-\operatorname{ad}^*_{\overline{\nu}}\dfrac{\delta l_\rho}{\delta\overline{\nu}}+\operatorname{div}^\gamma\left(\dfrac{\delta l_\rho}{\delta\gamma}\right),\quad \dfrac{\partial}{\partial t}\gamma+\mathbf{i}_{\mathbf{u}}B+\mathbf{d}^\gamma\overline{\nu}=0,\\ \dfrac{\partial}{\partial t}\rho+\operatorname{div}(\rho\mathbf{u})=0,\quad B=\mathbf{d}^\gamma\gamma,\quad \overline{\nu}=\gamma(\mathbf{u})+\nu.\end{cases}\tag{8.32}$$

This system of equations can be seen as a generalization of Eqs. (8.31). If l_ρ is hyperregular, then l is also hyperregular, and the associated Hamiltonian is given by

$$\int_{\mathcal{D}}\frac{1}{2\rho}\|\mathbf{m}-\kappa\cdot\gamma\|^2\mu+h_\rho(\kappa,\gamma),\tag{8.33}$$

where h_ρ is the Legendre transformation of l_ρ.

The case of Volovik–Dotsenko spin glasses. We now specialize the previous discussion to the case of the Volovik–Dotsenko spin glasses, that is, we consider the Lagrangian (8.29). In this case, the momenta are given by

$$\kappa=\epsilon\big(\gamma(\mathbf{u})+\nu\big)\quad\text{and}\quad\mathbf{m}=\rho\mathbf{u}^\flat+\kappa\cdot\gamma,$$

and Eqs. (8.32) read

$$\begin{cases}\dfrac{\partial}{\partial t}\mathbf{u}+\nabla_{\mathbf{u}}\mathbf{u}=-\operatorname{grad}\dfrac{1}{2}\|\gamma\|^2+\left[\left(\gamma^\sharp-\dfrac{1}{\rho}\kappa\mathbf{u}\right)\mathbf{i}_{\_}B\right]^\sharp,\\ \dfrac{\partial}{\partial t}\kappa+\operatorname{div}^\gamma\big(\rho\gamma^\sharp\big)=0,\quad \dfrac{\partial}{\partial t}\gamma+\mathbf{i}_{\mathbf{u}}B+\dfrac{1}{\epsilon}\mathbf{d}^\gamma\kappa^\sharp=0,\\ \dfrac{\partial}{\partial t}\rho+\operatorname{div}(\rho\mathbf{u})=0,\quad B=\mathbf{d}^\gamma\gamma.\end{cases}\tag{8.34}$$

In the case of the spin glass dynamics described by the equations above, the term $-\operatorname{grad}\frac{1}{2}\|\gamma\|^2$ is an analogue of the electrostatic force and the term $(\gamma^\sharp-\frac{1}{\rho}\kappa\mathbf{u})\mathbf{i}_{\_}B$ is an analogue of the Lorentz force.

Lagrangian reduction for the Volovik–Dotsenko spin glasses. A curve $(\eta,\chi)\in\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathcal{O})$ is a solution of the Euler–Lagrange equations associated to the Lagrangian $L_{(\rho_0,\gamma_0)}$ if and only if the curve

$$(\mathbf{u},\nu):=\big(\dot\eta\circ\eta^{-1},(TR_{\chi^{-1}}\dot\chi)\circ\eta^{-1}\big)$$

is a solution of the spin glass equations (8.34) with initial conditions (ρ_0,γ_0), and where $\kappa=\epsilon(\gamma(\mathbf{u})+\nu)$.

The evolution of the advected quantities is given by

$$\rho=J\big(\eta^{-1}\big)\big(\rho_0\circ\eta^{-1}\big)\quad\text{and}\quad\gamma=\eta_*\big(\operatorname{Ad}_\chi\gamma_0+\chi T\chi^{-1}\big),$$

and the evolution of the disclination density is

$$B = \eta_*\big(\mathrm{Ad}_\chi\, \mathbf{d}^{\gamma_0}\gamma_0\big).$$

The variable χ is the *orientation of Lagrangian particles in their reference configuration*.

By Legendre transforming $L_{(\rho_0,\gamma_0)}$ and l, we obtain the right-invariant Hamiltonian $H_{(\rho_0,\gamma_0)}$ and the reduced Hamiltonian h, given by

$$h(\mathbf{m},\kappa,\rho,\gamma) = \frac{1}{2}\int \frac{1}{\rho}\|\mathbf{m}-\kappa\cdot\gamma\|^2\mu + \frac{1}{2\epsilon}\int \|\kappa\|^2\mu + \frac{1}{2}\int \rho\|\gamma\|^2\mu,$$

see also (8.33). Note that this Hamiltonian differs by a sign in the first term from the Hamiltonian (2.26b) in [16]. This is due to our convention in the Hamiltonian structure which also differs from theirs. This justifies the choice we made for the Volovik–Dotsenko Lagrangian (8.29).

Hamiltonian reduction for the Volovik–Dotsenko spin glasses. A curve $(\mathbf{m}_\eta,\kappa_\chi)\in T^*[\mathrm{Diff}(\mathcal{D})\circledS \mathcal{F}(\mathcal{D},\mathcal{O})]$ is a solution of Hamilton's equations associated to the spin glass Hamiltonian $H_{(\rho_0,\gamma_0)}$ if and only if the curve

$$\big(\rho\mathbf{u}^\flat+\kappa\cdot\gamma,\kappa\big) =: (\mathbf{m},\kappa) := J\big(\eta^{-1}\big)\big(\mathbf{m}_\eta\circ\eta^{-1}, T^*R_{\chi\circ\eta^{-1}}\big(\kappa_\chi\circ\eta^{-1}\big)\big)$$

is a solution of the system (8.34) with initial conditions (ρ_0,γ_0).

The associated Poisson bracket is identical to that of Yang–Mills magnetohydrodynamics, except for the fact that the variable s is not present in this case. Notice that the variables have not the same physical meaning in the two theories; see [16].

The Kelvin–Noether theorem gives the complicated expression

$$\frac{d}{dt}\oint_{c_t}\left(\mathbf{u}^\flat+\frac{\kappa}{\rho}\gamma\right) = \oint_{c_t}\frac{1}{\rho}\big(-\kappa\mathbf{d}\nu+\mathrm{div}^\gamma(w)\gamma - w\cdot\mathbf{i}_{\_}B\big),$$

where

$$\nu := \frac{\kappa}{\epsilon}-\gamma(\mathbf{u}),\qquad w := \kappa\mathbf{u}-\rho\gamma^\sharp,$$

which can be rewritten in the simpler form

$$\frac{d}{dt}\oint_{c_t}\mathbf{u}^\flat = \oint_{c_t}\left(\gamma^\sharp-\frac{1}{\rho}\kappa\mathbf{u}\right)\mathbf{i}_{\_}B.$$

The γ-circulation gives

$$\frac{d}{dt}\oint_{c_t}\gamma = \oint_{c_t}\mathrm{ad}_{(\kappa/\epsilon)-\gamma(\mathbf{u})}\,\gamma.$$

8.10. Microfluids

Microfluids are fluids whose material points are *small deformable particles*. Examples of microfluids include *liquid crystals, blood, polymer melts, bubbly fluids, suspensions with deformable particles, biological fluids*. In this section we find the Hamiltonian structure of the equations governing the motion of non-dissipative microfluids in Eringen's formulation by showing that they appear by Euler–Poincaré and Lie–Poisson reduction.

We quickly recall from [7] some needed facts about microfluids. A material particle P in the fluid is characterized by its position X and by a vector Ξ attached to P that denotes the orientation and intrinsic deformation of P. Both X and Ξ have their own motions, $X \mapsto x = \eta(X,t)$ and $\Xi \mapsto \xi = \chi(X,\Xi,t)$, called respectively the *macromotion* and *micromotion*. Since the material particles are considered to be of very small size, a linear approximation in Ξ is permissible for the micromotion. Therefore, we can write

$$\xi = \chi(X,t)\Xi,$$

where $\chi(X,t) \in GL(3)^+ := \{A \in GL(3) \mid \det(A) > 0\}$. The classical Eringen theory considers only three possible groups in the description of the micromotion of the particles: $GL(3)^+ \supset CSO(3) \supset SO(3)$. These cases correspond to *micromorphic*, *microstretch*, and *micropolar* fluids, respectively, all of them discussed in detail below. The Lie group $CSO(3)$ is a certain closed subgroup of $GL(3)^+$ that is associated to rotations and stretch. Of course, the general theory developed in this paper admits other groups describing the micromotion.

Micromorphic fluids. A fluid in a domain $\mathcal{D}$ is called *micromorphic* if its macromotion and micromotion are described respectively by a diffeomorphism $\eta \in \mathrm{Diff}(\mathcal{D})$ and a function $\chi \in \mathcal{F}(\mathcal{D}, GL(3)^+)$. As a consequence of this description, the configuration manifold of micromorphic fluids is isomorphic to the product of the two groups $\mathrm{Diff}(\mathcal{D})$ and $\mathcal{F}(\mathcal{D}, GL(3)^+)$. We will show the remarkable facts that, as in the previous examples, the relevant group structure on the configuration manifold is given by the semidirect product $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, GL(3)^+)$.

The general equations for *micromorphic continua* are given by

$$\begin{cases} t_{kl,k} + \rho\left(f_l - \dfrac{D}{dt}\mathbf{u}_l\right) = 0, \\ m_{klm,k} + t_{ml} - s_{ml} + \rho(l_{lm} - \sigma_{lm}) = 0, \\ \dfrac{D}{dt}\rho + \rho \operatorname{div}\mathbf{u} = 0, \quad \dfrac{D}{dt} i_{kl} - i_{kr}v_{lr} - i_{lr}v_{kr} = 0, \end{cases} \tag{8.35}$$

where D/dt denotes the material derivative.

Since this is the first time this symbol D/dt appears, it is useful to make some comments and to place this discussion in the larger context of complex fluid motion on general Riemannian manifolds. The domain $\mathcal{D} \subset \mathbb{R}^3$ is endowed with the usual Riemannian metric given by the inner product in Euclidean space. This Riemannian metric has an associated Levi-Civita connection ∇ whose covariant derivative along an arbitrary vector field $\mathbf{u}$ is defined by $\nabla_{\mathbf{u}} := \mathbf{u} \cdot \nabla$. Of course, if $\mathcal{D}$ is replaced by a general Riemannian manifold, then $\nabla_{\mathbf{u}}$ is the covariant derivative of the Levi-Civita connection associated with the given metric and does not have this simple expression. The material derivative of a *vector field* $\mathbf{v}$ along $\mathbf{u}$ is given, in general, by

$$\frac{D}{dt}\mathbf{v} = \frac{\partial}{\partial t}\mathbf{v} + \nabla_{\mathbf{u}}\mathbf{v}.$$

On *functions* $f \in \mathcal{F}(\mathcal{D}, V)$, where V is a vector space, the material derivative is given by

$$\frac{D}{dt} f = \frac{\partial}{\partial t} f + \mathbf{d}f(\mathbf{u}),$$

which is metric independent.

In (8.35) the material derivative operator is applied to the *vector field* $\mathbf{u}$ and to the *functions* ρ and i_{kl}, even though i_{kl} are the components of the symmetric tensor i. Thus, one should not interpret the last equation in (8.35) as the covariant derivative of the tensor i. This convention is in force throughout the rest of the paper and we shall comment on this in the relevant places.

Eqs. (8.35) are given in [7, Eqs. (2.2.10), (2.2.11), (2.2.31), and (2.2.32)]. The variable ρ is the *mass density*, v_{kl} is the *microgyration tensor*, and the symmetric tensor i_{kl} is the *microinertia*. The vector field f_k is the *body force density* and l_{lm} is the *body couple density*. The *spin inertia* σ_{kl} is given by

$$\sigma_{kl} = i_{ml}\left(\frac{D}{dt}v_{km} + v_{kn}v_{nm}\right).$$

The tensors t_{kl}, m_{klm}, and s_{kl} are respectively the *stress tensor*, the *couple stress tensor*, and the *microstress tensor*; see Chapters 1, 2, and 17 in [7] for details. The four equations in (8.35) correspond respectively to the balance of momentum, the balance of momentum moments, the conservation of mass, and the conservation of microinertia.

The *constitutive equations* for *non-dissipative* micromorphic fluids are given by

$$t_{kl} = \frac{\partial \Psi}{\partial \rho^{-1}}\delta_{kl}, \quad m_{klm} = 0, \quad \text{and} \quad s_{kl} = \frac{\partial \Psi}{\partial \rho^{-1}}\delta_{kl} + 2\rho\frac{\partial \Psi}{\partial i_{rk}}i_{rl}, \tag{8.36}$$

see (3.4.6) and (3.4.7) in [7]. Here the function $\Psi = \Psi(\rho^{-1}, i) : \mathbb{R} \times Sym(3) \to \mathbb{R}$ is the *free energy* and $Sym(3)$ denotes the space of symmetric 3×3 matrices. It is usually assumed that

$$\Psi\left(\rho^{-1}, A^{-1}iA\right) = \Psi\left(\rho^{-1}, i\right), \quad \text{for all } A \in O(3).$$

This condition is imposed by the *axiom of objectivity*. Since i is symmetric, this implies that the free energy depends on i only through the quantities $\mathrm{Tr}(i)$, $\mathrm{Tr}(i^2)$ and $\mathrm{Tr}(i^3)$.

Using the constitutive equations (8.36) and assuming that $f_k = 0, l_{kl} = 0$, Eqs. (8.35) read

$$\begin{cases} \rho\dfrac{D}{dt}\mathbf{u}_l = \partial_l\dfrac{\partial \Psi}{\partial \rho^{-1}}, \quad \sigma_{lm} = -2\dfrac{\partial \Psi}{\partial i_{rm}}i_{rl}, \\ \dfrac{D}{dt}\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0, \quad \dfrac{D}{dt}i_{kl} - i_{kr}v_{lr} - i_{lr}v_{kr} = 0. \end{cases} \tag{8.37}$$

These are the equations for *non-dissipative micromorphic fluids*.

Microstretch fluids. A *microstretch fluid* is a micromorphic fluid whose micromotion χ takes values in the four-dimensional real Lie group $CSO(3)$ defined by

$$CSO(3) = \left\{A \in GL(3)^+ \,\middle|\, \text{there exists } \lambda \in \mathbb{R} \text{ such that } AA^T = \lambda I_3\right\}$$

and called the *conformal special orthogonal group*. The Lie algebra of $CSO(3)$ is

$$\mathfrak{cso}(3) = \left\{\nu \in \mathfrak{gl}(3) \,\middle|\, \text{there exists } \mu \in \mathbb{R} \text{ such that } \nu + \nu^T = \mu I_3\right\}.$$

Note that in the relations above we necessarily have $\lambda^3 = \det(A)^2$ and $3\mu = 2\,\mathrm{Tr}(\nu)$. Note also that each $\nu \in \mathfrak{cso}(3)$ decomposes uniquely as

$$\nu_{ij} = \nu_0\delta_{ij} - \varepsilon_{ijk}\boldsymbol{\nu}_k, \tag{8.38}$$

where $\nu_0 \in \mathbb{R}$, and $(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3) =: \boldsymbol{\nu}$ is a vector in $\mathbb{R}^3$. The equality (8.38) can be rewritten as

$$\nu = \nu_0 I_3 + \hat{\boldsymbol{\nu}},$$

where $\hat{\boldsymbol{\nu}} \in \mathfrak{so}(3)$ is the matrix whose entries are given by $\hat{\boldsymbol{\nu}}_{ij} = -\varepsilon_{ijk}\boldsymbol{\nu}_k$.

The material particles of microstretch fluids have seven degrees of freedom: three for *translations*, three for *rotations*, and one for *stretch*.

The general equations for *microstretch continua* are given by

$$\begin{cases} t_{kl,k} + \rho\left(f_l - \dfrac{D}{dt}\mathbf{u}_l\right) = 0, \\ m_{k,k} + t - s + \rho(l-\sigma) = 0, \quad m_{kl,k} + \varepsilon_{lmn}t_{mn} + \rho(l_l - \sigma_l) = 0, \\ \dfrac{D}{dt}\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0, \\ \dfrac{D}{dt}j_0 - 2j_0\nu_0 = 0, \quad \dfrac{D}{dt}j_{kl} - 2\nu_0 j_{kl} + (\varepsilon_{kpr}j_{lp} + \varepsilon_{lpr}j_{kp})\boldsymbol{\nu}_r = 0. \end{cases} \tag{8.39}$$

These equations are given in [7, Eqs. (2.2.38)–(2.2.41)]. Following the conventions introduced in the discussion following (8.35), the material derivative operator D/dt acts on the *vector field* $\mathbf{u}$ and on the *functions* ν_0, $\boldsymbol{\nu}_l$, ρ, j_0, and j_{kl}. As before, the variable ρ is the *mass density*. The variables ν_0 and $\boldsymbol{\nu}$ are respectively the *microstretch rate* and the *microrotation rate*, constructed from ν_{kl} as in (8.38). The *microstretch microinertia* j_0 and the *microinertia* j_{kl} are constructed from i_{kl} as follows:

$$j_0 := 2i_{kk} \quad \text{and} \quad j_{kl} := \frac{1}{2}j_0\delta_{kl} - i_{kl}.$$

The *microstretch spin inertia* σ and the *spin inertia* σ_k are given by

$$\sigma_k := j_{kl}\frac{D}{dt}\boldsymbol{\nu}_l + 2\nu_0 j_{kl}\boldsymbol{\nu}_l + \varepsilon_{klm}j_{mn}\boldsymbol{\nu}_l\boldsymbol{\nu}_n \quad \text{and} \quad \sigma := \frac{1}{2}\left(\frac{D}{dt}\nu_0 + \nu_0^2\right) - j_{kl}\boldsymbol{\nu}_k\boldsymbol{\nu}_l.$$

The *microstretch vector* m_k, the *couple stress tensor* m_{kl}, the *microstretch force density* l, and the *couple density* l are defined from m_{klm} and l_{kl} by the decompositions

$$m_{klm} = \frac{1}{3}m_k\delta_{lm} - \frac{1}{2}\varepsilon_{lmr}m_{kr} \quad \text{and} \quad l_{kl} = \frac{1}{3}l\delta_{kl} - \frac{1}{2}\varepsilon_{klr}l_r.$$

We also used the notations $t := t_{kk}$ and $s := s_{kk}$. Using these definitions and the fact that the fluid has the microstretch property, one can obtain Eqs. (8.39) from Eqs. (8.35).

The *constitutive equations* for *non-dissipative* microstretch fluids are given by

$$t_{kl} = \frac{\partial\Psi}{\partial\rho^{-1}}\delta_{kl}, \quad m_{kl} = 0, \quad m_k = 0, \quad \text{and} \quad s - t = 2\rho\left(\frac{\partial\Psi}{\partial j_{kl}}j_{kl} + \frac{\partial\Psi}{\partial j_0}j_0\right),$$

where $\Psi = \Psi(\rho^{-1}, j, j_0)$ is the *free energy*, see (3.4.16) in [7].

Micropolar fluids. A *micropolar fluid* is a micromorphic fluid whose micromotion χ takes values in the Lie group $SO(3)$. The material particles of micropolar fluids have six degrees of freedom: three for *translations* and three for *rotations*. Micropolar fluids are, therefore, a particular case of microstretch fluids.

The general equations for *micropolar continua* are given by

$$\begin{cases} t_{kl,k} + \rho\left(f_l - \dfrac{D}{dt}\mathbf{u}_l\right) = 0, \\ m_{kl,k} + \varepsilon_{lmn}t_{mn} + \rho(l_l - \sigma_l) = 0, \\ \dfrac{D}{dt}\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0, \\ \dfrac{D}{dt}j_{kl} + (\varepsilon_{kpr}j_{lp} + \varepsilon_{lpr}j_{kp})\boldsymbol{\nu}_r = 0, \end{cases} \tag{8.40}$$

where the variables and tensors have the same meaning as in the previous examples. These equations are given in [7, Eqs. (2.2.43)–(2.2.46)]. They are derived from those of microstretch continua, using that the microgyration tensor ν takes values in the Lie subalgebra $\mathfrak{so}(3)$ of $\mathfrak{csa}(3)$, that is, $\nu_0 = 0$. In particular, the *spin inertia* σ_k is given by

$$\sigma_k := j_{kl}\frac{D}{dt}\boldsymbol{v}_l + \varepsilon_{klm} j_{mn}\boldsymbol{v}_l\boldsymbol{v}_n = \frac{D}{dt}(j_{kl}\boldsymbol{v}_l).$$

The *constitutive equations* for *non-dissipative* micropolar fluids are given by

$$t_{kl} = \frac{\partial \Psi}{\partial \rho^{-1}}\delta_{kl} \quad \text{and} \quad m_{kl} = 0,$$

where $\Psi = \Psi(\rho^{-1}, j)$ is the *free energy*; see (3.4.27) in [7].

Lagrangian and Hamiltonian formulation for micromorphic fluids. We now show that the equations for *non-dissipative micromorphic fluids* can be obtained by Euler–Poincaré reduction associated to the semidirect product $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, GL(3)^+)$. The advected quantities are the *mass density* $\rho \in \mathcal{F}(\mathcal{D})$ and the *microinertia tensor* $i \in \mathcal{F}(\mathcal{D}, Sym(3))$. The symmetry group $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, GL(3)^+)$ acts *linearly* on the advected variables (ρ, i) by

$$(\rho, i) \mapsto \big(J\eta(\rho \circ \eta),\, \chi^T (i \circ \eta)\chi\big).$$

The choice of this group representation is dictated by the form of the advection equations for the mass density and the microinertia in Eqs. (8.37). The infinitesimal actions $\rho(\mathbf{u}, \nu) = \rho\mathbf{u}$ and $i(\mathbf{u}, \nu)$ of $(\mathbf{u}, \nu) \in \mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{gl}(3))$ on ρ and i are given, respectively, by

$$\rho\mathbf{u} = \mathrm{div}(\rho\mathbf{u}) \quad \text{and} \quad i(\mathbf{u}, \nu) = \mathbf{d}i(u) + \nu^T i + i\nu.$$

Given two matrices $a, b \in \mathfrak{gl}(3)$ we denote by ab their product and by $a \cdot b := \mathrm{Tr}(a^T b)$ the contraction. By identifying the dual of $\mathcal{F}(\mathcal{D}, Sym(3))$ with itself through the pairing

$$\langle m, i\rangle = \int_{\mathcal{D}} m(x) \cdot i(x)\mu,$$

we obtain the diamond operations

$$m \diamond_1 i = -m \cdot \mathbf{d}i \quad \text{and} \quad m \diamond_2 i = -2im.$$

In order to obtain the equations for micromorphic fluids we consider the Lagrangian l: $[\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{gl}(3))] \circledS [\mathcal{F}(\mathcal{D}) \oplus \mathcal{F}(\mathcal{D}, Sym(3))] \to \mathbb{R}$ given by

$$l(\mathbf{u}, \nu, \rho, i) := \frac{1}{2}\int_{\mathcal{D}} \rho\|\mathbf{u}\|^2\mu + \frac{1}{2}\int_{\mathcal{D}} \rho(i\nu \cdot \nu)\mu - \int_{\mathcal{D}} \rho\Psi\big(\rho^{-1}, i\big)\mu, \tag{8.41}$$

where the function Ψ represents the *free energy*, and where the norm in the first term is taken relative to a fixed Riemannian metric g on $\mathcal{D}$. The functional derivatives are

$$\frac{\delta l}{\delta \mathbf{u}} = \rho\mathbf{u}^\flat, \quad \frac{\delta l}{\delta \nu} = \rho i\nu, \quad \frac{\delta l}{\delta \rho} = \frac{1}{2}\|\mathbf{u}\|^2 + \frac{1}{2}i\nu \cdot \nu - \Psi\big(\rho^{-1}, i\big) + \frac{1}{\rho}\frac{\partial \Psi}{\partial \rho^{-1}}\big(\rho^{-1}, i\big) \quad \text{and}$$
$$\frac{\delta l}{\delta i} = \frac{1}{2}\rho\nu\nu^T - \rho\frac{\partial \Psi}{\partial i}\big(\rho^{-1}, i\big).$$

A computation, involving remarkable cancellations, shows that the Euler–Poincaré equations (1.2) associated to the Lagrangian (8.41) and to the group $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, GL(3)^+)$ acting linearly on $V^* = \mathcal{F}(\mathcal{D}) \oplus \mathcal{F}(\mathcal{D}, Sym(3))$ are given by

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = \dfrac{1}{\rho}\,\mathrm{grad}\,\dfrac{\partial \Psi}{\partial \rho^{-1}}, \\ i\left(\dfrac{\partial}{\partial t}\nu + \mathbf{d}\nu(\mathbf{u}) - \nu\nu\right) = 2i\dfrac{\partial \Psi}{\partial i}, \\ \dfrac{\partial}{\partial t}\rho + \mathrm{div}(\rho\mathbf{u}) = 0, \quad \dfrac{\partial}{\partial t} i + \mathbf{d}i(\mathbf{u}) + \nu^T i + i\nu = 0. \end{cases} \tag{8.42}$$

Thus we have recovered Eqs. (8.35) by Euler–Poincaré reduction, up to a change of variables, replacing ν by $-\nu^T$.

Consider the right-invariant Lagrangian $L_{(\rho_0, i_0)} : T[\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, GL(3)^+)] \to \mathbb{R}$ induced by the Lagrangian (8.41). A curve $(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, GL(3)^+)$ is a solution of the Euler–Lagrange equations associated to $L_{(\rho_0, i_0)}$ if and only if the curve

$$(\mathbf{u}, \nu) := \big(\dot\eta \circ \eta^{-1}, \dot\chi \chi^{-1} \circ \eta^{-1}\big) \in \mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}\big(\mathcal{D}, GL(3)^+\big)$$

is a solution of the micromorphic fluid equations (8.42) with initial conditions (ρ_0, i_0). Note that, in our approach, the relation $\nu = (\dot\chi \chi^{-1}) \circ \eta^{-1}$ between the microgyration tensor and the micro and macromotions is due to the passage from the Lagrangian to the spatial representation. This relation coincides with that given in [7], up to conventions. It is considered there as a definition of ν.

The evolution of the mass density ρ and the microinertia i is given by

$$\rho = J\big(\eta^{-1}\big)\big(\rho_0 \circ \eta^{-1}\big) \quad \text{and} \quad i = \big(\big(\chi^T\big)^{-1} i_0 \chi^{-1}\big) \circ \eta^{-1}.$$

Note that the evolution of the determinant of i is

$$\det(i) = \frac{\det(i_0)}{\det(\chi)^2} \circ \eta^{-1}.$$

Therefore, if the initial microinertia i_0 is invertible, then i is invertible for all time. Under this hypothesis we can take the Legendre transformation of the Lagrangian l and we obtain the Hamiltonian

$$h(\mathbf{m}, \kappa, \rho, i) = \frac{1}{2}\int_{\mathcal{D}} \frac{1}{\rho}\|\mathbf{m}\|^2 \mu + \frac{1}{2}\int_{\mathcal{D}} \frac{1}{\rho}\big(i^{-1}\kappa \cdot \kappa\big)\mu + \int_{\mathcal{D}} \rho \Psi\big(\rho^{-1}, i\big)\mu,$$

where $(\mathbf{m}, \kappa, \rho, i) \in \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathfrak{gl}(3)) \times \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, Sym(3))$, which consists of the sum of the kinetic energy due to macromotion and micromotion and the free energy. By Lie–Poisson reduction, we obtain that Eqs. (8.42) are Hamiltonian with respect to the Lie–Poisson bracket

$$\begin{aligned} \{f, g\}(\mathbf{m}, \kappa, \rho, i) = &\int_{\mathcal{D}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}}\right]\mu \\ &+ \int_{\mathcal{D}} \kappa \cdot \left(\mathrm{ad}_{\frac{\delta f}{\delta \kappa}} \frac{\delta g}{\delta \kappa} + \mathbf{d}\frac{\delta f}{\delta \kappa} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \kappa} \cdot \frac{\delta f}{\delta \mathbf{m}}\right)\mu \\ &+ \int_{\mathcal{D}} \rho\left(\mathbf{d}\left(\frac{\delta f}{\delta \rho}\right)\frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\left(\frac{\delta g}{\delta \rho}\right)\frac{\delta f}{\delta \mathbf{m}}\right)\mu \end{aligned}$$

$$+\int_{\mathcal{D}} i\cdot\left(\operatorname{div}\left(\frac{\delta f}{\delta i}\frac{\delta g}{\delta \mathbf{m}}\right)-\frac{\delta g}{\delta \kappa}\frac{\delta f}{\delta i}-\frac{\delta f}{\delta i}\left(\frac{\delta g}{\delta \kappa}\right)^{T}\right.$$

$$\left.-\operatorname{div}\left(\frac{\delta g}{\delta i}\frac{\delta f}{\delta \mathbf{m}}\right)+\frac{\delta f}{\delta \kappa}\frac{\delta g}{\delta i}+\frac{\delta g}{\delta i}\left(\frac{\delta f}{\delta \kappa}\right)^{T}\right)\mu. \tag{8.43}$$

The Kelvin–Noether circulation theorem applied to micromorphic fluids yields the simple relation

$$\frac{d}{dt}\oint_{c_t}\mathbf{u}^{\flat}=\oint_{c_t}\frac{\partial\Psi}{\partial i}\cdot\mathbf{d}i.$$

Lagrangian and Hamiltonian formulation for microstretch fluids. The symmetry group of microstretch fluid dynamics is $\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},CSO(3))$. As before, the advected quantities are the *mass density* $\rho\in\mathcal{F}(\mathcal{D})$ and the *microinertia tensor* $i\in\mathcal{F}(\mathcal{D},Sym(3))$, on which the symmetry group acts *linearly* as in the micromorphic case.

The Lagrangian of the microstretch fluid has the same expression as that of the micromorphic fluid, namely, $l:[\mathfrak{X}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathfrak{cso}(3))]\circledS[\mathcal{F}(\mathcal{D})\oplus\mathcal{F}(\mathcal{D},Sym(3))]\to\mathbb{R}$ is given by

$$l(\mathbf{u},\nu,\rho,i):=\frac{1}{2}\int_{\mathcal{D}}\rho\|\mathbf{u}\|^2\mu+\frac{1}{2}\int_{\mathcal{D}}\rho(i\nu\cdot\nu)\mu-\int_{\mathcal{D}}\rho\Psi\left(\rho^{-1},i\right)\mu.$$

Using the change of variables $i\mapsto j:=\operatorname{Tr}(i)I_3-i$, this Lagrangian reads

$$l(\mathbf{u},\nu,\rho,j)=\frac{1}{2}\int_{\mathcal{D}}\rho\|\mathbf{u}\|^2\mu+\frac{1}{2}\int_{\mathcal{D}}\rho\left(\frac{1}{2}j_0\nu_0^2+j\boldsymbol{\nu}\cdot\boldsymbol{\nu}\right)\mu-\int_{\mathcal{D}}\rho\Psi\left(\rho^{-1},j\right)\mu, \tag{8.44}$$

where $j_0:=\operatorname{Tr}(j)$. The functions $\nu_0\in\mathcal{F}(\mathcal{D})$ and $\boldsymbol{\nu}=(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2,\boldsymbol{\nu}_3)\in\mathcal{F}(\mathcal{D},\mathbb{R}^3)$ are defined as in Eq. (8.38). The expression (8.44) for the Lagrangian has the advantage of giving the formula of the energy due to microstretch and microrotation separately.

The associated Euler–Poincaré equations are the same as in (8.42). We now rewrite these equations, using the fact that, for microstretch fluids, the microgyration tensor ν has values in the Lie subalgebra $\mathfrak{cso}(3)$. We will make use of the following two lemmas.

Lemma 8.1. *Suppose that the free energy Ψ verifies the axiom of objectivity, that is,*

$$\Psi\left(\rho^{-1},A^{-1}iA\right)=\Psi\left(\rho^{-1},i\right),\quad \textit{for all } A\in O(3).$$

Then the matrix $i\frac{\partial\Psi}{\partial i}$ is symmetric.

Proof. Consider a curve $A(t)\in SO(3)$ such that $A(0)=I_3$ and $\dot{A}(0)=\xi\in\mathfrak{so}(3)$. Differentiating the equality $\Psi(A(t)^{-1}iA(t))=\Psi(i)$ at $t=0$, we obtain the condition

$$\mathbf{D}\Psi(i)(i\xi-\xi i)=0,\quad \text{for all } \xi\in\mathfrak{so}(3).$$

Using the equalities $\mathbf{D}\Psi(i)(i\xi-\xi i)=\operatorname{Tr}(\frac{\partial\Psi}{\partial i}(i\xi-\xi i))=\operatorname{Tr}((\frac{\partial\Psi}{\partial i}i-i\frac{\partial\Psi}{\partial i})\xi)$, we obtain that the matrix $\frac{\partial\Psi}{\partial i}i-i\frac{\partial\Psi}{\partial i}$ is symmetric. Since this matrix is clearly also antisymmetric, we obtain $\frac{\partial\Psi}{\partial i}i-i\frac{\partial\Psi}{\partial i}=0$. □

Lemma 8.2. *For $i \in \mathcal{F}(\mathcal{D}, Sym(3))$ define*

$$j := \mathrm{Tr}(i) I_3 - i \in \mathcal{F}\big(\mathcal{D}, Sym(3)\big).$$

For $\nu \in \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))$ define $\nu_0 \in \mathcal{F}(\mathcal{D})$ and $\boldsymbol{\nu} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3)$ by the condition

$$\nu = \nu_0 I_3 + \hat{\boldsymbol{\nu}}.$$

Then,

- *the equation*

$$i\left(\frac{D}{dt}\nu - \nu\nu\right) = 2i\frac{\partial \Psi}{\partial i} \tag{8.45}$$

  is equivalent to the system

$$\begin{cases} \dfrac{j_0}{2}\left(\dfrac{D}{dt}\nu_0 - \nu_0^2\right) + (j\boldsymbol{\nu})\cdot\boldsymbol{\nu} = 2\left(j_0\dfrac{\partial \Psi}{\partial j_0} + j\cdot\dfrac{\partial \Psi}{\partial j}\right), \\ j\dfrac{D}{dt}\boldsymbol{\nu} - 2\nu_0 j\boldsymbol{\nu} - (j\boldsymbol{\nu})\times\boldsymbol{\nu} = 0, \end{cases} \tag{8.46}$$

- *the equation*

$$\frac{D}{dt}i + \nu^T i + i\nu = 0$$

  is equivalent to the equation

$$\frac{D}{dt}j + 2\nu_0 j + [j, \hat{\boldsymbol{\nu}}] = 0.$$

Proof. The results follow by direct computations. The two equations in (8.46) are obtained by taking respectively the trace and the antisymmetric part of Eq. (8.45). For the computation of the trace, we use the equalities

$$\mathrm{Tr}(i\nu) = \nu_0 \mathrm{Tr}(i) = \frac{1}{2}\nu_0 j_0, \quad \mathrm{Tr}(i\nu\nu) = \frac{1}{2}\nu_0^2 - (j\boldsymbol{\nu})\cdot\boldsymbol{\nu} \quad \text{and} \quad \mathrm{Tr}\left(i\frac{\partial \Psi}{\partial i}\right) = j_0\frac{\partial \Psi}{\partial j_0} + j\cdot\frac{\partial \Psi}{\partial j}.$$

For the computation of the antisymmetric part, we use the equalities

$$i\nu - (i\nu)^T = \widehat{j\boldsymbol{\nu}},$$

$$i\nu\nu - (i\nu\nu)^T = 2\nu_0\big(i\hat{\boldsymbol{\nu}} - (i\hat{\boldsymbol{\nu}})^T\big) + \big(i\hat{\boldsymbol{\nu}}\hat{\boldsymbol{\nu}} - (i\hat{\boldsymbol{\nu}}\hat{\boldsymbol{\nu}})^T\big) = 2\nu_0\widehat{j\boldsymbol{\nu}} + \widehat{(j\boldsymbol{\nu})\times\boldsymbol{\nu}},$$

and the fact that the matrix $i\frac{\partial \Psi}{\partial i}$ is symmetric, by the preceding lemma. □

Using this lemma, we obtain that the Euler–Poincaré equations (8.42) become

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = \dfrac{1}{\rho}\operatorname{grad}\dfrac{\partial \Psi}{\partial \rho^{-1}}, \\ \dfrac{j_0}{2}\left(\dfrac{D}{dt}\nu_0 - \nu_0^2\right) + (j\boldsymbol{\nu})\cdot\boldsymbol{\nu} = 2\left(j_0\dfrac{\partial \Psi}{\partial j_0} + j\cdot\dfrac{\partial \Psi}{\partial j}\right), \\ j\dfrac{D}{dt}\boldsymbol{\nu} - 2\nu_0 j\boldsymbol{\nu} - (j\boldsymbol{\nu})\times\boldsymbol{\nu} = 0, \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \quad \dfrac{D}{dt}j + 2\nu_0 j + [j,\hat{\boldsymbol{\nu}}] = 0. \end{cases} \tag{8.47}$$

The third equation can be rewritten as

$$\frac{D}{dt}(j\boldsymbol{\nu}) = 0.$$

From the last equation, we deduce that the conservation law for j_0 reads

$$\frac{D}{dt}j_0 + 2\nu_0 j_0 = 0.$$

Thus we have recovered Eqs. (8.39) by Euler–Poincaré reduction, up to a change of variables, replacing ν by $-\nu^T$.

The Lagrangian reduction for microstretch fluids follows from that of micromorphic fluids and so we do not repeat it. Making use of the fact that the micromotion χ takes values in $CSO(3)$, we obtain that the evolution of the microinertia i with initial value i_0 is

$$i = \left(\left(\chi^T\right)^{-1} i_0 \chi^{-1}\right)\circ\eta^{-1} = \left(\frac{1}{\det(\chi)^{2/3}}\chi i_0 \chi^{-1}\right)\circ\eta^{-1}.$$

It follows that the evolution of the variable j has the same expression and that the evolution of the microstretch inertia j_0, with initial value $(j_0)_0$, is given by

$$j_0 = \frac{1}{\det(\chi)^{2/3}}(j_0)_0\circ\eta^{-1}.$$

As in the micromorphic case, if the initial microinertia tensor i_0 is invertible, we can obtain the equations and its associated Poisson bracket by Lie–Poisson reduction.

Lagrangian and Hamiltonian formulation for micropolar fluids. The symmetry group of micropolar fluid dynamics is $\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D}, SO(3))$. In terms of the variable j, the Lagrangian reads

$$l(\mathbf{u},\boldsymbol{\nu},\rho,j) = \frac{1}{2}\int_{\mathcal{D}}\rho\|\mathbf{u}\|^2\mu + \frac{1}{2}\int_{\mathcal{D}}\rho(j\boldsymbol{\nu}\cdot\boldsymbol{\nu})\mu - \int_{\mathcal{D}}\rho\Psi\left(\rho^{-1}, j\right)\mu.$$

Using that $\nu_0 = 0$ for micropolar fluids, we obtain that the Euler–Poincaré equations are given by

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = \dfrac{1}{\rho}\operatorname{grad}\dfrac{\partial \Psi}{\partial \rho^{-1}}, \quad j\dfrac{D}{dt}\boldsymbol{\nu} - (j\boldsymbol{\nu})\times\boldsymbol{\nu} = 0, \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \quad \dfrac{D}{dt}j + [j,\hat{\boldsymbol{\nu}}] = 0. \end{cases} \tag{8.48}$$

Note the analogy between the second equation and the equation for the rigid body.

The Lagrangian reduction for micropolar fluids follows from that of micromorphic fluids so we shall not repeat it. Making use of the fact that the micromotion χ takes values in $SO(3)$, we obtain that the evolution of the microinertia j with initial value j_0 is

$$j = \left(\chi\, j_0 \chi^{-1}\right) \circ \eta^{-1}.$$

Thus, the evolution of its determinant is given by

$$\det(j) = \det(j_0) \circ \eta^{-1},$$

which shows that j is invertible if and only if j_0 is invertible. Thus, if the initial microinertia tensor j_0 is invertible, the equations of motion (8.48) can be obtained by Lie–Poisson reduction. Using the equalities

$$\mathbf{m} := \frac{\delta l}{\delta \mathbf{u}} = \rho \mathbf{u}^\flat \quad \text{and} \quad \boldsymbol{\kappa} := \frac{\delta l}{\delta \boldsymbol{v}} = \rho\, j\boldsymbol{v},$$

the Legendre transformation yields the Hamiltonian

$$h(\mathbf{m}, \boldsymbol{\kappa}, \rho, j) = \frac{1}{2}\int_{\mathcal{D}} \frac{1}{\rho}\|\mathbf{m}\|^2 \mu + \frac{1}{2}\int_{\mathcal{D}} \frac{1}{\rho}\left(j^{-1}\boldsymbol{\kappa}\cdot\boldsymbol{\kappa}\right)\mu + \int_{\mathcal{D}} \rho\Psi\left(\rho^{-1}, j\right)\mu$$

representing the total energy of the system. In the particular case of micropolar fluids, and in terms of the variable j, the Poisson bracket (8.43) becomes

$$\begin{aligned}
\{f, g\}(\mathbf{m}, \boldsymbol{\kappa}, \rho, j) = {} & \int_{\mathcal{D}} \mathbf{m}\cdot\left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}}\right]\mu \\
& + \int_{\mathcal{D}} \kappa\cdot\left(\mathrm{ad}_{\frac{\delta f}{\delta \kappa}}\frac{\delta g}{\delta \kappa} + \mathbf{d}\frac{\delta f}{\delta \kappa}\cdot\frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \kappa}\cdot\frac{\delta f}{\delta \mathbf{m}}\right)\mu \\
& + \int_{\mathcal{D}} \rho\left(\mathbf{d}\left(\frac{\delta f}{\delta \rho}\right)\frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\left(\frac{\delta g}{\delta \rho}\right)\frac{\delta f}{\delta \mathbf{m}}\right)\mu \\
& + \int_{\mathcal{D}} j\cdot\left(\mathrm{div}\left(\frac{\delta f}{\delta j}\frac{\delta g}{\delta \mathbf{m}}\right) + \left[\frac{\delta f}{\delta j}, \frac{\delta g}{\delta \kappa}\right] - \mathrm{div}\left(\frac{\delta g}{\delta j}\frac{\delta f}{\delta \mathbf{m}}\right) - \left[\frac{\delta g}{\delta j}, \frac{\delta f}{\delta \kappa}\right]\right)\mu, \quad (8.49)
\end{aligned}$$

where the brackets in the last term denote the usual commutator bracket of matrices.

A quaternionic point of view on microstretch and micropolar fluids. We now show that, in the case of microstretch fluids, we can use the Lie group $\mathbb{H}^\times$ of invertible quaternions to describe the micromotion of the particles. To see this, recall that there is a 2 to 1 surjective group homomorphism

$$\pi : S^3 \cong SU(2) \to SO(3)$$

given by

$$\alpha + \mathbf{j}\beta \cong \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} \mapsto \begin{bmatrix} \mathrm{Re}(\alpha^2 - \beta^2) & \mathrm{Im}(\alpha^2 - \beta^2) & 2\,\mathrm{Re}(\alpha\overline{\beta}) \\ -\mathrm{Im}(\alpha^2 + \beta^2) & \mathrm{Re}(\alpha^2 + \beta^2) & -2\,\mathrm{Im}(\alpha\overline{\beta}) \\ -2\,\mathrm{Re}(\alpha\beta) & -2\,\mathrm{Im}(\alpha\beta) & |\alpha|^2 - |\beta|^2 \end{bmatrix},$$

where the universal covering group $S^3 = \{\alpha + \mathbf{j}\beta \in \mathbb{H} \mid |\alpha|^2 + |\beta|^2 = 1\} \subset \mathbb{H}^\times$ denotes the Lie group of unit quaternions. The tangent map of π at the identity is the Lie algebra isomorphism

$$\mathbf{p} = \mathbf{k}(\boldsymbol{\nu}_1 + \mathbf{i}\boldsymbol{\nu}_2 + \mathbf{j}\boldsymbol{\nu}_3) \in T_1S^3 \mapsto 2\hat{\boldsymbol{\nu}} \in \mathfrak{so}(3),$$

where $\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3)$ and the Lie algebra T_1S^3 consists of pure quaternions.

Remarkably, the map π extends to a 2 to 1 surjective group homomorphism

$$\pi : \mathbb{H}^\times \to CSO(3)$$

given by the same expression. Thus the group $\mathbb{H}^\times$ of invertible quaternions can be seen as the universal covering group of the conformal special orthogonal rotations. The Lie algebra isomorphism reads

$$\mathbf{p} = \nu_0 + \mathbf{k}(\boldsymbol{\nu}_1 + \mathbf{i}\boldsymbol{\nu}_2 + \mathbf{j}\boldsymbol{\nu}_3) \in \mathbb{H} \mapsto 2(\nu_0 I_3 + \hat{\boldsymbol{\nu}}) \in \mathfrak{cso}(3).$$

These observations show that the micromotion of a microstretch fluid can be described by a map $\chi : \mathcal{D} \to \mathbb{H}^\times$ with values in the group of invertible quaternions.

The same remark applies to micropolar fluids where one can use the group of unit quaternions S^3 instead of $SO(3)$. In both cases, the only difference in the equations of motion is that the variable ν takes values in $\mathbb{H}$ or in the Lie algebra of pure quaternions, respectively.

Remarks on the use of other groups and the anisotropic cases. For K an invertible and symmetric 3×3 matrix, we define the group

$$SO(K) = \left\{ A \in GL(3)^+ \;\middle|\; A^T K A = K \right\}.$$

This is a three-dimensional Lie group, called the *special K-orthogonal group* with Lie algebra given by

$$\mathfrak{so}(K) = \left\{ \nu \in \mathfrak{gl}(3) \;\middle|\; \nu^T K + K\nu = 0 \right\}.$$

Note that $SO(K)$ is the group of orthogonal linear maps relative to the (possibly indefinite) inner product $\boldsymbol{u}^T K \boldsymbol{v}$. The analogue of the hat map $\widehat{\ }: \mathbb{R}^3 \to \mathfrak{so}(3)$ is the map $\widehat{\ }^K : \mathbb{R}^3 \to \mathfrak{so}(K)$ defined by

$$\hat{\boldsymbol{u}}^K := K^{-1}\widehat{K\boldsymbol{u}} \in \mathfrak{so}(K).$$

The adjoint and coadjoint actions of $SO(K)$ on $\mathbb{R}^3$ are computed to be

$$\mathrm{Ad}_A \boldsymbol{u} = K^{-1}AK\boldsymbol{u} \quad \text{and} \quad \mathrm{Ad}^*_{A^{-1}} \boldsymbol{v} = KA^{-T}K^{-1}\boldsymbol{v}.$$

By differentiating the adjoint action we find the expression of the Lie bracket on $\mathbb{R}^3$ associated to the group $SO(K)$. This Lie bracket generalizes the cross-product $\boldsymbol{u} \times \boldsymbol{v}$ associated to $SO(3)$, thus we shall use the notation $\boldsymbol{u} \times_K \boldsymbol{v}$ and we call it the *cross-product associated to K*. We have

$$\begin{aligned} \boldsymbol{u} \times_K \boldsymbol{v} &:= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(t\hat{\boldsymbol{u}}^K)} \boldsymbol{v} = K^{-1}\hat{\boldsymbol{u}}^K K\boldsymbol{v} = K^{-2}\widehat{K\boldsymbol{u}}K\boldsymbol{v} \\ &= K^{-2}(K\boldsymbol{u} \times K\boldsymbol{v}) = \det(K)K^{-3}(\boldsymbol{u} \times \boldsymbol{v}), \end{aligned}$$

where we used the relation

$$K\boldsymbol{u} \times K\boldsymbol{v} = \det(K)K^{-1}(\boldsymbol{u} \times \boldsymbol{v}),$$

valid for any invertible and symmetric matrix K. To show this formula, we first note that it holds when K is a diagonal matrix. Then it suffices to write $K = B^T D B$, where $B \in SO(3)$ and D is the diagonal matrix of eigenvalues. Note that, by construction, we have the formula

$$\widehat{\boldsymbol{u} \times_K \boldsymbol{v}}^K = \big[\hat{\boldsymbol{u}}^K, \hat{\boldsymbol{v}}^K\big].$$

The infinitesimal coadjoint action is computed to be

$$\mathrm{ad}^*_{\boldsymbol{u}} \boldsymbol{w} = \det(K)\big(K^{-3}\boldsymbol{w}\big) \times \boldsymbol{u}.$$

Note that $\mathfrak{so}(K) = [\mathfrak{so}(K), \mathfrak{so}(K)]$, as is easily seen using the formula for $\times_K$. Thus, since $\dim \mathfrak{so}(K) = 3$, the Lie algebra $\mathfrak{so}(K)$ is simple (see [20, example at the end of §2, Chapter I]). According to the classification of all three-dimensional real Lie algebras (see [20, Problems 28–35 in §15, Chapter I]), it follows that if K is definite, then $\mathfrak{so}(K) \cong \mathfrak{so}(3)$ and if K is indefinite, then $\mathfrak{so}(K) \cong \mathfrak{sl}(2, \mathbb{R})$ (see also the discussion at the end of §14.6 in [23]).

A Casimir function for the Lie–Poisson bracket on $(\mathbb{R}^3, \times_K)$ is given by

$$C(\boldsymbol{w}) = \frac{1}{2}\boldsymbol{w}^T K^{-3} \boldsymbol{w},$$

since we have

$$\mathrm{ad}^*_{\delta C/\delta \boldsymbol{w}} \boldsymbol{w} = 0.$$

This shows that the coadjoint orbits of $SO(K)$ are ellipsoids (when K is definite) or hyperboloids (when K is indefinite).

When K is positive definite, the group $\mathcal{O} = SO(K)$ can be used for the description of the *anisotropic version* of the theory of micropolar fluids. In such micropolar fluids, the particles have preferred axes of deformations. Note that if $\boldsymbol{u} \in \mathbb{R}^3$ is given, then the $SO(K)$-orbit $\{A\boldsymbol{u} \mid A \in SO(K)\}$ through $\boldsymbol{u}$ in $\mathbb{R}^3$ coincides with the ellipsoid $\mathcal{E}_c := \{\boldsymbol{v} \in \mathbb{R}^3 \mid \boldsymbol{v}^T K \boldsymbol{v} = c\}$ containing $\boldsymbol{u}$. Typical cases are given by

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} \quad \text{or} \quad K = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \varepsilon \sim 0,$$

corresponding to extremely oblate and prolate ellipsoids.

As in the micropolar case, the Lagrangian of the *anisotropic micropolar fluid* is deduced from that of the micromorphic case. For the microrotation energy, we have identity

$$\mathrm{Tr}\big((i\nu)^T \nu\big) = j_K \boldsymbol{v} \cdot \boldsymbol{v},$$

where $\hat{\boldsymbol{v}}^K = \nu$ and where

$$j_K := \mathrm{Tr}\big(iK^{-2}\big)K^2 - i \in Sym(3)$$

is the anisotropic generalization of the relation

$$j = \mathrm{Tr}(i) I_3 - i$$

that was used in the micropolar case. Thus the Lagrangian of the anisotropic micropolar fluid can be written as

$$l(\mathbf{u}, \boldsymbol{v}, \rho, j_K) = \frac{1}{2}\int_{\mathcal{D}} \rho \|\mathbf{u}\|^2 \mu + \frac{1}{2}\int_{\mathcal{D}} \rho (j_K \boldsymbol{v} \cdot \boldsymbol{v}) \mu - \int_{\mathcal{D}} \rho \Psi\big(\rho^{-1}, j_K\big)\mu.$$

In terms of the variables $(\mathbf{u}, \nu, \rho, i)$ the associated motion equations are given by (8.42), except that now ν takes values in $\mathfrak{so}(K)$. One can write these equations in terms of the variables $(\mathbf{u}, \boldsymbol{\nu}, \rho, j_K)$ by simply replacing ν by $\hat{\boldsymbol{\nu}}^K$ and i by

$$\frac{1}{2}\operatorname{Tr}\big(j_K K^{-2}\big)K^2 - j_K$$

in the micromorphic fluid equations (8.42).

We also can consider an anisotropic version of the microstretch theory, by choosing as internal symmetry group the Lie group

$$CSO(K) = \big\{A \in GL^+(3) \,\big|\, \text{there exists } \lambda \in \mathbb{R} \text{ such that } A^T K A = \lambda K\big\}$$

called the *conformal special K-orthogonal group*. The Lie algebra of $CSO(K)$ is

$$\mathfrak{cso}(K) = \big\{\nu \in \mathfrak{gl}(3) \,\big|\, \text{there exists } \mu \in \mathbb{R} \text{ such that } \nu^T K + K\nu = \mu K\big\}.$$

In such *anisotropic microstretch fluids*, in addition to have preferred axes of deformations, the fluid particles can also stretch.

As in the microstretch case, we have, for all $\nu \in \mathfrak{cso}(K)$, the decomposition

$$\nu = \hat{\boldsymbol{\nu}}^K + \nu_0 I_3$$

for a unique $\boldsymbol{\nu} \in \mathbb{R}^3$ and $\nu_0 \in \mathbb{R}$. The adjoint action and Lie bracket read

$$\operatorname{Ad}_A\big(\hat{\boldsymbol{\nu}}^K + \nu_0 I_3\big) = \frac{1}{\det(A)^{1/3}}\big(K^{-1}AK\boldsymbol{\nu}\big)^{\widehat{}\,K} + \nu_0 I_3 \quad \text{and} \quad [\nu, \mu] = \widehat{\boldsymbol{\nu} \times_K \boldsymbol{\mu}}^K.$$

As in the microstretch case, the Lagrangian of the anisotropic microstretch fluid is deduced from that of the micromorphic case. In the anisotropic case the microrotation energy takes the complicate expression

$$\operatorname{Tr}\big((i\nu)^T \nu\big) = j_K \boldsymbol{\nu} \cdot \boldsymbol{\nu} - 2\nu_0 \operatorname{Tr}\big(j_K \hat{\boldsymbol{\nu}}^K\big) + \nu_0^2 \frac{1}{2}\operatorname{Tr}\big(j_K K^{-2}\big)\operatorname{Tr}\big(K^2\big) - \nu_0^2 \operatorname{Tr}(j_K),$$

which generalizes the formula

$$\operatorname{Tr}\big((i\nu)^T \nu\big) = j\boldsymbol{\nu} \cdot \boldsymbol{\nu} + \frac{1}{2} j_0 \nu_0^2$$

valid when $K = I_3$.

8.11. Liquid crystals

The liquid crystal state is a distinct phase of matter observed between the crystalline (solid) and isotropic (liquid) states. There are three main types of liquid crystal states, depending upon the amount of order in the material. The *nematic* liquid crystal phase is characterized by rod-like molecules that have no positional order but tend to point in the same direction. In the *cholesteric* (or *chiral nematic*) phase, molecules resemble helical springs, which may have opposite chiralities. As for nematics, the molecules exhibit a privileged direction, which is the axis of the helices. *Smectic* liquid crystals are essentially different from both nematics and cholesterics, in that they have one more degree of orientational order. Smectics generally form layers within which there is a loss of positional order, while orientational order is still preserved. See for example [4,5], and [28] for more information.

There are various approaches to the dynamics of liquid crystals. In this section we carry out the Lagrangian and Hamiltonian formulation for three of them:

- the *director theory* due to Oseen, Frank, Zöcher, Ericksen, and Leslie,
- the *micropolar* and *microstretch theories*, due to Eringen, which take into account the microinertia of the particles and which is applicable, for example, to *liquid crystal polymers*,
- the *ordered micropolar* approach, due to Lhuillier and Rey, which combines the director theory with the micropolar models.

For simplicity we suppose that the fluid container $\mathcal{D}$ is a domain in $\mathbb{R}^3$ and all boundary conditions are ignored. This means that in all integration by parts we assume that the boundary terms vanish.

8.11.1. Director theory

In this theory it is assumed that only the direction and not the sense of the molecules matter in the description of the physical phenomena. Thus, the preferred orientation of the molecules around a point is described by a unit vector $\mathbf{n}:\mathcal{D}\to S^2$, called the *director*, and $\mathbf{n}$ and $-\mathbf{n}$ are assumed to be equivalent.

This description is convenient for nematics and cholesterics. We will consider the director as a map $\mathbf{n}:\mathcal{D}\to\mathbb{R}^3$, and we will show that the condition $\|\mathbf{n}\|=1$ is preserved by the Euler–Poincaré dynamics.

We shall obtain the Ericksen–Leslie equations by Euler–Poincaré and Lie–Poisson reduction. For nematic and cholesteric liquid crystals, the order parameter Lie group is $\mathcal{O}=SO(3)$. In this paragraph we always identify the Lie algebra $\mathfrak{so}(3)$ with $\mathbb{R}^3$, that is, we have $\operatorname{ad}_u v=u\times v$. Identifying the dual through the canonical inner product on $\mathbb{R}^3$, we have $\operatorname{ad}^*_u w=w\times u$.

The symmetry group is the semidirect product $\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},SO(3))$. In the original Ericksen–Leslie approach, the liquid crystal flow is supposed to be incompressible. In this case the subgroup $\operatorname{Diff}(\mathcal{D})_{\mathrm{vol}}\circledS\mathcal{F}(\mathcal{D},SO(3))$ should be used. Here we treat general compressible flows. An element $(\eta,\chi)\in\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},SO(3))$ acts *linearly* on the advected quantities $(\rho,\mathbf{n})\in\mathcal{F}(\mathcal{D})\times\mathcal{F}(\mathcal{D},\mathbb{R}^3)$, by

$$(\rho,\mathbf{n})\mapsto\big(J\eta(\rho\circ\eta),\chi^{-1}(\mathbf{n}\circ\eta)\big).$$

The associated infinitesimal action and diamond operations are given by

$$\mathbf{n}\mathbf{u}=\nabla\mathbf{n}\cdot\mathbf{u},\quad \mathbf{n}\boldsymbol{v}=\mathbf{n}\times\boldsymbol{v},\quad \mathbf{m}\diamond_1\mathbf{n}=-\nabla\mathbf{n}^T\cdot\mathbf{m}\quad\text{and}\quad \mathbf{m}\diamond_2\mathbf{n}=\mathbf{n}\times\mathbf{m},$$

where $\boldsymbol{v},\mathbf{m},\mathbf{n}\in\mathcal{F}(\mathcal{D},\mathbb{R}^3)$. We use here and in the rest of the paper $\nabla\mathbf{n}$ instead of the usual derivative $\mathbf{dn}$ of the director $\mathbf{n}\in\mathcal{F}(\mathcal{D},\mathbb{R}^3)$, since this notation is standard in the liquid crystals literature. Thus $\nabla\mathbf{n}$ is a 3×3 matrix whose rows are the vectors $\nabla\mathbf{n}_1$, $\nabla\mathbf{n}_2$, $\nabla\mathbf{n}_3$ and hence its columns are $\partial_1\mathbf{n}$, $\partial_2\mathbf{n}$, $\partial_3\mathbf{n}$; $\nabla\mathbf{n}^T$ denotes the transpose of $\nabla\mathbf{n}$. The Euler–Poincaré equations (1.2) associated to the group $(\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},SO(3)))\circledS(\mathcal{F}(\mathcal{D})\times\mathcal{F}(\mathcal{D},\mathbb{R}^3))$ are

$$\begin{cases}\dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta\mathbf{u}}=-\boldsymbol{\pounds}_{\mathbf{u}}\dfrac{\delta l}{\delta\mathbf{u}}-\operatorname{div}\mathbf{u}\dfrac{\delta l}{\delta\mathbf{u}}-\dfrac{\delta l}{\delta\boldsymbol{v}}\cdot\mathbf{d}\boldsymbol{v}+\rho\,\mathbf{d}\dfrac{\delta l}{\delta\rho}-\left(\nabla\mathbf{n}^T\cdot\dfrac{\delta l}{\delta\mathbf{n}}\right)^\flat,\\[2ex] \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta\boldsymbol{v}}=\boldsymbol{v}\times\dfrac{\delta l}{\delta\boldsymbol{v}}-\operatorname{div}\left(\dfrac{\delta l}{\delta\boldsymbol{v}}\mathbf{u}\right)+\mathbf{n}\times\dfrac{\delta l}{\delta\mathbf{n}}.\end{cases}\tag{8.50}$$

Note that the first equation is in $\Omega^1(\mathcal{D})$, the dual of the Lie algebra $\mathfrak{X}(\mathcal{D})$. The advection equations are

$$\begin{cases}\dfrac{\partial}{\partial t}\rho+\operatorname{div}(\rho\mathbf{u})=0,\\[2ex] \dfrac{\partial}{\partial t}\mathbf{n}+(\nabla\mathbf{n})\mathbf{u}+\mathbf{n}\times\boldsymbol{v}=0.\end{cases}\tag{8.51}$$

The reduced Lagrangian for nematic and cholesteric liquid crystals is of the form

$$l(\mathbf{u}, \boldsymbol{\nu}, \rho, \mathbf{n}) := \frac{1}{2}\int_{\mathcal{D}} \rho\|\mathbf{u}\|^2\mu + \frac{1}{2}\int_{\mathcal{D}} \rho J\|\boldsymbol{\nu}\|^2\mu - \int_{\mathcal{D}} \rho F\big(\rho^{-1}, \mathbf{n}, \nabla\mathbf{n}\big)\mu, \tag{8.52}$$

where the constant J is the *microinertia constant* and F is the *free energy*. The axiom of objectivity requires that

$$F\big(\rho^{-1}, A^{-1}\mathbf{n}, A^{-1}\nabla\mathbf{n}A\big) = F\big(\rho^{-1}, \mathbf{n}, \nabla\mathbf{n}\big)$$

for all $A \in O(3)$ for nematics, or for all $A \in SO(3)$ for cholesterics.

A standard choice for F is the *Oseen–Zöcher–Frank free energy* given by

$$\begin{aligned}\rho F\big(\rho^{-1}, \mathbf{n}, \nabla\mathbf{n}\big) = K_2 \underbrace{(\mathbf{n}\cdot\operatorname{curl}\mathbf{n})}_{\text{chirality}} + \frac{1}{2}K_{11}\underbrace{(\operatorname{div}\mathbf{n})^2}_{\text{splay}} + \frac{1}{2}K_{22}\underbrace{(\mathbf{n}\cdot\operatorname{curl}\mathbf{n})^2}_{\text{twist}} \\ + \frac{1}{2}K_{33}\underbrace{\|\mathbf{n}\times\operatorname{curl}\mathbf{n}\|^2}_{\text{bend}},\end{aligned} \tag{8.53}$$

where $K_2 \neq 0$ for cholesterics and $K_2 = 0$ for nematics. The free energy can also contain additional terms due to external electromagnetic fields. The constants K_{11}, K_{22}, K_{33} are respectively associated to the three principal distinct director axis deformations in nematic liquid crystals, namely, splay, twist, and bend. In general, these constants are different, but the expression of the resulting equations of motion can be greatly simplified if we make the *one-constant approximation* $K_{11} = K_{22} = K_{33} = K$. In this case the free energy becomes, up to the addition of a divergence,

$$\rho F\big(\rho^{-1}, \mathbf{n}, \nabla\mathbf{n}\big) = \frac{1}{2}K\|\nabla\mathbf{n}\|^2.$$

The functional derivatives of the Lagrangian (8.52) are computed to be

$$\mathbf{m} := \frac{\delta l}{\delta\mathbf{u}} = \rho\mathbf{u}^\flat, \qquad \boldsymbol{\kappa} := \frac{\delta l}{\delta\boldsymbol{\nu}} = \rho J\boldsymbol{\nu},$$

and

$$\frac{\delta l}{\delta\rho} = \frac{1}{2}\|\mathbf{u}\|^2 + \frac{1}{2}J\|\boldsymbol{\nu}\|^2 - F + \frac{1}{\rho}\frac{\partial F}{\partial\rho^{-1}}, \qquad -\mathbf{h} := \frac{\delta l}{\delta\mathbf{n}} = -\rho\frac{\partial F}{\partial\mathbf{n}} + \partial_i\left(\rho\frac{\partial F}{\partial\mathbf{n}_{,i}}\right).$$

The vector field $\mathbf{h}$ is referred to as the *molecular field*. In the case of the free energy (8.53) for nematics ($K_2 = 0$), the vector $\mathbf{h}$ is given by

$$\mathbf{h} = K_{11}\operatorname{grad}\operatorname{div}\mathbf{n} - K_{22}\big(A\operatorname{curl}\mathbf{n} + \operatorname{curl}(A\mathbf{n})\big) + K_{33}\big(\mathbf{B}\times\operatorname{curl}\mathbf{n} + \operatorname{curl}(\mathbf{n}\times\mathbf{B})\big),$$

where $A := \mathbf{n}\cdot\operatorname{curl}\mathbf{n}$ and $\mathbf{B} := \mathbf{n}\times\operatorname{curl}\mathbf{n}$. In the case of the one-constant approximation we have $\mathbf{h} = -K\Delta\mathbf{n}$.

Using the Lagrangian (8.52), the Euler–Poincaré equations (8.50) become

$$\begin{cases}\rho\left(\dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u}\right) = \operatorname{grad}\dfrac{\partial F}{\partial\rho^{-1}} - \partial_i\left(\rho\dfrac{\partial F}{\partial\mathbf{n}_{,i}}\cdot\nabla\mathbf{n}\right), \\ \rho J\dfrac{D}{dt}\boldsymbol{\nu} = \mathbf{h}\times\mathbf{n},\end{cases} \tag{8.54}$$

where we have used the standard notation $\frac{D}{dt} := \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$ for the material derivative acting on every component of $\boldsymbol{\nu}$. The advection equations are

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \\ \dfrac{D}{dt}\mathbf{n} = \boldsymbol{\nu}\times\mathbf{n}. \end{cases} \tag{8.55}$$

The evolution of the advected quantities is given by

$$\rho = J\big(\eta^{-1}\big)\big(\rho_0\circ\eta^{-1}\big) \quad \text{and} \quad \mathbf{n} = (\chi\,\mathbf{n}_0)\circ\eta^{-1}.$$

We now show that under some conditions, Eqs. (8.54) and (8.55) are equivalent to the Ericksen–Leslie equations for liquid crystals. This will use the following lemma.

Lemma 8.3. *Let $\boldsymbol{\nu}$ and $\mathbf{n}$ be solutions of the Euler–Poincaré equations* (8.54) *and* (8.55)*. Then*:

(i) $\|\mathbf{n}_0\| = 1$ *implies* $\|\mathbf{n}\| = 1$ *for all time.*
(ii) $\frac{D}{dt}(\mathbf{n}\cdot\boldsymbol{\nu}) = 0$. *Therefore,* $\mathbf{n}_0\cdot\boldsymbol{\nu}_0 = 0$ *implies* $\mathbf{n}\cdot\boldsymbol{\nu} = 0$ *for all time.*
(iii) *Suppose that* $\mathbf{n}_0\cdot\boldsymbol{\nu}_0 = 0$ *and* $\|\mathbf{n}_0\| = 1$. *Then the second equation of* (8.55) *reads*

$$\boldsymbol{\nu} = \mathbf{n}\times\frac{D}{dt}\mathbf{n}$$

and the second equation of (8.54) *reads*

$$\rho J\frac{D^2}{dt^2}\mathbf{n} - 2q\mathbf{n} + \mathbf{h} = 0,$$

where $2q := \mathbf{n}\cdot\mathbf{h} - \rho J\|\frac{D\mathbf{n}}{dt}\|^2 = \mathbf{n}\cdot\mathbf{h} - \rho J\|\boldsymbol{\nu}\|^2$.

Proof. (i) This is clear from the evolution $\mathbf{n} = (\chi\,\mathbf{n}_0)\circ\eta^{-1}$, since $\chi\in SO(3)$.
(ii) Using the second equations in (8.54) and (8.55), we have

$$\frac{D}{dt}(\mathbf{n}\cdot\boldsymbol{\nu}) = \left(\frac{D}{dt}\mathbf{n}\right)\cdot\boldsymbol{\nu} + \mathbf{n}\cdot\left(\frac{D}{dt}\boldsymbol{\nu}\right) = (\boldsymbol{\nu}\times\mathbf{n})\cdot\boldsymbol{\nu} + \mathbf{n}\cdot\frac{1}{\rho J}(\mathbf{h}\times\mathbf{n}) = 0.$$

(iii) Using the first two results, we obtain

$$\mathbf{n}\times\frac{D}{dt}\mathbf{n} = \mathbf{n}\times(\boldsymbol{\nu}\times\mathbf{n}) = (\mathbf{n}\cdot\mathbf{n})\boldsymbol{\nu} - (\mathbf{n}\cdot\boldsymbol{\nu})\mathbf{n} = \boldsymbol{\nu},$$

which proves the first relation.

To prove the second, we take the material time-derivative of the relation above to rewrite the second equation of (8.54) as $\rho J(\mathbf{n}\times\frac{D^2}{dt^2}\mathbf{n}) = \mathbf{h}\times\mathbf{n}$. Taking the cross-product with $\mathbf{n}$ on the left we obtain the equation

$$\rho J\left(\mathbf{n}\cdot\frac{D^2}{dt^2}\mathbf{n}\right)\mathbf{n} - \rho J\frac{D^2}{dt^2}\mathbf{n} = \mathbf{h} - (\mathbf{n}\cdot\mathbf{h})\mathbf{n}.$$

Defining $2q := \mathbf{n}\cdot\mathbf{h} + \rho J(\mathbf{n}\cdot\frac{D^2}{dt^2}\mathbf{n})$, this equation reads

$$\rho J\frac{D^2}{dt^2}\mathbf{n} - 2q\mathbf{n} + \mathbf{h} = 0,$$

which is the second relation since $\mathbf{n}\cdot\frac{D^2}{dt^2}\mathbf{n} = -\|\frac{D\mathbf{n}}{dt}\|^2 = -\|\boldsymbol{\nu}\|^2$. The last two equalities are obtained by taking two consecutive material time-derivatives of $\|\mathbf{n}\|^2 = 1$ and using the identity $\boldsymbol{\nu} = \mathbf{n}\times\frac{D}{dt}\mathbf{n}$, respectively. □

Thus, we obtain the following theorem.

Theorem 8.4. *Let* $(\mathbf{u}, \boldsymbol{\nu}, \rho, \mathbf{n})$ *be a solution of Eqs.* (8.54) *and* (8.55)*, with the initial conditions* $\mathbf{n}_0$ *and* $\boldsymbol{\nu}_0$ *verifying*

$$\|\mathbf{n}_0\| = 1 \quad \textit{and} \quad \mathbf{n}_0\cdot\boldsymbol{\nu}_0 = 0.$$

Then $(\mathbf{u}, \rho, \mathbf{n})$ *is a solution of the Ericksen–Leslie equations*

$$\begin{cases} \rho\left(\dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u}\right) = \operatorname{grad}\dfrac{\partial F}{\partial \rho^{-1}} - \partial_j\left(\rho\dfrac{\partial F}{\partial \mathbf{n}_{,j}}\cdot\nabla\mathbf{n}\right), \\ \rho J\dfrac{D^2}{dt^2}\mathbf{n} - 2q\mathbf{n} + \mathbf{h} = 0, \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \end{cases} \tag{8.56}$$

where $2q := \mathbf{n}\cdot\mathbf{h} - \rho J\|\frac{D\mathbf{n}}{dt}\|^2$.

Conversely, let $(\mathbf{u}, \rho, \mathbf{n})$ *be a solution of the Ericksen–Leslie equations* (8.56) *such that* $\|\mathbf{n}\| = 1$*, and define*

$$\boldsymbol{\nu} := \mathbf{n}\times\frac{D}{dt}\mathbf{n}.$$

Then $(\mathbf{u}, \boldsymbol{\nu}, \rho, \mathbf{n})$ *is a solution of Eqs.* (8.54) *and* (8.55)*.*

Remark. One should think of the function q in the Ericksen–Leslie equation the way one regards the pressure in ideal incompressible homogeneous fluid dynamics, namely, the q is an unknown function determined by the imposed constraint $\|\mathbf{n}\| = 1$ in the following way. The dot product of the second equation of (8.56) with $\mathbf{n}$ yields the formula of q given in the statement of the theorem by imposing $\|\mathbf{n}\| = 1$.

This q does not appear in the Euler–Poincaré formulation relative to the variables $(\mathbf{u}, \boldsymbol{\nu}, \rho, \mathbf{n})$, since in this case, the constraint $\|\mathbf{n}\| = 1$ is automatically satisfied by Lemma 8.3(i).

As a consequence of Theorem 8.4, we obtain the Ericksen–Leslie equations for liquid crystals by Lagrangian reduction. Consider the right-invariant Lagrangian

$$L_{(\rho_0,\mathbf{n}_0)} : T\big[\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}\big(\mathcal{D}, SO(3)\big)\big] \to \mathbb{R}$$

induced by the Lagrangian (8.52), and assume that $\|\mathbf{n}_0\| = 1$ and $\boldsymbol{\nu}_0\cdot\mathbf{n}_0 = 0$. A curve $(\eta, \chi) \in \operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D}, SO(3))$ is a solution of the Euler–Lagrange equations associated to $L_{(\rho_0,\mathbf{n}_0)}$, with initial condition $\boldsymbol{\nu}_0$ if and only if the curve

$$(\mathbf{u}, \nu) := \big(\dot\eta\circ\eta^{-1}, \dot\chi\,\chi^{-1}\circ\eta^{-1}\big)$$

is a solution of the Ericksen–Leslie equations (8.56), where

$$\rho = J\big(\eta^{-1}\big)\big(\rho_0 \circ \eta^{-1}\big) \quad \text{and} \quad \mathbf{n} = (\chi \mathbf{n}_0) \circ \eta^{-1}.$$

As in the case of microfluids, the curve $\eta \in \mathrm{Diff}(\mathcal{D})$ describes the *Lagrangian motion of the fluid* or *macromotion*, and the curve $\chi \in \mathcal{F}(\mathcal{D}, SO(3))$ describes the *local molecular orientation relative to a fixed reference frame* or *micromotion*. A standard choice for the initial value of the director is

$$\mathbf{n}_0(x) := (0, 0, 1), \quad \text{for all } x \in \mathcal{D}.$$

In this case we obtain that

$$\mathbf{n} = \begin{pmatrix} \chi_{13} \\ \chi_{23} \\ \chi_{33} \end{pmatrix} \circ \eta^{-1}.$$

This relation is usually taken as a definition of the director, when the 3-axis is chosen as the reference axis of symmetry.

By the Legendre transformation, the Hamiltonian for liquid crystals is given by

$$h(\mathbf{m}, \boldsymbol{\kappa}, \rho, \mathbf{n}) := \frac{1}{2}\int_{\mathcal{D}} \frac{1}{\rho}\|\mathbf{m}\|^2 \mu + \frac{1}{2J}\int_{\mathcal{D}} \frac{1}{\rho}\|\boldsymbol{\kappa}\|^2 \mu + \int_{\mathcal{D}} \rho F\big(\rho^{-1}, \mathbf{n}, \nabla \mathbf{n}\big)\mu.$$

The Poisson bracket for liquid crystals is given by

$$\begin{aligned}
\{f, g\}(\mathbf{m}, \rho, \boldsymbol{\kappa}, \mathbf{n}) = {} & \int_{\mathcal{D}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}}\right]\mu \\
& + \int_{\mathcal{D}} \boldsymbol{\kappa} \cdot \left(\frac{\delta f}{\delta \boldsymbol{\kappa}} \times \frac{\delta g}{\delta \boldsymbol{\kappa}} + \mathbf{d}\frac{\delta f}{\delta \boldsymbol{\kappa}} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \boldsymbol{\kappa}} \cdot \frac{\delta f}{\delta \mathbf{m}}\right)\mu \\
& + \int_{\mathcal{D}} \rho\left(\mathbf{d}\frac{\delta f}{\delta \rho} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \rho} \cdot \frac{\delta f}{\delta \mathbf{m}}\right)\mu \\
& + \int_{\mathcal{D}} \left[\left(\mathbf{n} \times \frac{\delta f}{\delta \boldsymbol{\kappa}} + \nabla \mathbf{n} \cdot \frac{\delta f}{\delta \mathbf{m}}\right)\frac{\delta g}{\delta \mathbf{n}} - \left(\mathbf{n} \times \frac{\delta g}{\delta \boldsymbol{\kappa}} + \nabla \mathbf{n} \cdot \frac{\delta g}{\delta \mathbf{m}}\right)\frac{\delta f}{\delta \mathbf{n}}\right]\mu. \qquad (8.57)
\end{aligned}$$

The Kelvin circulation theorem for liquid crystals reads

$$\frac{d}{dt}\oint_{c_t} \mathbf{u}^\flat = \oint_{c_t} \frac{1}{\rho}\nabla \mathbf{n}^T \cdot \mathbf{h},$$

where $\mathbf{h}$ is the molecular field defined by

$$\mathbf{h} = \rho\frac{\partial F}{\partial \mathbf{n}} - \partial_i\left(\rho\frac{\partial F}{\partial \mathbf{n}_{,i}}\right).$$

8.11.2. Micropolar theory of liquid crystals

This approach is based on the equations for micropolar continua given in (8.40). The difference from the micropolar fluid treated before is that for liquid crystals the free energy Ψ depends also

on a new variable $\gamma = (\gamma_i^{ab}) \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$ called the *wryness tensor*. This variable is denoted by $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_i^a)$ when it is seen as a form with values in $\mathbb{R}^3$.

The *constitutive equations* in the *non-dissipative* case are given by

$$t_{kl} = \frac{\partial \Psi}{\partial \rho^{-1}} \delta_{kl} - \rho \frac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^a} \boldsymbol{\gamma}_l^a \quad \text{and} \quad m_{kl} = \rho \frac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^l},$$

according to (12.5.18) in [7], where the function $\Psi = \Psi(\rho^{-1}, j, \gamma) : \mathbb{R} \times Sym(3) \times \mathfrak{gl}(3) \to \mathbb{R}$ denotes the *free energy*. The axiom of objectivity requires that

$$\Psi\big(\rho^{-1}, A^{-1} j A, A^{-1} \boldsymbol{\gamma} A\big) = \Psi\big(\rho^{-1}, j, \boldsymbol{\gamma}\big)$$

for all $A \in O(3)$ (for nematics and nonchiral smectics), or for all $A \in SO(3)$ (for cholesterics and chiral smectics). See paragraphs 12.6, 12.8 and 12.9 in [7] for the choice of the free energy for nematics, cholesterics, and smectics, respectively.

Assuming that $f_l = 0$ and $l_k = 0$, Eqs. (8.40) for micropolar continua become

$$\begin{cases} \rho \dfrac{D}{dt} \mathbf{u}_l = \partial_l \dfrac{\partial \Psi}{\partial \rho^{-1}} - \partial_k \left(\rho \dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^a} \boldsymbol{\gamma}_l^a \right), \quad \rho \sigma_l = \partial_k \left(\rho \dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^l} \right) - \varepsilon_{lmn} \rho \dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}_m^a} \boldsymbol{\gamma}_n^a, \\ \dfrac{D}{dt} \rho + \rho \operatorname{div} \mathbf{u} = 0, \quad \dfrac{D}{dt} j_{kl} + (\varepsilon_{kpr} j_{lp} + \varepsilon_{lpr} j_{kp}) \boldsymbol{v}_r = 0. \end{cases} \tag{8.58}$$

These are the equations for *non-dissipative micropolar liquid crystals* as given in Section 12 of [7].

To these equations one needs to add the evolution of $\boldsymbol{\gamma}$ given by

$$\frac{D}{dt} \boldsymbol{\gamma}_l^a = \partial_l \boldsymbol{v}_a + v_{ab} \boldsymbol{\gamma}_l^b - \boldsymbol{\gamma}_r^a \partial_l \mathbf{u}_r, \tag{8.59}$$

which is Eq. (12.3.13) in [7]. Here $\frac{D}{dt}$ acts on the one-form $\boldsymbol{\gamma} \in \Omega^1(\mathcal{D}, \mathbb{R}^3)$ as $\frac{D}{dt} \boldsymbol{\gamma} := \frac{\partial}{\partial t} \boldsymbol{\gamma} + \pounds_{\mathbf{u}} \boldsymbol{\gamma}$.

Lagrangian and Hamiltonian formulation of micropolar liquid crystals. We now show that the system of Eqs. (8.58) and (8.59) can be obtained by affine Euler–Poincaré and affine Lie–Poisson reduction. As in the case of micropolar fluids, the symmetry group is the semidirect product $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, SO(3))$. The advected quantities are the *mass density* ρ, the *microinertia* tensor $j \in \mathcal{F}(\mathcal{D}, Sym(3))$, and the *wryness tensor* $\gamma \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$. The action of the symmetry group on the variables ρ and j is *linear* and is the same as for micropolar fluids. The action of (η, χ) on the wryness tensor γ is *affine* and is given by

$$\gamma \mapsto \chi^{-1} (\eta^* \gamma) \chi + \chi^{-1} T \chi.$$

The Lagrangian is the same as that for micropolar fluids, except for the fact that the free energy depends also on the wryness tensor. We thus have

$$l(\mathbf{u}, \boldsymbol{v}, \rho, j, \gamma) = \frac{1}{2} \int_{\mathcal{D}} \rho \|\mathbf{u}\|^2 \mu + \frac{1}{2} \int_{\mathcal{D}} \rho (j \boldsymbol{v} \cdot \boldsymbol{v}) \mu - \int_{\mathcal{D}} \rho \Psi\big(\rho^{-1}, j, \gamma\big) \mu. \tag{8.60}$$

The computation of the affine Euler–Poincaré equations is similar to that for micropolar fluids, except for the equation associated to the variable $\boldsymbol{v}$ for which we give some details below.

Using the Lagrangian (8.60) we find that the evolution for $\boldsymbol{\nu}$ is

$$\rho\frac{1}{2}\frac{D}{dt}\widehat{j\boldsymbol{\nu}} = 2\left(j\frac{\partial\Psi}{\partial j}\right)_A - \operatorname{div}\left(\rho\frac{\partial\Psi}{\partial\gamma}\right) - \left[\gamma_i, \rho\frac{\partial\Psi}{\partial\gamma_i}\right],$$

where the index A denotes the antisymmetric part of the matrix. According to our conventions, $\frac{D}{dt}\widehat{j\boldsymbol{\nu}}$ on the right-hand side means that $\frac{D}{dt}$ is applied to every entry of the matrix valued function $\widehat{j\boldsymbol{\nu}}: \mathcal{D} \to \mathfrak{so}(3)$. Using the equality

$$\widehat{\frac{\partial\Psi}{\partial\boldsymbol{\gamma}_i}} = 2\frac{\partial\Psi}{\partial\gamma_i},$$

we obtain

$$\rho\frac{D}{dt}(j\boldsymbol{\nu}) = 4\widetilde{\left(j\frac{\partial\Psi}{\partial j}\right)}_A - \operatorname{div}\left(\rho\frac{\partial\Psi}{\partial\boldsymbol{\gamma}}\right) - \boldsymbol{\gamma}_i \times \rho\frac{\partial\Psi}{\partial\boldsymbol{\gamma}_i},$$

where $\widetilde{\ }$ denotes the inverse of $\widehat{\ }$. We now use the axiom of objectivity to simplify this expression.

Lemma 8.5. *Suppose that the free energy Ψ verifies the axiom of objectivity, that is,*

$$\Psi\left(\rho^{-1}, A^{-1}iA, A^{-1}\boldsymbol{\gamma}A\right) = \Psi\left(\rho^{-1}, i, \boldsymbol{\gamma}\right), \quad \textit{for all } A \in SO(3)\ \left(\textit{or } O(3)\right).$$

Then

$$2\left(j\frac{\partial\Psi}{\partial j}\right)_A = \left(\left(\frac{\partial\Psi}{\partial\boldsymbol{\gamma}}\right)^T\boldsymbol{\gamma} - \boldsymbol{\gamma}\left(\frac{\partial\Psi}{\partial\boldsymbol{\gamma}}\right)^T\right)_A. \tag{8.61}$$

Proof. Consider a curve $A(t) \in SO(3)$ such that $A(0) = I_3$ and $\dot{A}(0) = \xi \in \mathfrak{so}(3)$. Differentiating the equality $\Psi(A(t)^{-1}jA(t), A(t)^{-1}\boldsymbol{\gamma}A(t)) = \Psi(j, \boldsymbol{\gamma})$ at $t = 0$, we obtain the condition

$$\mathbf{D}\Psi(j, \boldsymbol{\gamma})(j\xi - \xi j, \boldsymbol{\gamma}\xi - \xi\boldsymbol{\gamma}) = 0, \quad \text{for all } \xi \in \mathfrak{so}(3).$$

Using the equalities

$$\begin{aligned}\mathbf{D}\Psi(j, \boldsymbol{\gamma})(j\xi - \xi j, \boldsymbol{\gamma}\xi - \xi\boldsymbol{\gamma}) &= \operatorname{Tr}\left(\frac{\partial\Psi}{\partial j}(j\xi - \xi j)\right) + \operatorname{Tr}\left(\left(\frac{\partial\Psi}{\partial\boldsymbol{\gamma}}\right)^T(\boldsymbol{\gamma}\xi - \xi\boldsymbol{\gamma})\right)\\ &= \operatorname{Tr}\left(\left(\frac{\partial\Psi}{\partial j}j - j\frac{\partial\Psi}{\partial j} + \left(\frac{\partial\Psi}{\partial\boldsymbol{\gamma}}\right)^T\boldsymbol{\gamma} - \boldsymbol{\gamma}\left(\frac{\partial\Psi}{\partial\boldsymbol{\gamma}}\right)^T\right)\xi\right)\end{aligned}$$

and the identity $2(j\frac{\partial\Psi}{\partial j})_A = j\frac{\partial\Psi}{\partial j} - \frac{\partial\Psi}{\partial j}j$, we obtain the result. □

Note that Eq. (8.61), can be rewritten in $\mathbb{R}^3$ as

$$4\widetilde{\left(j\frac{\partial\Psi}{\partial j}\right)}_A = \boldsymbol{\gamma}^a \times \frac{\partial\Psi}{\partial\boldsymbol{\gamma}^a} + \boldsymbol{\gamma}_k \times \frac{\partial\Psi}{\partial\boldsymbol{\gamma}_k}.$$

Using these results for the equation for $\boldsymbol{\nu}$, the affine Euler–Poincaré equations associated to the Lagrangian (8.60) read

$$\begin{cases} \rho\left(\dfrac{\partial}{\partial t}\mathbf{u}+\nabla_{\mathbf{u}}\mathbf{u}\right)=\operatorname{grad}\dfrac{\partial\Psi}{\partial\rho^{-1}}-\partial_k\left(\rho\dfrac{\partial\Psi}{\partial\boldsymbol{\gamma}^a_k}\boldsymbol{\gamma}^a\right),\\ j\dfrac{D}{dt}\boldsymbol{\nu}-(j\boldsymbol{\nu})\times\boldsymbol{\nu}=-\dfrac{1}{\rho}\operatorname{div}\left(\rho\dfrac{\partial\Psi}{\partial\boldsymbol{\gamma}}\right)+\boldsymbol{\gamma}^a\times\dfrac{\partial\Psi}{\partial\boldsymbol{\gamma}^a},\\ \dfrac{\partial}{\partial t}\rho+\operatorname{div}(\rho\mathbf{u})=0,\quad \dfrac{D}{dt}j+[j,\hat{\boldsymbol{\nu}}]=0,\\ \dfrac{\partial}{\partial t}\gamma+\pounds_{\mathbf{u}}\gamma+\mathbf{d}^{\gamma}\hat{\boldsymbol{\nu}}=0. \end{cases} \tag{8.62}$$

Recall that $\mathbf{d}^{\gamma}\hat{\boldsymbol{\nu}}=\mathbf{d}\hat{\boldsymbol{\nu}}+[\gamma,\hat{\boldsymbol{\nu}}]$. Thus, we have recovered Eqs. (8.58) for non-dissipative micropolar liquid crystals, together with Eq. (8.59), up to a change of variables $\gamma\mapsto-\gamma$.

Consider the right-invariant Lagrangian $L_{(\rho_0,j_0,\gamma_0)}:T[\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},SO(3))]\to\mathbb{R}$ induced by the Lagrangian (8.60). A curve $(\eta,\chi)\in\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},SO(3))$ is a solution of the Euler–Lagrange equations associated to $L_{(\rho_0,j_0,\gamma_0)}$ if and only if the curve

$$(\mathbf{u},\hat{\boldsymbol{\nu}}):=\left(\dot{\eta}\circ\eta^{-1},\dot{\chi}\chi^{-1}\circ\eta^{-1}\right)\in\mathfrak{X}(\mathcal{D})\circledS\mathcal{F}\big(\mathcal{D},SO(3)\big)$$

is a solution of Eqs. (8.62) with initial conditions (ρ_0,j_0,γ_0). The evolution of the mass density ρ, the microinertia j, and the wryness tensor γ is given by

$$\rho=J\big(\eta^{-1}\big)\big(\rho_0\circ\eta^{-1}\big),\quad j=\big(\chi j_0\chi^{-1}\big)\circ\eta^{-1},\quad\text{and}\quad \gamma=\eta_*\big(\chi\gamma_0\chi^{-1}+\chi T\chi^{-1}\big).$$

If the initial value γ_0 is zero, then the evolution of γ is given by

$$\gamma=\eta_*\big(\chi T\chi^{-1}\big).$$

This relation is usually taken as a definition of γ when using Eq. (8.58). We consider γ as an independent variable and therefore the system (8.62) contains an evolution equation for γ.

The Legendre transformation and the Hamiltonian formulation can be carried out as in the case of micropolar fluids. The affine Lie–Poisson bracket consists of the sum of the Lie–Poisson bracket (8.49) with the term

$$\int_{\mathcal{D}}\left[\left(\mathbf{d}^{\gamma}\frac{\delta f}{\delta\kappa}+\pounds_{\frac{\delta f}{\delta\mathbf{m}}}\gamma\right)\cdot\frac{\delta g}{\delta\gamma}-\left(\mathbf{d}^{\gamma}\frac{\delta g}{\delta\kappa}+\pounds_{\frac{\delta g}{\delta\mathbf{m}}}\gamma\right)\cdot\frac{\delta f}{\delta\gamma}\right]\mu$$

due to the presence of the variable γ.

The Kelvin–Noether circulation theorem applied to micropolar liquid crystals yields the relation

$$\frac{d}{dt}\oint_{c_t}\mathbf{u}^{\flat}=\oint_{c_t}\frac{\partial\Psi}{\partial i}\cdot\mathbf{d}i+\frac{\partial\Psi}{\partial\gamma}\cdot\mathbf{i}_{-}\mathbf{d}\gamma-\frac{1}{\rho}\operatorname{div}\left(\rho\frac{\partial\Psi}{\partial\gamma}\right)\cdot\gamma.$$

The γ-circulation yields the relation

$$\frac{d}{dt}\oint_{c_t}\boldsymbol{\gamma}=\oint_{c_t}\boldsymbol{\nu}\times\boldsymbol{\gamma}$$

in $\mathbb{R}^3$.

Remark. According to Eq. (12.9.1) in [7], a liquid crystal flow is called *smectic* if the constraint

$$\operatorname{Tr}(\boldsymbol{\gamma}) = \sum_{i=1}^{3} \boldsymbol{\gamma}_i^i = 0$$

is satisfied. However, note that this constraint is not preserved by the evolution $\gamma = \eta_*(\chi\gamma_0\chi^{-1} + \chi T\chi^{-1})$ in general. This is consistent with the fact that the last equation in (8.62), which can be written equivalently in vectorial form as

$$\frac{\partial \boldsymbol{\gamma}}{\partial t} + \pounds_{\mathbf{u}}\boldsymbol{\gamma} + \mathbf{d}\boldsymbol{v} + \boldsymbol{\gamma} \times \boldsymbol{v} = 0,$$

does *not* imply that if the initial condition for $\boldsymbol{\gamma}$ has trace zero then $\operatorname{Tr}\boldsymbol{\gamma} = 0$ for all time.

8.11.3. Microstretch theory of polymeric liquid crystals

This approach is based on the equations for microstretch continua given in (8.39). The difference from the microstretch fluid treated before is that for polymeric liquid crystals the free energy Ψ depends also on the *wryness tensor* $\gamma \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$ and on the *microstrain* $e \in \Omega^1(\mathcal{D})$. The *constitutive equations* in the *non-dissipative* case are given by

$$t_{kl} = \frac{\partial \Psi}{\partial \rho^{-1}}\delta_{kl} - \rho\frac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^a}\boldsymbol{\gamma}_l^a - \rho\frac{\partial \Psi}{\partial e_k}e_l, \qquad m_{kl} = \rho\frac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^l},$$
$$m_k = \frac{\partial \Psi}{\partial e_k}, \quad \text{and} \quad s - t = 2\rho\left(\frac{\partial \Psi}{\partial j_{kl}}j_{kl} + \frac{\partial \Psi}{\partial j_0}j_0\right),$$

see Eqs. (16.3.13) and (16.3.13) in [7], where

$$\Psi = \Psi\left(\rho^{-1}, j, \gamma, e\right) : \mathbb{R} \times Sym(3) \times \mathfrak{gl}(3) \times \mathbb{R}^3 \to \mathbb{R}$$

denotes the *free energy*. The axiom of objectivity requires that

$$\Psi\left(\rho^{-1}, A^{-1}jA, A^{-1}\boldsymbol{\gamma}A, A^{-1}e\right) = \Psi\left(\rho^{-1}, j, \boldsymbol{\gamma}, e\right)$$

for all $A \in O(3)$ (for nematics and nonchiral smectics), or for all $A \in SO(3)$ (for cholesterics and chiral smectics). See paragraphs 16.4, 16.6 and 16.7 in [7] for the choice of the free energy for nematic, smectic, and cholesteric polymers, respectively.

Assuming that $f_l = 0$ and $l_k = 0$, Eqs. (8.39) for microstretch continua become

$$\begin{cases} \rho\dfrac{D}{dt}\mathbf{u}_l = \partial_l\dfrac{\partial \Psi}{\partial \rho^{-1}} - \partial_k\left(\rho\dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^a}\boldsymbol{\gamma}_l^a + \rho\dfrac{\partial \Psi}{\partial e_k}e_l\right), \\ \rho\sigma = \partial_k\left(\rho\dfrac{\partial \Psi}{\partial e_k}\right) - 2\rho\left(\dfrac{\partial \Psi}{\partial j_{kl}}j_{kl} + \dfrac{\partial \Psi}{\partial j_0}j_0\right), \\ \rho\sigma_l = \partial_k\left(\rho\dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^l}\right) - \varepsilon_{lmn}\rho\left(\dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}_m^a}\boldsymbol{\gamma}_n^a + \dfrac{\partial \Psi}{\partial e_m}e_n\right), \\ \dfrac{D}{dt}\rho + \rho\operatorname{div}\mathbf{u} = 0, \quad \dfrac{D}{dt}j_{kl} + (\varepsilon_{kpr}j_{lp} + \varepsilon_{lpr}j_{kp})\boldsymbol{v}_r = 0. \end{cases} \tag{8.63}$$

These are the equations for *non-dissipative microstretch polymeric liquid crystals* as studied in Section 16 of [7].

To these equations one adds the evolution of $\boldsymbol{\gamma} \in \Omega^1(\mathcal{D}, \mathbb{R}^3)$ and $e \in \Omega^1(\mathcal{D})$ given by

$$\frac{D}{dt}\boldsymbol{\gamma}_l^a = \partial_l \boldsymbol{v}_a + \hat{\boldsymbol{v}}_{ab}\boldsymbol{\gamma}_l^b - \boldsymbol{\gamma}_r^a \partial_l u_r$$

and

$$\frac{D}{dt}e_k = v_{0,k} + e_i \mathbf{u}_{i,k},$$

which is Eq. (16.3.8) in [7]. Like in Eq. (8.59), D/dt acts on the one-forms $\boldsymbol{\gamma}$ and e as $D/dt = \partial/\partial t + \boldsymbol{\pounds}_{\mathbf{u}}$.

Lagrangian and Hamiltonian formulation of polymeric liquid crystals. We now show that Eqs. (8.63) can be obtained by affine Euler–Poincaré and affine Lie–Poisson reduction. As in the case of microstretch fluids, the symmetry group is the semidirect product $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, CSO(3))$. The advected quantities are the *mass density* ρ, the *microinertia* tensor $i \in \mathcal{F}(\mathcal{D}, Sym(3))$, the *wryness tensor* $\gamma \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$, and the *microstrain* $e \in \Omega^1(\mathcal{D})$. The action of the symmetry group on the variables ρ and i is *linear* and is the same as for microstretch fluids. The action of $(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, CSO(3))$ on the wryness tensor γ is *affine* and is given by

$$\gamma \mapsto \overline{\chi}^{-1}(\eta^*\gamma)\overline{\chi} + \overline{\chi}^{-1}T\overline{\chi}, \tag{8.64}$$

where $\overline{\chi} \in \mathcal{F}(\mathcal{D}, SO(3))$ is defined by the equality

$$\chi = \det(\chi)^{1/3}\overline{\chi},$$

and is the *microrotation part* of $\chi \in \mathcal{F}(\mathcal{D}, CSO(3))$. The determinant $\det(\chi)^{1/3}$ can be seen as the *microstretch part* of χ. The action of (η, χ) on the microstrain e is also *affine* and is given by

$$e \mapsto \eta^* e + \frac{1}{3}\det(\chi)^{-1}\mathbf{d}\big(\det(\chi)\big). \tag{8.65}$$

We now explain why these affine actions are natural. The variables e and γ can be seen as the symmetric and antisymmetric parts of a *strain tensor* $\zeta \in \Omega^1(\mathcal{D}, \mathfrak{cso}(3))$. More precisely, we have

$$\zeta = eI_3 + \gamma,$$

where $3e := \mathrm{Tr}(\zeta)$ and $\gamma := \zeta_A$. The affine action of $(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, CSO(3))$ is the natural action of the automorphism group onto the connections of the trivial principal bundle $CSO(3) \times \mathcal{D} \to \mathcal{D}$, as defined in the general theory (see (3.6)), that is,

$$\zeta \mapsto \chi^{-1}\eta^*\zeta\chi + \chi^{-1}T\chi.$$

By taking the trace of this action we find the affine action (8.65), whereas the antisymmetric part gives the affine action (8.64).

We now give the associated right infinitesimal actions and diamond operations. We have

$$\gamma\mathbf{u} = \boldsymbol{\pounds}_{\mathbf{u}}\gamma, \qquad \gamma v = [\gamma, \hat{\boldsymbol{v}}], \qquad w \diamond_1 \gamma = (\mathrm{div}\, w)\cdot\gamma - w\cdot \mathbf{i}\_\mathbf{d}\gamma, \qquad w \diamond_2 \gamma = [\gamma_i, w_i],$$
$$e\mathbf{u} = \boldsymbol{\pounds}_{\mathbf{u}}e, \qquad ev = 0, \qquad f \diamond_1 e = (\mathrm{div}\, f)\cdot e - f\cdot \mathbf{i}\_\mathbf{d}e, \qquad f \diamond_2 e = 0.$$

Relative to the two group one-cocycles

$$C_1(\chi) = \overline{\chi}^{-1}T\overline{\chi} \quad \text{and} \quad C_2(\chi) = \frac{1}{3}\det(\chi)^{-1}\mathbf{d}\det(\chi)$$

we have

$$\mathbf{d}C_1(\nu) = \mathbf{d}\hat{\nu}, \qquad \mathbf{d}C_1^T(w) = -\operatorname{div} w,$$
$$\mathbf{d}C_2(\nu) = \mathbf{d}\nu_0, \qquad \mathbf{d}C_2^T(f) = -\frac{1}{3}(\operatorname{div} f)I_3.$$

The Lagrangian is the same as that for microstretch fluids, except for the fact that the free energy depends also on the wryness tensor γ and on the microstrain e. We thus have

$$l(\mathbf{u}, \nu, \rho, i, \gamma, e) = \frac{1}{2}\int_{\mathcal{D}} \rho\|\mathbf{u}\|^2\mu + \frac{1}{2}\int_{\mathcal{D}} \rho(i\nu\cdot\nu)\mu - \int_{\mathcal{D}} \rho\Psi\big(\rho^{-1}, i, \gamma, e\big)\mu, \tag{8.66}$$

where $i\nu\cdot\nu = \operatorname{Tr}((i\nu)^T\nu)$. The computation of the associated Euler–Poincaré equations involves the following generalization of Lemma 8.5.

Lemma 8.6. *Suppose that the free energy Ψ verifies the axiom of objectivity, that is,*

$$\Psi\big(\rho^{-1}, A^{-1}iA, A^{-1}\boldsymbol{\gamma}A, A^{-1}e\big) = \Psi\big(\rho^{-1}, i, \boldsymbol{\gamma}, e\big), \quad \text{for all } A\in SO(3)\ \big(\text{or } O(3)\big).$$

Then

$$2\left(j\frac{\partial\Psi}{\partial j}\right)_A = \left(\left(\frac{\partial\Psi}{\partial\boldsymbol{\gamma}}\right)^T\boldsymbol{\gamma} - \boldsymbol{\gamma}\left(\frac{\partial\Psi}{\partial\boldsymbol{\gamma}}\right)^T - e\otimes\frac{\partial\Psi}{\partial e}\right)_A. \tag{8.67}$$

Using this lemma, a direct (but long) verification shows that the affine Euler–Poincaré equations associated to the Lagrangian (8.66) are

$$\begin{cases} \rho\left(\dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u}\right) = \operatorname{grad}\dfrac{\partial\Psi}{\partial\rho^{-1}} - \partial_k\left(\rho\dfrac{\partial\Psi}{\partial\boldsymbol{\gamma}_k^a}\boldsymbol{\gamma}^a + \rho\dfrac{\partial\Psi}{\partial e_k}e\right), \\ \dfrac{j_0}{2}\left(\dfrac{D}{dt}\nu_0 - \nu_0^2\right) + (j\boldsymbol{\nu})\cdot\boldsymbol{\nu} = 2\left(j_0\dfrac{\partial\Psi}{\partial j_0} + j\cdot\dfrac{\partial\Psi}{\partial j}\right) - \dfrac{1}{\rho}\operatorname{div}\left(\rho\dfrac{\partial\Psi}{\partial e}\right), \\ j\dfrac{D}{dt}\boldsymbol{\nu} - 2\nu_0 j\boldsymbol{\nu} - (j\boldsymbol{\nu})\times\boldsymbol{\nu} = -\dfrac{1}{\rho}\operatorname{div}\left(\rho\dfrac{\partial\Psi}{\partial\boldsymbol{\gamma}}\right) + \boldsymbol{\gamma}^a\times\dfrac{\partial\Psi}{\partial\boldsymbol{\gamma}^a} + e\times\dfrac{\partial\Psi}{\partial e}, \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \quad \dfrac{D}{dt}j + 2\nu_0 j + [j, \hat{\boldsymbol{\nu}}] = 0, \\ \dfrac{\partial}{\partial t}\gamma + \pounds_{\mathbf{u}}\gamma + [\gamma, \hat{\boldsymbol{\nu}}] + \mathbf{d}\hat{\boldsymbol{\nu}} = 0, \quad \dfrac{\partial}{\partial t}e + \pounds_{\mathbf{u}}e + \mathbf{d}\nu_0 = 0. \end{cases} \tag{8.68}$$

Thus, we have recovered Eqs. (8.63) for *non-dissipative polymeric liquid crystals* together with the conservation laws for γ and e, up to a change of variables $\nu\mapsto -\nu^T$ and $\gamma\mapsto -\gamma$. Recall that the microstretch rate $\nu_0\in\mathcal{F}(\mathcal{D})$ and the microrotation rate $\boldsymbol{\nu}\in\mathcal{F}(\mathcal{D},\mathbb{R}^3)$ are constructed from the variable $\nu\in\mathcal{F}(\mathcal{D},\mathfrak{cso}(3))$ through the decomposition

$$\nu = \nu_0 I_3 + \hat{\boldsymbol{\nu}}.$$

As before, Lagrangian reduction can be carried out, by starting with the right-invariant Lagrangian $L_{(\rho_0,i_0,\gamma_0,e_0)}: T[\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D}, CSO(3))]\to\mathbb{R}$ induced by the Lagrangian (8.66). The evolution of the

linear advected quantities ρ and i is the same as in the case of microstretch fluids. The evolution of the affine advected quantities is given by

$$\gamma = \eta_*\big(\overline{\chi}\gamma_0\overline{\chi}^{-1} + \overline{\chi}T\overline{\chi}^{-1}\big) \quad \text{and} \quad e = \eta_*\bigg(e_0 + \frac{1}{3}\det(\chi)\mathbf{d}\big(\det(\chi)^{-1}\big)\bigg).$$

If the initial values are zero, then the evolution of γ and e is given by

$$\gamma = \eta_*\big(\overline{\chi}T\overline{\chi}^{-1}\big) \quad \text{and} \quad e = \frac{1}{3}\eta_*\big(\det(\chi)\mathbf{d}\big(\det(\chi)^{-1}\big)\big).$$

[7] takes these relations as definitions of γ and e.

The Legendre transformation and the Hamiltonian formulation can be carried out as in the case of microstretch fluids. The affine Lie–Poisson bracket consists of the sum of the Lie–Poisson bracket (8.43) with the two terms

$$\int_{\mathcal{D}}\bigg[\bigg(\mathbf{d}^{\gamma}\bigg(\frac{\delta f}{\delta\kappa}\bigg)_A + \pounds_{\frac{\delta f}{\delta\mathbf{m}}}\gamma\bigg)\cdot\frac{\delta g}{\delta\gamma} - \bigg(\mathbf{d}^{\gamma}\bigg(\frac{\delta g}{\delta\kappa}\bigg)_A + \pounds_{\frac{\delta g}{\delta\mathbf{m}}}\gamma\bigg)\cdot\frac{\delta f}{\delta\gamma}\bigg]\mu$$
$$+\int_{\mathcal{D}}\bigg[\bigg(\mathbf{d}\bigg(\frac{\delta f}{\delta\kappa}\bigg)_0 + \pounds_{\frac{\delta f}{\delta\mathbf{m}}}e\bigg)\frac{\delta g}{\delta e} - \bigg(\mathbf{d}\bigg(\frac{\delta g}{\delta\kappa}\bigg)_0 + \pounds_{\frac{\delta g}{\delta\mathbf{m}}}e\bigg)\frac{\delta f}{\delta e}\bigg]\mu$$

due to the presence of the variables γ and e. Here $\delta f/\delta\kappa \in \mathcal{F}(\mathcal{D}, \mathfrak{csl}(3))$ and so it has a unique decomposition $\delta f/\delta\kappa = (\delta f/\delta\kappa)_0 I_3 + (\delta f/\delta\kappa)_A$, where $(\delta f/\delta\kappa)_0 \in \mathcal{F}(\mathcal{D})$ and $(\delta f/\delta\kappa)_A \in \mathcal{F}(\mathcal{D}, \mathfrak{sl}(3))$.

The Kelvin–Noether circulation theorem applied to polymeric liquid crystals yields the relation

$$\frac{d}{dt}\oint_{c_t}\mathbf{u}^{\flat} = \oint_{c_t}\frac{\partial\Psi}{\partial i}\cdot\mathbf{d}i + \frac{\partial\Psi}{\partial\gamma}\cdot\mathbf{i}\_\mathbf{d}\gamma - \frac{1}{\rho}\operatorname{div}\bigg(\rho\frac{\partial\Psi}{\partial\gamma}\bigg)\cdot\gamma + \frac{\partial\Psi}{\partial e}\cdot\mathbf{i}\_\mathbf{d}e - \frac{1}{\rho}\operatorname{div}\bigg(\rho\frac{\partial\Psi}{\partial e}\bigg)e.$$

The γ-circulation yields the relations

$$\frac{d}{dt}\oint_{c_t}\boldsymbol{\gamma} = \oint_{c_t}\boldsymbol{v}\times\boldsymbol{\gamma}$$

in $\mathbb{R}^3$, and

$$\frac{d}{dt}\oint_{c_t}e = 0.$$

8.11.4. Ordered micropolar theory

This approach is developed in [21]. It is based on the micropolar theory and uses the Oseen–Zöcher–Frank free energy. As we will see, it gives a direct generalization of the Ericksen–Leslie equations.

Lhuillier and Rey consider the general equations for micropolar continua (8.40), together with the *constitutive relations* in the *non-dissipative case*,

$$t_{kl} = \frac{\partial F}{\partial\rho^{-1}}\delta_{kl} - \rho\frac{\partial F}{\partial\mathbf{n}^a{}_{,k}}\mathbf{n}^a{}_{,l} \quad \text{and} \quad m_{kl} = \varepsilon_{lab}\mathbf{n}_a\rho\frac{\partial F}{\partial\mathbf{n}^b{}_{,k}};$$

see Eqs. (10) in [21]. Note that here we adapted these relations to the compressible case. Thus, Eqs. (8.40) read

$$\begin{cases} \rho \dfrac{D}{dt}\mathbf{u}_l = \partial_l \dfrac{\partial F}{\partial \rho^{-1}} - \partial_k \left(\rho \dfrac{\partial F}{\partial \mathbf{n}^a_{,k}} \mathbf{n}^a_{,l} \right), \\ \rho \sigma_l = \partial_k \left(\varepsilon_{lab} \mathbf{n}_a \rho \dfrac{\partial F}{\partial \mathbf{n}^b_{,k}} \right) - \varepsilon_{lmn} \rho \dfrac{\partial F}{\partial \mathbf{n}^a_{,m}} \mathbf{n}^a_{,n}, \\ \dfrac{D}{dt}\rho + \rho \operatorname{div} \mathbf{u} = 0, \quad \dfrac{D}{dt} j_{kl} + (\varepsilon_{kpr} j_{lp} + \varepsilon_{lpr} j_{kp}) \boldsymbol{v}_r = 0. \end{cases} \tag{8.69}$$

To these equations one needs to add the evolution for $\mathbf{n}$

$$\frac{D}{dt}\mathbf{n} = \boldsymbol{v} \times \mathbf{n},$$

which is Eq. (9) in [21]. Recall that $\sigma = \frac{D}{dt}(j\boldsymbol{v})$.

Lagrangian and Hamiltonian formulation of ordered micropolar theory of liquid crystals. We now show that Eqs. (8.69) can be obtained by Euler–Poincaré and Lie–Poisson reduction. As in the case of micropolar fluids, the symmetry group is the semidirect product $\operatorname{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, SO(3))$. The advected quantities are the *mass density* ρ, the *microinertia* tensor $j \in \mathcal{F}(\mathcal{D}, Sym(3))$, and the *director* $\mathbf{n} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3)$. The representation of the symmetry group on the variables ρ and j is the same as for micropolar fluids. The representation on the director is the same as that for the director theory.

The Lagrangian is the same as that for micropolar fluids, except that it involves the elastic free energy $F(\rho^{-1}, \mathbf{n}, \nabla \mathbf{n})$, which is usually taken to be the Oseen–Zöcher–Frank free energy. We thus obtain

$$l(\mathbf{u}, \boldsymbol{v}, \rho, j, \mathbf{n}) = \frac{1}{2} \int_{\mathcal{D}} \rho \|\mathbf{u}\|^2 \mu + \frac{1}{2} \int_{\mathcal{D}} \rho (j\boldsymbol{v} \cdot \boldsymbol{v}) \mu - \int_{\mathcal{D}} \rho F\big(\rho^{-1}, \mathbf{n}, \nabla \mathbf{n}\big) \mu.$$

The associated Euler–Poincaré equations are

$$\begin{cases} \rho \left(\dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} \right) = \operatorname{grad} \dfrac{\partial F}{\partial \rho^{-1}} - \partial_i \left(\rho \dfrac{\partial F}{\partial \mathbf{n}_{,i}} \cdot \nabla \mathbf{n} \right), \\ \rho \dfrac{D}{dt}(j\boldsymbol{v}) = \mathbf{h} \times \mathbf{n}. \end{cases} \tag{8.70}$$

The advection equations are

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \dfrac{D}{dt} j + [j, \hat{\boldsymbol{v}}] = 0, \\ \dfrac{D}{dt}\mathbf{n} = \boldsymbol{v} \times \mathbf{n}. \end{cases} \tag{8.71}$$

The evolution of the advected quantities is given by

$$\rho = J\big(\eta^{-1}\big)\big(\rho_0 \circ \eta^{-1}\big), \quad j = \big(\chi\, j_0 \chi^{-1}\big) \circ \eta^{-1} \quad \text{and} \quad \mathbf{n} = (\chi\, \mathbf{n}_0) \circ \eta^{-1}.$$

Thus we have recovered Eqs. (8.69) together with the evolution of the director. One just needs to prove that the equations for the variable $\boldsymbol{v}$ are equivalent in (8.70) and (8.69). As shown in the following lemma, this is a consequence of the axiom of objectivity.

Lemma 8.7. *Suppose that the free energy F verifies the property*

$$F\big(\rho^{-1}, A^{-1}\mathbf{n}, A^{-1}\nabla\mathbf{n}A\big) = F\big(\rho^{-1}, \mathbf{n}, \nabla\mathbf{n}\big), \quad \text{for all } A \in SO(3)\ \big(\text{or } O(3)\big).$$

Then the matrix

$$\left(\frac{\partial F}{\partial \nabla\mathbf{n}}\right)^T(\nabla\mathbf{n}) - (\nabla\mathbf{n})\left(\frac{\partial F}{\partial \nabla\mathbf{n}}\right)^T - \mathbf{n}\left(\frac{\partial F}{\partial \mathbf{n}}\right)^T$$

is symmetric.

Note that in this expression the last term denotes the multiplication of a column vector with a row vector and the result of this operation is a 3×3 matrix whose (i, j) entry is $\mathbf{n}_i \frac{\partial F}{\partial \mathbf{n}_j}$.

Using this lemma, we obtain the equality

$$\partial_k\left(\varepsilon_{lab}\mathbf{n}_a\rho\frac{\partial F}{\partial \mathbf{n}^b_{,k}}\right) - \varepsilon_{lmn}\rho\frac{\partial F}{\partial \mathbf{n}^a_{,m}}\mathbf{n}^a_{,n} = (\mathbf{h}\times\mathbf{n})_l, \tag{8.72}$$

therefore, the equations associated to the variable $\boldsymbol{\nu}$ in (8.69) and (8.70) are equivalent.

The Hamiltonian and Lie–Poisson bracket can be computed as in the preceding examples. The Kelvin–Noether circulation theorem has the same form as that of the Ericksen–Leslie equations, namely

$$\frac{d}{dt}\oint_{c_t}\mathbf{u}^\flat = \oint_{c_t}\frac{1}{\rho}\nabla\mathbf{n}^T\cdot\mathbf{h}.$$

8.11.5. Comparison of the three theories

In this subsection we summarize the known relationships between the three theories for liquid crystals presented in this paper. We shall prove that the director theory of Ericksen–Leslie is a particular case of the ordered micropolar theory of Lhuillier–Rey. Therefore, one needs to compare the latter with the micropolar theory of Eringen. As will be shown, these two theories, while close, do not seem to be equivalent.

Theorem 8.8. *The Ericksen–Leslie equations are a particular case of the equations given by the ordered micropolar theory.*

Proof. As we have seen in Theorem 8.4, if $\|\mathbf{n}\| = 1$ and $\boldsymbol{\nu} = \mathbf{n}\times\frac{D}{dt}\mathbf{n}$, then the Ericksen–Leslie equations (8.56) are equivalent to Eqs. (8.54) and (8.55). It turns out that these equations are a particular case of Eqs. (8.70) and (8.71) given by the ordered micropolar theory. To see this, it suffices to assume that the microinertia j is constant and given by $j = JI_3$, where J is the microinertia constant appearing in the director theory, and I_3 is the identity 3×3 matrix. □

Thus, *the ordered micropolar theory is a generalization of the director theory, which takes into account the variation of microinertia.* We now compare the ordered director theory with the micropolar theory of Eringen for a particular choice of the initial condition for the microinertia j. This choice imposes the condition that j is a moment of inertia. This will use some technical lemmas.

Lemma 8.9. *Let j and $\mathbf{n}$ be solutions of Eqs.* (8.70) *and* (8.71)*. Let j_0 and $\mathbf{n}_0$ be the initial values and suppose that*

$$j_0 = J(I_3 - \mathbf{n}_0\otimes\mathbf{n}_0),$$

where J is a scalar constant. Then

$$j = J(I_3 - \mathbf{n}\otimes\mathbf{n})$$

for all time.

Proof. From the Lagrangian formulation, we know that the evolution of j and $\mathbf{n}$ is

$$\mathbf{n} = (\chi\,\mathbf{n}_0)\circ\eta^{-1} \quad \text{and} \quad j = \big(\chi\, j_0\chi^{-1}\big)\circ\eta^{-1}.$$

Since $\chi(\mathbf{n}\otimes\mathbf{n})\chi^{-1} = (\chi\mathbf{n})\otimes(\chi\mathbf{n})$ for all $\chi\in SO(3)$, we obtain

$$j = \big(\chi\, j_0\chi^{-1}\big)\circ\eta^{-1} = J\big(I_3 - (\chi\,\mathbf{n}_0)\otimes(\chi\,\mathbf{n}_0)\big) = J(I_3 - \mathbf{n}\otimes\mathbf{n}). \qquad \square$$

Lemma 8.10. *Let $\boldsymbol{v}$, j, and $\mathbf{n}$ be solutions of Eqs.* (8.70) *and* (8.71)*. Define*

$$\boldsymbol{\gamma} := (\nabla\mathbf{n})\times\mathbf{n} \in \Omega^1\big(\mathcal{D},\mathbb{R}^3\big),$$

that is, $\boldsymbol{\gamma}(\mathbf{v}) = [(\nabla\mathbf{n})\mathbf{v}]\times\mathbf{n} \in \mathcal{F}(\mathcal{D},\mathbb{R}^3)$ for every $\mathbf{v}\in\mathfrak{X}(\mathcal{D})$. Then $\boldsymbol{\gamma}$ verifies the equation

$$\frac{\partial}{\partial t}\boldsymbol{\gamma} + \pounds_{\mathbf{u}}\boldsymbol{\gamma} + \boldsymbol{\gamma}\times\boldsymbol{v} + \frac{j}{J}\mathbf{d}\boldsymbol{v} = 0.$$

Proof. Using the equation

$$\frac{D}{dt}\mathbf{n} = \boldsymbol{v}\times\mathbf{n}$$

and $\|\mathbf{n}\| = 1$, we compute

$$\begin{aligned}\frac{D}{dt}\boldsymbol{\gamma} &= \nabla\left(\frac{D}{dt}\mathbf{n}\right)\times\mathbf{n} + \nabla\mathbf{n}\times\frac{D}{dt}\mathbf{n} = (\mathbf{d}\boldsymbol{v}\times\mathbf{n})\times\mathbf{n} + (\boldsymbol{v}\times\nabla\mathbf{n})\times\mathbf{n} + \nabla\mathbf{n}\times(\boldsymbol{v}\times\mathbf{n})\\ &= (\mathbf{n}\cdot\mathbf{d}\boldsymbol{v})\mathbf{n} - \mathbf{d}\boldsymbol{v} + (\mathbf{n}\cdot\boldsymbol{v})\nabla\mathbf{n} - (\nabla\mathbf{n}\cdot\boldsymbol{v})\mathbf{n} = \boldsymbol{v}\times(\nabla\mathbf{n}\times\mathbf{n}) - \big(\mathbf{d}\boldsymbol{v} - (\mathbf{n}\cdot\mathbf{d}\boldsymbol{v})\mathbf{n}\big)\\ &= \boldsymbol{v}\times\boldsymbol{\gamma} - (I_3 - \mathbf{n}\otimes\mathbf{n})\mathbf{d}\boldsymbol{v}. \qquad \square\end{aligned}$$

Theorem 8.11. *Suppose* (*see the following lemma*) *that the free energy F can be written in terms of*

$$j := J(I_3 - \mathbf{n}\otimes\mathbf{n}) \quad \textit{and} \quad \boldsymbol{\gamma} := \nabla\mathbf{n}\times\mathbf{n},$$

that is, there exists a function $\Psi : \mathbb{R}\times Sym(3)\times\mathfrak{gl}(3)\to\mathbb{R}$ such that

$$F\big(\rho^{-1},\mathbf{n},\nabla\mathbf{n}\big) = \Psi\big(\rho^{-1}, j, \boldsymbol{\gamma}\big).$$

Let $(\mathbf{u}, \boldsymbol{v}, \rho, j, \mathbf{n})$ *be a solution of Eqs.* (8.70) *and* (8.71)*, and suppose that the initial conditions verify* $j_0 = J(I_3 - \mathbf{n}_0 \otimes \mathbf{n}_0)$*. Then* $(\mathbf{u}, \boldsymbol{v}, \rho, j, \boldsymbol{\gamma})$ *is a solution of the system*

$$\begin{cases} \rho\left(\dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u}\right) = \operatorname{grad}\dfrac{\partial \Psi}{\partial \rho^{-1}} - \partial_k\left(\rho\dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^a}\boldsymbol{\gamma}^a\right), \\ j\dfrac{D}{dt}\boldsymbol{v} - (j\boldsymbol{v}) \times \boldsymbol{v} = -\dfrac{1}{\rho}\operatorname{div}\left(\rho\dfrac{j}{J}\dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}}\right) + \boldsymbol{\gamma}^a \times \dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}^a}, \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \quad \dfrac{D}{dt}j + [j, \hat{\boldsymbol{v}}] = 0, \\ \dfrac{\partial}{\partial t}\boldsymbol{\gamma} + \pounds_{\mathbf{u}}\boldsymbol{\gamma} + \boldsymbol{\gamma} \times \boldsymbol{v} + \dfrac{j}{J}\mathbf{d}\boldsymbol{v} = 0. \end{cases} \tag{8.73}$$

This system is very close but distinct from the system (8.62) *studied by Eringen. The difference is due to the presence of the factor* $\frac{j}{J}$ *in the second and last equations.*

Proof. Let $(\mathbf{u}, \boldsymbol{v}, \rho, j, \mathbf{n})$ be a solution of Eqs. (8.70) and (8.71). The equations associated to the conservation of mass and microinertia in (8.73) are clearly verified since they are identical. Using that $j_0 = J(I_3 - \mathbf{n}_0 \otimes \mathbf{n}_0)$, we obtain $j = J(I_3 - \mathbf{n} \otimes \mathbf{n})$, by Lemma 8.9. Thus by Lemma 8.10, the last equation in (8.73) is verified.

Using that the relation $\boldsymbol{\gamma} = \nabla\mathbf{n} \times \mathbf{n}$ reads $\boldsymbol{\gamma}_i^a = \varepsilon_{bc}^a \mathbf{n}_{,i}^b \mathbf{n}^c$ in coordinates, we obtain

$$\frac{\partial F}{\partial \mathbf{n}_{,j}^b} = \frac{\partial \Psi}{\partial \boldsymbol{\gamma}_i^a}\frac{\partial \boldsymbol{\gamma}_i^a}{\partial \mathbf{n}_{,j}^b} = \frac{\partial \Psi}{\partial \boldsymbol{\gamma}_j^a}\varepsilon_{bc}^a \mathbf{n}^c.$$

This shows that

$$\partial_k\left(\rho\frac{\partial F}{\partial \mathbf{n}_{,k}^a}\mathbf{n}_{,l}^a\right) = \partial_k\left(\rho\frac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^a}\boldsymbol{\gamma}_l^a\right).$$

Thus, the first equation is verified.

We also have

$$\varepsilon_{lab}\mathbf{n}^a\frac{\partial F}{\partial \mathbf{n}_{,k}^b} = \varepsilon_{lab}\mathbf{n}^a\frac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^d}\varepsilon_{bc}^d\mathbf{n}^c = \mathbf{n}^a\mathbf{n}^l\frac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^a} - \frac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^l} = -(I_3 - \mathbf{n} \otimes \mathbf{n})_{la}\frac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^a} = -\left(\frac{j}{J}\frac{\partial \Psi}{\partial \boldsymbol{\gamma}}\right)_{lk}.$$

Using this equality and the relation (8.72), the second equation of (8.70) reads

$$\begin{aligned} \rho\frac{D}{dt}(j\boldsymbol{v})_l &= \partial_k\left(\varepsilon_{lab}\mathbf{n}^a\rho\frac{\partial F}{\partial \mathbf{n}_{,k}^b}\right) - \varepsilon_{lmn}\rho\frac{\partial F}{\partial \mathbf{n}_{,m}^a}\mathbf{n}_{,n}^a \\ &= -\partial_k\left(\rho\left(\frac{j}{J}\frac{\partial \Psi}{\partial \boldsymbol{\gamma}}\right)_{lk}\right) - \varepsilon_{lmn}\rho\frac{\partial \Psi}{\partial \boldsymbol{\gamma}_m^a}\boldsymbol{\gamma}_n^a. \end{aligned}$$

This can be written as

$$\rho\frac{D}{dt}(j\boldsymbol{v}) = -\operatorname{div}\left(\rho\frac{j}{J}\frac{\partial \Psi}{\partial \boldsymbol{\gamma}}\right) + \rho\boldsymbol{\gamma}^a \times \frac{\partial \Psi}{\partial \boldsymbol{\gamma}^a},$$

which is exactly the second equation in (8.73). □

No Lie–Poisson or Euler–Poincaré interpretation of the system (8.73) is known. This system would coincide with Eringen's system (8.62), if j/J could be taken equal to one. This, however, cannot be achieved since j is an advected variable; J is a constant. We interpret Theorem 8.11 as saying that the Lhuillier–Rey and Eringen formulations are very close but not equivalent.

We now show that the hypothesis made on the free energy F is verified in the case of the Oseen–Zöcher–Frank free energy.

Lemma 8.12. *Let* $\mathbf{n}$ *be a unit vector. Define*

$$\boldsymbol{\gamma} := \nabla\mathbf{n}\times\mathbf{n} \quad \textit{and} \quad j = I_3 - \mathbf{n}\otimes\mathbf{n}.$$

Then

$$\begin{aligned}
\operatorname{Tr}(\boldsymbol{\gamma}) &= \mathbf{n}\cdot\operatorname{curl}\mathbf{n},\\
\operatorname{Tr}\big(\boldsymbol{\gamma}^T\boldsymbol{\gamma}\big) &= \operatorname{Tr}\big(\nabla\mathbf{n}^T\nabla\mathbf{n}\big)\\
&= (\operatorname{div}\mathbf{n})^2 + (\mathbf{n}\cdot\operatorname{curl}\mathbf{n})^2 + \|\mathbf{n}\times\operatorname{curl}\mathbf{n}\|^2 + \operatorname{div}\big((\mathbf{n}\cdot\nabla)\mathbf{n} - \mathbf{n}\operatorname{div}\mathbf{n}\big),\\
\operatorname{Tr}\big(\boldsymbol{\gamma}^T\boldsymbol{\gamma}(\mathbf{n}\otimes\mathbf{n})\big) &= \|\mathbf{n}\times\operatorname{curl}\mathbf{n}\|^2.
\end{aligned}$$

Thus,

$$\operatorname{Tr}\big(\boldsymbol{\gamma}^T\boldsymbol{\gamma}\, j\big) = (\operatorname{div}\mathbf{n})^2 + (\mathbf{n}\cdot\operatorname{curl}\mathbf{n})^2 + \operatorname{div}\big((\mathbf{n}\cdot\nabla)\mathbf{n} - \mathbf{n}\operatorname{div}\mathbf{n}\big).$$

Proof. Note that the relation $\boldsymbol{\gamma} = \nabla\mathbf{n}\times\mathbf{n}$ reads

$$\boldsymbol{\gamma}_i = \partial_i\mathbf{n}\times\mathbf{n} = \begin{pmatrix} n^3\partial_i n^2 - n^2\partial_i n^3\\ n^1\partial_i n^3 - n^3\partial_i n^1\\ n^2\partial_i n^1 - n^1\partial_i n^2\end{pmatrix}.$$

Thus, we obtain

$$\begin{aligned}
\mathbf{n}\cdot\operatorname{curl}\mathbf{n} &= n^1\big(\partial_2 n^3 - \partial_3 n^2\big) + n^2\big(\partial_3 n^1 - \partial_1 n^3\big) + n^3\big(\partial_1 n^2 - \partial_2 n^1\big)\\
&= \big(n^3\partial_1 n^2 - n^2\partial_1 n^3\big) + \big(n^1\partial_2 n^3 - n^3\partial_2 n^1\big) + \big(n^2\partial_3 n^1 - n^1\partial_3 n^2\big)\\
&= \boldsymbol{\gamma}_1^1 + \boldsymbol{\gamma}_2^2 + \boldsymbol{\gamma}_3^3 = \operatorname{Tr}(\boldsymbol{\gamma}).
\end{aligned}$$

This shows the first equality.

In order to check the second assertion, we compute $\boldsymbol{\gamma}^T\boldsymbol{\gamma}$ in terms of $\mathbf{n}$. We have

$$\boldsymbol{\gamma}^T(\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot\boldsymbol{\gamma}(\mathbf{v}) = \mathbf{u}\cdot\big(\nabla\mathbf{n}(\mathbf{v})\times\mathbf{n}\big) = \nabla\mathbf{n}(\mathbf{v})\cdot(\mathbf{n}\times\mathbf{u}) = \mathbf{v}\cdot\nabla\mathbf{n}^T(\mathbf{n}\times\mathbf{u}),$$

therefore, we have $\boldsymbol{\gamma}^T(\mathbf{u}) = \nabla\mathbf{n}^T(\mathbf{n}\times\mathbf{u})$ and we can compute

$$\begin{aligned}
\boldsymbol{\gamma}^T\boldsymbol{\gamma}(\mathbf{u}) &= \nabla\mathbf{n}^T\big[\mathbf{n}\times\big(\boldsymbol{\gamma}(\mathbf{u})\big)\big] = \nabla\mathbf{n}^T\big[\mathbf{n}\times\big(\nabla\mathbf{n}(\mathbf{u})\times\mathbf{n}\big)\big]\\
&= \nabla\mathbf{n}^T\big[(\mathbf{n}\cdot\mathbf{n})\nabla\mathbf{n}(\mathbf{u}) - \big(\mathbf{n}\cdot\nabla\mathbf{n}(\mathbf{u})\big)\mathbf{n}\big]\\
&= \nabla\mathbf{n}^T\nabla\mathbf{n}(\mathbf{u}).
\end{aligned}$$

We have obtained the equality $\boldsymbol{\gamma}^T\boldsymbol{\gamma} = \nabla\mathbf{n}^T\nabla\mathbf{n}$. The formula

$$\operatorname{Tr}\big(\nabla\mathbf{n}^T\nabla\mathbf{n}\big) = (\operatorname{div}\mathbf{n})^2 + (\mathbf{n}\cdot\operatorname{curl}\mathbf{n})^2 + \|\mathbf{n}\times\operatorname{curl}\mathbf{n}\|^2 + \operatorname{div}\big((\mathbf{n}\cdot\nabla)\mathbf{n} - \mathbf{n}\operatorname{div}\mathbf{n}\big)$$

can be checked directly, see for example Lemma 3.3 in [28]. This proves the second equality.

The third assertion follows from the equalities

$$\operatorname{Tr}\big(\boldsymbol{\gamma}^T\boldsymbol{\gamma}(\mathbf{n}\otimes\mathbf{n})\big) = \operatorname{Tr}\big((\boldsymbol{\gamma}\mathbf{n})^T\boldsymbol{\gamma}\mathbf{n}\big) = \|\boldsymbol{\gamma}\mathbf{n}\|^2 = \big\|(\nabla\mathbf{n})\mathbf{n}\times\mathbf{n}\big\|^2 = \big\|(\nabla\mathbf{n})\mathbf{n}\big\|^2 = \|\mathbf{n}\times\operatorname{rot}\mathbf{n}\|^2,$$

where the last one follows from the identity $(\nabla\mathbf{n})\mathbf{n} = -\mathbf{n}\times\operatorname{rot}\mathbf{n}$; see Lemma 3.3 in [28]. □

As a consequence of this lemma, the expressions associated to chirality, twist, splay, and bend appearing in the Oseen–Zöcher–Frank free energy can be expressed in terms of the variables j and $\boldsymbol{\gamma}$. We have

$$\mathbf{n}\cdot\operatorname{curl}\mathbf{n} = \operatorname{Tr}(\boldsymbol{\gamma}), \qquad (\mathbf{n}\cdot\operatorname{curl}\mathbf{n})^2 = \operatorname{Tr}(\boldsymbol{\gamma})^2,$$
$$(\operatorname{div}\mathbf{n})^2 = \operatorname{Tr}\big(\boldsymbol{\gamma}^T\boldsymbol{\gamma}\, j\big) - \operatorname{Tr}(\boldsymbol{\gamma})^2, \quad \text{modulo a divergence},$$
$$\|\mathbf{n}\times\operatorname{curl}\mathbf{n}\|^2 = \operatorname{Tr}\big(\boldsymbol{\gamma}^T\boldsymbol{\gamma}\big) - \operatorname{Tr}\big(\boldsymbol{\gamma}^T\boldsymbol{\gamma}\, j\big).$$

Thus, in terms of j and $\boldsymbol{\gamma}$, the Oseen–Zöcher–Frank free energy reads

$$\rho\Psi\big(\rho^{-1}, j, \boldsymbol{\gamma}\big) = K_2\operatorname{Tr}(\boldsymbol{\gamma}) + \frac{1}{2}K_{11}\big(\operatorname{Tr}\big(\boldsymbol{\gamma}^T\boldsymbol{\gamma}\, j\big) - \operatorname{Tr}(\boldsymbol{\gamma})^2\big) + \frac{1}{2}K_{22}\operatorname{Tr}(\boldsymbol{\gamma})^2$$
$$+ \frac{1}{2}K_{33}\big(\operatorname{Tr}\big(\boldsymbol{\gamma}^T\boldsymbol{\gamma}\big) - \operatorname{Tr}\big(\boldsymbol{\gamma}^T\boldsymbol{\gamma}\, j\big)\big),$$

up to the addition of a divergence. This functional clearly satisfies the axiom of objectivity.

8.11.6. Remark on the use of other groups

From a mathematical point of view, the previous approaches generalize to any order parameter Lie group $\mathcal{O}\subset GL(3)^+$.

$\boldsymbol{\gamma}$-theory. In the case of Eringen's theory, it suffices to consider the group $\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathcal{O})$ acting on the advected quantities $(\rho, i, \zeta)\in\mathcal{F}(\mathcal{D})\oplus Sym(3)\oplus\Omega^1(\mathcal{D},\mathfrak{o})$ as

$$\rho\mapsto J\eta(\rho\circ\eta), \qquad i\mapsto\chi^T(i\circ\eta)\chi, \qquad \zeta\mapsto \operatorname{Ad}_{\chi^{-1}}\eta^*\zeta + \chi^{-1}T\chi.$$

If $\mathcal{O} = SO(3)$ then $\zeta = \gamma$ is the *wryness tensor*, and we recover the theory of micropolar liquid crystals (see Section 8.11.2). If $\mathcal{O} = CSO(3)$ then $\zeta = (\gamma, e)$, where γ is the *wryness tensor* and e is the *microstrain* and we recover the microstretch theory of polymeric liquid crystals (see Section 8.11.3). In general, we obtain a theory of "*$\mathcal{O}$-liquid crystals*" whose Lagrangian given by

$$l(\mathbf{u}, v, \rho, i, \zeta) = \frac{1}{2}\int_{\mathcal{D}}\rho\|\mathbf{u}\|^2\mu + \frac{1}{2}\int_{\mathcal{D}}\rho(iv\cdot v)\mu - \int_{\mathcal{D}}\rho\Psi\big(\rho^{-1}, i, \zeta\big)\mu,$$

and the variable ζ is interpreted as a connection on the trivial $\mathcal{O}$-principal bundle over $\mathcal{D}$. The associated affine Euler–Poincaré equations (3.13) are

$$\begin{cases} \rho\left(\dfrac{\partial}{\partial t}\mathbf{u}+\nabla_{\mathbf{u}}\mathbf{u}\right)=\operatorname{grad}\dfrac{\partial\Psi}{\partial\rho^{-1}}-\partial_k\left(\rho\dfrac{\partial\Psi}{\partial\zeta_k^a}\zeta^a\right), \\ P\left(i\left(\dfrac{\partial}{\partial t}\nu+\mathbf{d}\nu(\mathbf{u})-\nu\nu-2\dfrac{\partial\Psi}{\partial i}\right)\right)=0, \\ \dfrac{\partial}{\partial t}\rho+\operatorname{div}(\rho\mathbf{u})=0, \quad \dfrac{\partial}{\partial t}i+\mathbf{d}i(\mathbf{u})+\nu^T i+i\nu=0, \\ \dfrac{\partial}{\partial t}\zeta+\pounds_{\mathbf{u}}\zeta+[\zeta,\nu]+\mathbf{d}\nu=0, \end{cases} \tag{8.74}$$

where $P:\mathfrak{gl}(3)\to\mathfrak{o}$ denotes the orthogonal projection onto the Lie algebra $\mathfrak{o}$, associated to the inner product

$$a\cdot b=\operatorname{Tr}(a^T b),\quad a,b\in\mathfrak{o}.$$

n-theory. Recall that in the case of the Ericksen–Leslie and Lhuillier–Rey theories the director is a map $\mathbf{n}:\mathcal{D}\to\mathbb{R}^3$, on which the group $\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},SO(3))$ acts linearly as

$$\mathbf{n}\mapsto\chi^{-1}(\mathbf{n}\circ\eta). \tag{8.75}$$

This representation can be generalized to other groups in two ways. First the action (8.75) clearly still makes sense for any matrix Lie group $\mathcal{O}$. In this case, the advection equation for $\mathbf{n}$ reads

$$\frac{D}{dt}\mathbf{n}=\nu\mathbf{n},$$

where $\nu\in\mathcal{F}(\mathcal{D},\mathfrak{o})$, and the variable $\mathbf{n}$ evolves by

$$\mathbf{n}=(\chi\,\mathbf{n}_0)\circ\eta^{-1}.$$

Note that, in general, the norm $\|\mathbf{n}(x)\|$ is not constant. For example, if $\mathcal{O}=SO(K)$, for K positive definite, then $\mathbf{n}$ describes an ellipsoid. As in the case of microfluids, the typical cases to consider are

$$K_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&\varepsilon\end{pmatrix}\quad\text{or}\quad K_2=\begin{pmatrix}\varepsilon&0&0\\0&\varepsilon&0\\0&0&1\end{pmatrix},\quad \varepsilon\sim 0.$$

The group $SO(K_1)$ should be useful for the description of smectic liquid crystals. If $\mathcal{O}=CSO(3)$, then the norm $\|\mathbf{n}\|$ evolves as

$$\big\|\mathbf{n}\big(\eta(x)\big)\big\|=\det\big(\chi(x)\big)^{1/3}\big\|\mathbf{n}_0(x)\big\|.$$

Therefore, the norm depends on the "stretch part" $\det(\chi)^{1/3}$ of $\chi\in CSO(3)$. More generally, we can use the group $CSO(K)$ to model the theory of *anisotropic microstretch liquid crystals*. This constitutes a first way of generalizing the representation (8.75) to other groups.

A second way to generalize the representation (8.75) consists in replacing the director $\mathbf{n}:\mathcal{D}\to\mathbb{R}^3$ by a Lie algebra valued variable $n:\mathcal{D}\to\mathfrak{o}$ on which the group $\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathcal{O})$ acts as

$$n\mapsto\operatorname{Ad}_{\chi^{-1}}(n\circ\eta). \tag{8.76}$$

In this case, the advection equation reads

$$\frac{D}{dt}n = \mathrm{ad}_v\, n,$$

and the variable n evolves as

$$n = (\mathrm{Ad}_\chi\, n_0) \circ \eta^{-1}.$$

Note that when $\mathcal{O} = SO(3)$, both approaches coincide because the birth representation and the adjoint representation of $SO(3)$ on $\mathbb{R}^3$ are identical. When $\mathcal{O} = CSO(3)$, (microstretch case) then we can write $n = \hat{\mathbf{n}} + mI_3$ and the action (8.76) reads

$$(\mathbf{n}, m) \mapsto \big(\overline{\chi}^{-1}(\mathbf{n} \circ \eta), m \circ \eta\big),$$

where

$$\overline{\chi} := \frac{1}{\det(\chi)^{1/3}}\chi : \mathcal{D} \to SO(3),$$

and the evolutions are given by

$$\mathbf{n} = (\overline{\chi}\,\mathbf{n}_0) \circ \eta^{-1} \quad \text{and} \quad m = m_0 \circ \eta^{-1}.$$

Therefore $\mathbf{n}$ can be seen as a director, since $\|\mathbf{n}(x)\|$ is constant in time. Since the evolution of the variable m does not depend on the micromotion and the evolution of $\mathbf{n}$ depends only on the $SO(3)$ part of the micromotion $\chi \in CSO(3)$, this approach cannot be used for the description of microstretch liquid crystals. This shows that the first generalization seems physically more interesting.

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