Weak Convergence of Sequences in Function Spaces

J. W. Brace*

Department of Mathematics, University of Maryland, College Park, Maryland 20742

AND

G. D. Friend

Department of Mathematics, Lehigh University, Bethlehem, Pennsylvania 18015

Communicated by Ralph Phillips
Received December 10, 1968

1. INTRODUCTION

This paper is concerned with weak convergence of sequences of vector valued functions. The functions are to be members of a locally convex space. Thus it is meaningful to speak of the weak topology and weak convergence. Our interest comes from the fact that conditions for the weak convergence of bounded sequences are usually weaker than those for filters and nets. Theorems 2–4 give such results. A familiar example for scalar valued functions is the space $C(K)$ in which a sequence $\{x_n\}$ converges weakly to $x$ if and only if $\{x_n\}$ is bounded and converges pointwise to $x$ ([7] p. 265, Cor. 4).

We consider a linear space $G(X, F)$ of functions defined on a set $X$ with ranges in a locally convex Hausdorff space $F$. Conditions for weak convergence are obtained relative to a given $\Phi$-topology or $\mathcal{G}$-topology. The results are used to obtain three theorems on weak compactness. One of them, Theorem 6, gives a necessary and sufficient condition and improves on Theorem 17.12 of [9] when restricted to scalar valued functions (see Remark 3). The last section contains an application to the problem of approximating compact operators.

When the functions are scalar valued our space is denoted by $G(X)$. In Theorem 1 of [5] we showed that every locally convex topology

* This author was partially supported by a grant from the National Science Foundation, NSF GP-7492.
on $G(x)$ is a topology of convergence on a family of filters in the linear span of $X$. With some loss of generality we assume that $\Phi$ is composed of filters in $X$. On $G(X)$ we place the $\Phi$-topology (the topology of convergence on the members of $\Phi$) as developed in [4] and [5]. The topology is assumed to be locally convex and the space is denoted by $G_\Phi(X)$.

Let $\mathcal{G}$ be a family of subsets of $X$, $\mathcal{G}$ covering $X$ and closed under finite unions. The space $G_{\mathcal{G}}(X)$ will be assumed to have a locally convex topology of uniform convergence on members of $\mathcal{G}$ ([8], p. 195).

If $\Phi$ is taken to be all filters containing a member of $\mathcal{G}$, the $\mathcal{G}$-topology and the $\Phi$-topology are the same. Thus a $\mathcal{G}$-topology is always a special case of $\Phi$-topology.

Similar notation is used for topologies on $G(X, F)$. For a given $G_{\mathcal{G}}(X, F)$ or $G_\Phi(X, F)$ the weak topology is denoted by $\sigma(G, G')$. Our notation is patterned after [8].

2. CONVERGENCE OF SEQUENCES

Because of our concern with sequences we combine Definition 1.2 and Theorem 2.1 of [4] to give the following proposition.

**Proposition 1.** A sequence $(g_n)$ in $G_\Phi(X, F)$ converges to $g_0$ if and only if for each neighborhood $V$ of 0 in $F$ and filter $\mathcal{F}$ in $\Phi$ there exists a positive integer $n_0$ such that for each $n > n_0$ there is a $D_n$ in $\mathcal{F}$ such that $g_n(x) - g_0(x)$ is in $V$ for all $x$ in $D_n$.

The starting point is our characterization of the weak topology, Theorem 2 of [5], and the following theorem which can be found as Theorem 17.11 of [9] and implicitly in [10].

**Theorem 1.** Let $C$ be a compact (or countably compact) subset of a linear topological space $E$, and let $\{f_n\}$ be a sequence of continuous linear functionals on $E$ which is uniformly bounded on $C$. If, for each $x$ in $C$, $\lim_n f_n(x) = 0$, then the same equality holds for each $x$ in the closed convex extension of $C$.

When Theorem 1 is interpreted in terms of weak convergence we obtain Theorem 2.

**Theorem 2.** In $G_\Phi(X)$ a sequence $\{g_n\}$ $\sigma(G, G')$-converges to a function $g_0$ in $G_\Phi(X)$ if and only if $\{g_n\}$ is $\Phi$-bounded and converges to $g_0$ on all ultra-filters which are refinements of members of $\Phi$. 
Proof. Consider the space $G_\Phi(X)$ as a space of linear scalar valued functions defined on the algebraic dual $G_\sigma(X)^\ast$. Let $\mathcal{A}$ be the family of all subsets of $G_\sigma(X)$ which are of the form $\{e(D) : D \in \mathcal{F}\}$, $\mathcal{F}$ ranging through $\Phi$. Let $e$ be the natural mapping of $X$ into $G(X)^\ast$. The closure of $e(D)$ is in the $\sigma(G^\ast, G)$-topology. Because the $\Phi$-topology is linear we know that for each $g$ in $G_\sigma(X)$ and $\mathcal{F} \in \Phi$ there is a $D \in \mathcal{F}$ such that $g(D)$ is a bounded set of scalars (see [5], Def. 2 and Prop. 1). Thus the members of $\mathcal{A}$ are totally bounded and closed subsets of the algebraic dual. Such sets are compact. In the proof of Proposition 2 from [5] it is shown that the members of $\mathcal{A}$ are not empty. The $\Phi$-topology is uniform convergence on members of $\mathcal{A}$ (see Proposition 2 of [5]). From Proposition 3 (p. 204 of [8]) we see that $\sigma(G, G')$ is pointwise convergence on the closed convex hulls of members of $\mathcal{A}$. The application of Theorem 1 says that pointwise convergence of the sequence on members of $\mathcal{A}$ implies $\sigma(G, G')$-convergence. In the proof of Theorem 2 and Proposition 2 of [5] it was shown that pointwise convergence on members of $\mathcal{A}$ is equivalent to convergence on all ultra filters which are refinements of members of $\Phi$.

**Corollary 1.** In $G_\Xi(X)$ a sequence $\{g_n\}$ $\sigma(G, G')$-converges to a function $g_0$ in $G_\Xi(X)$ if and only if $\{g_n\}$ is $\Xi$-bounded and converges almost uniformly to $g_0$ on members of $\Xi$. (Special cases of the corollary are contained in Theorems 5.3 and 5.4 on p. 445 of [10].) (For the definition of almost uniform convergence see Def. 2 of [1].)

Proof. Change the $\Xi$-topology to a $\Phi$-topology by letting $\Phi$ be all filters in $X$ which contain a member of $\Xi$. Theorem 2 now says that $\sigma(G, G')$-convergence is equivalent to convergence on all ultra-filters which are refinements of members $\Phi$. This is the same as convergence on all ultra-filters which contain at least one member of $\Xi$. Such a convergence is equivalent to almost uniform convergence on member of $\Xi$ (see [4], Cor. 2.4).

**Remark 1.** Corollaries can also be obtained for Theorems 3, 5, and 7 by replacing $G_\Phi(X, F)$ by $G_\Xi(X, F)$ and changing convergence on all ultra-filters which are refinements of members of $\Phi$ to almost uniform convergence on members of $\Xi$. The proofs would be similar to the proof of Corollary 1.

For vector valued functions the condition in Theorem 2 is only a sufficient condition. This is stated in Theorem 3. The example at the end of the paper shows that the converse of Theorem 3 is usually false. An exception is given in Remark 2 below.
THEOREM 3. In \( G_\Phi(X, F) \) a sequence \( \{g_n\} \) \( \sigma(G, G') \)-converges to a function \( g_0 \) in \( G_\Phi(X, F) \) if \( \{g_n\} \) is \( \Phi \)-bounded and converges to \( g_0 \) on all ultra-filters which are refinements of members of \( \Phi \).

In the proof of Theorem 3, as well as Theorems 4, 6, 7, and 8, it is necessary to consider \( G(X, F) \) as a linear space of linear scalar valued functions. To achieve this, let \( E \) be a linear space having \( X \) as its Hamel base. Extend the members of \( G(X, F) \) over the domain \( E \) so that they become linear functions mapping \( E \) into \( F \). Now replace \( G(X, F) \) by a new space \( G(E, F) \) composed of all the linear extensions. Each \( g \) in \( G(E, F) \) is a bilinear form on \( E \times F' \). The tensor product of \( E \otimes F' \) is defined so that \( g \) will become a linear map with domain \( E \otimes F' \) (see page 366 of [S]). For each \( g \) in \( G(E, F) \) and arbitrary \( \sum_{k=1}^n x_k \otimes y_k' \) in \( E \otimes F' \), \( g(\sum_{k=1}^n x_k \otimes y_k') = \sum_{k=1}^n g(x_k, y_k') \). We will refer to \( G(X, F) \) as a subset of the algebraic dual of \( E \otimes F' \) without mention of the injection.

Proof of Theorem 3. In order to apply Theorem 2 we consider \( G(X, F) \) as a space of linear forms defined on \( E \otimes F' \). The \( \Phi \)-topology must be represented as convergence on a family of filters in \( E \otimes F' \). The desired family \( \Phi \otimes \Psi \) is composed of filters with a base \( \{ D \otimes B : D \in \mathcal{F}, B \in \mathcal{G} \} \), where \( \mathcal{F} \) and \( \mathcal{G} \) range through member of \( \Phi \) and \( \Psi \) respectively. The family \( \Psi \) is composed of all filters having a single equicontinuous subset of \( F' \) as a base. The set \( D \otimes B \) is \( \{ x \otimes y' : x \in D, y' \in B \} \). In reference [6], proposition 3.2, (iii) and (v), it is shown that \( \Phi \)-topology on the vector valued functions is the same as the \( \Phi \otimes \Psi \)-topology on the linear forms.

Let \( \mathcal{U} \) be the family of all ultra filters in \( X \) which are a refinement of a member of \( \Phi \). An argument similar to the one above implies that convergence on members of \( \mathcal{U} \), as given in the hypothesis, is equivalent to convergence on members \( \mathcal{U} \otimes \Psi \). This is stronger than \( \sigma(G, G') \)-convergence which Theorem 2 determines to be convergence on all ultra filters in \( E \otimes F' \) which are refinements of members of \( \Phi \otimes \Psi \).

REMARK 2. When \( F \) has the \( \sigma(F, F') \) topology, \( G_\Phi(X, F) \) behaves very much like \( G_\Phi(X) \). Under this condition Theorem 3.3 (ii) from [6] implies that \( \sigma(G, G') \)-convergence in \( G_\Phi(X, F) \) is a sufficient condition for convergence on all ultra-filters which are refinements of members of \( \Phi \). Thus the converse of Theorem 3 is true when \( F \) has the \( \sigma(F, F') \) topology. Under the same condition the converses of Theorems 5 and 7 are true.

THEOREM 4. In \( G_\Xi(X, F) \) a sequence \( \{g_n\} \) \( \sigma(G, G') \)-converges to 0 if
and only if \( \{g_n\} \) is uniformly bounded on each member of \( \mathcal{S} \) and 
\[
\lim_{n} \lim_i \inf \langle g_n(x_i), y_i' \rangle = 0 \quad \text{for every pair } \{(x_i), (y_i')\} \text{ of sequences,}
\]
where \( \{x_i\} \) is from some member of \( \mathcal{S} \) and \( \{y_i'\} \) is an equicontinuous sequence from \( F' \).

**Proof.** As in Theorem 3, consider \( G_\mathcal{S}(X, F) \) as a linear space of linear forms defined on \( E \otimes F' \). On the linear forms the \( \mathcal{S} \)-topology becomes uniform convergence on the family 
\[
B \text{ ranging through equicontinuous subsets of } F'
\]
(see 3.2 vii of [6]). Apply Corollary 1 to see that \( \sigma(G, G') \)-convergence of \( \{g_n\} \) is almost uniform convergence on the members of the family 
\[
\mathcal{S} \otimes \mathcal{B} \equiv \{A \otimes B : A \in \mathcal{S}, B \text{ ranging through equicontinuous subsets of } F'\}
\]
(see 5.2 of [10] and 2.3 of [3]).

### 3. Compactness

The three theorems of the section are concerned with sequential compactness, compactness, and countable compactness in the stated order.

**Theorem 5.** A set \( H \) in \( G_\mathcal{S}(X, F) \) is (relatively) sequentially \( \sigma(G, G') \)-compact if \( H \) is \( \Phi \)-bounded and (relatively) sequentially compact for the topology of convergence on all ultra-filters which are refinements of members of \( \Phi \).

The proof follows immediately from Theorem 3.

**Theorem 6.** In \( G_\mathcal{S}(X, F) \) let \( H \) be a bounded set whose closed convex hull is complete. Then the following are equivalent:

1. The set \( H \) is relatively \( \sigma(G, G') \)-compact.
2. For each sequence \( \{g_n\} \) from \( H \) and each pair \( \{(x_i), (y_i')\} \) of sequences, \( \{x_i\} \) from some member of \( \mathcal{S} \) and \( \{y_i'\} \) an equicontinuous sequence form \( F' \), it is true that 
\[
\lim_n \lim_i \langle g_n(x_i), y_i' \rangle = \lim_i \lim_n \langle g_n(x_i), y_i' \rangle
\]

whenever both of the limits exist.

**Proof.** For this proof it is not necessary to require that \( \mathcal{S} \) covers \( X \).
or is closed under finite unions. Instead we will assume only that \( \{ A : A \in \mathcal{B} \} \) distinguishes between members of \( G_{\mathcal{B}}(X, F) \), in other words, the \( \mathcal{B} \)-topology is Hausdorff.

As in the proof of Theorem 4 we view \( G_{\mathcal{B}}(X, F) \) as a linear space of linear forms defined on \( E \otimes F' \). The \( \mathcal{B} \)-topology is now the \( \mathcal{B} \otimes \mathcal{B} \)-topology, i.e., uniform convergence on members \( \mathcal{B} \otimes \mathcal{B} \). The space \( G_{\mathcal{B}}(X, F) \) is to be embedded in a product of Banach spaces as in Ref. ([8], 6.3, 6.4 on p. 46 and Exercise D on page 52). For each \( A \otimes B \) in \( \mathcal{B} \otimes \mathcal{B} \) define the factor space \( L(A \otimes B) \) to be all linear forms defined on the linear span of \( A \otimes B \) and bounded on \( A \otimes B \). Give \( L(A \otimes B) \) the supremum norm over \( A \otimes B \) to make it a Banach space. The projection of the linear form \( g, g \) in \( G_{\mathcal{B}}(X, F) \), into \( L(A \otimes B) \) is its restriction to the linear span of \( A \otimes B \). These projections combine to form a topological isomorphism of \( G_{\mathcal{B}}(X, F) \) onto a subspace of \( X[L(A \otimes B) : A \otimes B \in \mathcal{B} \otimes \mathcal{B}] \). This is obtained from the embedding Lemma 6.3 in [9] except that the relative openness of the injection follows from the fact that for each neighborhood \( U \) of 0 in \( G_{\mathcal{B}}(X, F) \) there exist factor spaces \( (L_i : i = 1, \ldots, n) \) with projections \( (p_i : i = 1, \ldots, n) \) and neighborhoods \( (V_i : i = 1, \ldots, n \) of 0) in the respective space such that \( \bigcap \{ p_i - 1(V_i) : i = 1, \ldots, n \} \subset U \).

The injection is also a topological isomorphism for the weak topologies. Recall that the weak topology on the product space is the product of the weak topologies on the factor spaces ([8], p. 268, Prop. 3).

If \( H \) is relatively compact for the \( \sigma(G, G') \) topology its projection in each \( L(A \otimes B) \) is relatively compact for the \( \sigma(L, L') \) topology. The \( \sigma(L, L') \) topology is the topology of almost uniform convergence on the convex hull of \( A \otimes B \) (see Theorem 5.5 and 5.6 of [7]). This implies statement (ii) of the theorem (see 4.3 (xii) of [3]).

For the converse observe that the image of \( H \) in \( L(A \otimes B) \) is relatively sequentially compact for the topology of almost uniform convergence on \( A \otimes B \) (see 4.2 and 4.3 (x) and (xi) of [3]). The image of \( H \) is now relatively sequentially \( \sigma(L, L') \)-compact because of Theorem 5. Because \( L(A \otimes B) \) is a Banach space, the image of \( H \) is relatively \( \sigma(L, L') \) compact. We now return to the product space

\[ X[L(A \otimes B) : A \otimes B \in \mathcal{B} \otimes \mathcal{B}] \]

and observe that \( H \) is a subset of a weakly compact set. The closed convex hull of \( H \) is complete and thus closed for the weak topology both as a subset of \( G_{\mathcal{B}}(X, F) \) and as a subset of the product space. Thus \( H \) is a relatively \( \sigma(G, G') \)-compact set.
**Remark 3.** The above theorem is also useful for scalar valued functions. For example, let $E$ be a Banach space. Consider $E$ as linear forms defined on $E'$ and obtain the norm topology as the topology of uniform convergence on the extreme points of the unit ball of $E'$. Theorem 6 says that a bounded subset $H$ of $E$ is $\sigma(E, E')$-compact if and only if for each sequence $\{g_n\}$ from $H$ and sequence $\{x_i\}$ of extreme points of the unit ball of $E'$, it is true that

$$\lim_{n} \lim_{i} \langle g_n, x_i \rangle = \lim_{i} \lim_{n} \langle g_n, x_i \rangle$$

whenever both limits exist. If Theorem 17.12 of [9] had been used instead we would have had to let $\{x_i\}$ range over all equicontinuous sequences in $E'$.

In order to prove Theorem 7 on countable compactness it is necessary to have the following lemma.

**Lemma.** If the family $\{g_i : i = 0, 1, \ldots\}$ of bounded scalar valued functions defined on a set $S$ is relatively countably compact and $g_0$ is a cluster point of the sequence $\{g_i : i, 1, \ldots\}$ for the topology of almost uniform convergence on $S$, then there is a subsequence which converges almost uniformly to $g_0$ on $S$.

**Proof.** Replace $S$ by the closure $D$ of its image in

$$X\{g_i(S) : i = 1, 2, \ldots\}$$

for the product topology. The set $D$ is separable because it is compact and metrizable. For the family $\{g_i : i = 0, 1, \ldots\}$ the topology of almost uniform convergence on $S$ is the topology of pointwise convergence on $D$ for the continuous extensions ([2], p. 986). The countable compactness of the family makes it simply equicontinuous ([3], Theorem 4.3). Thus the topology is metrizable (see Cor. 4.4 [4]).

**Theorem 7.** A subset $H$ of $G_\Xi(X, F)$ is (relatively) countably $\sigma(G, G')$-compact if it is $\Xi$-bounded and (relatively) countably compact for the topology of almost uniform convergence on members of $\Xi$.

**Proof.** As in the previous proof consider $G_\Xi(X, F)$ as a linear space of linear forms defined on $E \otimes F'$. The topology is now uniform convergence on members of $\Xi \otimes \mathcal{F}$. The $\sigma(G, G')$ topology is convergence on all ultra filters which contain the convex hull of some member of $\Xi \otimes \mathcal{F}$ (see [5] Theorem 2 and [4] Cor. 2.4).

Consider a sequence $\{g_n\}$ from $H$ having $g_0$ as a cluster point for
the topology of almost uniform convergence on members of \( \mathcal{G} \), in other words, the topology of convergence on all ultra-filters which contain some member of \( \mathcal{G} \). Let \( W \) be a \( \sigma(G, G') \)-neighborhood of \( g_0 \). Assume \( W \) is a member of the local base presented in Proposition 1 of [5]. Thus \( W \) is determined by a positive number \( \varepsilon \) and a finite collection of ultra filters, each ultra filter containing the convex hull of some member of \( \mathcal{G} \otimes \mathcal{B} \). Since \( \mathcal{G} \otimes \mathcal{B} \) is closed under finite unions there is a member \( A \otimes B \) whose convex hull belongs to each ultra filter of the finite collection.

On the family consisting of \( g_0 \) and the sequence \( \{g_n\} \) place the topology of almost uniform convergence on \( A \otimes B \). The function \( g_0 \) is a cluster point of the sequence for this topology, and our lemma tells us that there is a subsequence converging to \( g_0 \). Theorem 2 of this paper and Theorem 2 of [5] combine to say that the subsequence converges to \( g_0 \) on every ultra-filter containing the convex hull of \( A \otimes B \). Thus the subsequence is eventually in the \( \sigma(G, G') \)-neighborhood \( W \) of \( g_0 \). Since the same result can be obtained for every \( W \) from the local base, it follows that \( g_0 \) is a cluster point of \( \{g_n\} \) for the \( \sigma(G, G') \)-topology. Thus \( H \) is (relatively) countably \( \sigma(G, G') \)-compact.

**Remark 4.** It is possible to prove a theorem which differs from Theorem 7 only in that \( G(X, F) \) is replaced by \( G_{\Phi}(X, F) \) and almost uniform convergence on members of \( \mathcal{G} \) is replaced by convergence on all ultrafilters which are refinements of members of \( \Phi \).

4. **An Application and an Example**

**Theorem 8.** Let \( X \) and \( Y \) be Banach spaces, \( X \) reflexive, and let \( T_0 \) be a compact operator (a continuous linear operator which maps the unit ball of \( X \) onto a compact subset of \( Y \)). The set \( X' \otimes Y \) will be all continuous linear operators defined on \( X \) with finite dimensional range in \( Y \). The following statements are equivalent:

(i) There is a sequence \( \{T_n\} \) in \( X' \otimes Y \) such that

\[
\lim_{n} \|T_n - T_0\| = 0.
\]

(ii) There is a sequence in \( X' \otimes Y \) which converges to \( T_0 \) for the weak operator topology ([7] page 476, def. 3).

(iii) There is a subset \( M \) of \( X' \otimes Y \) such that for the weak
operator topology, $M$ is relatively countably compact and $T_0$ is in the closure of $M$.

Proof. Statement (i) implies (ii) and (ii) implies (iii). We now show that (iii) implies (i).

Let $S'$ be the unit ball of $Y'$ and consider the operators as linear forms on $X \otimes Y'$ where the norm topology is now uniform convergence on $S \otimes S'$, $S$ being the unit ball of $X$. The weak operator topology is pointwise convergence on $S \otimes S'$. Let $K$ be all compact operators. When we view $K$ as a collection of linear forms, $K$ is all linear forms whose restrictions are continuous on $S \otimes S'$ for the relative topology obtained from $\sigma(X \otimes Y', X' \otimes Y')$ (see [6], Theorem 2.4). The set $S \otimes S'$ is compact for this topology. (This is the point where the reflexivity of $X$ is used. If $X$ was not reflexive the proof could be continued by replacing $X$ with $X''$, and strengthening the weak operator topology.)

Because of the compactness of the domain and the continuity of the operators, the weak operator topology on $K$ is the topology of almost uniform convergence on $S \otimes S'$ [2]. Thus $M$ is relatively countably compact for this topology and its closure $\overline{M}$ in $K$ is countably compact and compact for the same topology ([3] Def. 3.1 and Theorem 3.3). Since the norm topology on $K$ is uniform convergence on $S \otimes S'$, Theorem 7 says that $\overline{M}$ is countably $\sigma(K, K')$-compact. The set $\overline{M}$ is also $\sigma(K, K')$-compact because we are working with the weak topology on a Banach space ([II] p. 185, Cor. 2). The set $\overline{M}$ is now compact for two Hausdorff topologies. The topologies must be identical, and thus $T_0$ is in the $\sigma(K, K')$-closure of $M$. The operator $T_0$ is also in the $\sigma(K, K')$-closure of the convex set $X' \otimes Y$. This gives statement (i) because the norm closure of $X' \otimes Y$ is the same as the weak closure (see [9] p. 154, Cor. 17.2).

In applications where one must verify either statement (ii) or (iii), statement (iii) may be the most useable. This is because in (iii) the operator $T_0$ must be shown to be the limit of a filter or net instead of the limit of a sequence.

AN EXAMPLE. In the course of the above proof we saw that a sequence of compact operators defined on a reflexive Banach converges for the $\sigma(K, K')$ topology if and only if the sequence converges for the weak operator topology. The strong operator topology is almost uniform convergence on the unit ball of $X$ because of the continuity of the compact operators and the weak compactness of the unit ball ([I] p. 647, Theorem 4.2).
Thus a sequence of compact operators which converges in the weak operator topology and not for the strong operator topology is an example of a bounded sequence which converges for the $\sigma(K, K')$-topology but not for the topology of almost uniform convergence on the unit ball of $X$. The converse of Theorem of 3 is not true.

References