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The Free Completely Regular Semigroup on a Set

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By a completely regular ("c.r.") semigroup we mean a semigroup which is a union of groups. For an account of these, see [1, Sect. 4.2] or [9, Chap. IV].

D. B. McAlister [8, Sect. 3] showed that if C is one of the following four classes of semigroups, and X is any set, there exists a free C-semigroup on X: (i) inverse semigroups, (ii) c.r. semigroups, (iii) semilattices of groups, (iv) completely simple ("c.s.") semigroups. In each of the four cases, he expresses the free C-semigroup on X as the direct limit of a sequence $\{F_n, \lambda_n\}$ of semigroups F_n and homomorphisms $\lambda_n: F_n \to F_{n+1}$, with F_1 the ordinary free semisemigroup on X.

That free inverse semigroups exist also follows from the fact, first published by B. M. Schein [12], that inverse semigroups, considered as algebras with two operations (multiplication and inversion) form a variety (equational class). (See [13] for an historical survey and further references.) The same applies to class (iii), which is easily seen to be a subvariety (in the foregoing sense) of class (i). That the argument applies also to class (ii) was pointed out by Mario Petrich [10]; his argument is repeated in Section 1 below for the sake of completeness. Finally, Professor Petrich pointed out in a letter to the author that class (iv) is a subvariety of class (ii), and so has free objects; see Section 7 below. We denote by $\mathscr{F}_X^{\mathrm{cr}}[\mathscr{F}_X^{\mathrm{cs}}]$ the free c.r. [c.s.] semigroup on a set X. For any variety of algebras, there is a standard procedure for constructing the free members of the variety; see [2] or [3]. A modification of this is developed in Sections 2 and 3 to describe $\mathscr{F}_X^{\mathrm{cr}}$.

Let us say that a semigroup S has exponent r (r an integer $\geqslant 1$) if $x^{r+1} = x$ for all x in S. The class of semigroups of exponent r is a subvariety of the variety of all c.r. semigroups, and so has a free member $\mathscr{F}_X^{\exp(r)}$ on any set X. Green's relations $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{D}$ on $\mathscr{F}_X^{\exp(r)}$ were described (although not in those terms) by J. A. Green and D. Rees [4]; see also J. M. Howie [6; IV.4]. Their results are generalized to $\mathscr{F}_X^{\mathrm{cr}}$ in Sections 4–5.

The structure of free inverse semigroups was given explicitly by H. E. Scheiblich [11], and that of free semilattices of groups by S. A. Liber [7]. The author had hoped to do the same for classes (ii) and (iv). For the latter, he

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succeeded (Section 7); but, for the former, he succeeded only in the case when X has cardinal 2 (Section 6).

The result of Section 6 is less than fascinating, but it contains one interesting piece of news: the maximal subgroups of $\mathscr{F}_X^{\mathrm{cr}}$ are free groups when $|X| = 2$. Is this true for arbitrary X? (It would suffice to prove it for all finite X.) Perhaps this question can be answered without knowing the precise structure of $\mathscr{F}_X^{\mathrm{cr}}$ From the result of Section 7, it is seen to be true for $\mathscr{F}_X^{\mathrm{cs}}$, for arbitrary X. One may also ask: are the maximal subgroups of $\mathscr{F}_X^{\exp(r)}$ free groups of exponent r?

The author wishes to express his thanks to Professor Petrich for suggesting this interesting problem, and for a number of valuable comments in addition to those mentioned above; likewise to Professor B. M. Schein for a number of corrections.

1. Preliminaries

If S is a c.r. semigroup, $x \in S$, and x^* denotes the inverse of x in the maximal subgroup of S containing x, then

$$xx^*x = x, \qquad xx^* = x^*x, \qquad (x^*)^* = x. \tag{1.1}$$

Conversely, if a semigroup S admits a unary operation $x \mapsto x^*$ satisfying (1.1), then x^* is an inverse of x commuting with x, so x belongs to a subgroup of S [9, Proposition IV.1.2], and hence S is c.r. We may thus regard a c.r. semigroup as an algebra equipped with an associative binary operation and a unary operation satisfying (1.1). The class of c.r. semigroups so regarded is equationally defined, hence a variety, and so has a free member $\mathscr{F}_X^{\mathrm{cr}}$ for each set X[2, p. 170; 3, p. 167]. Since a homomorphism ϕ of one c.r. semigroup S into another one preserves $*$, that is, $(x\phi)^* = x^*\phi$ for all x in S, it is clear that $\mathscr{F}_X^{\mathrm{cr}}$ is also a free c.r. semigroup on X in the category of semigroups. (The foregoing is due to M. Petrich [10].)

We shall find it convenient to introduce a less restrictive class of algebra. By a ***unary semigroup*** we shall mean an algebra $S(\cdot, *)$ having an associative binary operation $\cdot$ and a unary operation $*$, with no restriction at all on $*$. If S and T are unary semigroups, a mapping $\phi\colon S \to T$ will be called a ***unary homomorphism*** if $(uv)\phi = (u\phi)(v\phi)$ and $u^*\phi = (u\phi)^*$ for all u, v in S. A subset T of a unary semigroup S is called a ***unary subsemigroup*** if $u, v \in T$ imply $uv \in T$ and $u^* \in T$. A subset T of S ***generates*** S if no proper unary subsemigroup of S contains T. The free unary semigroup on a set X will be denoted by $\mathscr{F}_X^*$.

By a slight modification of the standard procedure for describing free algebras (see [2] or [3]), we can regard the elements of $\mathscr{F}_X^*$ as obtained from X by a finite sequence of operations, whereby we either apply $*$ to an expression previously formed, or take a finite sequence of such expressions. An example is

$$w = x_1(x_2x_3(x_4)^*x_5(x_5x_6x_7(x_8x_{10})^*x_{11}(x_{12})^*x_{13})^*x_{14}x_{15})^*, \tag{1.2}$$

where $x_1, x_2, \ldots, x_{15}$ are elements of X. Such an expression will be called a *polynomial* although this use of the term is a little different from that of Gratzer [3, Sect. 8]. We then get $\mathscr{F}_X^{\mathrm{cr}}$ as $\mathscr{F}_X^*/\rho$, where ρ is the smallest congruence on $\mathscr{F}_X^*$ containing the pairs (ww^*w, w), (ww^*, w^*w), and (w^{**}, w) for all w in $\mathscr{F}_X^*$.

It will be necessary in Section 5 below to consider initial or final segments of words like (1.2), for example $x_1(x_2x_3(x_4$ or $x_{13})^*x_{14}x_{15})^*$, which are meaningless in the above context. We shall therefore give a different procedure that arrives at the same destination. We define $\mathscr{F}_X^*$ as consisting of formal expressions like (1.2), prove that it is indeed the free unary semigroup on X, and then show that its elements can be interpreted as polynomials in the above sense.

2. Construction of the Free Unary Semigroup $\mathscr{F}_X^*$ on a Set X

We begin by adjoining two elements to the set X, namely the symbols (and)*, and then form the free semigroup $\mathscr{F}_{\bar{X}}$ on the enlarged set $\bar{X}$. We name these symbols *paren* and *parenstar*, respectively. We define $\mathscr{F}_X^*$ to be the set of all well-formed words in $\mathscr{F}_{\bar{X}}$. By a *well-formed word* we mean an element w of $\mathscr{F}_{\bar{X}}$ satisfying the following conditions:

(WF1) the number of occurrences of paren in w is equal to the number of occurrences of parenstar;

(WF2) in each initial segment of w, the number of parens is at least as great as the number of parenstars:

(WF3) the symbol paren is never immediately followed by parenstar.

If u and v are well-formed words on $\bar{X}$, so is their concatenation uv (product in $\mathscr{F}_{\bar{X}}$), and so is $(u)^*$. Hence $\mathscr{F}_X^*$ is a subsemigroup of $\mathscr{F}_{\bar{X}}$, and is moreover a unary semigroup, with the unary operation $w \mapsto (w)^*$.

A word in $\mathscr{F}_{\bar{X}}$ satisfying only (WF2) and (WF3) will be called *pre-well-formed*. Thus, any initial segment of a well-formed word is pre-well-formed, and has a perfectly good meaning in $\mathscr{F}_{\bar{X}}$. Any final segment is *epi-well-formed* in the sense that in each final segment there are at least as many parenstars as parens.

By the length $|w|$ of a word w in $\mathscr{F}_{\bar{X}}$ we mean its length as a sequence, counting parens and parenstars as well as elements of X. In the example (1.2), $|w| = 25$. We shall often abbreviate $(w)^*$ to w^*, and $((w)^*)^*$ to w^{**}, but note that $|w^*| = |w| + 2$.

We identify the element x of X with the word of length 1 whose only term is x. Clearly the word x is well-formed, hence belongs to $\mathscr{F}_X^*$. To summarize, $\mathscr{F}_X^*$ is a unary semigroup containing X and having length-function $|\cdot|: \mathscr{F}_X^* \to \mathbb{N}$ (the natural numbers) satisfying $|uv| = |u| + |v|$ and $|u^*| = |u| + 2$, for all u, v in $\mathscr{F}_X^*$.

An element w of $\mathscr{F}_X^*$ is called *reducible* if $w = uv$ for some u, v in $\mathscr{F}_X^*$, and otherwise *irreducible.*

LEMMA 2.1. (i) *The irreducible elements of $\mathscr{F}_X^*$ are just the elements of X and the elements of the form w^* with w in $\mathscr{F}_X^*$.*

(ii) *Each element of $\mathscr{F}_X^*$ is uniquely expressible as a product of irreducible elements.*

(iii) *If $w = w_1 w_2 \cdots w_m$ with $w_1, \ldots, w_m$ irreducible, and if $w = uv$ for some u, v in $\mathscr{F}_X^*$, then there exists k in $\mathbb{N}$ $(1 \leqslant k < m)$, such that $u = w_1 \cdots w_k$ and $v = w_{k+1} \cdots w_m$.*

Proof. Let $w \in \mathscr{F}_X^*$. Let w_1 be the shortest initial segment of w which is well-formed (hence in $\mathscr{F}_X^*$), then let w_2 be the shortest well-formed initial segment of the part of w following w_1, etc. We arrive at a factorization $w = w_1 w_2, \ldots, w_m$ of w with each w_i irreducible, and clearly no other such factorization is possible. Thus (ii) holds, and (i) and (iii) are easy consequences of (ii).

THEOREM 2.2. *For any set X, $\mathscr{F}_X^*$ is the free unary semigroup on X.*

Proof. First we show that X generates $\mathscr{F}_X^*$. Let T be a unary subsemigroup of $\mathscr{F}_X^*$ containing X. Let F_n be the set of all elements of $\mathscr{F}_X^*$ of length n or less. We show by induction that $F_n \subseteq T$ for all $n \in \mathbb{N}$, hence $\mathscr{F}_X^* \subseteq T$. Since $F_1 = X$, this is true for $n = 1$. Assume $n > 1$ and $F_{n-1} \subseteq T$, and let $w \in F_n \backslash F_{n-1}$. By Lemma 2.1, either $w = u^*$ or $w = uv$ for some u, v in $\mathscr{F}_X^*$. Clearly $u, v \in F_{n-1} \subseteq T$, and hence $w \in T$.

Now let ϕ be a mapping of X into a unary semigroup S. We are to show that there exists a unique unary homomorphism $\psi: \mathscr{F}_X^* \to S$ extending ϕ. We first show by induction that there exists a sequence $\{\psi_n\}$ of mappings $\psi_n: F_n \to S$, with $\psi_1 = \phi$, having the following properties:

(i) ψ_n extends ψ_m for every $m < n$;

(ii) if $u, v, uv \in F_n$, then $(uv)\psi_n = (u\psi_n)(v\psi_n)$;

(iii) if $u, u^* \in F_n$, then $u^*\psi_n = (u\psi_n)^*$.

(We also use $*$ to denote the unary operation on S.) Let $n > 1$ and assume that ψ_{n-1} has been constructed so as to have these three properties. We proceed to define $\psi_n: F_n \to S$. If $w \in F_{n-1}$, define $w\psi_n = w\psi_{n-1}$. Let $w \in F_n \backslash F_{n-1}$, so $|w| = n$. If w is irreducible, then $w = u^*$, for some u in $\mathscr{F}_X^*$, by Lemma 2.1. Moreover, u is uniquely determined. For $(u)^* = (v)^*$ implies $u = v$, since we are in the free semigroup $\mathscr{F}_{\bar{X}}$, so we can cancel paren on the left and parenstar on the right. We define $w\psi_n = (u\psi_{n-1})^*$, which exists since $u \in F_{n-2}$.

If w is reducible, then $w = w_1 w_2 \cdots w_m$, with $m > 1$, with each w_i irreducible $(i = 1, \ldots, m)$, and $(w_1, \ldots, w_m)$ uniquely determined, by Lemma 2.1. Clearly

$|w_i| < n$ for each i, so we can define $w\psi_n = (w_1\psi_{n-1}) \cdots (w_m\psi_{n-1})$. Thus, in both cases, we have defined $w\psi_n$ unambiguously.

Since ψ_{n-1} satisfies (i), and ψ_n extends ψ_{n-1}, it is clear that ψ_n satisfies (i). Let $u, v, uv \in F_n$. Since ψ_{n-1} satisfies (ii), we may assume $uv \in F_n \backslash F_{n-1}$. Since uv is reducible, $uv = w_1 w_2 \cdots w_m$ with $m > 1$ and each w_i irreducible. By the last part of Lemma 2.1, $u = w_1 \cdots w_k$ and $v = w_{k+1} \cdots w_m$ for some k such that $1 \leqslant k < m$. Then, by definition of ψ_n, and the hypothesis that ψ_{n-1} satisfies (ii),

$$\begin{aligned}(uv)\psi_n &= (w_1\psi_{n-1}) \cdots (w_m\psi_{n-1}) = (w_1 \cdots w_k)\psi_{n-1} \cdot (w_{k+1} \cdots w_m)\psi_{n-1} \\ &= (u\psi_{n-1})(v\psi_{n-1}) = (u\psi_n)(v\psi_n).\end{aligned}$$

Hence ψ_n satisfies (ii). If $u, u^* \in F_n$, then we may assume $u^* \in F_n \backslash F_{n-1}$, and, by definition of ψ_n,

$$u^*\psi_n = (u\psi_{n-1})^* = (u\psi_n)^*.$$

Hence (iii) holds, and this concludes the inductive argument.

Because of (i) we can define $\psi\colon \mathscr{F}_X^* \to S$ unambiguously by $w\psi = w\psi_n$ if $w \in F_n$. For any u, v in F_X^*, there exists n in $\mathbb{N}$ such that u, v, uv, and u^* all belong to F_n. Hence $(uv)\psi = (u\psi)(v\psi)$ and $u^*\psi = (u\psi)^*$. ψ is unique, for if ψ' is any unary homomorphism of F_X^* into S extending ϕ, the set $\{w \in \mathscr{F}_X^*\colon w\psi = w\psi'\}$ is a unary subsemigroup of $\mathscr{F}_X^*$ containing X, and we have shown above that the only such is $\mathscr{F}_X^*$ itself.

If $w \in \mathscr{F}_{\bar{X}}$, we define the *content* $C(w)$ of w to be the set of elements of X appearing in w. Clearly $C(uv) = C(u) \cup C(v)$ and $C((u)^*) = C(u)$ for all u, v in $\mathscr{F}_{\bar{X}}$.

Let $w \in \mathscr{F}_X^*$. We proceed to "mate" the parens and parenstars in w. We mate the first parenstar in w, reading from left to right, with the nearest paren to its left, then the next parenstar with the nearest paren to its left, not counting the paren already mated, etc. In general, each parenstar in w is mated with the nearest unmated paren to its left. Again using the example (1.2), mates are indicated by assigning them the same index number:

$$w = x_1 \overset{1}{(} x_2x_3 \overset{2}{(} x_4 \overset{2}{)}{}^* x_5 \overset{3}{(} x_6x_7x_8 \overset{4}{(} x_9x_{10} \overset{4}{)}{}^* x_{11} \overset{5}{(} x_{12} \overset{5}{)}{}^* x_{13} \overset{3}{)}{}^* x_{14}x_{15} \overset{1}{)}{}^*$$

From the mating procedure, it is clear that the subword u of w lying between two mated symbols is well-formed, and so represents an element of $\mathscr{F}_X^*$. The subword $(u)^*$ of w is the image of u under the unary operation $*$ on $\mathscr{F}_X^*$. It follows that w can be obtained from $C(w)$ by a finite sequence of operations of two kinds: forming products of words already formed, and performing the unary operation $*$ on a word already formed. In other words, we may "read" w as though it were a polynomial (sect. 1).

Proposition 2.3. *Let Y be a subset of X, and let $\langle Y \rangle$ be the unary subsemigroup of $\mathscr{F}_X^*$ generated by Y. Then* (i) $\langle Y \rangle = \{w \in \mathscr{F}_X^*\colon C(w) \subseteq Y\}$, *and* (ii) $\langle Y \rangle \cong \mathscr{F}_Y^*$.

Proof. (i) Let $T = \{w \in \mathscr{F}_X^*\colon C(w) \subseteq Y\}$. Clearly, T is a unary subsemigroup of $\mathscr{F}_X^*$ containing Y, and hence $T \supseteq \langle Y \rangle$. Conversely, let $w \in T$. From the remark preceding the proposition, it follows that $w \in \langle C(w) \rangle$. From $C(w) \subseteq Y$, we then have $w \in \langle Y \rangle$, so $T \subseteq \langle Y \rangle$.

(ii) Let ϕ be any mapping of Y into a unary semigroup S. Extend ϕ in any way to a mapping $\bar{\phi}$ of X into S. By Theorem 2.2, there exists a unary homomorphism $\bar{\psi}$ of $\mathscr{F}_X^*$ into S extending $\bar{\phi}$. But then $\psi \mid \langle Y \rangle$ is a unary homomorphism of $\langle Y \rangle$ into S extending ϕ, and is unique since Y generates $\langle Y \rangle$. Hence $\langle Y \rangle$ is freely generated by Y, and therefore isomorphic with $\mathscr{F}_Y^*$.

3. Construction of $\mathscr{F}_X^{\mathrm{cr}}$

Define the relations ρ_1, ρ_2, ρ_3 on $\mathscr{F}_X^*$ as follows:

$$\rho_1 = \{(ww^*w, w)\colon w \in \mathscr{F}_X^*\},$$

$$\rho_2 = \{(ww^*, w^*w)\colon w \in \mathscr{F}_X^*\},$$

$$\rho_3 = \{(w^{**}, w)\colon w \in \mathscr{F}_X^*\}.$$

Let ρ be the smallest congruence on $\mathscr{F}_X^*$ containing $\rho_1 \cup \rho_2 \cup \rho_3$, and define $\mathscr{F}_X^{\mathrm{cr}} = \mathscr{F}_X^*/\rho$. $\mathscr{F}_X^{\mathrm{cr}}$ is c.r. since each element $w\rho$ of $\mathscr{F}_X^{\mathrm{cr}}$ (w in $\mathscr{F}_X^*$) has an inverse $w^*\rho$ which commutes with it. We denote by $\rho^\natural$ the natural homomorphism of $\mathscr{F}_X^*$ onto $\mathscr{F}_X^{\mathrm{cr}}$.

If $u, v \in \mathscr{F}_X^*$, then $u\rho v$ if and only if we can transform u into v by a finite sequence of elementary transitions [1, p. 18] of the three types corresponding to ρ_1, ρ_2, and ρ_3. These do not change the content of the word being transformed. Hence $u\rho v$ implies $C(u) = C(v)$, and we can apply the term "content" to the elements of $\mathscr{F}_X^{\mathrm{cr}}$ without ambiguity.

In particular, if $x, y \in X$ and $x\rho y$, then $\{x\} = C(x) = C(y) = \{y\}$, and we conclude $x = y$. This shows that $\eta = \rho^\natural \mid X$ is an injection of X into $\mathscr{F}_X^{\mathrm{cr}}$.

THEOREM 3.1. *The pair $(\mathscr{F}_X^{\mathrm{cr}}, \eta)$ is a free completely regular semigroup on the set X.*

Proof. We have just seen that $\eta\colon X \to \mathscr{F}_X^{\mathrm{cr}}$ is injective. Since X generates $\mathscr{F}_X^*$ as a unary semigroup, $X\eta = \{x\rho\colon x \in X\}$ generates $\mathscr{F}_X^{\mathrm{cr}} = \mathscr{F}_X^*/\rho$ as a unary semigroup, hence also as a c.r. semigroup.

Assume that ϕ is a mapping of X into a c.r. semigroup S. We are to show that there is a unique (unary) homomorphism θ of $\mathscr{F}_X^{\mathrm{cr}}$ into S such that $\eta \circ \theta = \phi$.

Since S is a unary semigroup, and $\mathscr{F}_X^*$ the free unary semigroup on X (Theorem 2.2), there exists a unary homomorphism $\psi\colon \mathscr{F}_X^* \to S$ extending ϕ. If $w \in \mathscr{F}_X^*$, then $(ww^*w)\psi = (w\psi)(w\psi)^*(w\psi) = w\psi$, since S is c.r. Hence

$\rho_1 \subseteq \ker \psi$. Similarly, $\rho_2 \subseteq \ker \psi$ and $\rho_3 \subseteq \ker \psi$. Since ρ is the smallest congruence on $\mathscr{F}_X^*$ containing $\rho_1 \cup \rho_2 \cup \rho_3$, we conclude that $\rho \subseteq \ker \psi$. Hence $u\rho v$ implies $u\psi = v\psi$, and consequently we can define $\theta: \mathscr{F}_X^{\mathrm{cr}} \to S$ by $(u\rho)\theta = u\psi$. It is easily seen that θ is a (unary) homomorphism, and by its definition, we have $\rho \circ \theta = \psi$. Restricting this to X, we find $\eta \circ \theta = \phi$. The uniqueness of ψ follows from the fact noted above that X generates $\mathscr{F}_X^{\mathrm{cr}}$.

PROPOSITION 3.2. *With the notation of Theorem* 3.1, *let Y be a subset of X, and let $\langle Y\eta\rangle$ be the c.r. subsemigroup of $\mathscr{F}_X^{\mathrm{cr}}$ generated by $Y\eta$. Then $\langle Y\eta\rangle$ consists of all $w\rho$ in $\mathscr{F}_X^{\mathrm{cr}}$ such that $C(w) \subseteq Y$, and $(\langle Y\eta\rangle, \eta \mid Y)$ is a free c.r. semigroup on Y.*

Proof. $\langle Y\rangle\rho^\natural$ is a c.r. subsemigroup of $\mathscr{F}_X^{\mathrm{cr}}$ containing $Y\eta$, hence containing $\langle Y\eta\rangle$. $\langle Y\eta\rangle\rho^{\natural-1} = \{w \in \mathscr{F}_X^*: w\rho^\natural \in \langle Y\eta\rangle\}$ is a unary subsemigroup of $\mathscr{F}_X^*$ containing Y, hence containing $\langle Y\rangle$. Consequently

$$\langle Y\eta\rangle \supseteq \langle Y\eta\rangle\rho^{\natural-1}\rho^\natural \supseteq \langle Y\rangle\rho^\natural \supseteq \langle Y\eta\rangle,$$

and we conclude that $\langle Y\rangle\rho^\natural = \langle Y\eta\rangle$. The proposition now follows from Proposition 2.3 by applying the homomorphism $\rho^\natural$.

PROPOSITION 3.3. *With the notation of Theorem* 3.1 *and Proposition* 3.2, *for each x in X we have*:

(i) $\langle x\eta\rangle = \{w\rho \in \mathscr{F}_X^{\mathrm{cr}}: C(w) = \{x\}\}$,

(ii) $\langle x\eta\rangle \cong \mathscr{F}_{\{x\}}^{\mathrm{cr}}$, *and*

(iii) $\langle x\eta\rangle$ *is an infinite cyclic group.*

Proof. (i) and (ii) are immediate from Proposition 3.2, taking $Y = \{x\}$. To show (iii), let H be the $\mathscr{H}$-class of $\langle x\eta\rangle$ containing $x\eta$. Since H is a group, it is a c.r. subsemigroup of $\langle x\eta\rangle$ containing $x\eta$. Since $x\eta$ generates $\langle x\eta\rangle$, we conclude that $H = \langle x\eta\rangle$, and hence $\langle x\eta\rangle$ is a group. By (ii), it is freely generated by $x\eta$, so must be an infinite cyclic group.

4. GREEN'S RELATION $\mathscr{D}$ ON $\mathscr{F}_X^{\mathrm{cr}}$

The next proposition is due to A. Horn and N. Kimura [5; Theorem 4.3]. We include a proof for the sake of completeness.

PROPOSITION 4.1. *Let X be a set, and let P be the semilattice of all non-empty finite subsets of X under union. Define $\kappa: X \to P$ by $x\kappa = \{x\}$. Then (P, κ) is a free semilattice on X. (We write $\mathscr{F}_X^{sl}$ for P.)*

Proof. Clearly κ is injective and $X\kappa$ generates P. Let ϕ be any mapping of X into a semilattice Ω. For each element $Y = \{x_1 ,..., x_n\}$ of P, define $Y\psi =$

$(x_1\phi)(x_2\phi)\cdots(x_n\phi)$. Since Ω is commutative, $Y\psi$ is independent of the order in which the elements of Y are listed. Since Ω is also idempotent, it is clear that $(Y\psi)(Z\psi) = (Y \cup Z)\psi$ for all Y, Z in P. Hence ψ is a homomorphism of P into Ω such that $x(\kappa \circ \psi) = \{x\}\psi = x\phi$ for all x in X, so $\kappa \circ \psi = \phi$.

THEOREM 4.2. *Two elements of $\mathscr{F}_X^{\mathrm{cr}}$ are $\mathscr{D}$-equivalent if and only if they have the same content.*

Proof. If S is any c.r. semigroup, then [1; Th. 4.6] Green's relations $\mathscr{D}$ and $\mathscr{J}$ coincide, and $\mathscr{D}$ is the finest semilattice congruence on S. The mapping C sending an element $w\rho$ of $\mathscr{F}_X^{\mathrm{cr}}$ into its content $C(w\rho) = C(w)$ is a homomorphism of $\mathscr{F}_X^{\mathrm{cr}}$ onto the semilattice P of finite subsets of X under union. Write $(w\rho)C$ for $C(w\rho)$. The statement of the theorem is equivalent to the assertion $\mathscr{D} = \ker C$, and this will follow as soon as we show that any homomorphism θ of $\mathscr{F}_X^{\mathrm{cr}}$ into a semilattice Ω must factor through C.

By Proposition 4.1, there exists a (unique) homomorphism $\psi: P \to \Omega$ such that $\kappa \circ \psi = \eta \circ \theta$. For any x in X we have $x(\eta \circ C) = C(x\eta) = C(x) = \{x\} = x\kappa$, so $\eta \circ C = \kappa$ and $\eta \circ C \circ \psi = \kappa \circ \psi = \eta \circ \theta$. Hence the two homomorphisms $C \circ \psi$ and θ of $\mathscr{F}_X^{\mathrm{cr}}$ into Ω agree on $X\eta$. Since $X\eta$ generates $\mathscr{F}_X^{\mathrm{cr}}$, we conclude that $C \circ \psi = \theta$.

COROLLARY 4.3. *To each non-empty finite subset Y of X there corresponds a $\mathscr{D}$-class D_Y of $\mathscr{F}_X^{\mathrm{cr}}$ consisting of all elements of $\mathscr{F}_X^{\mathrm{cr}}$ having content Y, and the mapping which sends Y to D_Y is an isomorphism of the free semilattice $\mathscr{F}_X^{\mathrm{sl}}$ on X onto the structure semilattice $\mathscr{F}_X^{\mathrm{cr}}/\mathscr{D}$ of $\mathscr{F}_X^{\mathrm{cr}}$.*

5. GREEN'S RELATIONS $\mathscr{R}$ AND $\mathscr{L}$ ON $\mathscr{F}_X^{\mathrm{cr}}$

In this section we write $\sim$ for ρ and $\tilde{w}$ for $w\rho$. Let $w \in \mathscr{F}_X^*$, and let w_1 be an initial segment of w, regarded as a word in $\mathscr{F}_{\bar{X}}$. As noted in Section 2, w_1 is pre-well-formed. By the *excess* of w_1 we mean the number of parens in w_1 minus the number of parenstars; it is the number of unmated parens in w_1 (see Section 2). Let $\bar{w}_1$ be the well-formed word which results when the unmated parens in w_1 are removed.

LEMMA 5.1. *With the above notation, $w \sim \bar{w}_1 v$ for some v in $\mathscr{F}_X^*$. Dually, if w_2 is a final segment of w, and $\bar{w}_2$ is the element of $\mathscr{F}_X^*$ which results when the unmated parenstars in w_2 are removed, then $w \sim u\bar{w}_2$ for some u in $\mathscr{F}_X^*$.*

Proof. We need prove only the first part. If w_1 is an initial segment of w, then w factors into $w_1 v_1$ in $\mathscr{F}_{\bar{X}}$. If the excess of w_1 is 0, then $\bar{w}_1 = w_1$ and $v_1 \in \mathscr{F}_X^*$, so we can take $v = v_1$. Assuming that the excess of w_1 is greater than 0, w_1 has the form $w_1 = p(q$, where (is the last unmated paren in w_1, and its mate lies in

v_1, say $v_1 = r)^*s$. Note that q and r are well-formed, except that r may be empty, that is, $v_1 =)^*s$. We now have $w = p(qr)^*s$, with qr in $\mathscr{F}_X^*$. Then $w \sim pqr(qr)^*(qr)^*s$, and so $w \sim w_2v_2$ with $w_2 = pq$ and $v_2 = r(qr)^*(qr)^*s$. The element w_2 of $\mathscr{F}_X$ differs from w_1 only in that one unmated paren has been removed. This process can be continued until all unmated parens have been removed. We then have $w \sim w_nv_n$ with w_n and v_n both in $\mathscr{F}_X^*$, and so $w_n = \bar{w}_1$ and we take $v = v_n$.

Let $w \in \mathscr{F}_X^*$. Let w_1 be the shortest initial segment of w such that $C(w_1) = C(w)$. By the *left indicator* of w we mean the element $\bar{w}_1$ which results from w_1 when we remove all the unmated parens in w_1. It has the form ux, where $u \in \mathscr{F}_X^{*1}$, $x \in X$, and $C(u) = C(w)\backslash\{x\}$. In the terminology of Green and Rees [4], u is the "initial" and x is the "initial mark" of w.

For example, let x_1, x_2, x_3, x_4 be distinct elements of X, and let

$$w = x_1(x_2((x_4x_1)^*x_3)^*x_4(x_3x_2)^*x_4)^*.$$

Then $w_1 = x_1(x_2((x_4x_1)^*x_3$ and the left indicator of w is ux_3, where $u = x_1x_2(x_4x_1)^*$.

Dually, by the *right indicator* of w we mean the element yv of $\mathscr{F}_X^*$ obtained as follows. Reading from right to left, y is the first occurrence of the last member of $C(w)$ to appear in w, and yv results from the final segment of w beginning with y by removing unmated parenstars. In the above example, the right indicator of w is x_1v, where $v = x_3x_4(x_3x_2)^*x_4$.

The following is immediate from Lemma 5.1.

Corollary 5.2. *If $w \in \mathscr{F}_X^*$ and $ux[yv]$ is its left [right] indicator, then $w \sim uxs[w \sim tyv]$ for some $s[t]$ in $\mathscr{F}_X^{*1}$.*

The next proposition shows that the notion of left indicator can be applied to $\mathscr{F}_X^{\mathrm{cr}}$.

Proposition 5.3. *Let $w, w' \in \mathscr{F}_X^*$, and let $ux, u'x'$ be their respective left indicators. Then $w \sim w'$ implies $u \sim u'$ and $x = x'$.*

Proof. Clearly it suffices to prove $u \sim u'$ and $x = x'$ when w' is obtained from w by a single elementary transition.

Let $w = w_1w_2$, where (as before) w_1 is the shortest initial segment of w with content $C(w)$. Then ux is the word resulting from w_1 when we remove all the unmated parens in w_1.

Consider first an elementary transition of the form $v \to v(v)^*v$. If v is a subword of w_1 not involving x, the result is to change u into a word $u' \sim u$, leaving x unaltered. If v is a subword of w_2, u and x are unaffected. If v involves x, then $(v)^*v$ is inserted in w_2, while u and x are unchanged. A similar discussion holds for the transition $v(v)^*v \to v$, with "inserted in w_2 "replaced by "removed from w_2" in the last case.

Now consider $(v)^*v \to v(v)^*$. If $(v)^*v$ is a subword of w_1 not involving x, then u is changed into a word $u' \sim u$. If it is a subword of w_2, then u and x are unchanged. Suppose it involves x. Then $v = pxq$ (in $\mathscr{F}_X$), where p and/or q may be empty, and $w = \cdots (pxq)^*pxq \cdots \to \cdots pxq(pxq)^* \cdots$. Since $w_1 = \cdots (px$, we see that the only effect of this transition on w_1 is to remove the unmated paren in front of p. This, of course, has no effect at all on the left indicator of w. A similar discussion holds for $v(v)^* \to (v^*)v$, with "remove" replaced by "insert" in the last case.

The transitions $v \to ((v)^*)^*$ and $((v)^*)^* \to v$ have similar effects; when v involves x, two unmated parens are inserted or removed.

Of course the left-right dual of Proposition 5.3 holds.

THEOREM 5.4. *Two elements of $\mathscr{F}_Y^{\mathrm{cr}}$ are $\mathscr{R}[\mathscr{L}]$-equivalent if and only if they have the same left [right] indicator.*

Proof. Let $w \in \mathscr{F}_X^*$, and let ux be its left indicator. Then $\tilde{u}\tilde{x}$ is the left indicator of $\tilde{w}$ (Proposition 5.3). By Corollary 5.2, $w \sim uxt$ for some t in $\mathscr{F}_X^*$. Now $C(ux) = C(w)$, so $\tilde{u}\tilde{x} \mathscr{D} \tilde{w}$ by Theorem 4.2. Since each $\mathscr{D}$-class is a completely simple semigroup, and $a \mathscr{R} ab$ for any two elements a, b of such a semigroup, it follows that $\tilde{u}\tilde{x} \mathscr{R} \tilde{u}\tilde{x}\tilde{t} = \tilde{w}$. Hence each element of $\mathscr{F}_X^{\mathrm{cr}}$ is $\mathscr{R}$-equivalent to its left indicator. It follows that two elements with the same left indicator must be $\mathscr{R}$-equivalent.

Conversely, let $\tilde{w}_1 \mathscr{R} \tilde{w}_2$, and let $\tilde{u}_1\tilde{x}_1[\tilde{u}_2\tilde{x}_2]$ be the left indicator of $\tilde{w}_1[\tilde{w}_2]$. Since $\tilde{w}_1 \mathscr{R} \tilde{u}_1\tilde{x}_1$ and $\tilde{w}_2 \mathscr{R} \tilde{u}_2\tilde{x}_2$, we have $\tilde{u}_1\tilde{x}_1 \mathscr{R} \tilde{u}_2\tilde{x}_2$, and hence $\tilde{u}_2\tilde{x}_2 = \tilde{u}_1\tilde{x}_1 z$ for some $\tilde{t}$ in the completely simple semigroup $D_{\tilde{w}_1}$. Going back to $\mathscr{F}_X^*$, u_1x_1 is the left indicator of u_1x_1t, u_2x_2 is that of itself, and $u_1x_1t \sim u_2x_2$. By Proposition 5.3, $u_1 \sim u_2$ and $x_1 = x_2$, hence $\tilde{u}_1\tilde{x}_1 = \tilde{u}_2\tilde{x}_2$.

Let $X_n = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of X. Write $X_{n/i}$ for $X_n\backslash\{x_i\}$. Write $D_n[D_{n/i}]$ for the $\mathscr{D}$-class of $\mathscr{F}_X^{\mathrm{cr}}$ consisting of all elements of content $X_n[X_{n/i}]$. The $\mathscr{R}[\mathscr{L}]$-classes of D_n are labelled (one-to-one) by the elements of $\mathscr{F}_X^{\mathrm{cr}}$ of the form $\tilde{u}_1\tilde{x}_1, \ldots, \tilde{u}_n\tilde{x}_n[\tilde{x}_1\tilde{v}_1, \ldots, \tilde{x}_n\tilde{v}_n]$, where $C(\tilde{u}_i) = C(\tilde{v}_i) = X_{n/i}$. The mapping $\tilde{u}_i \mapsto \tilde{u}_i\tilde{x}_i$ is an injection (for each $i = 1, \ldots, n$) of $D_{n/i}$ into D_n. The sum over i of these mappings is an injection of $\bigcup_{i=1}^n D_{n/i}$ into D_n the image of which is a transversal of the $\mathscr{R}$-classes of D_n. Similar remarks apply to $\tilde{v}_i \mapsto \tilde{x}_i\tilde{v}_i$.

The $\mathscr{H}$-classes of D_n are labelled by the set of all pairs $(\tilde{u}_i\tilde{x}_i, \tilde{x}_j\tilde{v}_j)$, i and j ranging over $\{1, 2, \ldots, n\}$, and $\tilde{u}_i$ and $\tilde{v}_i$ over $D_{n/i}$. The product $\tilde{u}_i\tilde{x}_i\tilde{x}_j\tilde{v}_j$ belongs to the $\mathscr{H}$-class labelled by $(\tilde{u}_i\tilde{x}_i, \tilde{x}_j\tilde{v}_j)$, and so $\tilde{u}_i\tilde{x}_i\tilde{x}_j\tilde{v}_j(\tilde{u}_i\tilde{x}_i\tilde{x}_j\tilde{v}_j)^*$ is the idempotent element in this $\mathscr{H}$-class.

COROLLARY 5.5. *Every idempotent element of $\mathscr{F}_X^{\mathrm{cr}}$ can be represented by a word in $\mathscr{F}_X^*$ of the form $uxyv\,(uxyv)^*$, where $x, y \in X$ and u, v are words in $\mathscr{F}_X^*$ such that*

$$x \notin C(u), \qquad y \notin C(v), \qquad \textit{and} \qquad C(u) \cup \{x\} = C(v) \cup \{y\}.$$

6. The Structure of $\mathscr{F}_X^{\mathrm{cr}}$ when $|X| = 2$

It follows from Proposition 3.2 that we would know the structure of $\mathscr{F}_X^{\mathrm{cr}}$ for any set X if we know it for every finite set Y. For if $u, v \in \mathscr{F}_X^{\mathrm{cr}}$, and $Y = C(u) \cup C(v)$, then $u, v \in \langle Y \rangle \simeq \mathscr{F}_Y^{\mathrm{cr}}$. We make a start on this difficult problem by determining the structure of $\mathscr{F}_{\{x,y\}}^{\mathrm{cr}}$. We denote the latter by F_2 for the sake of brevity. Moreover, we identify $\tilde{x}$ with x and $\tilde{y}$ with y, and write $=$ instead of $\sim$. This will lead to no confusion since we shall have no need for inductive arguments involving elementary transitions.

By Theorem 4.2, F_2 has three $\mathscr{D}$-classes: D_x, D_y, and D_{xy}. By Proposition 3.3, D_x and D_y are infinite cyclic groups generated by x and y, respectively. We shall write $D_x = \{x^i : i \in \mathbb{Z}\}$, $D_y = \{y^i : i \in \mathbb{Z}\}$; thus $x^{-1} = x^*$, $x^0 = xx^*$, etc. D_{xy} consists of all words in F_2 of content $\{x, y\}$, and is a completely simple subsemigroup of F_2. In fact D_{xy} is an ideal of F_2 (the Suschkewitsch kernel), and F_2 is an ideal extension of D_{xy} by the 0-direct union of two infinite cyclic groups.

By Theorem 5.4, the $\mathscr{R}$-classes of D_{xy} are in one-to-one correspondence with all possible left indicators, and the latter are readily seen to be the elements x^iy $(i \in \mathbb{Z})$ and y^ix $(i \in \mathbb{Z})$. Dually, the $\mathscr{L}$-classes are given by the possible right indicators yx^i and xy^i $(i \in \mathbb{Z})$.

Let $\bar{\mathbb{Z}}$ be a copy of the set $\mathbb{Z}$ of integers, disjoint from $\mathbb{Z}$. Elements of $\mathbb{Z}$ will be denoted by $i, j, k, l, \ldots$; those of $\bar{\mathbb{Z}}$ by $\bar{\imath}, \bar{\jmath}, \bar{k}, \bar{l}, \ldots$; those of $\mathbb{Z} \cup \bar{\mathbb{Z}}$ by $\alpha, \beta, \gamma, \delta, \ldots$. We index the $\mathscr{R}$- and $\mathscr{L}$-classes of F_2 by the set $\mathbb{Z} \cup \bar{\mathbb{Z}}$ as follows:

$$\begin{aligned} R_i &= x^iyF_2, & R_{\bar{\imath}} &= y^ixF_2, \\ L_i &= F_2yx^i, & L_{\bar{\imath}} &= F_2xy^i. \end{aligned}$$

The $\mathscr{H}$-classes of D_{xy} are then denoted as follows:

$$\begin{aligned} H_{ij} &= R_i \cap L_j & H_{i\bar{\jmath}} &= R_i \cap L_{\bar{\jmath}} \\ H_{\bar{\imath}j} &= R_{\bar{\imath}} \cap L_j & H_{\bar{\imath}\bar{\jmath}} &= R_{\bar{\imath}} \cap L_{\bar{\jmath}}. \end{aligned}$$

We proceed to obtain a Rees matrix representation for F_2 over the group $H_{0\bar{0}}$. Let e denote the identity element of $H_{0\bar{0}}$. Inverses in $H_{0\bar{0}}$ will usually be denoted by superscripts -1 rather than $*$. Thus $x^0y^0(x^0y^0)^{-1} = x^0y^0(x^0y^0)^* = e$.

Following the usual procedure, we select, for each α in $\mathbb{Z} \cup \bar{\mathbb{Z}}$, representative elements q_α in $H_{0\alpha}$ and r_α in $H_{\alpha\bar{0}}$, and set

$$(a; \alpha, \beta) = r_\alpha a q_\beta \qquad (\text{all } a \text{ in } H_{0\bar{0}} \text{ and } \alpha, \beta \text{ in } \mathbb{Z} \cup \bar{\mathbb{Z}}). \tag{6.1}$$

Then

$$(a; \alpha, \beta)(b; \gamma, \delta) = r_\alpha a q_\beta r_\gamma b q_\delta = (a p_{\beta\gamma} b; \alpha, \delta),$$

and $F_2 = \mathscr{M}(H_{0\bar{0}}; \mathbb{Z} \cup \bar{\mathbb{Z}}, \mathbb{Z} \cup \bar{\mathbb{Z}}; P)$ with sandwich matrix $P = (p_{\beta\gamma})$, where $p_{\beta\gamma} = q_\beta r_\gamma$.

Our choice of the q_α and r_α is the following:

$$\begin{aligned} q_i &= ex^i \in H_{0i}, \qquad & q_{\bar{i}} &= ey^i \in H_{0\bar{i}}, \\ r_i &= x^i e \in H_{i\bar{0}}, \qquad & r_{\bar{i}} &= y^i e \in H_{\bar{i}\bar{0}}. \end{aligned} \tag{6.2}$$

Notice that $q_{\bar{0}} = r_0 = e$. Then

$$\begin{aligned} p_{ij} &= q_i r_j = ex^{i+j}e, \\ p_{i\bar{j}} &= q_i r_{\bar{j}} = ex^i y^j e, \\ p_{\bar{i}j} &= q_{\bar{i}} r_j = ey^i x^j e, \\ p_{\bar{i}\bar{j}} &= q_{\bar{i}} r_{\bar{j}} = ey^{i+j}e. \end{aligned} \tag{6.3}$$

Although P is not normalized, we do have $p_{\bar{0}0} = q_{\bar{0}} r_0 = e$. Hence

$$(a; 0, \bar{0})(b; 0, \bar{0}) = (ab; 0; \bar{0}),$$

$(e; 0, \bar{0})$ is the identity element e of $H_{0\bar{0}}$, and $(a; 0, \bar{0})^{-1} = (a^{-1}; 0, \bar{0})$.
The computations needed below make frequent use of the following:

$$y^i e x^j = y^i x^j \qquad (\text{all } i, j \text{ in } \mathbb{Z}). \tag{6.4}$$

We have

$$\begin{aligned} y^0 e x^0 &= y^0 x^0 y^0 (x^0 y^0)^* x^0 \\ &= (y^0 x^0)^* y^0 x^0 y^0 x^0 y^0 (x^0 y^0)^* x^0 \\ &= (y^0 x^0)^* y^0 x^0 y^0 x^0 = y^0 x^0. \end{aligned}$$

Hence

$$y^i e x^j = y^i y^0 e x^0 x^j = y^i y^0 x^0 x^j = y^i x^j.$$

We also make frequent use of

$$x^i p_{0\bar{0}} y^j = x^i y^j \qquad (\text{all } i, j \text{ in } \mathbb{Z}). \tag{6.5}$$

This is evident from $p_{0\bar{0}} = ex^0 y^0 e = x^0 y^0$. We now define

$$p_i = ex^i e, \qquad p_{\bar{i}} = ey^i e. \tag{6.6}$$

We proceed to express the structure of F_2 in terms of the elements p_α ($\alpha \in \mathbb{Z} \cup \bar{\mathbb{Z}}$) and $p_{0\bar{0}}$ of the group $H_{0\bar{0}}$: equations (6.7) below give the sandwich matrix of D_{xy}, (6.8) give the set products $D_x D_y$ and $D_y D_x$, and (6.9) give the action of D_x and D_y on D_{xy}. They are easy consequences of (6.1)–(6.6). Equations (6.7) and (6.8) are almost immediate, while each of the equations (6.9) involves a simple computation, for example:

$$x^k(a;\bar{\imath},\beta) = x^k r_{\bar{\imath}} a q_\beta = x^k y^i e a q_\beta = x^k p_{0\bar{0}} y^i e a q_\beta$$
$$= x^k e p_{0\bar{0}} e y^i e a q_\beta = r^k p_{0\bar{0}} p_{\bar{\imath}} a q_\beta = (p_{0\bar{0}} p_{\bar{\imath}} a; k, \beta).$$

$$\left.\begin{aligned} p_{ij} &= p_{i+j}, \; p_{i\bar{\jmath}} = p_i p_{0\bar{0}} p_{\bar{\jmath}}, \\ p_{\bar{\imath}j} &= p_{\bar{\imath}} p_j, \; p_{\bar{\imath}\bar{\jmath}} = p_{\overline{i+j}}, \end{aligned}\right\} \quad (i, j \in \mathbb{Z}). \tag{6.7}$$

$$x^i y^j = (p_{0\bar{0}}\,; i, \bar{\jmath}),\; y^i x^j = (e; \bar{\imath}, j),\; (i, j \in \mathbb{Z}). \tag{6.8}$$

$$\begin{aligned} x^k(a; i, \beta) &= (a; k+i, \beta), & (a; \alpha, j)x^k &= (a; \alpha, j+k), \\ y^k(a; i, \beta) &= (p_i a; \bar{k}, \beta), & (a; \alpha, j)y^k &= (a p_j p_{0\bar{0}}\,; \alpha, \bar{k}), \\ x^k(a; \bar{\imath}, \beta) &= (p_{0\bar{0}} p_{\bar{\imath}} a; k, \beta), & (a; \alpha, \bar{\jmath})x^k &= (a p_{\bar{\jmath}}\,; \alpha, k), \\ y^k(a; \bar{\imath}, \beta) &= (a; \overline{k+i}, \beta), & (a; \alpha, \bar{\jmath})y^k &= (a; \alpha, \overline{j+k}), \end{aligned} \tag{6.9}$$
$$(a \in H_{0\bar{0}}\,; i, j, k \in \mathbb{Z};\; \alpha, \beta \in \mathbb{Z} \cup \bar{\mathbb{Z}}).$$

In the following lemma (which we use twice), the symbols x, y, p_α, etc., are not restricted to their meaning so far in this section.

LEMMA 6.1. *Let K be a (c.r.) semigroup which is a disjoint union $D_x \cup D_y \cup M$, where $D_x[D_y]$ is an infinite cyclic group generated by $x[y]$, and M is a Rees matrix semigroup $\mathcal{M}(G; \mathbb{Z} \cup \bar{\mathbb{Z}}, \mathbb{Z} \cup \bar{\mathbb{Z}}; P)$ over a group G, with sandwich matrix $P = (p_{\alpha\beta})$, $(\alpha, \beta \in \mathbb{Z} \cup \bar{\mathbb{Z}})$. Assume that $P_1 = \{p_\alpha : \alpha \in (\mathbb{Z}\backslash 0) \cup (\bar{\mathbb{Z}}\backslash\bar{0})\} \cup \{p_{0\bar{0}}\}$ is a subset of G such that the components $p_{\alpha\beta}$ of P can be expressed in terms of the members p_α and $p_{0\bar{0}}$ of P_1 by equations (6.7), with $p_0 = p_{\bar{0}} = e$, the identity element of G. Assume also that equations (6.8) and (6.9) hold. Let G_1 be the subgroup of G generated by P_1, and let K_1 be the c.r. subsemigroup of K generated by x and y. Then*

$$K_1 = D_x \cup D_y \cup \mathcal{M}(G_1; \mathbb{Z} \cup \bar{\mathbb{Z}}, \mathbb{Z} \cup \bar{\mathbb{Z}}; P).$$

Proof. Let $M_1 = \mathcal{M}(G_1; \mathbb{Z} \cup \bar{\mathbb{Z}}, \mathbb{Z} \cup \bar{\mathbb{Z}}; P)$. We show first that $K_1 \cap M \subseteq M_1$. From (6.8), it is clear that all the products $x^i y^j$ and $y^i x^j$ belong to M_1. Assume $(a; \alpha, \beta) \in M_1$. From (6.9) it is clear that the products of $(a; \alpha, \beta)$ with x^k or y^k on either side also belong to M_1. Putting these two facts together, we conclude that $K_1 \cap M \subseteq M_1$.

To show $M_1 \subseteq K_1 \cap M$ it suffices to show that $M_1 \subseteq K_1$. By (6.8), K_1 contains $(e; \bar{\imath}, j)$ and $(p_{0\bar{0}}; i, \bar{\jmath})$. Since $p_{\bar{0}0} = e$, K_1 contains $(x^0 y^0)^{-1} = (p_{0\bar{0}}^{-1}; 0, \bar{0})$, hence also $(p_{0\bar{0}}; i, \bar{0})(p_{0\bar{0}}^{-1}; 0, \bar{0}) = (e; i, \bar{0})$. We see successively that K_1 contains the following:

$$(e; i, \bar{0})(e; \bar{0}, j) = (e; i, j),$$
$$(p_{0\bar{0}}^{-1}; 0, \bar{0})(p_{0\bar{0}}\,; 0, \bar{\jmath}) = (e; 0, \bar{\jmath}),$$
$$(e; i, \bar{0})(e; 0, \bar{\jmath}) = (e; i, \bar{\jmath}),$$

$$(e; \bar{\imath}, 0)(e; 0, \bar{\jmath}) = (e; \bar{\imath}, \bar{\jmath}),$$

$$(e; 0, i)(e; 0, \bar{0}) = (p_i ; 0, \bar{0}),$$

$$(e; 0, \bar{\imath})(e; \bar{0}, \bar{0}) = (p_{\bar{\imath}} ; 0, \bar{0}).$$

Since K_1 also contains $(p_{0\bar{0}}; 0, \bar{0})$, it contains $(a; 0, \bar{0})$ for all a in P_1, hence for all a in G_1. That $K_1 \supseteq M_1$ now follows from the two facts that K_1 contains a full $\mathscr{H}$-class of M_1, and meets all $\mathscr{H}$-classes of M_1. Hence $K_1 \cap M = M_1$, and the conclusion of the lemma follows.

COROLLARY 6.2. *In $F_2 = \mathscr{F}^{\mathrm{cr}}_{\{x,y\}}$, the group $H_{0\bar{0}}$ is generated by the set $P_1 = \{p_\alpha : \alpha \in (\mathbb{Z}\backslash 0) \cup (\bar{\mathbb{Z}}\backslash\bar{0})\} \cup \{p_{0\bar{0}}\}$.*

We proceed now to construct a model for F_2. Let

$$\Pi_1 = \{\pi_\alpha : \alpha \in (\mathbb{Z}\backslash 0) \cup (\bar{\mathbb{Z}}\backslash\bar{0})\} \cup \{\pi_{0\bar{0}}\}$$

be a set in one-to-one correspondence with $(\mathbb{Z}\backslash 0) \cup (\bar{\mathbb{Z}}\backslash\bar{0}) \cup \{(0, \bar{0})\}$, and indexed thereby as shown. Let Γ be the free group on Π_1. Let ϵ be the identity element of Γ, and define $\pi_0 = \pi_{\bar{0}} = \epsilon$. Let $\Pi = (\pi_{\alpha\beta})$ be the $(\mathbb{Z} \cup \bar{\mathbb{Z}}) \times (\mathbb{Z} \cup \bar{\mathbb{Z}})$-matrix over Γ defined by equations (6.7), replacing p_α by π_α and $p_{\alpha\beta}$ by $\pi_{\alpha\beta}$. Let Δ be the Rees matrix semigroup $\mathscr{M}(\Gamma; \mathbb{Z} \cup \bar{\mathbb{Z}}, \mathbb{Z} \cup \bar{\mathbb{Z}}; \Pi)$. Let Δ_ξ and Δ_η be infinite cyclic groups generated by ξ and η, respectively, disjoint from each other and from Δ. Let $\Phi_2 = \Delta_\xi \cup \Delta_\eta \cup \Delta$. Define a binary operation on Φ_2, extending those already defined on Δ_ξ, Δ_η, and Δ, by equations (6.9), replacing x by ξ, y by η, p_α by π_α, and $p_{0\bar{0}}$ by $\pi_{0\bar{0}}$.

THEOREM 6.3. *Φ_2 is a completely regular semigroup, and there exists a unique isomorphism θ of F_2 $(= \mathscr{F}^{\mathrm{cr}}_{\{x,y\}})$ onto Φ_2 such that $x\theta = \xi$ and $y\theta = \eta$.*

Proof. It is tedious but straightforward to check the associativity of the binary relation defined above on Φ_2. It is then obvious that Φ_2 is c.r. Since F_2 is the free c.r. semigroup on $\{x, y\}$, the mapping of $\{x, y\}$ onto $\{\xi, \eta\}$ sending x to ξ and y to η can be extended uniquely to a homomorphism θ of F_2 into Φ_2. Since Π_1 generates Γ, it follows from Lemma 6.1 that $\{\xi, \eta\}$ generates Φ_2, and hence θ is surjective.

Denote by $H^\Delta_{\alpha\beta}$ the $\mathscr{H}$-class $\{(a; \alpha, \beta) : a \in \Gamma\}$ of Φ_2. From $x^i y^j \in H_{i\bar{\jmath}}$ and $(x^i y^j)\theta = \xi^i\eta^j = (\pi_{0\bar{0}}; i, \bar{\jmath}) \subseteq H^\Delta_{i\bar{\jmath}}$, we see that $H_{i\bar{\jmath}}\theta \subseteq H^\Delta_{i\bar{\jmath}}$. Similarly, $H_{\bar{\imath}j}\theta \subseteq H^\Delta_{\bar{\imath}j}$. Since θ preserves the relations $\mathscr{R}$ and $\mathscr{L}$, we also have $H_{ij}\theta \subseteq H^\Delta_{ij}$ and $H_{\bar{\imath}\bar{\jmath}}\theta \subseteq H^\Delta_{\bar{\imath}\bar{\jmath}}$. Since θ is surjective, we have $H_{\alpha\beta}\theta = H^\Delta_{\alpha\beta}$, for all α, β in $\mathbb{Z} \cup \bar{\mathbb{Z}}$.

In particular, $H_{0\bar{0}}\theta = H^\Delta_{0\bar{0}}$, so $e\theta = (\epsilon; 0, \bar{0})$. By (6.3), $p_{0\bar{0}} = ex^0y^0e = x^0y^0$; and hence, by the Greek version of (6.8),

$$p_{0\bar{0}}\theta = (x^0y^0)\theta = \xi^0\eta^0 = (\pi_{0\bar{0}}; 0, \bar{0}).$$

By (6.6) and the Greek versions of (6.7) and (6.9),

$$p_i\theta = (ex^i e)\theta = (\epsilon; 0, \bar{0})\, \xi^i(\epsilon; 0, \bar{0})$$
$$= (\epsilon; 0, \bar{0})(\epsilon; i, \bar{0}) = (\pi_{\bar{0}i}; 0, \bar{0}) = (\pi_i; 0, \bar{0}).$$

Similarly, $p_{\bar{i}}\theta = (\pi_{\bar{i}}; 0, 0)$.

Since $\pi_{\bar{0}0} = \pi_{\bar{0}}\pi_0 = \epsilon$, the mapping $\gamma \mapsto (\gamma; 0, \bar{0})$ is an isomorphism of Γ onto $H^{\Delta}_{0\bar{0}}$. Hence $H^{\Delta}_{0\bar{0}}$ is freely generated by the elements $(\pi_\alpha; 0, 0)$ and $(\pi_{0\bar{0}}; 0, 0)$, $(\alpha \in (\mathbb{Z}\backslash 0) \cup (\bar{\mathbb{Z}}\backslash \bar{0}))$. Since θ induces a homomorphism of $H_{0\bar{0}}$ onto $H^{\Delta}_{0\bar{0}}$ mapping p_α onto $(\pi_\alpha; 0, \bar{0})$ and $p_{0\bar{0}}$ onto $(\pi_{0\bar{0}}; 0, \bar{0})$, these elements p_α and $p_{0\bar{0}}$ (which generate $H_{0\bar{0}}$ by Corollary 6.2) must generate $H_{0\bar{0}}$ freely. It is then clear that θ induces an isomorphism of $H_{0\bar{0}}$ onto $H^{\Delta}_{0\bar{0}}$. Being injective on one $\mathscr{H}$-class of F_2, θ is injective on all of F_2, and hence is an isomorphism.

7. The Free Completely Simple Semigroup $\mathscr{F}_X^{cs}$ on a Set X

It was pointed out to the author by M. Petrich that the class of completely simple ("c.s.") semigroups is the same as the class of c.r. semigroups S satisfying the identity

$$uu^* = uvu(uvu)^*, \qquad (\text{all } u, v \text{ in } S), \tag{7.1}$$

and consequently the same as the class of all unary semigroups satisfying (1.1) and (7.1). The class of all c.s. semigroups is therefore a variety, and so has a free member $\mathscr{F}_X^{cs}$ on any set X. The existence of $\mathscr{F}_X^{cs}$ was first shown by D. B. McAlister [8, Sect. 3]. The purpose of this concluding section is to describe $\mathscr{F}_X^{cs}$.

Let σ be the smallest congruence on $\mathscr{F}_X^*$ containing the relations ρ_1, ρ_2, ρ_3 of Section 3 and the relation

$$\rho_4 = \{(uu^*, uvu(uvu)^*: u, v \in \mathscr{F}_X^*\}.$$

Let $F_1 = \mathscr{F}_X^*/\sigma$, and let $\sigma^\natural$ be the natural homomorphism of $\mathscr{F}_X^*$ onto F_1. Let $\zeta = \sigma^\natural \mid X$. We proceed to show that ζ is one-to-one.

Let E be the rectangular band on $X \times X$. Let $\phi: X \to E$ be defined by $x\phi = (x, x)$. Since E is a unary semigroup with $(x, y)^* = (x, y)$, ϕ can be extended uniquely to a unary homomorphism $\bar{\phi}: \mathscr{F}_X^* \to E$. Since E is c.s., $\rho_i \subseteq \ker\bar{\phi}$ $(i = 1, 2, 3, 4)$, hence $\sigma \subseteq \ker\bar{\phi}$. If $x, y \in X$, and $x\zeta = y\zeta$, then $x\sigma y$, and hence $x\bar{\phi} = y\bar{\phi}$. But $x\bar{\phi} = x\phi = (x, x)$ and $y\bar{\phi} = (y, y)$, so we conclude $x = y$. Hence ζ is one-to-one.

It now follows from the obvious analog of Theorem 3.1 that (F_1, ζ) is a free c.s. semigroup on X. Applying this to the above mapping ϕ of X into E, there exists a unique homomorphism ψ of F_1 into E such that $\zeta \circ \psi = \phi$. Let $x, y \in X$.

If $x\zeta \mathscr{R} y\zeta$, then $x\zeta\psi \mathscr{R} y\zeta\psi$, hence $x\phi \mathscr{R} y\phi$, hence $(x, x) \mathscr{R} (y, y)$. Again this implies $x = y$, and we conclude that $x\zeta \mathscr{R} y\zeta \rightarrow x = y$. Dually, $x\zeta \mathscr{L} y\zeta \rightarrow x = y$. We have thus shown the following.

PROPOSITION 7.1. *If x and y are distinct elements of X, then the elements $x\zeta$ and $y\zeta$ lie in distinct $\mathscr{R}$- and $\mathscr{L}$-classes of F_1.*

Let $w \in \mathscr{F}_X^*$, and let x be the first element of X appearing in w, reading from left to right. By Lemma 5.1, $w \sim xv$ for some v in $\mathscr{F}_X^*$, and hence $w\sigma = (xv)\sigma = (x\sigma)(v\sigma)$. Since $F_1 = \mathscr{F}_X^*/\sigma$ is c.s., this implies $w\sigma \mathscr{R} x\sigma$ $(= x\zeta)$. If $w' \in \mathscr{F}_X^*$, with x' the first element of X in w', and if $w\sigma w'$, we conclude that $x\zeta \mathscr{R} x'\zeta$, hence $x = x'$ by Proposition 7.1. Consequently it is unambiguous to say that the element $w\sigma$ of F_1 *begins with* x, and the following proposition is clear.

PROPOSITION 7.2. *There is a one-to-one correspondence between X and the set of $\mathscr{R}$- [$\mathscr{L}$-] classes of F_1 such that the $\mathscr{R}$- [$\mathscr{L}$-] class of F_1 corresponding to an element x of X consists of all elements of F_1 beginning [ending] with x.*

For the remainder of this section, we identify $x\zeta$ with x for each x in X, and F_1 with $\mathscr{F}_X^{\text{cs}}$. We shall write $X = \{x_i : i \in I\}$, with $x_i \neq x_j$ if $i \neq j$ in the index set I.

By Proposition 7.2, we can also use I to index the $\mathscr{R}$- and $\mathscr{L}$-classes of F_1. Thus $R_i[L_i]$ is the $\mathscr{R}$- [$\mathscr{L}$-] class of F_1 consisting of all elements of F_1 beginning [ending] with x_i. As usual, we write $H_{ij} = R_i \cap L_j$. Since $x_ix_j \in H_{ij}$, the identity element of the group H_{ij} is $e_{ij} = x_ix_j(x_ix_j)^*$. Of course $e_{ii} = x_ix_i^*$, and we may write e_i for e_{ii}.

LEMMA 7.3. *Let $S = \mathscr{M}(G; I, I; T)$ be a Rees $I \times I$-matrix semigroup without zero over a group G with sandwich matrix $T = (t_{jk})$, $(j, k \in I)$. Assume T normalized* [1; p. 95]; *that is, $t_{j1} = t_{1j} = e$ for all j in I, where e is the identity element of G, and 1 is some fixed element of I. For each i in I, let $u_i \in G$, let $\xi_i = (u_i; i, i)$, and let S_1 be the c.r. subsemigroup of S generated by the set $\{\xi_i : i \in I\}$. Let G_1 be the subgroup of G generated by the set $\{t_{jk} : j, k \in I\} \cup \{u_i : i \in I\}$. Then $S_1 = \mathscr{M}(G_1; I, I; T)$.*

Proof. Since each $\xi_i \in \mathscr{M}(G_1; I, I; T)$, we have $S_1 \subseteq \mathscr{M}(G_1; I, I; T)$. Since S_1 contains $\xi_i\xi_j = (u_it_{ij}u_j; i, j)$, it contains an element in each $\mathscr{H}$-class of $\mathscr{M}(G_1; I, I; T)$, and hence it suffices to show that it contains all the elements $(u_i; 1, 1)$ and $(t_{jk}; 1, 1)$. Bearing in mind that the sandwich matrix T is normalized in the 1-row and the 1-column, we see that S_1 contains the following elements of S:

$$(u_1 ; 1, 1)^{-1} = (u_1^{-1}; 1, 1),$$

$$(u_1 ; 1, 1)(u_1^{-1}; 1, 1) = (e; 1, 1),$$

$$(e; 1, 1)(u_i ; i, i)(e; 1, 1) = (u_i ; 1, 1),$$

$$(u_j^{-1};\, 1,\, 1)(u_j\,;\, j, j) = (e;\, 1, j),$$
$$(u_k\,;\, k,\, k)(u_k^{-1};\, 1,\, 1) = (e;\, k,\, 1),$$
$$(e;\, 1, j)(e;\, k,\, 1) = (t_{jk}\,;\, 1,\, 1).$$

Hence $S_1 \supseteq \mathscr{M}(G_1; I, I; T)$, and equality follows.

THEOREM 7.4. *Let* $X = \{x_i : i \in I\}$ *be a set . Choose and fix an element* 1 *of* I, *and let* $I' = I\backslash\{1\}$. *Let*

$$Q = \{q_i : i \in I\} \cup \{p_{jk} : j, k \in I'\}$$

be a set in one-to-one correspondence with $I \cup (I' \times I')$, *and indexed thereby as shown. Let* F_Q *be the free group on* Q. *Define* $p_{j1} = p_{1k} = 1_F$ *(the identity element of* F_Q*) for all* j, k *in* I, *and let* P *be the* $I \times I$*-matrix* (p_{jk}) *over* F_Q . *Then there is an isomorphism* θ *of* $\mathscr{F}_X^{\text{cs}}$ *onto the Rees matrix semigroup* $\mathscr{M}(F_Q; I, I; P)$ *such that* $x_i\theta = (q_i : i, i)$ *for all* i *in* I.

Proof. Following the usual procedure for representing $F_1 = \mathscr{F}_X^{\text{cs}}$ as a Rees $I \times I$-matrix semigroup over H_{11} , we select, for each i in I, the element $e_{i1}[e_{1i}]$ as the representative of $H_{i1}[H_{1i}]$, and set

$$(a; i, j) = e_{i1}ae_{1j} \qquad \text{(all } a \text{ in } H_{11} \text{ and } i, j \text{ in } I\text{)}.$$

Then

$$(a; i, j)(b; k, l) = (at_{jk}b; i, l)$$

with $t_{jk} = e_{1j}e_{k1}$, and $F_1 = \mathscr{M}(H_{11}; I, I; T)$, where $T = (t_{jk})$. Note that the sandwich matrix T is normalized: $t_{j1} = e_{1j}e_{11} = e_{11} = e_1$, since the idempotents e_{1j} and e_{11} are in the same $\mathscr{R}$-class, and similarly $t_{1j} = e_1$, (all j in I).

For each i in I, let $u_i = e_1x_ie_1$. We note that $u_i \in H_{11}$ and

$$(u_i; i, i) = e_{i1}e_1x_ie_1e_{1i} = e_{i1}x_ie_{1i} = x_i\,.$$

By Lemma 7.3, H_{11} is generated by the set

$$U = \{u_i : i \in I\} \cup \{t_{jk} : j, k \in I'\}.$$

Let M denote the Rees matrix semigroup $\mathscr{M}(F_Q; I, I; P)$ defined in the statement of the theorem. Since F_1 is the free completely simple semigroup on X, there exists a homomorphism $\theta: F_1 \to M$ such that $x_i\theta = (q_i; i, i)$ for all i in I. By Lemma 7.3, the elements $(q_i; i, i)$ generate M, and hence θ is surjective.

Let H_{ij}^M denote the $\mathscr{H}$-class of M consisting of all $(a; i, j)$ with a in F_Q . Since $x_i\theta = (q_i; i, i)$, and homomorphisms preserve the $\mathscr{R}$-, $\mathscr{L}$-, and $\mathscr{H}$-relations, we see that $H_{ij}\theta \subseteq H_{ij}^M$. Since θ is surjective, $H_{ij}\theta = H_{ij}^M$. In particular, θ induces a group homomorphism of H_{11} onto H_{11}^M. If we can show that this is one-to-one, then θ must be one-to-one on all of F_1, and consequently an isomorphism.

Since $e_{1j}[e_{k1}]$ is the identity element of $H_{1j}[H_{k1}]$, $e_{1j}\theta[e_{k1}\theta]$ must be the identity element of $H_{1j}^M[H_{k1}^M]$, so

$$e_{1j}\theta = (1_F; 1, j) \qquad \text{and} \qquad e_{k1}\theta = (1_F; k, 1).$$

Hence

$$t_{jk}\theta = (e_{1k}e_{k1})\theta = (1_F; 1, j)(1_F; k, 1) = (p_{jk}; 1, 1).$$

Also

$$u_i\theta = (e_1x_ie_1)\theta = (1_F; 1, 1)(q_i; i, i)(1_F; 1, 1) = (q_i; 1, 1).$$

Since the set Q freely generates F_Q, this shows that the set U must freely generate H_{11}, and so θ induces an isomorphism of H_{11} onto H_{11}^M.

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