Three-connected graphs whose maximum nullity is at most three

Hein van der Holst

Department of Mathematics and Computer Science, Eindhoven University of Technology,
5600 MB Eindhoven, The Netherlands

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Abstract

For a graph $G = (V, E)$ with vertex-set $V = \{1, 2, \ldots, n\}$, let $\mathcal{S}(G)$ be the set of all $n \times n$ real-valued symmetric matrices $A$ which represent $G$. The maximum nullity of a graph $G$, denoted by $M(G)$, is the largest possible nullity of any matrix $A \in \mathcal{S}(G)$. Fiedler showed that a graph $G$ has $M(G) \leq 1$ if and only if $G$ is a path. Johnson et al. gave a characterization of all graphs $G$ with $M(G) \leq 2$. Independently, Hogben and van der Holst gave a characterization of all 2-connected graphs with $M(G) \leq 2$.

In this paper, we show that $k$-connected graphs $G$ have $M(G) \geq k$, that $k$-connected partial $k$-graphs $G$ have $M(G) = k$, and that for 3-connected graphs $G$, $M(G) \leq 3$ if and only if $G$ is a partial 3-path.

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1. Introduction

Let $G = (V, E)$ be a graph with $V = \{1, 2, \ldots, n\}$. (In this paper all graphs are assumed to be simple.) Define $\mathcal{S}(G)$ as the set of all $n \times n$ real-valued symmetric matrices $A = [a_{i,j}]$ with $a_{i,j} \neq 0$, $i \neq j$ if and only if $ij \in E$. The maximum nullity of $G$, denoted by $M(G)$, is the largest possible nullity of any matrix $A \in \mathcal{S}(G)$. For example, $M(K_n) = n - 1$, $n \geq 2$, and a matrix that attains this value is the matrix all whose entries are 1. By $mr(G)$ we denote the

E-mail address: H.v.d.Holst@tue.nl

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smallest possible rank of any matrix \( A \in \mathcal{S}(G) \). If the graph \( G \) has \( n \) vertices, then \( M(G) + \text{mr}(G) = n \).

Fiedler [9] showed that the paths are the only graphs \( G \) for which \( M(G) \leq 1 \). Johnson et al. [12] characterized all graphs \( G \) with \( M(G) \leq 2 \). Independently, Hogben and van der Holst [10] characterized all 2-connected graphs \( G \) with \( M(G) \leq 2 \). It is easy to see that a graph \( G \) has \( \text{mr}(G) \leq 1 \) if and only if \( G \) is the union of a complete graph and possibly some isolated vertices. Barrett et al. [3] characterized all graphs \( G \) for which \( \text{mr}(G) \leq 2 \) as those for which six specific graphs do not occur as induced subgraphs.

In this paper we first show that \( k \)-connected graphs \( G \) have \( M(G) \geq k \) (in fact we prove a theorem stronger than this) and that \( k \)-connected partial \( k \)-paths \( G \) have \( M(G) = k \); see Section 3 for the definition of partial \( k \)-paths. Then we characterize all 3-connected graphs \( G \) with \( M(G) \leq 3 \). We will see that these graphs are exactly the 3-connected partial 3-paths. Above, we mentioned already that 3-connected partial 3-paths \( G \) have \( M(G) = 3 \). An outline of the reverse direction is as follows. If a graph \( G \) has \( M(G) \leq 3 \), then \( \xi(G) \leq 3 \), where \( \xi(G) \) is the graph parameter introduced by Barioli et al. in [2]. If \( \xi(G) \leq 3 \), then \( G \) has no minor isomorphic to a graph in a certain collection of five graphs. Finally, if \( G \) is 3-connected and has no minor isomorphic to a graph in this collection of five graphs, then \( G \) is a partial 3-path.

The outline of the paper is as follows. In Section 2, the graph parameter \( \xi(G) \) is studied. We prove that \( k \)-connected graphs \( G \) have \( \xi(G) \geq k \) and present the values of \( \xi(G) \) on some graphs \( G \). In Section 3, we study partial \( k \)-paths. An important result here is that \( k \)-connected partial \( k \)-paths \( G \) have \( M(G) = \xi(G) = k \). As a corollary of this result, we show that certain graphs are not partial 3-paths. In Section 4, we give the characterizations of 3-connected graphs \( G \) that have \( M(G) \leq 3 \).

2. The graph parameter \( \xi(G) \)

In the proof of the characterization of all 3-connected graphs \( G \) with \( M(G) \leq 3 \), we use the graph parameter \( \xi(G) \), introduced by Barioli, Hogben, and Fallat in [2]. The definition of this parameter depends on the Strong Arnold property. The definition of the Strong Arnold property is as follows. Let \( N_G \) denote the set of all \( n \times n \) symmetric matrices \( X = [x_{i,j}] \) with \( x_{i,i} = 0 \) for all \( i \in V \) and \( x_{i,j} = 0 \) for all \( ij \in E \). A matrix \( A \in \mathcal{S}(G) \) has the Strong Arnold property if \( X \in N_G \) and \( AX = 0 \) implies that \( X = 0 \). The parameter \( \xi(G) \) is defined as the maximum nullity over all matrices \( A \in \mathcal{S}(G) \) having the Strong Arnold property.

Let \( G \) be a graph. If \( e \) is an edge of \( G \), then contracting \( e \) means that we delete \( e \) and identify the two endpoints of \( e \). A minor of a graph \( G \) is a graph that can be obtained from a subgraph of \( G \) by contracting a collection of edges. If \( G \) has a minor isomorphic to \( H \), we also say that \( G \) has an \( H \)-minor. One of the properties of \( \xi(G) \), which \( M(G) \) lacks, is stated in the following theorem.

**Theorem 1** [2, Corollary 2.5]. If \( G' \) is a minor of \( G \), then \( \xi(G') \leq \xi(G) \).

The parameter \( \nu_1^S(G) \), which was introduced by Colin de Verdière in [7], is defined as the maximum nullity over all positive semi-definite matrices \( A \in \mathcal{S}(G) \) having the Strong Arnold property. Also \( \nu_1^S(G) \) has the property that \( \nu_1^S(G') \leq \nu_1^S(G) \) if \( G' \) is a minor of \( G \). Another parameter introduced by Colin de Verdière is \( \mu(G) \), see [5,6]. Each of the parameters \( \mu(G) \) and \( \nu_1^S(G) \) forms a lower bound for \( \xi(G) \).
Since \( \xi(G) \leq M(G) \), the graph parameter \( \xi(G) \) can be used to find a lower bound for \( M(G) \). For example, if \( G \) has a \( K_k \)-minor, \( k \geq 2 \), then \( M(G) \geq k - 1 \), as \( \xi(K_k) = k - 1 \leq \xi(G) \).

For a graph \( G = (V, E) \) and \( S \subseteq V \), \( G - S \) denotes the graph obtained from \( G \) by deleting all vertices in \( S \).

A graph \( G \) is connected if every two vertices of \( G \) are connected by a path. A graph \( G = (V, E) \) is \( k \)-connected if \( |V| > k \) and \( G - S \) is connected for each \( S \subseteq V \) with \( |S| < k \). For example, \( K_{2,2,2} \), the graph with vertex-set \( \{v_1, v_2, \ldots, v_6\} \) such that each pair of vertices \( i, j \) with \( i \) and \( j \) in distinct sets in \( \{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\}\} \) is connected by an edge, is 4-connected.

In the proof of Theorem 14, we will use the following theorem.

**Theorem 2** Menger’s Theorem cf. [8]. Let \( G = (V, E) \) be a \( k \)-connected. Then for any \( B \subseteq V \) and \( a \in V \setminus B \), there are \( k \) vertex-disjoint paths between \( a \) and \( B \).

An orthogonal representation of \( G = (V, E) \) in \( \mathbb{R}^d \) is an assignment \( f : V \to \mathbb{R}^d \) such that \( f(i) \) and \( f(j) \) are orthogonal for every pair of distinct nonadjacent vertices \( i \) and \( j \). An orthogonal representation is in general position if every set of \( d \) representing vectors is linear independent. An orthogonal representation \( f \) is faithful if \( f(i) \) and \( f(j) \) are orthogonal if and only if \( i \) and \( j \) are nonadjacent. Lovász, Saks, and Schrijver proved the following theorem, which is essentially a combination of Theorem 1.2 and Corollary 1.4 in [13] (see [14] for a correction of [13]):

**Theorem 3.** For a graph \( G \) with \( n \) vertices, \( G \) is \( (n - d) \)-connected if and only if \( G \) has a general-position faithful orthogonal representation in \( \mathbb{R}^d \).

If the graph \( G \) has vertex-set \( V = \{1, 2, \ldots, n\} \), then a faithful orthogonal representation \( f \) gives rise to a positive semi-definite matrix \( A = [a_{i,j}] \in \mathcal{S}(G) \) whose entries are defined by \( a_{i,j} = f(i)^T f(j) \). Hence, from Theorem 3 it follows that there is a positive semi-definite matrix \( A \in \mathcal{S}(G) \) with nullity \( \geq k \) if \( G \) is \( k \)-connected. In fact, we can prove more. A similar proof for \( \psi_1^C(G) \geq k \) if \( G \) is \( k \)-connected can be found in [11].

**Theorem 4.** Let \( G = (V, E) \) be a graph with \( V = \{1, 2, \ldots, n\} \). If \( G \) is a \( k \)-connected graph, then \( \xi(G) \geq \psi_1^R(G) \geq k \).

**Proof.** Since \( G \) is \( k \)-connected, there is a general-position faithful orthogonal representation \( f \) of \( G \) in \( \mathbb{R}^{n-k} \). Define \( A = [a_{i,j}] \in \mathcal{S}(G) \) by \( a_{i,j} = f(i)^T f(j) \). We show that \( A \) has the Strong Arnold property. This then implies that \( \psi_1^R(G) \geq k \). Let \( X = [x_{i,j}] \in N_G \) such that \( AX = 0 \). Since \( f \) is in general-position, for each subset of \( n - k \) vertices \( \{v_1, v_2, \ldots, v_{n-k}\} \) of \( V \), the set \( \{f(v_1), f(v_2), \ldots, f(v_{n-k})\} \) is linearly independent. Hence each nonzero vector \( x \in \ker(A) \) has at least \( n - k + 1 \) nonzero entries. So each nonzero column of \( X \) has at least \( n - k + 1 \) nonzero entries. Suppose \( X \) has a nonzero column, say the \( i \)th column. Then \( x_{i,i} = 0 \) and \( x_{i,j} = 0 \) for each vertex \( j \) adjacent to \( i \). Since \( G \) is \( k \)-connected, vertex \( i \) has degree at least \( k \). Hence the \( i \)th column of \( X \) has at least \( k + 1 \) zero. This contradicts that the \( i \)th column has at least \( n - k + 1 \) nonzero entries. Thus, \( X \) is the all-zero matrix, and so \( A \) has the Strong Arnold property. \( \square \)

A \( \Delta Y \)-transformation on a triangle \( C \) in a graph \( G \) is the transformation which deletes the edges of \( C \), adds a new vertex \( v \) and connects \( v \) to each of the vertices of \( C \) by an edge. If we apply a \( \Delta Y \)-transformation on \( K_{2,2,2} \), we obtain a graph denoted by \( Q_3 Y \Delta \). The graph \( Q_3 \) can
be obtained from $Q_3Y\Delta$ by another $\Delta Y$-transformation; see Fig. 1 for a picture of the graphs $Q_3$, $Q_3Y\Delta$, and $K_{2,2,2}$.

Lemma 5 [10]. Let $G$ be a graph and let $G'$ be obtained from $G$ by a $\Delta Y$-transformation. Then $\xi(G') \geq \xi(G)$.

Lemma 6 [2, Observation 1.7 and Example 1.12]. $\xi(K_5) = 4$, $\xi(K_4) = 3$, $\xi(K_{3,3}) = 4$, and $\xi(K_{2,3}) = 3$.

By $T_3$ we denote the graph obtained from $K_{2,2,2}$ by deleting the edges of a triangle.

Lemma 7 ([10, Lemma 2.2]). $\xi(T_3) = 3$.

Lemma 8. $\xi(K_{2,2,2}) = 4$, $\xi(Q_3) = 4$, and $\xi(Q_3\Delta) = 4$.

Proof. Since $K_{2,2,2}$ is 4-connected, $\xi(K_{2,2,2}) \geq 4$. By [3], $\text{mr}(K_{2,2,2}) = 2$, that is, $M(K_{2,2,2}) = 4$, and so $\xi(K_{2,2,2}) = 4$. By Lemma 5, $\xi(Q_3\Delta) \geq 4$ and $\xi(Q_3) \geq 4$. To see that $\xi(Q_3) \leq 4$, suppose to the contrary that $\xi(Q_3) > 4$. Then there exists a matrix $B \in S(G)$ with nullity $> 4$. Let $S$ be the vertices of a cycle of size 4 in $Q_3$. Since $B$ has nullity $> 4$, there is a nonzero vector $x \in \ker(B)$ with $x_S = 0$. Let $v$ be a vertex with $x_v \neq 0$, and let $u \in S$ be the vertex adjacent to $v$. Let $B_u$ be the $u$th row of $B$. From $B_u x = 0$, it follows that $x_v = 0$. This contradiction shows that each $B \in S(G)$ has nullity $\leq 4$. Hence $\xi(Q_3) = 4$. Hence $\xi(Q_3\Delta) = 4$ also. □

3. Partial $k$-paths

A $k$-tree is defined recursively as follows:

1. A complete graph with $k + 1$ vertices is a $k$-tree.
2. If $G = (V, E)$ is a $k$-tree and $v_1, \ldots, v_k$ form a clique in $G$ with $k$ vertices, then $H = (V \cup \{v\}, E \cup \{(v_i, v)| 1 \leq i \leq k\})$ with $v$ a new vertex, is a $k$-tree.

A partial $k$-tree is a subgraph of a $k$-tree. A graph has tree-width $\leq k$ if it is a partial $k$-tree.
Equivalently, the tree-width of a graph \( G \) can also be defined as follows. A tree-decomposition of a graph \( G = (V, E) \) is a pair \( (T, \mathcal{W}) \) where \( T \) is a tree and \( \mathcal{W} = \{ W_t | t \in V(T) \} \) is a family of subsets of \( V \) with the properties.

(i) \( \bigcup \{ W_t | t \in V(T) \} = V \),

(ii) every edge of \( G \) has both ends in some \( W_t \), and

(iii) if \( t_1, t_2, t_3 \in V(T) \) and \( t_2 \) lies on the path from \( t_1 \) to \( t_3 \), then \( W_{t_1} \cap W_{t_3} \subseteq W_{t_2} \).

The subsets \( W_t \) are called the bags of the tree-decomposition. The width of a tree-decomposition is \( \max(\{ |W_t| - 1 | t \in V(T) \}) \), and the tree-width of \( G \) is the minimum width of any tree-decomposition of \( G \).

A tree-decomposition \( (T, \mathcal{W}) \) of width \( k \) is called smooth if for all \( t \in V(T) \), \( |W_t| = k + 1 \), and for all \( st \in E(T), |W_s \cap W_t| = k \); see e.g. [4]. Any tree-decomposition of width \( k \) of a graph \( G \) can be transformed to a smooth tree-decomposition of width \( k \) by applying the following transformations on the tree-decomposition until none is possible.

- If \( W_s \subseteq W_t \) for some \( st \in E(T) \), then contract the edge \( st \) in \( T \) and take for the new vertex \( t' \), \( W_{t'} := W_t \).
- If for a vertex \( t \in V(T), |W_t| < k + 1 \), then choose a vertex \( v \in W_s \setminus W_t \), where \( s \) is adjacent to \( t \), and add \( v \) to \( W_t \).
- If for adjacent vertices \( s, t \in V(T), |W_s \setminus W_t| > 1 \), then subdivide the edge \( st \), let \( r \) be the new vertex, choose a vertex \( v \in W_s \setminus W_t \) and a vertex \( w \in W_t \setminus W_s \), and let \( W_r := (W_s \setminus \{ v \}) \cup \{ w \} \).

If \( (T, \mathcal{W}) \) is smooth and for each \( W_t \) and each pair of vertices \( \{ v, w \} \) in \( W_t \) we add an edge between \( v \) and \( w \) if there is none, then the resulting graph is a \( k \)-tree. If \( G \) is a \( k \)-tree, as defined at the beginning of this section, then a smooth tree-decomposition of width \( k \) can be obtained by using as the bags each set \( V_1, \ldots, V_k \), \( v \) used in the construction of \( G \) as a \( k \)-tree.

If \( G \) has tree-width \( \leq k \), then each of its minors has tree-width \( \leq k \). Hence, if \( H \) is a minor of \( G \) and \( H \) has tree-width \( \geq k \), then \( G \) has tree-width \( \geq k \). For example, \( K_3 \) has tree-width 2, and so each graph that has a \( K_3 \)-minor has tree-width \( \geq 2 \). Conversely, if a graph has no \( K_3 \)-minor, then it has no cycles, so is a forest. A forest has tree-width \( \leq 1 \). Hence, a graph \( G \) has tree-width \( \leq 1 \) if and only if \( G \) has no \( K_3 \)-minor. For graphs with tree-width \( \leq 2 \), we have the following: a graph \( G \) has tree-width \( \leq 2 \) if and only if \( G \) has no \( K_4 \)-minor. For graphs with tree-width \( \leq 3 \), the following theorem holds; see Fig. 2 for a picture of the graphs \( V_8 \) and \( C_5 \times K_2 \).

**Theorem 9** [1]. A graph \( G = (V, E) \) has tree-width \( \leq 3 \) if and only if \( G \) has no \( K_5, K_{2,2,2}, V_8 \), and no \( C_5 \times K_2 \)-minor.

At the moment of this writing no such characterization is known for graphs that have tree-width \( \leq 4 \). For results on graphs that have tree-width \( \leq 4 \), we refer to Sanders [15].

A \( k \)-path is a \( k \)-tree with at most \( k + 1 \) vertices or exactly two vertices of degree \( k \). A partial \( k \)-path is a subgraph of a \( k \)-path. A 2-connected partial 2-path is the same as a linear singly edge articulated cycle graph (LSEAC), a type of graphs introduced by Johnson et al. [12], and it is the same as a linear 2-tree, a type of graphs introduced by Hogben and van der Holst [10].

If \( H = (W, F) \) is a subgraph of a graph \( G = (V, E) \), we denote by \( N_G(H) \) the set of all vertices in \( V \setminus W \) that are adjacent to a vertex in \( W \). If \( v \) is a vertex of \( G \), then by \( N_G(v) \) we denote the set of all vertices in \( V \setminus \{ v \} \) that are adjacent to \( v \).

We use the following lemma in Theorem 14.
Lemma 10. Let \((T, \mathcal{W})\) be a smooth tree-decomposition of width \(k\) of a \(k\)-connected graph \(G\). Let \(W_t\) be a bag of \((T, \mathcal{W})\). Then \(|N_G(K)| = k\) for each component \(K\) of \(G - W_t\).

Proof. Let \(H\) be the graph obtained from \((T, \mathcal{W})\) by adding for each bag \(W_t\) and each pair of vertices \(\{v, w\}\) of \(W_t\) an edge between \(v\) and \(w\) if there is none. Then \(H\) is a \(k\)-tree and \(G\) is a subgraph of \(H\). Since each component \(K\) of \(G - W_t\) is a subgraph of a component \(L\) of \(H - W_t\), and \(N_G(K) \subseteq N_H(L)\), we see that \(|N_G(K)| \leq |N_H(L)| \leq k\). Since \(G\) is \(k\)-connected, \(|N_G(K)| = k\) for each component \(K\) of \(G - W_t\). □

Lemma 11. If \(G = (V, E)\) is a \(k\)-connected partial \(k\)-path, then \(G\) is a subgraph of a \(k\)-path \(H = (W, F)\) with \(W = V\).

Proof. Let \(H\) be a \(k\)-path which has \(G\) as a subgraph.

If \(v\) is a vertex of degree \(k\) in \(H\), then \(v\) is a vertex of \(G\), for otherwise we could take \(H - v\) for \(H\).

Suppose \(v\) is a vertex of degree \(> k\) in \(H\) and \(v\) is not a vertex in \(G\). Let \(v_1\) and \(v_2\) be the vertices of degree \(k\) in \(H\). There is a vertex-cut \(S\) of size \(< k\) in \(H - v\) such that \(v_1\) and \(v_2\) belong to different components of \(H - S\). Then \(v_1\) and \(v_2\) also belong to different components in \(G - S\). Since \(S\) is a vertex-cut of size \(< k\) in \(G\), this contradicts the \(k\)-connectivity of \(G\). □

Theorem 12. If \(G = (V, E)\) is a \(k\)-connected partial \(k\)-path, then \(M(G) = \xi(G) = k\).

Proof. By Theorem 4, \(\xi(G) \geq k\).

We now show that \(M(G) \leq k\). From this it follows that \(k \leq \xi(G) \leq M(G) \leq k\), and so \(k = \xi(G) = M(G)\).

As \(G\) is a partial \(k\)-path, it is a subgraph of a \(k\)-path \(H = (W, F)\). By Lemma 11, we may assume that \(W = V\). Let \(v\) be a vertex of degree \(k\) in \(H\). Then \(v\) has also degree \(k\) in \(G\). Let \(S\) be a subset of \(N_G(v)\) of size \(k - 1\).

Suppose, to the contrary, that \(M(G) > k\). Then there is an \(A = [a_{i,j}] \in \mathcal{S}(G)\) with nullity \(> k\). We can find a nonzero vector \(x \in \ker(A)\) with \(x_v = 0\) and \(x_S = 0\). We will show that \(x = 0\), contradicting that \(x\) is nonzero.
Let $A_v$ be the $v$th row of $A$. Since $x_v = 0$, it follows from $A_v x = 0$ that $x_{NG(v)} = 0$.

We can order the $(k + 1)$-cliques in $H$ as $C_1, \ldots, C_t$ such that $v \in C_1$ and building up $H$ we sequentially add $C_2, \ldots, C_t$. Let $C_i$ be a $(k + 1)$-clique in $H$ such that $x_{C_i} \neq 0$, while $x_{C_j} = 0$ for $j < i$. First suppose that $i = t$. Let $u$ be a vertex in $C_i$ such that $x_u = 0$, and let $w$ be the vertex in $C_i$ such that $x_w \neq 0$. In $G$, $u$ is adjacent to $w$, for otherwise $G$ would not be $k$-connected. However, from $A_u x = 0$, it follows that $x_w = 0$, a contradiction. Suppose now that $i < t$; let $R = C_{i-1} \cap C_{i+1}$. So $x_R = 0$. In $G$ there is an edge connecting the two vertices $u, w$ of $C_i \setminus R$, for otherwise $G$ would not be $k$-connected. We may assume that $u \in C_{i-1}$, and so $x_u = 0$. From $A_u x = 0$, it follows that $x_w = 0$, contradicting that $x_{C_i} \neq 0$. □

Since $K_5, K_{2,2,2}, K_{3,3}, Q_3,$ and $Q_3 Y \Delta$ are 3-connected, we obtain from Theorem 12 and Lemmas 6 and 8.

**Corollary 13.** None of the graphs $K_5, K_{2,2,2}, K_{3,3}, Q_3,$ and $Q_3 Y \Delta$ is a partial 3-path.

### 4. Characterization of 3-connected graphs $G$ with $M(G) \leq 3$

We are now ready for the characterization of 3-connected graphs $G$ with $M(G) \leq 3$. From Theorem 4 it follows that 3-connected graphs $G$ with $M(G) \leq 3$ have $M(G) = 3$.

**Theorem 14.** For a 3-connected graph $G = (V, E)$ the following are equivalent:

(i) $G$ is a partial 3-path;

(ii) $M(G) = \xi(G) = 3$;

(iii) $G$ has no $K_5$, $K_{2,2,2}$, $K_{3,3}$, $Q_3$, and no $Q_3 Y \Delta$-minor.

**Proof.** (i)⇒(ii) If $G$ is a partial 3-path, then, by Theorem 12, $M(G) = \xi(G) = 3$.

(ii)⇒(iii) If $\xi(G) = 3$, then, by Lemmas 6 and 8, none of the graphs $K_5, K_{2,2,2}, K_{3,3}, Q_3,$ and $Q_3 Y \Delta$ is isomorphic to a minor of $G$.

(iii)⇒(i) Suppose that $G$ is a graph with no $K_5$, $K_{2,2,2}$, $K_{3,3}$, $Q_3$, and no $Q_3 Y \Delta$-minor. As $K_{3,3}$ is a minor of $V_8$ and $Q_3$ is a minor of $C_5 \times C_2$, $G$ has no $K_5, K_{2,2,2}, V_8$, and no $C_5 \times C_2$-minor. Hence $G$ has a tree-decomposition $(T, \mathcal{Y})$ of width 3, by Theorem 9; we may assume that $(T, \mathcal{Y})$ is smooth. We call a bag $W_s$ bad if $G - W_s$ has more than two components. Take a smooth tree-decomposition $(T, \mathcal{Y})$ such that the number of bad bags is minimal. Suppose to the contrary that this number is not zero; take a bad bag $W_s$.

By Lemma 10, $|N_G(K)| = 3$ for each component $K$ of $G - W_s$.

Suppose that there are distinct components $K_1$ and $K_2$ of $G - W_s$ such that $N_G(K_1) = N_G(K_2)$. Let $w$ be the vertex of $W_s - N_G(K_1)$ since $G$ is 3-connected, there are three vertex-disjoint paths of length $\geq 1$ from $w$ to $N_G(K_1)$ by Menger’s theorem. Contracting each of these paths to an edge, and contracting $K_1$ and $K_2$ each to a vertex shows that $G$ has a $K_{3,3}$-minor. This contradiction shows that there are at most four components in $G - W_s$.

If there are four components $K_1, K_2, K_3, K_4$ in $G - W_s$, then contracting each $K_i$ to a vertex shows that $G$ has a $Q_3$-minor. Hence there are at most three components in $G - W_s$.

Suppose now that there are three components $K_1, K_2, K_3$ in $G - W_s$. For $i = 1, 2, 3$, let $A_i$ be the subgraph induced by $K_i \cup N_G(K_i)$. Let $w$ be the common vertex of $A_1, A_2, A_3$. First suppose that the subgraphs $A_i - \{w\}, i = 1, 2, 3$, contain a cycle. In each $A_i - \{w\}$, choose a cycle $C_i$. Since $G$ is 3-connected, there are vertex-disjoint paths $P_i^1, P_i^2, P_i^3$ from $C_i$ to $N_G(K_i)$,
by Menger’s theorem; let \( P_i^1 \) and \( P_i^2 \) be the paths connecting \( C_i \) to \( N_G(K_i) \setminus \{w\} \), and let \( P_i^3 \) be the path connecting \( C_i \) to \( w \). Remove all edges from \( A_i \) that do not belong to \( C_i, P_i^1, P_i^2, \) and \( P_i^3 \). Contracting each of the edges on the paths \( P_i^1 \) and \( P_i^2 \), and contracting all but one edge on the path \( P_i^3 \) yields a graph that contains a \( Q_3 \Delta \)-minor.

Hence at least one of the subgraphs \( A_1 - \{w\} \), \( A_2 - \{w\} \), \( A_3 - \{w\} \) contains no cycle; without loss of generality we may assume that \( A_1 - \{w\} \) contains no cycle. Hence \( A_1 - \{w\} \) is a tree. Since \( G \) is 3-connected, \( A_1 - \{w\} \) is a path \( u_1u_2\ldots u_m \) connecting the vertices of \( N_G(K_1) \setminus \{w\} \). We assume that \( u_1 \in N_G(K_2) \) and \( u_m \in N_G(K_3) \). Let \( v \) be the vertex in \( N_G(K_2) \cap N_G(K_3) \setminus \{w\} \). Let \( T_1, \ldots, T_r \) be the components of \( T - s \) such that for each vertex \( t \in V(T_i), i = 1, \ldots, r, \) \( W_t \subseteq A_2 \). For \( i = 1, \ldots, r \), let \( t_i \) be the vertex of \( T_i \) adjacent to \( s \). Define similarly \( T'_1, \ldots, T'_r \) and \( t'_1, \ldots, t'_r \), except with \( A_3 \) instead of \( A_2 \). Let \( S \) be the tree obtained from \( T_1, \ldots, T_r, T'_1, \ldots, T'_r \) and a path \( P = p_1p_2\ldots p_{m-1} \) of length \( m - 1 \) by connecting the vertices \( t_i, i = 1, \ldots, r \) to \( p_1 \), and the vertices \( t'_i, i = 1, \ldots, r' \) to \( p_{m-1} \). Define \( W'_{p_i} = \{u_i, u_{i+1}, v, w\} \) for \( i = 1, \ldots, m - 1 \) and \( W'_t = W_t \) for \( t \in V(T), t \neq s \). Let \( \mathcal{W} = \{W'_t | t \in V(S)\} \). Then \( (S, \mathcal{W}) \) is a smooth tree-decomposition of \( G \) with fewer bad bags, contradicting the assumption that \( (T, \mathcal{W}) \) is a tree-decomposition with a minimum number of bad bags.

Hence \( (T, \mathcal{W}) \) has no bad bags. For each bag \( W_t \), add an edge between each pair of vertices of \( W_t \) if there is none. The graph we obtain is a 3-path, and so \( G \) is a partial 3-path.

References