J. Math. Anal. Appl. 355 (2009) 577–593

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Stochastic population dynamics under regime switching II

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ARTICLE INFO

Article history:
Received 4 July 2007
Available online 13 February 2009
Submitted by K. Gopalsamy

Keywords:
Brownian motion
Markov chain
Stochastic differential equation
Itô's formula
Extinction
Average in time

ABSTRACT

This is a continuation of our paper [Q. Luo, X. Mao, Stochastic population dynamics under regime switching, J. Math. Anal. Appl. 334 (2007) 69–84] on stochastic population dynamics under regime switching. In this paper we still take both white and color environmental noise into account. We show that a sufficient large white noise may make the underlying population extinct while for a relatively small noise we give both asymptotically upper and lower bound for the underlying population. In some special but important situations we precisely describe the limit of the average in time of the population.

1. Introduction

Population systems are often subject to environmental noise and there are various types of environmental noise e.g. white or color noise (see e.g. [2,6,8,21,24]) and it has been shown that the presence of such noise affects population systems significantly. For example, Takeuchi et al. [26] consider a predator–prey Lotka–Volterra model with Markovian switching between two regimes of environment. Although the predator–prey Lotka–Volterra model in each regime (without switching) develops periodically [9,25], Takeuchi et al. [26] reveal a very interesting and surprising result: switching between two regimes makes the system become neither permanent nor dissipative [4,5,7,14,15,27]. However, the situation changes if the white noise is taken into account. In our previous paper [16], we show that under both telegraph and white noise, the general Lotka–Volterra model will not explode to infinity or become extinct at any finite time with probability one. Moreover, the model is not only stochastically ultimately bounded but the time-average of the second moment is also bounded. These nice properties indicate that taking both telegraph and white noise into account produces more desired results. It is in this spirit that we will consider both telegraph and white noise in this paper as well. However, the type of white noise considered here is different from that in our pervious paper [16].

To explain, let us consider the Lotka–Volterra model under regime switching for a system with n interacting components, namely

$$\dot{x}(t) = \operatorname{diag}\big(x_1(t), \dots, x_n(t)\big)\big[b\big(r(t)\big) + A\big(r(t)\big)x(t)\big], \tag{1.1}$$

where $r(t)$ is a Markov chain on the state space $\mathbb{S}$ as defined in the next section, $x = (x_1, \dots, x_n)^T$ and for each $i \in \mathbb{S}$,

$$b(i) = \big(b_1(i), \dots, b_n(i)\big)^T, \qquad A(i) = \big(a_{jk}(i)\big)_{n\times n}.$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.02.010

Recall that the parameter $b_j(i)$ represents the intrinsic growth rate of species j in regime i. In practice we usually estimate it by an average value plus an error term. If we still use $b_j(i)$ to denote the average growth rate, then the intrinsic growth rate becomes $b_j(i) + \mathrm{error}_j(i)$. Let us consider a small subsequent time interval dt, during which $x_j(t)$ changes to $x_j(t) + dx_j(t)$. Accordingly, Eq. (1.1) becomes

$$dx_j(t) = x_j(t)\left(b_j\big(r(t)\big) + \sum_{k=1}^{n} a_{jk}\big(r(t)\big)x_k(t)\right)dt + x_j(t)\,\mathrm{error}_j\big(r(t)\big)\,dt \tag{1.2}$$

for $1 \leqslant j \leqslant n$. According to the well-known central limit theorem, the error term $\mathrm{error}_j(i)\,dt$ may be approximated by a normal distribution with mean zero and variance $v_j^2(i)\,dt$. In terms of mathematics, $\mathrm{error}_j(i)\,dt \sim N(0, v_j^2(i)\,dt)$, which can be written as $\mathrm{error}_j(i)\,dt \sim v_j(i)\,dB(t)$, where $dB(t) = B(t+dt) - B(t)$ is the increment of a Brownian motion (to be defined in the next section) that follows $N(0, dt)$. Hence Eq. (1.2) becomes the Itô stochastic differential equation (SDE)

$$dx_j(t) = x_j(t)\left[\left(b_j\big(r(t)\big) + \sum_{k=1}^{n} a_{jk}\big(r(t)\big)x_k(t)\right)dt + v_j\big(r(t)\big)\,dB(t)\right]$$

for $1 \leqslant j \leqslant n$. That is, in the matrix form,

$$dx(t) = \mathrm{diag}\big(x_1(t), \dots, x_n(t)\big)\big(\big[b\big(r(t)\big) + A\big(r(t)\big)x(t)\big]\,dt + v\big(r(t)\big)\,dB(t)\big), \tag{1.3}$$

where $v(i) = (v_1(i), \dots, v_n(i))^T$ is the vector of the standard deviations of the errors, known as the noise intensities. These intensities may or may not depend on population sizes. If they are dependent on population sizes, we may have

$$v_j(i) = \sum_{k=1}^{n} \sigma_{jk}(i)x_k(t)$$

and hence Eq. (1.2) becomes

$$dx(t) = \mathrm{diag}\big(x_1(t), \dots, x_d(t)\big)\big(\big[b\big(r(t)\big) + A\big(r(t)\big)x(t)\big]\,dt + \sigma\big(r(t)\big)x(t)\,dB(t)\big),$$

where $\sigma(i) = (\sigma_{jk}(i))_{n\times n}$. This is the stochastic Lotka–Volterra model which we have discussed in our previous paper [16].

However, if the noise intensities are independent of population sizes, we can write $v(i)$ as a constant vector $\sigma(i) = (\sigma_1(i), \dots, \sigma_n(i))^T$. As a result, Eq. (1.2) becomes an SDE under regime switching of the form

$$dx(t) = \mathrm{diag}\big(x_1(t), \dots, x_n(t)\big)\big(\big[b\big(r(t)\big) + A\big(r(t)\big)x(t)\big]\,dt + \sigma\big(r(t)\big)\,dB(t)\big), \tag{1.4}$$

and this is the model which we will discuss in this paper. As this is a population system, our natural aims of this paper are:

- to establish a sufficient condition on $A(i)$ only under which the solution of Eq. (1.4) starting from anywhere in $\mathbb{R}_+^n$ will remain in $\mathbb{R}_+^n$ with probability one;
- to show that the solution is ultimately bounded in mean and the average in time of the variance of the solution is bounded too;
- to reveal that a large white noise will force the population to become extinct;
- to give upper and lower bound for the solution under a relatively small white noise and, in some special but important cases, to describe $\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,ds$ precisely.

2. Stochastic Lotka–Volterra model under regime switching

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t)$, $t \geqslant 0$ be a scalar Brownian motion defined on this probability space. Let $r(t)$, $t \geqslant 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \dots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\big\{r(t+\delta) = j \,\big|\, r(t) = i\big\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\delta + o(\delta) & \text{if } i = j, \end{cases}$$

where $\delta > 0$. Here γ_{ij} is the transition rate from i to j and $\gamma_{ij} > 0$ if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. We also fix the initial value $r(0) = i_0 \in \mathbb{S}$ arbitrarily so the Markov chain is fixed too. It is well known that almost every sample path of $r(\cdot)$ is a right continuous step function with a finite number of sample jumps in any finite subinterval of $\mathbb{R}_+ := [0, \infty)$.

We will need a few more notations. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\operatorname{trace}(A^T A)}$ whilst its operator norm is denoted by $\|A\| = \sup\{|Ax|:\ |x| = 1\}$. We also introduce the positive corn $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n:\ x_i > 0 \text{ for all } 1 \leqslant i \leqslant n\}$. Please note the difference between $\mathbb{R}^1_+ = (0, \infty)$ and $\mathbb{R}_+ = [0, \infty)$.

In this paper we will use a lot of quadratic functions of the form $x^T Ax$ for the state $x \in \mathbb{R}^n_+$ only. Therefore, for a symmetric $n \times n$ matrix $A = (a_{jk})_{n\times n}$, we recall the following definition

$$\lambda^+_{\max}(A) = \sup_{x\in\mathbb{R}^n_+,\ |x|=1} x^T Ax,$$

which was introduced by Bahar and Mao [2]. Let us emphasise that this is different from the largest eigenvalue $\lambda_{\max}(A)$ of the matrix A. To see this more clearly, let us recall the nice property of the largest eigenvalue:

$$\lambda_{\max}(A) = \sup_{x\in\mathbb{R}^n,\ |x|=1} x^T Ax.$$

It is therefore clear that we always have

$$\lambda^+_{\max}(A) \leqslant \lambda_{\max}(A).$$

In many situations we even have $\lambda^+_{\max}(A) < \lambda_{\max}(A)$. For example, for

$$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

we have $\lambda^+_{\max}(A) = -1 < \lambda_{\max}(A) = 0$. On the other hand, $\lambda^+_{\max}(A)$ does have many similar properties as $\lambda_{\max}(A)$ has. For example, it follows straightforward from the definition that

$$x^T Ax \leqslant \lambda^+_{\max}(A)|x|^2, \quad \forall x \in \mathbb{R}^n_+$$

and

$$\lambda^+_{\max}(A) \leqslant \|A\|.$$

Moreover

$$\lambda^+_{\max}(A + C) \leqslant \lambda^+_{\max}(A) + \lambda^+_{\max}(C)$$

if C is another symmetric $n \times n$ matrix. For more properties of $\lambda^+_{\max}(A)$ please see [2,22].

As the mth state $x_m(t)$ of Eq. (1.4) is the size of the mth component in the system, it should be non-negative. Moreover, the coefficients of Eq. (1.4) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of Eq. (1.4) may explode to infinity at a finite time [17,18,20]. It is therefore useful to establish some conditions under which the solution of Eq. (1.4) is not only positive but will also not explode to infinity at any finite time.

Theorem 2.1. *Assume that there are positive numbers $c_1(i), \dots, c_n(i)$, $i \in \mathbb{S}$ such that*

$$\lambda^+_{\max}\big(C(i)A(i) + A^T(i)C(i)\big) \leqslant 0, \tag{2.1}$$

where $C(i) = \operatorname{diag}(c_1(i), \dots, c_n(i))$. Then for any system parameters $b(\cdot)$, $\sigma(\cdot)$, and any given initial value $x_0 \in \mathbb{R}^n_+$, there is a unique solution $x(t)$ to Eq. (1.4) on $t \geqslant 0$ and the solution will remain in $\mathbb{R}^n_+$ with probability one, namely $x(t) \in \mathbb{R}^n_+$ for all $t \geqslant 0$ almost surely.

Proof. Since the coefficients of the equation are locally Lipschitz continuous, it is known (see e.g. [16, Theorem A.2]) that for any given initial value $x_0 \in \mathbb{R}^n_+$ there is a unique maximal local solution $x(t)$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To show this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $k_0 > 0$ be sufficiently large for every component of x_0 lying within the interval $[1/k_0, k_0]$. For each integer $k \geqslant k_0$, define the stopping time

$$\tau_k = \inf\big\{t \in [0, \tau_e):\ x_m(t) \notin (1/k, k) \text{ for some } m = 1, \dots, n\big\},$$

where throughout this paper we set $\inf\emptyset = \infty$ (as usual, $\emptyset =$ the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leqslant \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $x(t) \in \mathbb{R}^n_+$ a.s. for all $t \geqslant 0$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. For if this statement is false, then there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leqslant T\} > \varepsilon.$$

Hence there is an integer $k_1 \geqslant k_0$ such that

$$\mathbb{P}\{\tau_k \leqslant T\} \geqslant \varepsilon \quad \text{for all } k \geqslant k_1. \tag{2.2}$$

Define a function $V : \mathbb{R}^n_+ \times \mathbb{S} \to \mathbb{R}_+$ by

$$V(x,i) = \sum_{m=1}^{n} c_m(i)\big[x_m - 1 - \log(x_m)\big].$$

By the generalized Itô formula (see e.g. [22, p. 48] or [23, p. 104]), we have, for any $t \in [0,T]$,

$$\mathbb{E}V\big(x(t\wedge\tau_k), r(t\wedge\tau_k)\big) = V(x_0,i_0) + \mathbb{E}\int_0^{t\wedge\tau_k} LV\big(x(s),r(s)\big)\,ds. \tag{2.3}$$

Here LV is a mapping from $\mathbb{R}^n_+ \times \mathbb{S}$ to $\mathbb{R}$ defined by

$$LV(x,i) = x^T C(i)b(i) + \frac{1}{2}x^T\big[C(i)A(i) + A^T(i)C(i)\big]x - \bar{C}(i)\big(b(i) + A(i)x\big) + \frac{1}{2}\sigma^T(i)C(i)\sigma(i) + \sum_{j=1}^{N}\gamma_{ij}V(x,j), \tag{2.4}$$

where $\bar{C}(i) = (c_1(i), \dots, c_n(i))$. By condition (2.1), it is easy to see that there is a constant K_1 such that

$$LV(x,i) \leqslant K_1\big(1 + |x|\big) + \sum_{j=1}^{N}\gamma_{ij}V(x,j). \tag{2.5}$$

Let

$$K_2 = \max\left\{\frac{c_m(i)}{c_m(j)}:\ 1 \leqslant m \leqslant n \text{ and } i,j \in \mathbb{S}\right\}$$

and

$$K_3 = \min\big\{c_m(i):\ 1 \leqslant m \leqslant n,\ i \in \mathbb{S}\big\}.$$

Then, by the definition of V, for any $i, j \in \mathbb{S}$, we have

$$K_2V(x,i) = \sum_{m=1}^{n} K_2c_m(i)\big[x_m - 1 - \log(x_m)\big] \geqslant \sum_{m=1}^{n} c_m(j)\big[x_m - 1 - \log(x_m)\big] = V(x,j)$$

and

$$\begin{aligned}|x| &\leqslant \sum_{m=1}^{n} x_m \leqslant \sum_{m=1}^{n}\big[2\big(x_m - 1 - \log(x_m)\big) + 2\big] \leqslant 2n + \frac{2}{K_3}\sum_{m=1}^{n} c_m(i)\big(x_m - 1 - \log(x_m)\big) \\ &= 2n + \frac{2}{K_3}V(x,i).\end{aligned}$$

Hence there is a constant $K_4 > 0$ such that

$$LV(x,i) \leqslant K_4\big(1 + V(x,i)\big). \tag{2.6}$$

It then follows from (2.3) that

$$\begin{aligned}\mathbb{E}V\big(x(t\wedge\tau_k), r(t\wedge\tau_k)\big) &\leqslant V(x_0,i_0) + \mathbb{E}\int_0^{t\wedge\tau_k} K_4(1 + V\big(x(s),r(s)\big)\,ds \\ &\leqslant K_5 + K_4\int_0^{t} EV\big(x(s\wedge\tau_k), r(s\wedge\tau_k)\big)\,ds,\end{aligned} \tag{2.7}$$

where $K_5 = V(x_0,i_0) + K_4T$. The Gronwall inequality yields that

$$\mathbb{E}V\big(x(T\wedge\tau_k), r(T\wedge\tau_k)\big) \leqslant K_6 := K_5e^{K_4T}. \tag{2.8}$$

From here we can show that $\tau_\infty = \infty$ almost surely in the same way as in the proof of [16, Theorem 2.2]. $\square$

3. Ultimate boundedness

Theorem 2.1 shows that under condition (2.1) the solutions of Eq. (1.4) will remain in the positive cone $\mathbb{R}_+^n$. These properties of positivity and non-explosion are essential for a population system. On the other hand, due to the limit of resource, the property of ultimate boundedness is more desired. To be precise, let us now give the definition of ultimate boundedness.

Definition 3.1. Eq. (1.4) is said to be *ultimately bounded in mean* if there is a positive constant H such that for any initial value $x_0 \in \mathbb{R}_+^n$, the solution $x(t)$ of Eq. (1.4) has the property that

$$\limsup_{t\to\infty} \mathbb{E}\big|x(t)\big| \leqslant H. \tag{3.1}$$

The following theorem shows that a little bit stronger condition than (2.1) guarantees the ultimate boundedness in mean.

Theorem 3.2. *If condition* (2.1) *is strengthened by*

$$-\lambda := \max_{i\in\mathbb{S}} \lambda_{\max}^+\big(C(i)A(i) + A^T(i)C(i)\big) < 0, \tag{3.2}$$

then Eq. (1.4) *is ultimately bounded in mean.*

Proof. By Theorem 2.1, the unique solution $x(t)$ will remain in $\mathbb{R}_+^n$ for all $t \geqslant 0$ with probability 1. Define

$$V(x,t,i) = e^t \bar{C}(i)x, \quad (x,t,i) \in \mathbb{R}_+^n \times \mathbb{R}_+ \times \in \mathbb{S},$$

where $\bar{C}(i) = (c_1(i), \ldots, c_n(i))$. For each integer $k \geqslant |x_0|$, define the stopping time

$$\rho_k = \inf\big\{t \in \mathbb{R}_+\colon\ \big|x(t)\big| \geqslant k\big\}.$$

By the generalized Itô formula, we have, for any $t \geqslant 0$,

$$\mathbb{E}V\big(x(t\wedge\rho_k), t\wedge\rho_k, r(t\wedge\rho_k)\big) = V(x_0, 0, i_0) + \mathbb{E}\int_0^{t\wedge\rho_k} LV\big(x(s), s, r(s)\big)\, ds, \tag{3.3}$$

where LV is a mapping from $\mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{S}$ to $\mathbb{R}$ defined by

$$LV(x,t,i) = V(x,t,i) + e^t\big(x^T C(i)b(i) + x^T C(i)A(i)x\big) + \sum_{j=1}^N \gamma_{ij} V(x,t,j). \tag{3.4}$$

By condition (3.2),

$$x^T C(i)A(i)x = \frac{1}{2}x^T\big[C(i)A(i) + A^T(i)C(i)\big]x \leqslant -\lambda|x|^2.$$

Therefore

$$LV(x,t,i) \leqslant e^t\big(\big[\big|\bar{C}(i)\big| + \big|C(i)b(i)\big| + \gamma_i\big]|x| - \lambda|x|^2\big) \leqslant K_7 e^t,$$

where $\gamma_i = \sum_{j\neq i}\gamma_{ij}|\bar{C}(j)|$ and

$$K_7 = \max_{i\in\mathbb{S}} \frac{1}{4\lambda}\big[\big|\bar{C}(i)\big| + \big|C(i)b(i)\big| + \gamma_i\big]^2. \tag{3.5}$$

It then follows from (3.3) that

$$\mathbb{E}V\big(x(t\wedge\rho_k), t\wedge\rho_k, r(t\wedge\rho_k)\big) \leqslant V(x_0, 0, i_0) + K_7 e^t.$$

Letting $k \to \infty$ gives

$$\mathbb{E}V\big(x(t), t, r(t)\big) \leqslant V(x_0, 0, i_0) + K_7 e^t.$$

Noting that

$$\big|x(t)\big| \leqslant \frac{V(x(t), t, r(t))}{K_3 e^{\alpha t}},$$

where K_3 has been defined in the proof of Theorem 2.1, we obtain

$$\limsup_{t\to\infty} \mathbb{E}|x(t)| \leqslant \frac{K_7}{K_3},$$

which means Eq. (1.4) is ultimately bounded in mean. □

Theorem 3.3. *Under condition* (3.2), *there is a positive constant* $\bar{H}$ *such that for any initial value* $x_0 \in \mathbb{R}^n_+$, *the solution* $x(t)$ *of Eq.* (1.4) *has the property that*

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}|x(s)|^2\, ds \leqslant \bar{H}. \tag{3.6}$$

Proof. We use the same notations as in the proof of Theorem 3.2. Applying the generalized Itô formula to $\bar{C}(r(t))x(t)$, we can show in the same way as there that

$$\begin{aligned} 0 &\leqslant C(r_0)x_0 + \mathbb{E}\int_0^{t\wedge\rho_k} \left(\left[\left|C(r(s))b(r(s))\right| + \gamma_{r(s)}\right]\left|x(s)\right| - \lambda\left|x(s)\right|^2\right) ds \\ &\leqslant C(r_0)x_0 + K_8 t - \frac{\lambda}{2}\mathbb{E}\int_0^{t\wedge\rho_k} \left|x(s)\right|^2 ds, \end{aligned} \tag{3.7}$$

where

$$K_8 = \max_{i\in\mathbb{S}} \frac{1}{2\lambda}\left[\left|C(i)b(i)\right| + \gamma_i\right]^2. \tag{3.8}$$

Hence

$$\frac{\lambda}{2}\mathbb{E}\int_0^{t\wedge\rho_k} \left|x(s)\right|^2 ds \leqslant C(r_0)x_0 + K_8 t.$$

Letting $k \to \infty$ and then dividing both sides by $\lambda t/2$, letting $t \to \infty$, we obtain

$$\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left|x(s)\right|^2 ds \leqslant \frac{2K_8}{\lambda}.$$

Finally, by the well-known Fubini theorem, we get assertion (3.6) with $\bar{H} = 2K_8/\lambda$. □

Theorem 3.3 shows that the average in time of the second moment, hence the variance, of the solutions will be bounded.

4. Extinction

In this section we will discuss another important issue in the study of population dynamics, namely the problem of extinction. As pointed out in Section 1, Takeuchi et al. [26] have recently revealed a very interesting and surprising result: the telegraph may make a population system become neither permanent nor dissipative. We will in this section reveal another important fact: the white noise may make a population system become extinct with probability one. For this purpose, we impose a new assumption in this section.

Assumption 4.1. The Markov chain $r(\cdot)$ is irreducible.

This is a very reasonable assumption as it means that the system will switch from any regime to any other regime. It is known (see e.g. [1]) that the irreducibility implies that the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_N) \in \mathbb{R}^{1\times N}$ which can be determined by solving the following linear equation

$$\pi\Gamma = 0 \tag{4.1}$$

subject to

$$\sum_{i=1}^{N} \pi_i = 1 \quad \text{and} \quad \pi_i > 0, \quad \forall i \in \mathbb{S}.$$

Theorem 4.2. *Let Assumption* 4.1 *hold. Assume that there are positive numbers $c_1, \ldots, c_n$ such that*

$$\lambda^+_{\max}\big(CA(i) + A^T(i)C\big) \leqslant 0, \quad \forall i \in \mathbb{S}, \tag{4.2}$$

where $C = \mathrm{diag}(c_1, \ldots, c_n)$. Then for any initial value $x_0 \in \mathbb{R}^+_n$, the solution $x(t)$ of Eq. (1.4) *has the property that*

$$\limsup_{t\to\infty} \frac{1}{t} \log\big(|x(t)|\big) \leqslant \frac{1}{2} \sum_{i\in\mathbb{S}} \pi_i \lambda^+_{\max}\big(Q(i)\big) \quad a.s., \tag{4.3}$$

where $Q(i) = b(i)\vec{1} + \vec{1}^T b(i) - \sigma(i)\sigma^T(i)$ for $i \in \mathbb{S}$ with $\vec{1} = (1, \ldots, 1)$. In particular, if

$$\sum_{i\in\mathbb{S}} \pi_i \lambda^+_{\max}\big(Q(i)\big) < 0, \tag{4.4}$$

then the population will become extinct exponentially with probability one.

Proof. By Theorem 2.1, the solution $x(t)$ will remain in R^n_+ for all $t \in \mathbb{R}_+$ with probability one. Let $\bar{C} = (c_1, \ldots, c_n)$ and define

$$V(x) = \bar{C}x = \sum_{i=1}^{n} c_i x_i \quad \text{for } x \in R^n_+. \tag{4.5}$$

By the Itô formula, we have

$$\begin{aligned} d\big[\log\big(V(x(t))\big)\big] = {} & \frac{1}{V(x(t))} x^T(t)C\big(\big[b\big(r(t)\big) + A\big(r(t)\big)x(t)\big]\,dt + \sigma\big(r(t)\big)\,dB(t)\big) \\ & - \frac{1}{2V^2(x(t))}\big|x^T(t)C\sigma\big(r(t)\big)\big|^2\,dt. \end{aligned} \tag{4.6}$$

But, by condition (4.2),

$$x^T(t)CA\big(r(t)\big)x(t) \leqslant 0,$$

while

$$\begin{aligned} \frac{1}{V(x(t))} x^T(t)Cb\big(r(t)\big) - \frac{1}{2V^2(x(t))}\big|x^T(t)C\sigma\big(r(t)\big)\big|^2 &= \frac{1}{2V^2(x(t))}\big[2x^T(t)Cb\big(r(t)\big)Cx(t) - x^T(t)C\sigma\big(r(t)\big)\sigma^T\big(r(t)\big)Cx(t)\big] \\ &= \frac{1}{2V^2(x(t))}\big[2x^T(t)Cb\big(r(t)\big)\vec{1}Cx(t) - x^T(t)C\sigma\big(r(t)\big)\sigma^T\big(r(t)\big)Cx(t)\big] \\ &= \frac{1}{2V^2(x(t))} x^T(t)CQ\big(r(t)\big)Cx(t) \\ &\leqslant \frac{1}{2}\lambda^+_{\max}\big(Q\big(r(t)\big)\big). \end{aligned}$$

Substituting these into (4.6) yields

$$d\big[\log\big(V(x(t))\big)\big] \leqslant \frac{1}{2}\lambda^+_{\max}\big(Q\big(r(t)\big)\big)\,dt + \frac{x^T(t)C\beta}{V(x(t))}\,dB(t). \tag{4.7}$$

This implies

$$\log\big(V(x(t))\big) \leqslant \log\big(V(x_0)\big) + \frac{1}{2}\int_0^t \lambda^+_{\max}\big(Q\big(r(s)\big)\big)\,ds + M(t), \tag{4.8}$$

where $M(t)$ is a martingale defined by

$$M(t) = \int_0^t \frac{x^T(s)C\sigma(r(s))}{V(x(s))}\,dB(s).$$

It is easy to show by the strong law of large numbers for martingales (see e.g. [19, Theorem 1.3.4, p. 12]) that

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \quad \text{a.s.}$$

Dividing t on the both sides of (4.8), letting $t \to \infty$ and then applying the ergodic property of the Markov chain, we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \log\bigl(V\bigl(x(t)\bigr)\bigr) \leqslant \frac{1}{2} \lim_{t\to\infty} \frac{1}{t} \int_0^t \lambda_{\max}^{+}\bigl(Q\bigl(r(s)\bigr)\bigr)\, ds = \frac{1}{2} \sum_{i\in\mathbb{S}} \pi_i \lambda_{\max}^{+}\bigl(Q(i)\bigr) \quad \text{a.s.}$$

which yields the required assertion (4.3) immediately. □

The following corollary shows that if the noise intensities are sufficiently large, then (4.4) will hold whence the population will become extinct with probability one.

Corollary 4.3. *Under Assumption* 4.1 *and condition* (4.2)*, if moreover the noise intensities are strong enough in the sense that for each* $i \in \mathbb{S}$,

$$\sigma_j(i)\sigma_k(i) > b_j(i) + b_k(i), \quad 1 \leqslant j, k \leqslant n, \tag{4.9}$$

then the population system (1.4) *will become extinct exponentially with probability one.*

Proof. For each $i \in \mathbb{S}$, the jkth element of the matrix $Q(i)$ defined in Theorem 4.2 is

$$b_j(i) + b_k(i) - \sigma_j(i)\sigma_k(i),$$

which is negative by condition (4.9). It is then easy to show that

$$\lambda_{\max}^{+}\bigl(Q(i)\bigr) = -\min_{1\leqslant j\leqslant n}\bigl[\sigma_j^2(i) - 2b_j(i)\bigr] < 0,$$

whence (4.4) holds and the assertion follows from Theorem 4.2. □

The above corollary requires that condition (4.9) holds for every i, but this is unnecessary. The following example illustrates this point clearly.

Example 4.4. To make it simple, we classify the environment of a population system for 2 interacting species as "good" and "bad" seasons. Assume that in the good season, the environmental noise has little effect on the system so we can describe the system by a deterministic Lotka–Volterra model

$$\dot{x}(t) = \operatorname{diag}\bigl(x_1(t), x_2(t)\bigr)\bigl[b(1) + A(1)x(t)\bigr], \tag{4.10}$$

where $b(1) = (b_1(1), b_2(1))^T$ and $A(1) = (a_{jk}(1))_{2\times 2}$. Assume that the system parameters obey

$$b_1(1), b_2(1) > 0; \qquad a_{11}(1), a_{22}(1) < 0; \qquad a_{12}(1), a_{21}(1) > 0; \qquad 4a_{11}(1)a_{22}(1) \geqslant \bigl(a_{12}(1) + a_{21}(1)\bigr)^2.$$

It is therefore well known (see e.g. [10–12]) that the population system (4.10) is persistent. On the other hand, in the bad season, the environmental noise has a significant effect on the system so we describe the system by a stochastic Lotka–Volterra model

$$dx(t) = \operatorname{diag}\bigl(x_1(t), x_2(t)\bigr)\bigl(\bigl[b(2) + A(2)x(t)\bigr]dt + \sigma(2)\, dB(t)\bigr), \tag{4.11}$$

where $b(2) = (b_1(2), b_2(2))^T$, $A(2) = (a_{jk}(2))_{2\times 2}$ and $\sigma(2) = (\sigma_1(2), \sigma_2(2))^T$. Assume that the system parameters obey

$$b_1(2), b_2(2) > 0; \qquad a_{11}(2), a_{22}(2) < 0; \qquad a_{12}(2), a_{21}(2) > 0; \qquad 4a_{11}(2)a_{22}(2) \geqslant \bigl(a_{12}(2) + a_{21}(2)\bigr)^2$$

and

$$\sigma_1^2(2) > 2b_1(2); \qquad \sigma_2^2(2) > 2b_2(2); \qquad \sigma_1(2)\sigma_2(2) > b_1(2) + b_2(2).$$

Assume furthermore that the switching between two seasons is governed by a Markovian chain $r(t)$ on the state space $\mathbb{S} = \{1, 2\}$ with the generator

$$\Gamma = \begin{pmatrix} -\gamma_{12} & \gamma_{12} \\ \gamma_{21} & -\gamma_{21} \end{pmatrix},$$

where states 1 and 2 stand for the good and bad season, respectively, while $\gamma_{12} > 0$ and $\gamma_{21} > 0$. Hence the system with the Markovian switching between two seasons can be described by

$$dx(t) = \mathrm{diag}\big(x_1(t), x_2(t)\big)\big(\big[b\big(r(t)\big) + A\big(r(t)\big)x(t)\big]\,dt + \sigma\big(r(t)\big)\,dB(t)\big), \tag{4.12}$$

where we set $\sigma(0) = (0,0)^T$. It is easy to see that the Markov chain has its stationary probability distribution $\pi = (\pi_1, \pi_2)$ given by

$$\pi_1 = \frac{\gamma_{21}}{\gamma_{12} + \gamma_{21}} \quad \text{and} \quad \pi_2 = \frac{\gamma_{12}}{\gamma_{12} + \gamma_{21}}.$$

To apply Theorem 4.2, we let C be the 2×2 identity matrix. By the conditions listed above, it is easy to see

$$\lambda^+_{\max}\big(CA(i) + A^T(i)C\big) \leqslant \lambda_{\max}\big(A(i) + A^T(i)\big) \leqslant 0, \quad i = 1, 2.$$

It is also easy to show that $Q(i)$'s defined in Theorem 4.2 obey

$$\lambda^+_{\max}\big(Q(1)\big) = 2\big[b_1(1) \vee b_2(1)\big] \quad \text{and} \quad \lambda^+_{\max}\big(Q(2)\big) = -\big[\sigma_1^2(2) - 2b_1(2)\big] \wedge \big[\sigma_2^2(2) - 2b_2(2)\big].$$

Hence, by Theorem 4.2, the solution of Eq. (4.12) has the property

$$\limsup_{t\to\infty} \frac{1}{t}\log\big(\big|x(t)\big|\big) \leqslant \pi_1\big[b_1(1) \vee b_2(1)\big] - \frac{\pi_2}{2}\big[\sigma_1^2(2) - 2b_1(2)\big] \wedge \big[\sigma_2^2(2) - 2b_2(2)\big] \quad \text{a.s.}$$

We can therefore conclude that if

$$\frac{\lambda_{21}}{\lambda_{12}} < \frac{[\sigma_1^2(2) - 2b_1(2)] \wedge [\sigma_2^2(2) - 2b_2(2)]}{2[b_1(1) \vee b_2(1)]},$$

then the population system (4.12) will become extinct with probability one.

5. Small noise

We have shown in the previous section that a strong environmental noise could make the population system become extinct. The interesting question is: what happens if the noise is not so strong? In this section we will discuss the asymptotic properties of the solution of Eq. (1.4) when the noise is relatively small. Let us begin with the following theorem.

Theorem 5.1. *Assume that condition* (4.2) *holds. Assume also that for each $i \in \mathbb{S}$,*

$$\hat{b}(i) := \min_{1\leqslant m\leqslant n} b_m(i) > \frac{1}{2}\max_{1\leqslant m\leqslant n} \sigma_m^2(i) := \frac{1}{2}\check{\sigma}^2(i). \tag{5.1}$$

Then for any given initial value $x(0) \in \mathbb{R}^n_+$, the solution $x(t)$ of Eq. (1.4) *has the property that*

$$\liminf_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \geqslant -\max_{i\in\mathbb{S}}\left(\frac{\check{\sigma}(i)^2}{2\hat{b}(i) - \check{\sigma}^2(i)}\right) \quad \text{a.s.} \tag{5.2}$$

Proof. The proof is very technical so we divide it into three steps.

Step 1. By Theorem 2.1, the solution $x(t)$ will remain in $\mathbb{R}^n_+$ for all $t \geqslant 0$ with probability one. Let $V(x)$ be the same as defined by (4.5). Define

$$y(t) = \frac{1}{V(x(t))} \quad \text{and} \quad z(t) = 1 + y(t) \quad \text{on } t \geqslant 0.$$

We claim that for any θ obeying

$$0 < \theta < \min_{i\in\mathbb{S}}\left(\frac{2\hat{b}(i)}{\check{\sigma}^2(i)} - 1\right), \tag{5.3}$$

there is a positive constant $K = K(\theta)$ such that

$$\limsup_{t\to\infty} \mathbb{E}\big[z^\theta(t)\big] \leqslant K. \tag{5.4}$$

To show this, we compute, by the Itô formula,

$$dV\big(x(t)\big) = x^T C\big[b\big(r(t)\big) + A\big(r(t)\big)x(t)\big]\,dt + x^T(t)C\sigma\big(r(t)\big)\,dB(t); \tag{5.5}$$

and then

$$\begin{aligned} dy(t) &= -y^2(t)dV\big(x(t)\big) + y^3(t)\big[x^T(t)C\sigma\big(r(t)\big)\big]^2\,dt \\ &= \big(-y^2(t)x^TC\big[b\big(r(t)\big) + A\big(r(t)\big)x(t)\big] + y^3(t)\big[x^T(t)C\sigma\big(r(t)\big)\big]^2\big)\,dt - y^2(t)x^T(t)C\sigma\big(r(t)\big)\,dB(t); \end{aligned} \tag{5.6}$$

and furthermore

$$\begin{aligned} dz^\theta(t) &= \theta z^{\theta-1}(t)\,dy(t) - \frac{1}{2}\theta(1-\theta)z^{\theta-2}(t)y^4(t)\big[x^T(t)C\sigma\big(r(t)\big)\big]^2\,dt \\ &= \theta z^{\theta-2}(t)\bigg(z(t)\big\{-y^2(t)x^TC\big[b\big(r(t)\big) + A\big(r(t)\big)x(t)\big] + y^3(t)\big[x^T(t)C\sigma\big(r(t)\big)\big]^2\big\} \\ &\quad - \frac{1}{2}(1-\theta)y^4(t)\big[x^T(t)C\sigma\big(r(t)\big)\big]^2\bigg)\,dt - \theta z^{\theta-1}(t)y^2(t)x^T(t)C\sigma\big(r(t)\big)\,dB(t). \end{aligned} \tag{5.7}$$

Dropping t from $x(t)$, etc. we compute that

$$\begin{aligned} & z\big\{-y^2x^TC\big[b(r) + A(r)x\big] + y^3\big[x^TC\sigma(r)\big]^2\big\} - \frac{1}{2}(1-\theta)y^4\big[x^TC\sigma(r)\big]^2 \\ &\quad = -y^2x^TCb(r) - y^2x^TCA(r)x - y^3x^TCb(r) - y^3x^TCA(r)x + y^3\big[x^TC\sigma(r)\big]^2 + \frac{1}{2}(1+\theta)y^4\big[x^TC\sigma(r)\big]^2 \\ &\quad \leqslant -\frac{x^TCA(r)x}{V^2(x)} + \frac{(x^TC\sigma(r))^2 - x^TCA(r)x}{V^2(x)}y - \bigg(\frac{x^TCb(r)}{V(x)} - \frac{1}{2}(1+\theta)\frac{(x^TC\sigma(r))^2}{V^2(x)}\bigg)y^2. \end{aligned}$$

It is easy to see that for all $(x,r) \in \mathbb{R}^n_+ \times \mathbb{S}$,

$$-\frac{x^TCA(r)x}{V^2(x)} \leqslant K_1 \quad \text{and} \quad \frac{x^TCA(r)x + (x^TC\sigma(r))^2}{V^2(x)} \leqslant K_1,$$

where K_1 is a positive constant, while

$$\frac{x^TCb(r)}{V(x)} \geqslant \hat{b}(r) \quad \text{and} \quad \frac{(x^TC\sigma(r))^2}{V^2(x)} \leqslant \check{\sigma}^2(r).$$

Hence

$$z\big\{-y^2x^TC\big[b(r) + A(r)x\big] + y^3\big[x^TC\sigma(r)\big]^2\big\} - \frac{1}{2}(1-\theta)y^4\big[x^TC\sigma(r)\big]^2 \leqslant K_1(1+y) - \bigg[\hat{b}(r) - \frac{1}{2}(1+\theta)\check{\sigma}^2(r)\bigg]y^2.$$

Substituting this into (5.7) yields

$$\begin{aligned} dz^\theta(t) &\leqslant \theta z^{\theta-2}(t)\bigg(K_1\big(1+y(t)\big) - \bigg[\hat{b}\big(r(t)\big) - \frac{1}{2}(1+\theta)\check{\sigma}^2\big(r(t)\big)\bigg]y^2(t)\bigg)\,dt \\ &\quad - \theta z^{\theta-1}(t)y^2(t)x^T(t)C\sigma\big(r(t)\big)\,dB(t). \end{aligned} \tag{5.8}$$

Now, choose $\varepsilon > 0$ sufficiently small for

$$\frac{\varepsilon}{\theta} < \min_{i\in\mathbb{S}}\bigg[\hat{b}(i) - \frac{1}{2}(1+\theta)\check{\sigma}^2(i)\bigg]. \tag{5.9}$$

Then, by the Itô formula,

$$\begin{aligned} d\big[e^{\varepsilon t}z^\theta(t)\big] &= e^{\varepsilon t}\big[\varepsilon z^\theta(t)\,dt + dz^\theta(t)\big] \leqslant \theta\varepsilon^{\varepsilon t}z^{\theta-2}(t)\bigg(\frac{\varepsilon}{\theta}\big(1+y(t)\big)^2 + K_1\big(1+y(t)\big) - \bigg[\hat{b} - \frac{1}{2}(1+\theta)\check{\sigma}^2\bigg]y^2(t)\bigg)\,dt \\ &\quad - \varepsilon^{\varepsilon t}\theta z^{\theta-1}(t)y^2(t)x^T(t)C\sigma\big(r(t)\big)\,dB(t). \end{aligned}$$

It is easy to see that there is a constant K_2 such that

$$\theta(1+y)^{\theta-2}\bigg(\frac{\varepsilon}{\theta}(1+y)^2 + K_1(1+y) - \bigg[\hat{b}(r) - \frac{1}{2}(1+\theta)\check{\sigma}^2(r)\bigg]y^2\bigg) \leqslant K_2 \tag{5.10}$$

for all $(y,r) \in \mathbb{R}_+ \times \mathbb{S}$. Thus

$$d\big[e^{\varepsilon t}z^\theta(t)\big] \leqslant K_2\varepsilon^{\varepsilon t}\,dt - \varepsilon^{\varepsilon t}\theta z^{\theta-1}(t)y^2(t)x^T(t)C\sigma\big(r(t)\big)\,dB(t).$$

This implies

$$\mathbb{E}\big[e^{\varepsilon t}z^\theta(t)\big] \leqslant z^\theta(0) + \frac{K_2}{\varepsilon}\varepsilon^{\varepsilon t},$$

and (5.4) follows by setting $K = K_2/\varepsilon$.

Step 2. Using (5.10), we observe from (5.8) that

$$dz^\theta(t) \leqslant K_2\,dt - \theta z^{\theta-1}(t)y^2(t)x^T(t)C\sigma\big(r(t)\big)\,dB(t).$$

This implies that

$$\mathbb{E}\Big(\sup_{t\leqslant u\leqslant t+1} z^\theta(u)\Big) \leqslant \mathbb{E}\big[z^\theta(t)\big] + K_2 + \mathbb{E}\left(\sup_{t\leqslant u\leqslant t+1}\left|\int_t^u \theta z^{\theta-1}(s)y^2(s)x^T(s)C\sigma\big(r(s)\big)\,dB(s)\right|\right). \tag{5.11}$$

But, by the well-known Burkholder–Davis–Gundy inequality (see e.g. [22, p. 76]), we compute

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\leqslant u\leqslant t+1}\left|\int_t^u \theta z^{\theta-1}(r)y^2(s)x^T(s)C\sigma\big(r(s)\big)\,dB(s)\right|\right) &\leqslant 3\theta\mathbb{E}\left(\int_t^{t+1} z^{2\theta-2}(s)y^4(s)x^T(s)C\sigma\big(r(s)\big)^2\,ds\right)^{\frac{1}{2}} \\
&\leqslant 3\theta\mathbb{E}\left(\int_t^{t+1} z^{2\theta}(s)\frac{x^T(s)C\sigma\big(r(s)\big)^2}{V^2(x(s))}\,ds\right)^{\frac{1}{2}} \\
&\leqslant 3\theta\check{\sigma}\,\mathbb{E}\left(\Big[\sup_{t\leqslant s\leqslant t+1} z^\theta(s)\Big]\int_t^{t+1} z^\theta(s)\,ds\right)^{\frac{1}{2}} \\
&\leqslant \frac{1}{2}\mathbb{E}\Big[\sup_{t\leqslant s\leqslant t+1} z^\theta(s)\Big] + \frac{9\theta^2\check{\sigma}^2}{2}\mathbb{E}\int_t^{t+1} z^\theta(s)\,ds,
\end{aligned}$$

where $\hat{\sigma}^2 = \max_{i\in\mathbb{S}}\check{\sigma}^2(i)$. Substituting this into (5.11) gives

$$\mathbb{E}\Big(\sup_{t\leqslant u\leqslant t+1} z^\theta(u)\Big) \leqslant 2\mathbb{E}\big[z^\theta(t)\big] + 2K_2 + 9\theta^2\check{\sigma}^2\int_t^{t+1}\mathbb{E}z^\theta(s)\,ds.$$

Letting $t\to\infty$ and using (5.4) we obtain that

$$\limsup_{t\to\infty}\mathbb{E}\Big(\sup_{t\leqslant u\leqslant t+1} z^\theta(u)\Big) \leqslant 2K_2 + K\big(2+9\theta^2\check{\sigma}^2\big) := K_3. \tag{5.12}$$

Step 3. We observe from (5.12) that there is a positive constant K_4 such that

$$\mathbb{E}\Big(\sup_{k\leqslant t\leqslant k+1} z^\theta(t)\Big) \leqslant K_4, \quad k=1,2,\ldots.$$

Let $\bar{\varepsilon}>0$ be arbitrary. Then, by the well-known Chebyshev inequality, we have

$$\mathbb{P}\Big\{\sup_{k\leqslant t\leqslant k+1} z^\theta(t) > k^{1+\bar{\varepsilon}}\Big\} \leqslant \frac{K_4}{k^{1+\bar{\varepsilon}}}, \quad k=1,2,\ldots.$$

Applying the well-known Borel–Cantelli lemma (see e.g. [19]), we obtain that for almost all $\omega\in\Omega$,

$$\sup_{k\leqslant t\leqslant k+1} z^\theta(t) \leqslant k^{1+\bar{\varepsilon}} \tag{5.13}$$

holds for all but finitely many k. Hence, there exists a $k_0(\omega)$, for almost all $\omega\in\Omega$, for which (5.13) holds whenever $k\geqslant k_0$. Consequently, for almost all $\omega\in\Omega$, if $k\geqslant k_0$ and $k\leqslant t\leqslant k+1$,

$$\frac{\log(z^\theta(t))}{\log t} \leqslant \frac{(1+\bar{\varepsilon})\log k}{\log k} = 1+\bar{\varepsilon}.$$

Therefore

$$\limsup_{t\to\infty}\frac{\log(z(t))}{\log t} \leqslant \frac{1+\bar{\varepsilon}}{\theta} \quad \text{a.s.}$$

Letting $\bar{\varepsilon}\to 0$ and recalling that $z(t)=1+y(t)$ we obtain

$$\limsup_{t\to\infty}\frac{\log(y(t))}{\log t} \leqslant \frac{1}{\theta} \quad \text{a.s.}$$

This implies, by recalling that $y(t) = 1/V(x(t))$,

$$\limsup_{t\to\infty} \frac{\log(V(x(t)))}{\log t} \geqslant -\frac{1}{\theta} \quad \text{a.s.}$$

Since $V(x(t)) \leqslant |\bar{C}||x(t)|$, we then have

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \geqslant -\frac{1}{\theta} \quad \text{a.s.}$$

But this holds for any θ that obeys (5.3), we must therefore have the assertion (5.2). $\square$

If we set the right-hand side term of (5.2) to be $\hat{\theta}$, Theorem 5.1 shows that the solution will not decay faster than $t^{-(\hat{\theta}+\varepsilon)}$ asymptotically for any $\varepsilon > 0$, with probability one. In the following theorem, we will show that the solution will not grow faster than $t^{1+\varepsilon}$.

Theorem 5.2. *Under condition* (3.2)*, the solution $x(t)$ of Eq.* (1.4) *obeys*

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \leqslant 1 \quad \textit{a.s.} \tag{5.14}$$

for any initial value $x(0) \in \mathbb{R}^n_+$.

Proof. By Theorem 3.2, the solution obeys (3.1). Define

$$V(x,i) = \bar{C}(i)x \quad \text{for } x \in \mathbb{R}^n_+ \times \mathbb{S},$$

where $\bar{C}(i) = (C_1(i), \dots, C_n(i))$. By the generalized Itô formula, we have

$$dV\big(x(t), r(t)\big) = LV\big(x(t), r(t)\big)\,dt + x^T(t)C\big(r(t)\big)\sigma\big(r(t)\big)\,dB(t),$$

where $LV : \mathbb{R}^n_+ \times \mathbb{S} \to \mathbb{R}$ is defined by

$$LV(x,i) = x^T C(i)\big[b(i) + A(i)x\big] + \sum_{j\in\mathbb{S}} \gamma_{ij} V(x,j).$$

By condition (3.2), it is easy to show that there is a constant $H_1 > 0$ such that

$$LV(x,i) \leqslant H_1, \quad \forall (x,i) \in \mathbb{R}^n_+ \times \mathbb{S}.$$

Hence

$$dV\big(x(t), r(t)\big) \leqslant H_1\,dt + x^T(t)C\big(r(t)\big)\sigma\big(r(t)\big)\,dB(t). \tag{5.15}$$

From here, we can show, in the same way as in Step 2 of the proof of Theorem 5.1, that there is a constant $H_2 > 0$ such that

$$\mathbb{E}\Big(\sup_{k\leqslant t\leqslant k+1} \big|x(t)\big|\Big) \leqslant H_2, \quad k = 1, 2, \dots, \tag{5.16}$$

and then show the required assertion (5.14) in the same way as in Step 3 of the proof of Theorem 5.1. $\square$

We should point out that Theorem 5.2 holds for arbitrary growth rates $b_m(i)$ and noise intensities $\sigma_m(i)$. In other words, under condition (3.2), the population will not grow faster than $t^{1+\varepsilon}$ asymptotically for any $\varepsilon > 0$ with probability one no matter the growth rates and noise intensities are small and large.

Theorem 5.3. *Let Assumption* 4.1 *and condition* (5.1) *hold. Assume that there are positive numbers $c_1, \dots, c_n$ such that*

$$-\lambda := \max_{i\in\mathbb{S}} \lambda^+_{\max}\big(CA(i) + A^T(i)C\big) < 0, \tag{5.17}$$

where $C = \operatorname{diag}(c_1, \dots, c_n)$. Then for any given initial value $x(0) \in \mathbb{R}^n_+$, the solution $x(t)$ of Eq. (1.4) *obeys*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \big|x(u)\big|\,du \leqslant \frac{2|\bar{C}|}{\lambda} \sum_{i\in\mathbb{S}} \pi_i \bigg[\check{b}(i) - \frac{1}{2}\hat{\sigma}^2(i)\bigg] \quad \textit{a.s.} \tag{5.18}$$

and

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t |x(u)|\,du \geqslant \frac{2\hat{c}}{\check{\lambda}}\sum_{i\in\mathbb{S}}\pi_i\left[\hat{b}(i)-\frac{1}{2}\check{\sigma}^2(i)\right]>0 \quad a.s., \tag{5.19}$$

where $\bar{C}=(c_1,\dots,c_n)$, $\check{b}(i)=\max_{1\leqslant m\leqslant n} b_m(i)$, $\hat{\sigma}^2(i)=\min_{1\leqslant m\leqslant n}\sigma_m^2(i)$, $\hat{c}=\min_{1\leqslant m\leqslant n} c_m$ *and*

$$\check{\lambda}=\max_{i\in\mathbb{S}}\left[\lambda^+_{\max}\left(-CA(i)-A^T(i)C\right)\right]\geqslant\lambda>0.$$

Proof. Let $V(x)=\bar{C}x$ for $x\in\mathbb{R}^n_+$. It is easy to observe from Theorems 5.1 and 5.2 that

$$\lim_{t\to\infty}\frac{1}{t}\log\left(V\left(x(t)\right)\right)=0 \quad \text{a.s.} \tag{5.20}$$

By the Itô formula, we can derive from (5.5) that

$$d\left[\log\left(V\left(x(t)\right)\right)\right]=\left(\frac{x^T(t)Cb(r(t))}{V(x(t))}+\frac{x^T(t)CA(r(t))x(t)}{V(x(t))}-\frac{(x^T(t)C\sigma(r(t)))^2}{2V^2(x(t))}\right)dt+\frac{x^T(t)C\sigma(r(t))}{V(x(t))}\,dB(t). \tag{5.21}$$

By condition (5.17) and the definitions of $\check{b}(i)$ and $\hat{\sigma}^2(i)$, we see that

$$d\left[\log\left(V\left(x(t)\right)\right)\right]\leqslant\left(\check{b}\left(r(t)\right)-\frac{\lambda}{2|\bar{C}|}\left|x(t)\right|-\frac{1}{2}\hat{\sigma}^2\left(r(t)\right)\right)dt+\frac{x^T(t)C\sigma(r(t))}{V(x(t))}\,dB(t). \tag{5.22}$$

Hence

$$\log\left(V\left(x(t)\right)\right)+\frac{\lambda}{2|\bar{C}|}\int_0^t\left|x(u)\right|du\leqslant\log\left(V\left(x(0)\right)\right)+\int_0^t\left(\check{b}\left(r(s)\right)-\frac{1}{2}\hat{\sigma}^2\left(r(s)\right)\right)ds+\int_0^t\frac{x^T(s)C\sigma(r(s))}{V(x(s))}\,dB(s). \tag{5.23}$$

However, it is straightforward to show by the strong law of large numbers of martingales (see e.g. [19, Theorem 5.4, p. 12]) that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{x^T(s)C\sigma(r(s))}{V(x(s))}\,dB(s)=0 \quad \text{a.s.}$$

We can therefore divide both sides of (5.23) by t and then let $t\to\infty$ to obtain

$$\frac{\lambda}{2|\bar{C}|}\left(\limsup_{t\to\infty}\frac{1}{t}\int_0^t\left|x(u)\right|du\right)\leqslant\lim_{t\to\infty}\frac{1}{t}\int_0^t\left(\check{b}\left(r(s)\right)-\frac{1}{2}\hat{\sigma}^2\left(r(s)\right)\right)ds=\sum_{i\in\mathbb{S}}\pi_i\left[\check{b}(i)-\frac{1}{2}\hat{\sigma}^2(i)\right] \quad \text{a.s.}$$

which implies the required assertion (5.18).

To prove another assertion, we observe from (5.21) that

$$d\left[\log\left(V\left(x(t)\right)\right)\right]\geqslant\left(\hat{b}\left(r(t)\right)-\frac{\check{\lambda}|x(t)|^2}{2V(x(t))}-\frac{1}{2}\check{\sigma}^2\left(r(t)\right)\right)dt+\frac{x^T(t)C\sigma(r(t))}{V(x(t))}\,dB(t). \tag{5.24}$$

Note that

$$\lambda^+_{\max}\left(-CA(i)-A^T(i)C\right)\geqslant-\lambda^+_{\max}\left(CA(i)+A^T(i)C\right)$$

for every $i\in\mathbb{S}$, whence

$$\hat{\lambda}\geqslant\max_{i\in\mathbb{S}}\left[-\lambda^+_{\max}\left(CA(i)+A^T(i)C\right)\right]=-\min_{i\in\mathbb{S}}\lambda^+_{\max}\left(CA(i)+A^T(i)C\right)\geqslant\lambda>0.$$

It is also easy to see that $V(x(t))\geqslant\hat{c}|x(t)|$. It then follows from (5.24) that

$$\log\left(V\left(x(t)\right)\right)+\frac{\hat{\lambda}}{2\hat{c}}\int_0^t\left|x(u)\right|du\geqslant\log\left(V\left(x(0)\right)\right)+\int_0^t\left(\hat{b}\left(r(s)\right)-\frac{1}{2}\check{\sigma}^2\left(r(s)\right)\right)ds+\int_0^t\frac{x^T(s)C\sigma(r(s))}{V(x(s))}\,dB(s). \tag{5.25}$$

Dividing both sides by t and then letting $t\to\infty$ gives the another assertion (5.19). □

This theorem shows that the average in time of the norm of the solution of Eq. (1.4) is asymptotically bounded by a pair of positive constants with probability one.

6. Case studies

Let us now discuss two important cases for illustrations.

6.1. A single species population system

Let us consider a single species population system under regime switching described by the following SDE

$$dx(t) = x(t)\big(\big[b\big(r(t)\big) - a\big(r(t)\big)x(t)\big]\,dt + \sigma\big(r(t)\big)\,dB(t)\big), \tag{6.1}$$

where $b(i), a(i)$ and $\sigma(i)$ are now all positive numbers for $i \in \mathbb{S}$. Assume that the Markov chain $r(\cdot)$ obeys Assumption 4.1. Given an initial value $x(0) > 0$, the solution will remain positive for all $t \geqslant 0$. Set $y(t) = 1/x(t)$. Then

$$dy(t) = \big[a\big(r(t)\big) + \big[-b\big(r(t)\big) + \sigma^2\big(r(t)\big)\big]y(t)\big]\,dt - \sigma\big(r(t)\big)y(t)\,dB(t).$$

By the variation-of-constants formula (see e.g. [19, p. 96]), we can show that this equation has its explicit solution

$$y(t) = G(t)y(0) + \int_0^t a\big(r(s)\big)G(t-s)\,ds,$$

where

$$G(t) = \exp\left(\int_0^t \left[-b\big(r(s)\big) + \frac{1}{2}\sigma^2\big(r(s)\big)\right]ds - \int_0^t \sigma\big(r(s)\big)\,dB(s)\right).$$

Hence, Eq. (6.1) has the explicit solution

$$x(t) = \frac{1}{G(t)/x(0) + \int_0^t a(r(s))G(t-s)\,ds}. \tag{6.2}$$

By Theorem 5.2, the solution of Eq. (6.1) obeys

$$\limsup_{t\to\infty} \frac{\log(x(t))}{\log t} \leqslant 1 \quad \text{a.s.} \tag{6.3}$$

Let us furthermore assume that $b(i) > \frac{1}{2}\sigma^2(i)$ for all $i \in \mathbb{S}$. Then, by Theorem 5.1, the solution of Eq. (6.1) also obeys

$$\liminf_{t\to\infty} \frac{\log(x(t))}{\log t} \geqslant -\max_{i\in\mathbb{S}}\left(\frac{\sigma^2(i)}{2b(i) - \sigma^2(i)}\right) \quad \text{a.s.} \tag{6.4}$$

Consequently,

$$\lim_{t\to\infty} \frac{1}{t}\log\big(x(t)\big) = 0 \quad \text{a.s.} \tag{6.5}$$

By the Itô formula, it is easy to show that

$$\log\big(x(t)\big) = \log\big(x(0)\big) + \int_0^t \left(b\big(r(s)\big) - \frac{1}{2}\sigma^2\big(r(s)\big)\right)ds - \int_0^t a\big(r(s)\big)x(s)\,ds + \int_0^t \sigma\big(r(s)\big)\,dB(s).$$

Dividing both sides by t and then letting $t \to \infty$ we obtain

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t a\big(r(s)\big)x(s)\,ds = \sum_{i\in\mathbb{S}} \pi_i\left[b(i) - \frac{1}{2}\sigma^2(i)\right] \quad \text{a.s.} \tag{6.6}$$

This implies

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,ds \leqslant \frac{\sum_{i\in\mathbb{S}} \pi_i[b(i) - \frac{1}{2}\sigma^2(i)]}{\min_{i\in\mathbb{S}} a(i)} \quad \text{a.s.} \tag{6.7}$$

while

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,ds \geqslant \frac{\sum_{i\in\mathbb{S}} \pi_i[b(i)-\frac{1}{2}\sigma^2(i)]}{\max_{i\in\mathbb{S}} a(i)} \quad \text{a.s.} \tag{6.8}$$

In particular, if $a(i) \equiv a > 0$ then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,ds = \frac{1}{a}\sum_{i\in\mathbb{S}} \pi_i\left[b(i) - \frac{1}{2}\sigma^2(i)\right] \quad \text{a.s.} \tag{6.9}$$

6.2. Multi-dimensional system of facultative mutualism

Let us now return to Eq. (1.4) but impose the following condition:

$$-A(i) \text{ is a non-singular } M\text{-matrix for every } i \in \mathbb{S}. \tag{6.10}$$

By the definition of M-matrices (see e.g. [3,22]), we observe that $A(i) = (a_{ik})i))_{n\times n}$ has negative diagonal elements and non-negative off-diagonal elements, that is

$$a_{jj}(i) < 0, \qquad a_{jk}(i) \geqslant 0, \quad 1 \leqslant j,k \leqslant n,\ \ j \neq k. \tag{6.11}$$

In such a system, each species enhances the growth of the other, as the parameters $a_{jk}(i) \geqslant 0$ ($j \neq k$). This type of ecological interaction is known as facultative mutualism (see e.g. [11,21]).

Non-singular M-matrices have many very nice properties (see e.g. [3,22]). In particular, (6.11) is equivalent to the following statement:

- For each $i \in \mathbb{S}$, there is a positive-definite diagonal matrix $C(i) = \operatorname{diag}(c_1(i), \ldots, c_n(i))$ such that $C(i)A + A^T C(i)$ is a negative-definite matrix.

This implies that

$$\lambda^+_{\max}\big(C(i)A + A^T C(i)\big) \leqslant \lambda_{\max}\big(C(i)A + A^T C(i)\big) < 0.$$

By Theorems 2.1 and 5.2, we observe that, under condition (6.10), Eq. (1.4) has a unique solution $x(t) \in \mathbb{R}^n_+$ on $t \in \mathbb{R}_+$ which obeys

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \leqslant 1 \quad \text{a.s.} \tag{6.12}$$

Moreover, we note that, for each $1 \leqslant m \leqslant n$,

$$dx_m(t) = x_m(t)\left[\left(b_m\big(r(t)\big) + \sum_{j=1}^n a_{mj}\big(r(t)\big)x_j(t)\right)dt + \sigma_m\big(r(t)\big)\,dB(t)\right].$$

Introduce the corresponding stochastic differential equation

$$d\xi_m(t) = \xi_m(t)\big[\big(b_m\big(r(t)\big) + a_{mm}\big(r(t)\big)\xi_m(t)\big)\,dt + \sigma_m\big(r(t)\big)\,dB(t)\big]$$

with initial value $\xi_m(0) = x_m(0)$. By the classical comparison theorem (see e.g. [13]) we have $x_m(t) \geqslant \xi_m(t)$ a.s. for all $t \geqslant 0$. If we further assume that

$$b_m(i) > \frac{1}{2}\sigma_m^2(i), \quad 1 \leqslant m \leqslant n,\ \ i \in \mathbb{S}, \tag{6.13}$$

then, by Theorem 5.1, we have

$$\liminf_{t\to\infty} \frac{\log(x_m(t))}{\log t} \geqslant \liminf_{t\to\infty} \frac{\log(\xi_m(t))}{\log t} \geqslant -\max_{i\in\mathbb{S}}\left(\frac{\sigma_m^2(i)}{2b_m(i) - \sigma_m^2(i)}\right) \quad \text{a.s.} \tag{6.14}$$

Combining the results above we can conclude that, under conditions (6.10) and (6.13), the solution of Eq. (1.4) obeys

$$\lim_{t\to\infty} \frac{1}{t}\log\big(x_m(t)\big) = 0 \quad \text{a.s.},\ \ 1 \leqslant m \leqslant n. \tag{6.15}$$

Define $\eta_m(t) = \log(x_m(t))$ $(1 \leqslant m \leqslant n)$ and $\eta(t) = (\eta_1(t), \ldots, \eta_n(t))^T$. It is easy to show that

$$d\eta(t) = \big(\zeta\big(r(t)\big) + A\big(r(t)\big)x(t)\big)\,dt + \sigma\big(r(t)\big)\,dB(t), \tag{6.16}$$

where $\zeta(i) = (\zeta_1(i), \ldots, \zeta_n(i))^T$ with $\zeta_m(i) = b_m(i) - \frac{1}{2}\sigma_m^2(i)$. Hence

$$\frac{1}{t}\big(\eta(t) - \eta(0)\big) = \frac{1}{t}\int_0^t \zeta\big(r(s)\big)\,ds + \frac{1}{t}\int_0^t A\big(r(s)\big)x(s)\,ds + \frac{1}{t}\int_0^t \sigma\big(r(s)\big)\,dB(s).$$

Letting $t \to \infty$, by (6.15) and the large number theory of martingales, the left-hand side term and the last term on the right-hand side tend to zero while, by the ergodic theory, the first term on the right-hand side tends to $\sum_{i\in\mathbb{S}} \pi_i \zeta(i)$, whence

$$\sum_{i\in\mathbb{S}} \pi_i \zeta(i) + \lim_{t\to\infty} \frac{1}{t}\int_0^t A\big(r(s)\big)x(s)\,ds = 0 \quad \text{a.s.} \tag{6.17}$$

In particular, if $A(i) \equiv A$ with $-A$ being a non-singular M-matrix, then the above equality becomes

$$\sum_{i\in\mathbb{S}} \pi_i \zeta(i) + A\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,ds\right) = 0 \quad \text{a.s.}$$

which implies

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,ds = (-A)^{-1}\sum_{i\in\mathbb{S}} \pi_i \zeta(i) \quad \text{a.s.} \tag{6.18}$$

It should be pointed out that by condition (6.13), $\sum_{i\in\mathbb{S}} \pi_i \zeta(i) \in \mathbb{R}_+^n$; whence, by the theory of M-Matrices (see e.g. [3,22]), $(-A)^{-1}\sum_{i\in\mathbb{S}} \pi_i \zeta(i) \in \mathbb{R}_+^n$. We form the above result as a theorem to conclude this section.

Theorem 6.1. *Let Assumption* 4.1 *and condition* (6.13) *hold. Assume also that $A(i) \equiv A$ for all $i \in \mathbb{S}$ with $-A$ being a non-singular M-matrix. Then for any initial value $x(0) \in \mathbb{R}_+^n$, the solution of Eq.* (1.4) *obeys* (6.18)*.*

7. Conclusions

As in our previous paper [16], we consider a population system under both telegraph and white noise. The white noise considered here is independent of the population size so is different from that considered in [16]. We show that this type of white noise has some significant effects on the population system. In particular, we reveal that a large white noise will force the population to become extinct while the population may be persistent under a relatively small white noise.

Acknowledgments

The authors also wish to thank the financial supports from the National Natural Science Foundation of China (No. 60874110) as well as the EPSRC (UK), the London Mathematical Society and the Edinburgh Mathematical Society.

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