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High-order P-stable multistep methods *

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Abstract: Recently numerical integration of the special IVP $y'' = f(t, y)$ whose solutions are of periodic type have been considered of great interest. P-stability is the appropriate requirement for numerical methods integrating periodic problems. As established by Lambert and Watson (1976) a P-stable linear multistep method has order no greater than two. In this paper, two-step multi-stage linear methods are proposed for numerical integration of periodic problems, and P-stable methods of this type (obtained in a simple form) up to order eight are shown. Finally, those methods are successfully tested.

Keywords: Linear multistep methods, P-stability.

1. Introduction

In this paper, we are interested in the numerical integration of the second-order special initial-value problem

$$\begin{aligned} &y'' = f(t, y), \\ &y(0) = y_0, \quad y'(0) = y'_0, \end{aligned} \tag{1.1}$$

whose solution is a priori known to be periodic. Many problems of this type appear in orbital mechanics; they have as a common feature the fact that, usually, there is only interest in obtaining the values of the dependent variable $y(t)$, forgetting the values of the derivative $y'(t)$. Generally, the most effective way to solve this problem consists in using an initial or starting method and after that, integrating the problem. This is done by means of a direct integration multistep method, which gives the values of $y(t_n)$ in a point net, forgetting the values of y'.

Methods of this type are the classical Störmer–Cowell formulae; but, it has been observed in practice [5] that, when more than two steps are used, the numerical solution spirals inwards. Stiefel and Bettis [6] refer to this phenomenon as *orbital unstability*. When problems like (1.1), with periodic solution, are integrated numerically, it is desirable that the numerical solution is also periodic, with similar period as the analytic one. An appropriate requirement for numerical methods which integrate periodic problems like (1.1) is *P-stability* in the sense given by Lambert

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and Watson [5]. Besides, according to these authors, the maximum order reached by a P-stable multistep method is two. To investigate P-stability properties of methods employed to solve problems of the type (1.1), the scalar test equation

$$y'' + \lambda^2 y = 0 \tag{1.2}$$

is used.

When direct integration methods are applied to the test equation (1.2), the next difference equation is obtained:

$$q_k(z)y_{n+k} + q_{k-1}(z)y_{n+k-1} + \cdots + q_0(z)y_n = 0, \quad z = \mathrm{i}\lambda h, \tag{1.3}$$

where $q_0, q_1, \ldots, q_k$ are polynomials in z, h is the integration step and y_n is the numerical solution approximated to $y(t_n)$ $(n = 0, 1, 2, \ldots)$. The general solution to the difference equation (1.3) is

$$y_n = A_1\zeta_1^n + A_2\zeta_2^n + \cdots + A_k\zeta_k^n,$$

where A_j $(j = 0, 1, \ldots, k)$ are constants which can be determined from the initial conditions, and ζ_j $(j = 0, 1, \ldots, k)$ are the roots of the stability polynomial given by:

$$\Pi(\zeta, z) = q_k(z)\zeta^k + q_{k-1}(z)\zeta^{k-1} + \cdots + q_0(z). \tag{1.4}$$

From now on, we will denote by ζ_1 and ζ_2 the principal roots of this stability polynomial.

A numerical method to solve the problem (1.1) is said to have a *periodicity interval* $(0, H_0)$ (see [5]) if, for all $H \in (0, H_0)$ the roots of the stability polynomial (1.4) satisfy

$$\zeta_1 = \mathrm{e}^{\mathrm{i}\theta(H)}, \qquad \zeta_2 = \mathrm{e}^{-\mathrm{i}\theta(H)} \quad \text{and} \quad |\zeta_j| \leqslant 1, \ j = 3, 4, \ldots, k, \tag{1.5}$$

where θ is a real function of $H = \lambda h$. Likewise, the method is said to be *P-stable* if its periodicity interval is $(0, \infty)$.

2. Construction of P-stable methods

In this section, we study the construction of high-order P-stable multistep methods, i.e., P-stable methods whose order exceeds the bound given by Lambert and Watson [5], mentioned in Section 1. From now on, we will suppose that (1.4) satisfies the following beginning hypothesis:

(i) $q_k(z)\zeta^k + q_{k-1}(z)\zeta^{k-1} + \cdots + q_0(z)$ is irreducible;

(ii) $k > 1$, $q_0(0) \neq 0$;

(iii) $\Pi(1, 0) = 0, \qquad \dfrac{\partial \Pi}{\partial \zeta}(1, 0) = 0, \qquad \dfrac{\partial^2 \Pi}{\partial \zeta^2}(1, 0) \neq 0.$

Definition 1. The solution of the characteristic equation $\Pi(\zeta, z) = 0$ is said to be *of order p* $(p \geqslant 1)$, if one of the principal roots of $\Pi(\zeta, z)$ (e.g., $\zeta_1(z)$) satisfies

$$\mathrm{e}^z - \zeta_1(z) = Cz^{p+1} + \mathrm{O}(z^{p+2}) \quad \text{for } z \to 0,$$

where C $(\neq 0)$ is the error constant of $\zeta_1(z)$.

The meaning of this definition is related to the order of the method given by the difference equation (1.3) and is reflected in the following property [4].

Proposition 2. *If the solution of the equation $\Pi(\zeta, z) = 0$ is of order p $(p \geqslant 1)$ with error constant C, then*

$$\Pi(\mathrm{e}^z,\ z) = \frac{\partial^2 \Pi}{\partial \zeta^2}(1, 0) C z^{p+2} + \mathrm{O}(z^{p+3}) \quad \textit{for } z \to 0.$$

Let us now consider the following algebraic equation:

$$\Pi(\zeta,\ \omega) = \left(P_m(\omega) P_m(-\omega)\right)\zeta^2 - \left(P_m^2(\omega) + P_m^2(-\omega)\right)\zeta + \left(P_m(\omega) P_m(-\omega)\right), \tag{2.1}$$

with $\omega \in \mathbb{C}$ and P_m given by the expression

$$P_m(\omega) = 1 + \frac{m}{2m}\omega + \frac{m(m-1)}{2m(2m-1)}\frac{\omega^2}{2!} + \cdots + \frac{m(m-1)\cdots 1}{2m(2m-1)\cdots(m+1)}\frac{\omega^m}{m!}.$$

The roots of equation (2.1) are given by

$$\zeta_1(\omega) = \zeta_2(\omega)^{-1} = \frac{P_m(\omega)}{P_m(-\omega)},$$

i.e., we are dealing with the (m,m)-diagonal Padé approximant to $\exp(z)$. So, if we take $\omega = \mathrm{i}\lambda h$, those roots are going to be conjugate complex numbers and always lie in the unity circumference. They are the solution of order $2m$ of the equations (2.1):

$$\mathrm{e}^{\omega} - \zeta_1(\omega) = C\omega^{2m+1} + \mathrm{O}(\omega^{2m+2}) \quad \text{for } \omega \to 0.$$

In this paper, we propose symmetric linear multistep methods of two steps and several stages (function evaluations) such as

$$\begin{aligned} &\alpha_0 y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h^2\left(\beta_0 f_{n+2}^{(1)} + \beta_1 f_{n+1} + \beta_0 f_n\right), \\ &y_{n+2}^{(s)} = y_{n+2} - h^2\left(\beta_{0s} f_{n+2}^{(s+1)} + \beta_{1s} f_{n+1} + \beta_{0s} f_n\right), \quad s = 1, 2, \ldots, m-1, \end{aligned} \tag{2.2}$$

where $f_n^{(m)} = f(t_n, y_n)$, $f_n^{(s)} = f(t_n, y_n^{(s)})$ $(s = 1, 2, \ldots, m-1)$.

If we apply method (2.2) to scalar test equation (1.2), we obtain the stability polynomial of the method

$$q_0(z)\zeta^2 + q_1(z)\zeta + q_0(z), \quad z = \mathrm{i}\lambda h, \tag{2.3}$$

where polynomials in z, q_0 and q_1, are written as

$$q_0(z) = \alpha_0 - \beta_0 z^2 + \beta_0\beta_{01} z^4 - \beta_0\beta_{01}\beta_{02} z^6 + \cdots \pm \beta_0\beta_{01}\cdots\beta_{0m-1} z^{2m}, \tag{2.4a}$$

$$q_1(z) = \alpha_1 - \beta_1 z^2 + \beta_0\beta_{11} z^4 - \beta_0\beta_{01}\beta_{12} z^6 + \cdots \pm \beta_0\beta_{01}\cdots\beta_{1m-1} z^{2m}. \tag{2.4b}$$

If we impose the conditions

$$q_0(z) = P_m(z) P_m(-z), \tag{2.5a}$$

$$q_1(z) = -\left(P_m^2(z) + P_m^2(-z)\right), \tag{2.5b}$$

identifying terms in both of them, we obtain a system of $2(m+1)$ equations and $2(m+1)$ unknowns. Its solution determines the coefficients of a linear symmetric method of two steps and m stages, P-stable and of order $2m$. Thus, in a simple and recurrent way, we can obtain multistep linear methods with good periodicity properties to integrate periodic problems of type (1.1). At the same time, the methods proposed in (2.2) may be considered as a generalization of methods proposed in the works of Lambert and Watson [5], Hairer [4], Cash [1], Thomas [7] and Chawla and Rao [2].

Continuing this we deduce P-stable methods with orders 4, 6, 8, and their corresponding error constants.

Order 4 and two stages (m = 2)

$$P_2(z)P_2(-z) = 1 - \tfrac{1}{12}z^2 + \tfrac{1}{144}z^4,$$
$$P_2^2(z) + P_2^2(-z) = 2 + \tfrac{5}{6}z^2 + \tfrac{1}{72}z^4.$$

Taking into account expressions (2.4a,b) and (2.5a,b) we obtain the coefficients

$$\alpha_0 = 1, \qquad \alpha_1 = -2, \qquad \beta_0 = \tfrac{1}{12}, \qquad \beta_1 = \tfrac{5}{6}, \qquad \beta_{01} = \tfrac{1}{12}, \qquad \beta_{11} = -\tfrac{1}{6},$$

resulting in the method

$$y_{n+2} - 2y_{n+1} + y_n = h^2\left(\tfrac{1}{12}f_{n+2}^{(1)} + \tfrac{5}{6}f_{n+1} + \tfrac{1}{12}f_n\right),$$
$$y_{n+2}^{(1)} = y_{n+2} - h^2\left(\tfrac{1}{12}f_{n+2} - \tfrac{1}{6}f_{n+1} + \tfrac{1}{12}f_n\right),$$

and the error constant $C_{(4)} = \frac{1}{360}$.

Order 6 and three stages (m = 3)

Proceeding in the same way as in the preceding case,

$$P_3(z)P_3(-z) = 1 - \tfrac{1}{20}z^2 + \tfrac{1}{600}z^4 - \tfrac{1}{14400}z^6,$$
$$P_3^2(z) + P_3^2(-z) = 2 + \tfrac{9}{10}z^2 + \tfrac{11}{300}z^4 + \tfrac{1}{7200}z^6,$$

the method coefficients are

$$\alpha_0 = 1, \qquad \alpha_1 = -2, \qquad \beta_0 = \tfrac{1}{20}, \qquad \beta_1 = \tfrac{9}{10},$$
$$\beta_{01} = \tfrac{1}{30}, \qquad \beta_{11} = -\tfrac{11}{15}, \qquad \beta_{02} = \tfrac{1}{24}, \qquad \beta_{12} = \tfrac{1}{12},$$

and the method will become

$$y_{n+2} - 2y_{n+1} + y_n = h^2\left(\tfrac{1}{20}f_{n+2}^{(1)} + \tfrac{9}{10}f_{n+1} + \tfrac{1}{20}f_n\right),$$
$$y_{n+2}^{(1)} = y_{n+2} - h^2\left(\tfrac{1}{30}f_{n+2}^{(2)} - \tfrac{11}{15}f_{n+1} + \tfrac{1}{30}f_n\right),$$
$$y_{n+2}^{(2)} = y_{n+2} - h^2\left(\tfrac{1}{24}f_{n+2} + \tfrac{1}{12}f_{n+1} + \tfrac{1}{24}f_n\right),$$

being $C_{(6)} = -\frac{1}{50400}$ the error constant.

Order 8 and four stages (m = 4)

As in the other cases, we obtain

$$P_4(z)P_4(-z) = 1 - \tfrac{1}{28}z^2 + \tfrac{3}{3920}z^4 - \tfrac{1}{70560}z^6 + \tfrac{1}{2822400}z^8,$$
$$P_4^2(z) + P_4^2(-z) = 2 + \tfrac{13}{14}z^2 + \tfrac{289}{5880}z^4 + \tfrac{19}{35280}z^6 + \tfrac{1}{1411200}z^8,$$

the coefficients

$$\alpha_0 = 1, \quad \alpha_1 = -2, \quad \beta_0 = \tfrac{1}{28}, \quad \beta_1 = \tfrac{13}{14}, \quad \beta_{01} = \tfrac{3}{140},$$
$$\beta_{11} = -\tfrac{289}{210}, \quad \beta_{02} = \tfrac{1}{54}, \quad \beta_{12} = \tfrac{19}{27}, \quad \beta_{03} = \tfrac{1}{40}, \quad \beta_{13} = -\tfrac{1}{20},$$

and the method

$$y_{n+2} - 2y_{n+1} + y_n = h^2\left(\tfrac{1}{28}f_{n+2}^{(1)} + \tfrac{13}{14}f_{n+1} + \tfrac{1}{28}f_n\right),$$
$$y_{n+2}^{(1)} = y_{n+2} - h^2\left(\tfrac{3}{140}f_{n+2}^{(2)} - \tfrac{289}{210}f_{n+1} + \tfrac{3}{140}f_n\right),$$
$$y_{n+2}^{(2)} = y_{n+2} - h^2\left(\tfrac{1}{54}f_{n+2}^{(3)} + \tfrac{19}{27}f_{n+1} + \tfrac{1}{54}f_n\right),$$
$$y_{n+2}^{(3)} = y_{n+2} - h^2\left(\tfrac{1}{40}f_{n+2} - \tfrac{1}{20}f_{n+1} + \tfrac{1}{30}f_n\right),$$

with error constant $C_{(8)} = \frac{1}{12700800}$.

3. Phase properties for the methods

In this section, we study methods (2.2) confronted with a scalar test equation more general than that given in (1.2), the test equation used by Gladwell and Thomas [3]:

$$y'' + \lambda^2 y = v\ \mathrm{e}^{\mathrm{i}\omega t}, \tag{3.1}$$

where λ, v and ω are real parameters and in which there appears a forced oscillatory term, that excites the phase oscillations. The general solution to this equation is

$$y(t) = C_1\ \mathrm{e}^{\mathrm{i}\lambda t} + \overline{C}_1\ \mathrm{e}^{-\mathrm{i}\lambda t} + Q\ \mathrm{e}^{\mathrm{i}\omega t},$$

where C_1 is a constant determined by the initial conditions and $Q\mathrm{e}^{\mathrm{i}\omega t}$ is a particular solution with $Q = v/(\lambda^2 - \omega^2)$. When methods (2.2) are applied to equation (3.1), we obtain a recurrence relation as follows:

$$r_2 y_{n+2} + r_1 y_{n+1} + r_0 y_n = vh^2\left\{\omega_2\ \mathrm{e}^{\mathrm{i}\omega t_{n+2}} + \omega_1\ \mathrm{e}^{\mathrm{i}\omega t_{n+1}} + \omega_0\ \mathrm{e}^{\mathrm{i}\omega t_n}\right\}, \tag{3.2}$$

where coefficients r_i, ω_i $(i = 1,2)$ depend on $z = \lambda h$. The general solution to difference equation (3.2) is

$$y_n = A_1\zeta_1^n + A_2\zeta_2^n + Q_2\ \mathrm{e}^{\mathrm{i}\omega nh},$$

where A_1 and A_2 are constant numbers (which depend on the initial conditions and the employed initial or starting method) and $Q_2\mathrm{e}^{\mathrm{i}\omega nh}$ is a particular solution to difference equation (3.2), with Q_2 satisfying

$$Q_2\left\{r_2\ \mathrm{e}^{2\mathrm{i}\omega h} + r_1\ \mathrm{e}^{\mathrm{i}\omega h} + r_0\right\} = vh^2\left\{w_2\ \mathrm{e}^{2\mathrm{i}\omega h} + w_1\ \mathrm{e}^{\mathrm{i}\omega h} + w_0\right\}. \tag{3.3}$$

Since Q is real, the forced numerical oscillation *is in phase* [3] with its corresponding analytic solution if Q_2 is also real. Then, method (2.3) is in phase if and only if $\mathrm{Im}(Q_2) = 0$, which is equivalent to

$$(w_2 r_0 - w_0 r_2)\sin(2\omega h) + (w_2 r_1 - w_1 r_2 + w_1 r_0 - w_0 r_1)\sin(\omega h) = 0. \tag{3.4}$$

Taking into account that for symmetric methods of type (2.2) we have

$$r_0 = r_2 = \alpha_0 - \beta_0 z^2 + \beta_0\beta_{01}z^4 - \beta_0\beta_{01}\beta_{02}z^6 + \cdots \pm \beta_0\beta_{01} \cdots \beta_{0m-1}z^{2m},$$
$$r_1 = \alpha_1 - \beta_1 z^2 + \beta_0\beta_{11}z^4 - \beta_0\beta_{01}\beta_{12}z^6 + \cdots \pm \beta_0\beta_{01} \cdots \beta_{1m-1}z^{2m},$$
$$w_0 = w_2 = \beta_0 - \beta_0\beta_{01}z^2 + \beta_0\beta_{01}\beta_{02}z^4 - \cdots \pm \beta_0\beta_{01} \cdots \beta_{0m-1}z^{2m-2},$$
$$w_1 = \beta_1 - \beta_0\beta_{11}z^2 + \beta_0\beta_{01}\beta_{12}z^4 - \cdots \pm \beta_0\beta_{01} \cdots \beta_{1m-1}z^{2m-2},$$

with $z = \mathrm{i}\lambda h$, it is obvious that these methods are always in phase in the sense of Gladwell and Thomas [3], for they always satisfy relation (3.4) for any number of stages m.

4. Computational aspects and numerical results

In this section, we study how to obtain the solution of the implicit algebraic equation defined by method (2.2) in Section 2. To analyze this problem, we will suppose that function $f(t, y)$ satisfies the Lipschitz condition given by

$$\| f(t, y) - f(t, y^*) \| \leqslant L \| y - y^* \|$$

for all y and y^*, where $0 \leqslant L < 1$. If we write equations (2.2) as

$$y_{n+2} = \Phi(y_{n+2}), \tag{4.1}$$

the Lipschitz constant associated to function Φ is

$$\beta_0 h^2 L - \beta_0\beta_{01}h^4L^2 + \cdots \pm \beta_0\beta_{01} \cdots \beta_{0m-1}h^{2m}L^m. \tag{4.2}$$

We will follow the functional iteration scheme

$$y_{n+2}^{(p+1)} = \Phi\left(y_{n+2}^{(p)}\right), \quad y_{n+2}^{(0)} \text{ known},$$

that converges to the only solution for (4.1), if it is verified that

$$\| \beta_0 h^2 L - \beta_0\beta_{01}h^4L^2 + \cdots \pm \beta_0\beta_{01} \cdots \beta_{0m-1}h^{2m}L^m \| < 1, \tag{4.3}$$

and this leads to an inequality which must be satisfied by the method integration step h. It may happen in some cases that the problem has a Lipschitz constant L large enough; in these cases, to solve the implicit equations, resulting in applying method (2.2), the Newton iteration method modified in the following way may be used:

$$J_{2m}\left(y_{n+2}^{(p+1)} - y_{n+2}^{(p)}\right) = F\left(y_{n+2}^{(p)}\right), \quad p \geqslant 0,$$

where

$$J_{2m} = \alpha_0 I - \beta_0 h^2\left(\frac{\partial f}{\partial y}\right) + \beta_0\beta_{01}h^4\left(\frac{\partial f}{\partial y}\right)^2 - \cdots \pm \beta_0\beta_{01} \cdots \beta_{0m-1}h^{2m}\left(\frac{\partial f}{\partial y}\right)^m,$$
$$F\left(y_{n+2}^{(p)}\right) = -\alpha_0 y_{n+2}^{(p)} + \alpha_1 y_{n+1} - \alpha_0 y_n + h^2\left(\beta_0 f\left(t_{n+2}, y_{n+2}^{(1)(p)}\right) + \beta_1 f_{n+1} + \beta_0 f_n\right),$$

where

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}\left(t_{n+2}, y_{n+2}^{(p)}\right)$$

is the Jacobian matrix of f with respect to y.

For the particular case of the order-4 method, the Jacobian matrix of the modified Newton iterative process will be

$$J_4 = I - \tfrac{1}{12}h^2\frac{\partial f}{\partial y} + \tfrac{1}{144}h^4\left(\frac{\partial f}{\partial y}\right)^2.$$

Since matrix products are costly for large systems, from the computational point of view, and even if $(\partial f/\partial y)$ is a sparse matrix, $(\partial f/\partial y)^2$ will be so in a less degree. So, this is not an attractive scheme from the computational point of view. In these cases, matrix J_4 is factorized, leading to the system

$$\left(I - h^2 r_+ \frac{\partial f}{\partial y}\right)\left(I - h^2 r_- \frac{\partial f}{\partial y}\right)\left(y_{n+2}^{(p+1)} - y_{n+2}^{(p)}\right) = F\left(y_{n+2}^{(p)}\right),$$

where $r_+ = \bar{r}_- = r = 6(1 + \mathrm{i}\sqrt{3})$.

To implement this algorithm, we will solve the system

$$\left(I - h^2 r \frac{\partial f}{\partial y}\right) Z = F\left(y_{n+2}^{(p)}\right),$$

using, for instance, an LU factorization for the matrix of the system. Then, we can calculate $y_{n+2}^{(p)}$ as

$$y_{n+2}^{(p+1)} = y_{n+2}^{(p)} + \frac{\operatorname{Im}(rZ)}{\operatorname{Im}(r)}$$

like Thomas does [7].

In the case of order-6 and order-8 methods, the Jacobian matrices for the modified Newton iteration are given by

$$J_6 = I - \tfrac{1}{20}h^2\frac{\partial f}{\partial y} + \tfrac{1}{600}h^4\left(\frac{\partial f}{\partial y}\right)^2 - \tfrac{1}{14400}h^6\left(\frac{\partial f}{\partial y}\right)^3,$$

$$J_8 = I - \tfrac{1}{28}h^2\frac{\partial f}{\partial y} + \tfrac{3}{3920}h^4\left(\frac{\partial f}{\partial y}\right)^2 - \tfrac{1}{70560}h^6\left(\frac{\partial f}{\partial y}\right)^3 + \tfrac{1}{2822400}h^8\left(\frac{\partial f}{\partial y}\right)^4,$$

respectively. To implement the corresponding algorithms, we may proceed as in the case of the order-4 method, only with obtaining the corresponding factorization roots by means of a numerical algorithm (which should calculate the roots of a polynomial).

To conclude this section, we present some numerical results related to several test problems.

We have considered our P-stable methods (given by (2.2)) versus three other methods: the Störmer–Cowell classical formulae, the methods obtained by Lambert and Watson [5] ($a = 0$) and the Cash formulae [1] (for that case, only order 6 has been considered, since formulae for order 8 are not available) to solve numerically the following linear and nonlinear problems. The initialization of the methods was carried out, in each case, with the exact solution of the corresponding problem.

Example 3. The quasi periodic problem

$$z'' + z = \epsilon\, \mathrm{e}^{\mathrm{i}\omega t}, \quad t \in [0, 40\pi],$$
$$z(0) = 1, \qquad z'(0) = \mathrm{i},$$

Table 1

	Störmer–Cowell (8)	Lambert–Watson (8)	Cash (8)	P-stable (8)
$h=\frac{1}{36}\pi$	0.575×10^{-9}	0.168×10^{-9}	not available	0.412×10^{-10}
$h=\frac{1}{24}\pi$	0.148×10^{-7}	0.434×10^{-8}	not available	0.859×10^{-10}
$h=\frac{1}{16}\pi$	0.384×10^{-6}	0.114×10^{-6}	not available	0.240×10^{-9}
$h=\frac{1}{12}\pi$	0.389×10^{-5}	0.118×10^{-5}	not available	0.223×10^{-8}
$h=\frac{1}{8}\pi$	0.103×10^{-3}	0.336×10^{-4}	not available	0.179×10^{-6}
$h=\frac{1}{6}\pi$	0.108×10^{-2}	0.403×10^{-3}	not available	0.423×10^{-5}

Table 2

	Störmer–Cowell (6)	Lambert–Watson (6)	Cash (6)	P-stable (6)
$h=\frac{1}{36}\pi$	0.102×10^{-6}	0.437×10^{-7}	0.764×10^{-9}	0.525×10^{-9}
$h=\frac{1}{24}\pi$	0.116×10^{-5}	0.500×10^{-6}	0.873×10^{-8}	0.624×10^{-8}
$h=\frac{1}{16}\pi$	0.134×10^{-4}	0.575×10^{-5}	0.100×10^{-6}	0.728×10^{-7}
$h=\frac{1}{12}\pi$	0.764×10^{-4}	0.328×10^{-4}	0.585×10^{-6}	0.431×10^{-6}
$h=\frac{1}{8}\pi$	0.897×10^{-3}	0.388×10^{-3}	0.804×10^{-5}	0.636×10^{-5}
$h=\frac{1}{6}\pi$	0.530×10^{-2}	0.232×10^{-2}	0.651×10^{-4}	0.560×10^{-4}

whose exact solution is

$$z(t)=u(t)+\mathrm{i}v(t),\quad u,\ v\in\mathbb{R},$$

$$u(t)=\frac{1-\epsilon-\omega^2}{1-\omega^2}\cos(t)+\frac{\epsilon}{1-\omega^2}\cos(\omega t),$$

$$v(t)=\frac{1-\epsilon\omega-\omega^2}{1-\omega^2}\sin(t)+\frac{\epsilon}{1-\omega^2}\sin(\omega t).$$

This exact solution represents a perturbed circular motion in the complex plane. We have written this problem as a coupled set of real differential equations for $u(t)$ and $v(t)$. The numerical results have been computed for $t=40\pi$ with integration steps $h=\frac{1}{36}\pi,\ \frac{1}{24}\pi,\ \frac{1}{16}\pi,\ \frac{1}{12}\pi,\ \frac{1}{8}\pi,\ \frac{1}{6}\pi$ and parameter values $\epsilon=0.001$, $\omega=0.01$. These results are given in Tables 1 and 2, in which the error between the exact solution and the numerical solution in the form $\|z(t)-z_n\|_2$ is presented.

Table 3

	Störmer–Cowell (8)	Lambert–Watson (8)	Cash (8)	P-stable (8)
$h=\frac{1}{36}\pi$	0.314×10^{-9}	0.327×10^{-11}	not available	0.274×10^{-13}
$h=\frac{1}{24}\pi$	0.158×10^{-7}	0.984×10^{-11}	not available	0.222×10^{-11}
$h=\frac{1}{16}\pi$	0.449×10^{-6}	0.502×10^{-7}	not available	0.190×10^{-9}
$h=\frac{1}{12}\pi$	0.579×10^{-5}	0.691×10^{-6}	not available	0.435×10^{-8}
$h=\frac{1}{8}\pi$	0.229×10^{-3}	0.100×10^{-5}	not available	0.222×10^{-6}
$h=\frac{1}{6}\pi$	0.239×10^{-2}	0.465×10^{-3}	not available	0.658×10^{-5}

Table 4

	Störmer–Cowell (6)	Lambert–Watson (6)	Cash (6)	P-stable (6)
$h=\frac{1}{36}\pi$	0.331×10^{-7}	0.853×10^{-8}	0.157×10^{-9}	0.115×10^{-9}
$h=\frac{1}{24}\pi$	0.889×10^{-6}	0.901×10^{-8}	0.321×10^{-9}	0.313×10^{-9}
$h=\frac{1}{16}\pi$	0.958×10^{-5}	0.252×10^{-5}	0.555×10^{-7}	0.427×10^{-7}
$h=\frac{1}{12}\pi$	0.713×10^{-4}	0.192×10^{-4}	0.476×10^{-6}	0.385×10^{-6}
$h=\frac{1}{8}\pi$	0.172×10^{-2}	0.935×10^{-3}	0.492×10^{-5}	0.489×10^{-5}
$h=\frac{1}{6}\pi$	0.871×10^{-2}	0.268×10^{-2}	0.115×10^{-3}	0.104×10^{-3}

Example 4. The second-order linear system

$$y''=y+4z,$$
$$z''=-2y-5z,$$

which has the exact solution $y=2\cos(t)$, $z=-\cos(t)$. We have calculated the numerical solution in $t=40\pi$, for the integration steps $h=\frac{1}{36}\pi, \frac{1}{24}\pi, \frac{1}{16}\pi, \frac{1}{12}\pi, \frac{1}{8}\pi, \frac{1}{6}\pi$. The absolute errors (exact solution − numerical solution) in the $\|\cdot\|_2$-norm are shown in Tables 3 and 4.

Example 5. The nonlinear problem

$$z''+\left(1+\gamma+\gamma\delta\,\mathrm{e}^{-2\mathrm{i}t}\right)z=\gamma\,\mathrm{e}^{-\mathrm{i}t}z^2,$$
$$z(0)=1+\delta, \qquad z'(0)=\mathrm{i}(1-\delta),$$

with $\gamma\geqslant 0$, $0\leqslant\delta\leqslant 1$. The exact solution is $z(t)=\mathrm{e}^{\mathrm{i}t}+\delta\mathrm{e}^{-\mathrm{i}t}$ which represents an ellipse, where δ is a distortion parameter and γ a nonlinearity parameter. The problem has been solved as a coupled system of real differential equations for $t=10\pi$ and $h=\frac{1}{12}\pi$, $\gamma=0.1\cdot10^{-5}$ and different values of δ. The absolute errors are tabulated (in $\|\cdot\|_2$-norm) in Table 5.

5. Conclusions

The results presented in Tables 1–5 were obtained with numerical methods of the same order in each case, i.e., methods comparable in terms of local approximation. From these results, we deduce the following conclusions:

Table 5

	Störmer–Cowell (8)	Lambert–Watson (8)	Cash (6)	P-stable (8)
$\delta=0.0$	0.938×10^{-6}	0.239×10^{-6}	0.148×10^{-6}	0.452×10^{-7}
$\delta=0.1$	0.975×10^{-6}	0.264×10^{-6}	0.132×10^{-6}	0.327×10^{-7}
$\delta=0.2$	0.102×10^{-5}	0.235×10^{-6}	0.117×10^{-6}	0.295×10^{-7}
$\delta=0.3$	0.107×10^{-5}	0.205×10^{-6}	0.103×10^{-6}	0.225×10^{-7}
$\delta=0.4$	0.113×10^{-5}	0.176×10^{-6}	0.899×10^{-7}	0.172×10^{-7}
$\delta=0.5$	0.119×10^{-5}	0.147×10^{-6}	0.769×10^{-7}	0.153×10^{-7}

(1) Methods which have orbital stability or periodicity properties (Lambert–Watson, Cash and our P-stable methods) yield better results than those which do not have these properties (Störmer–Cowell).

(2) P-stable methods (Cash and our P-stable methods) provide better approximations to the solution than methods which have a finite periodicity interval (Lambert and Watson).

(3) Our P-stable methods provide a slightly better approximation than the Cash methods (we only have compared sixth-order methods), but the first ones are simpler in formulation and cheaper in time-consuming than the second ones.

(4) In Example 5, it is observed that Störmer–Cowell formulae degenerate when the torsion parameter δ grows, whilst our P-stable methods, Lambert–Watson and Cash methods improve their approximation.

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