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# On the structure and probabilistic interpretation of Askey–Wilson densities and polynomials with complex parameters

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**Abstract**

We give equivalent forms of the Askey–Wilson polynomials expressing them with the help of the Al-Salam–Chihara polynomials. After restricting parameters of the Askey–Wilson polynomials to complex conjugate pairs we expand the Askey–Wilson weight function in the series similar to the Poisson–Mehler expansion formula and give its probabilistic interpretation. In particular this result can be used to calculate explicit forms of '$q$-Hermite' moments of the Askey–Wilson density, hence enabling calculation of all moments of the Askey–Wilson density. On the way (by setting certain parameter $q$ to 0) we get some formulae useful in the rapidly developing so-called 'free probability'.


*Keywords:* Askey–Wilson density; Askey–Wilson polynomials; $q$-Hermite polynomials; Al-Salam–Chihara polynomials; Poisson–Mehler expansion

## 1. Introduction

The aim of this paper is to present some properties of the Askey–Wilson (briefly AW) polynomials and their weight function. This is the function that makes these polynomials orthogonal. As it is well known (see e.g. [15]) the AW polynomials are characterized by 5 parameters one of

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which is special, traditionally denoted by $q$, often called a base. In majority of cases $-1 < q \leqslant 1$. The parameter $q$ plays a special rôle that will be exposed in the sequel. The remaining 4 parameters can be either real or complex but forming conjugate pairs. If products of all pairs of these 4 parameters have absolute values less than 1 then there exists a positive measure on a compact segment that makes the AW polynomials orthogonal. If absolute values of all parameters are less than 1 then this measure has a density. If all 4 parameters are complex and are in conjugate pairs then the AW weight can be scaled to be the probability density having nice probabilistic interpretation. This density for $q = 1$ is nothing else but one of the conditional densities of certain 3-dimensional jointly Normal distribution. We will explain it in the sequel. Because of these interpretations our main concern will be with the complex parameter case. In particular our main result will allow expansion of the AW density in certain series of the so-called $q$-Hermite polynomials. The expansion is analogous to the Poisson–Mehler series.

However to present briefly and clearly our results we have to refer to the $q$-series theory and some of its basic notions. Although the $q$-series theory has links with combinatorics, non-commutative analysis and probability theory it is not widely known. That is why we will recall some notions and facts concerning it. Our considerations and calculations are simple and in fact elementary.

Traditionally the AW polynomials (see e.g. [4] or [15] or [18]) are defined through the finite $q$-hypergeometric series. More precisely the $n$-th AW polynomial $D_n$ is defined by

$$D_n(x|a,b,c,d,q) = \frac{(ab,ac,ad)_n}{a^n(abcdq^{n-1})_n}\, {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{-i\theta}, ae^{i\theta} \\ ab, ac, ad, q \end{matrix}\middle| q, q\right),$$

where ${}_4\phi_3$ is the $q$-hypergeometric series defined by

$${}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s,q \end{matrix}\middle| q, y\right) = \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r)_k}{(b_1,\ldots,b_s,q)_k}(-1)^{s+1-r} q^{(s+1-r)\binom{k}{2}} y^k,$$

where $\binom{n}{k}$ is the binomial coefficient and $x = \cos\theta$. $(a_1,\ldots,a_r)_k$ and $(b_1,\ldots,b_s,q)_k$ as well as $(ab,ac,ad)_n$ and $(abcdq^{n-1})_n$ are explained at the beginning of the next section.

The above mentioned form is difficult to use and analyze by those who do not work in the special function theory. On the other hand due to pioneering works of Bożejko et al. [5] and also of Bryc et al. [6–11], the Askey–Wilson polynomials and some of their subclasses have nice, clear and classical probabilistic interpretation. Hence interest in this family of polynomials has not only been among specialists in special functions or those working in the theory orthogonal polynomials, but also among specialists in the probability theory both non-commutative and classical. Not to mention people working in quantum mechanics or quantum groups (see e.g. [14]).

By setting $q = 0$ we enter the world of rapidly developing so-called 'free probability' (see e.g. [22,23,19]).

The family of probabilistic models, where the AW polynomials and densities appear, has 5 parameters and is very versatile. Hence it can be used in a brief descriptions of various, complicated statistical models.

We will express the Askey–Wilson polynomials as certain combinations of simpler polynomials (the Al-Salam–Chihara polynomials introduced in Section 2). Especially simple form of the AW polynomials will be obtained in the special case of complex, grouped in conjugate pairs, parameters.

The paper is organized as follows. Notation and known results of the $q$-series theory that will be of help in further calculations are presented in Section 2. Next Section 3 presents our main results. The following short Section 4 presents some immediate open problems. The lengthy proofs of some of the results are collected in the last Section 5.

## 2. Auxiliary results

Assume that $-1 < q \leqslant 1$. We will use traditional notation of the $q$-series theory i.e. $[0]_q = 0$, $[n]_q = 1 + q + \cdots + q^{n-1}$, $[n]_q! = \prod_{i=1}^n [i]_q$, with $[0]_q! = 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{[n]_q!}{[n-k]_q![k]_q!}, & n \geqslant k \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

It will be also helpful to use the so-called $q$-Pochhammer symbol defined for $n \geqslant 1$ by

$$(a;q)_n = \prod_{i=0}^{n-1}\left(1 - aq^i\right), \quad \text{with } (a;q)_0 = 1,\ (a_1, a_2, \ldots, a_k; q)_n = \prod_{i=1}^{k}(a_i;q)_n.$$

Often $(a;q)_n$ as well as $(a_1, a_2, \ldots, a_k; q)_n$ will be abbreviated to $(a)_n$ and $(a_1, a_2, \ldots, a_k)_n$, if it will not cause misunderstanding.

In particular it is easy to notice that $(q)_n = (1-q)^n [n]_q!$ and that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_{n-k}(q)_k}, & n \geqslant k \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let us remark that $[n]_1 = n$, $[n]_1! = n!$, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_1 = \binom{n}{k}$, $(a;1)_n = (1-a)^n$ and

$$[n]_0 = \begin{cases} 1 & \text{if } n \geqslant 1, \\ 0 & \text{if } n = 0, \end{cases} \qquad [n]_0! = 1, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_0 = 1, \qquad (a;0)_n = \begin{cases} 1 & \text{if } n = 0, \\ 1-a & \text{if } n \geqslant 1. \end{cases}$$

In the sequel we will use the following two simple properties of the $q$-Pochhammer symbol.

**Lemma 1.**

i) *For* $-1 < q \leqslant 1$, $a \in \mathbb{R}$, $n \geqslant 0$: $\sum_{i=0}^n \left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_q a^i (a)_{n-i} = 1$,

ii) *For* $-1 < q \leqslant 1$, $a, b \in \mathbb{R}$, $n \geqslant 0$: $\sum_{i=0}^n (-1)^i q^{\binom{i}{2}} \left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_q (a)_i b^i (abq^i)_{n-i} = (b)_n$.

**Proof.** An easy proof based on the so-called $q$-binomial theorem (compare Thm. 10.2.1 of [2] or Thm. 12.2.5 of [15]) is shifted to Section 5. □

Let us define the following sets of polynomials.

The $q$-Hermite polynomials defined by

$$h_{n+1}(x|q) = 2xh_n(x|q) - \left(1 - q^n\right)h_{n-1}(x|q), \tag{2.1}$$

for $n \geqslant 1$, with $h_{-1}(x|q) = 0$, $h_0(x|q) = 1$. The polynomials $h_n$ are also often called continuous $q$-Hermite polynomials. However we will more frequently use the following transformed form of polynomials $h_n$, namely the polynomials:

$$H_n(x|q) = (1-q)^{-n/2} h_n\left(\frac{x\sqrt{1-q}}{2}\Big|q\right).$$

We will call them also $q$-Hermite. The name is justified since one can easily show that $H_n(x|1) = H_n(x)$, where $H_n$ denotes the $n$-th ordinary, so-called probabilistic Hermite polynomial. More precisely the polynomials $\{H_n\}_{n\geqslant -1}$ satisfy 3-term recurrence (2.2), below:

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x), \tag{2.2}$$

with $H_0(x) = H_1(x) = 1$. Hence they are orthogonal with respect to the measure with the density equal to $\exp(-x^2/2)/\sqrt{2\pi}$.

The polynomials $\{H_n(x|q)\}$ satisfy the following 3-term recurrence

$$H_{n+1}(x|q) = xH_n(x|q) - [n]_q H_{n-1}(x), \tag{2.3}$$

with $H_{-1}(x|q) = 0$, $H_1(x|q) = 1$.

We shall also use the following polynomials called Al-Salam–Chihara (ASC polynomials). As before, in the literature connected with the special functions or orthogonal polynomials as the ASC polynomials function polynomials defined recursively:

$$\begin{aligned} Q_{n+1}(x|a,b,q) = {} & \left(2x - (a+b)q^n\right)Q_n(x|a,b,q) \\ & - \left(1 - abq^{n-1}\right)\left(1-q^n\right)Q_{n-1}(x|a,b,q), \end{aligned} \tag{2.4}$$

with $Q_{-1}(x|a,b,q) = 0$, $Q_0(x|a,b,q) = 1$.

We will more often use these polynomials re-scaled, with new parameters $\rho$ and $y$ defined by

$$a = \frac{\sqrt{1-q}}{2}\rho_1\left(y - i\sqrt{\frac{4}{1-q} - y^2}\right), \qquad b = \frac{\sqrt{1-q}}{2}\rho_1\left(y + i\sqrt{\frac{4}{1-q} - y^2}\right),$$

such that $y^2 \leqslant 4/(1-q)$, $|\rho| < 1$. In the formula above $i$ stands for the imaginary unit.

More precisely we will consider the polynomials

$$\begin{aligned} P_n(x|y,\rho,q) = Q_n\Bigg(& x\sqrt{1-q}/2\Big|\frac{\sqrt{1-q}}{2}\rho\left(y - i\sqrt{\frac{4}{1-q} - y^2}\right), \\ & \frac{\sqrt{1-q}}{2}\rho\left(y + i\sqrt{\frac{4}{1-q} - y^2}\right), q\Bigg). \end{aligned} \tag{2.5}$$

One shows that polynomials $\{P_n\}$ satisfy the following 3-term recurrence:

$$P_{n+1}(x|y,\rho,q) = \left(x - \rho y q^n\right)P_n(x|y,\rho,q) - \left(1 - \rho^2 q^{n-1}\right)[n]_q P_{n-1}(x|y,\rho,q), \tag{2.6}$$

with $P_{-1}(x|y,\rho,q)=0$, $P_0(x|y,\rho,q)=1$. The polynomials $\{P_n\}$ have nice probabilistic interpretation see e.g. [7]. To support intuition let us remark that

$$P_n(x|y,\rho,1)=\left(1-\rho^2\right)^{n/2}H_n\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right).$$

The polynomials (2.3) satisfy the following very useful identity originally formulated for the continuous $q$-Hermite polynomials $h_n$ (can be found in e.g. [15, Thm. 13.1.5]) and here, below presented for the polynomials $H_n$:

$$H_n(x|q)H_m(x|q)=\sum_{j=0}^{\min(n,m)}\begin{bmatrix}m\\j\end{bmatrix}_q\begin{bmatrix}n\\j\end{bmatrix}_q[j]_q!H_{n+m-2j}(x|q). \tag{2.7}$$

Let us denote for simplicity the following real subsets:

$$S(q)=\begin{cases}[-2/\sqrt{1-q},2/\sqrt{1-q}\,] & \text{if } |q|<1,\\ \mathbb{R} & \text{if } q=1,\end{cases} \tag{2.8}$$

and the following family of quadratic, auxiliary, polynomials:

$$w_k(x,y|\rho,q)=\left(1-\rho^2q^{2k}\right)^2-(1-q)\rho q^k\left(1+\rho^2q^{2k}\right)xy+(1-q)\rho^2\left(x^2+y^2\right)q^{2k}, \tag{2.9}$$

$k=0,1,2,\ldots$. Notice that $\forall k\geqslant 0$: $w_k(x,y|\rho,q)=w_0(x,y|\rho q^k,q)$ and that $w_k(x,y|0,q)=1$.

It is known (see e.g. [6], but also [15, Thm. 13.1.3] with an obvious modification for the polynomials $H_n$) that the $q$-Hermite polynomials are monic and orthogonal with respect to the measure that has the density given by

$$f_N(x|q)=\frac{\sqrt{1-q}(q)_\infty}{2\pi\sqrt{4-(1-q)x^2}}\prod_{k=0}^{\infty}\left(\left(1+q^k\right)^2-(1-q)x^2q^k\right)I_{S(q)}(x), \tag{2.10}$$

defined for $|q|<1$, $x\in\mathbb{R}$, where

$$I_A(x)=\begin{cases}1 & \text{if } x\in A,\\ 0 & \text{if } x\notin A.\end{cases}$$

We will set also

$$f_N(x|1)=\frac{1}{\sqrt{2\pi}}\exp\left(-x^2/2\right). \tag{2.11}$$

Similarly it is known (e.g. from [7] and also [15, formula (15.1.5)] after re-scaling polynomials $Q_n$ to $P_n$) that the polynomials $\{P_n(x|y,\rho,q)\}_{n\geqslant -1}$ are monic and orthogonal with respect to the measure that for $q\in(-1,1]$ and $|\rho|<1$ has the density. For $|q|<1$ this density is given by

$$f_{CN}(x|y,\rho,q)=f_N(x|q)\prod_{k=0}^{\infty}\frac{(1-\rho^2q^k)}{w_k(x,y|\rho,q)}I_{S(q)}(x), \tag{2.12a}$$

for $x \in \mathbb{R}$, $y \in S(q)$ and for $q = 1$ is given by

$$f_{CN}(x|y,\rho,1) = \frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left(-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right),$$

with $x, y \in \mathbb{R}$.

It is known (see e.g. [15, formula (13.1.10)]) that for $|q| < 1$:

$$\sup_{x\in S(q)}\left|H_n(x|q)\right| \leqslant s_n(q)(1-q)^{-n/2}, \tag{2.13}$$

where

$$s_n(q) = \sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}_q. \tag{2.14}$$

We will be studying the following density

$$\phi(x|y,\rho_1,z,\rho_2,q) = f_N(x|q)\frac{(\rho_1^2,\rho_2^2)_\infty}{(\rho_1^2\rho_2^2)_\infty}\prod_{k=0}^{\infty}\frac{w_k(y,z|\rho_1\rho_2,q)}{w_k(x,y|\rho_1,q)w_k(x,z|\rho_2,q)}, \tag{2.15}$$

where the polynomials $w_k(s,t|\rho,q)$ are defined by (2.9).

For $q = 1$ we set

$$\phi(x|y,\rho_1,z,\rho_2,1) = \frac{1}{\sqrt{2\pi\frac{(1-\rho_1^2)(1-\rho_2^2)}{1-\rho_1^2\rho_2^2}}}\exp\left(-\frac{(x-\frac{y\rho_1(1-\rho_2^2)+z\rho_2(1-\rho_1^2)}{1-\rho_1^2\rho_2^2})^2}{2\frac{(1-\rho_1^2)(1-\rho_2^2)}{1-\rho_1^2\rho_2^2}}\right), \tag{2.16}$$

that is $\phi(x|y,\rho_1,z,\rho_2,1)$ is the density of the Normal distribution

$$N\left(\frac{y\rho_1(1-\rho_2^2)+z\rho_2(1-\rho_1^2)}{1-\rho_1^2\rho_2^2}, \frac{(1-\rho_1^2)(1-\rho_2^2)}{1-\rho_1^2\rho_2^2}\right).$$

We have the following important but easy remark.

**Remark 1.** i) $\phi(x|y,\rho_1,z,\rho_2,q) = \frac{f_{CN}(z|x,\rho_2,q)f_{CN}(x|y,\rho_1,q)f_N(y|q)}{f_{CN}(z|y,\rho_1\rho_2,q)f_N(y|q)}$, hence in particular $\phi(x|y, 0,z,\rho_2,q) = f_{CN}(x|z,\rho_2,q)$.

ii) $\phi(x|y,\rho_1,z,\rho_2,q) = \psi(\frac{\sqrt{1-q}}{2}x|a,b,c,d,q)$ where

$$a = \frac{\sqrt{1-q}}{2}\rho_1\left(y - i\sqrt{\frac{4}{1-q}-y^2}\right), \tag{2.17}$$

$$b = \frac{\sqrt{1-q}}{2}\rho_1\left(y + i\sqrt{\frac{4}{1-q}-y^2}\right), \tag{2.18}$$

$$c=\frac{\sqrt{1-q}}{2}\rho_2\left(z-i\sqrt{\frac{4}{1-q}-z^2}\right), \tag{2.19}$$

$$d=\frac{\sqrt{1-q}}{2}\rho_2\left(z+i\sqrt{\frac{4}{1-q}-z^2}\right) \tag{2.20}$$

and $\psi(t|a,b,c,d,q)$ is a normalized (that is multiplied by a constant so that its integral is 1) weight function of the AW polynomials. Compare e.g. [4] or [15]. Again in the formulae (2.17), ..., (2.20) $i$ denotes the imaginary unit.

From assertion i) of the remark above it follows that the properties of the density $\phi$ are closely related to the properties of the densities $f_{CN}$ and $f_N$. Hence now we will recall properties of these densities and related to them families of the polynomials $\{H_n(x|q)\}_{n\geqslant -1}$ and $\{P_n(x|y,\rho,q)\}_{n\geqslant -1}$ that are crucial for the main results of this paper. We will collect them in the following two propositions:

**Proposition 1.**

i) *For* $n,m\geqslant 0$:

$$\int\limits_{S(q)} H_n(x|q)H_m(x|q)f_N(x|q)\,dx=\begin{cases}0 & \text{when } n\neq m,\\ [n]_q! & \text{when } n=m.\end{cases}$$

ii) *For* $n\geqslant 0$:

$$\int\limits_{S(q)} H_n(x|q)f_{CN}(x|y,\rho,q)\,dx=\rho^n H_n(y|q).$$

iii) *For* $n,m\geqslant 0$:

$$\int\limits_{S(q)} P_n(x|y,\rho,q)P_m(x|y,\rho,q)f_{CN}(x|y,\rho,q)\,dx=\begin{cases}0 & \text{when } n\neq m,\\ (\rho^2)_n[n]_q! & \text{when } n=m.\end{cases}$$

iv)
$$\int\limits_{S(q)} f_{CN}(x|y,\rho_1,q)f_{CN}(y|z,\rho_2,q)f_N(y|q)\,dy=f_{CN}(x|z,\rho_1\rho_2,q).$$

v) *For* $|t|,|q|<1$:

$$\sum_{i=0}^{\infty}\frac{s_i(q)t^i}{(q)_i}=\frac{1}{(t)_\infty^2},\qquad \sum_{i=0}^{\infty}\frac{s_i^2(q)t^i}{(q)_i}=\frac{(t^2)_\infty}{(t)_\infty^4},$$

*convergence is absolute, where* $s_i(q)$ *is defined by* (2.14).

vi) *For* $(1-q)\max(x^2, y^2) \leqslant 4$, $|\rho| < 1$:

$$f_{CN}(x|y,\rho,q) = f_N(x|q)\sum_{n=0}^{\infty}\frac{\rho^n}{[n]_q!}H_n(x|q)H_n(y|q), \tag{2.21}$$

*convergence is absolute in* $\rho$, $y$ *and* $x$ *and uniform in* $x$ *and* $y$.

vii) $\forall x, y \in S(q)$:

$$0 < C(y,\rho,q) \leqslant \frac{f_{CN}(x|y,\rho,q)}{f_N(x|q)} \leqslant \frac{(\rho^2)_\infty}{(\rho)_\infty^4}.$$

**Proof.** i) It is formula (13.1.11) of [15] with an obvious modification for the polynomials $H_n$ instead of $h_n$ (compare (2.1)) and the normalized weight function (i.e. $f_N$). ii) Exercise 15.7 of [15] also in [6]. iii) Formula (15.1.5) of [15] with an obvious modification for the polynomials $P_n$ instead of $Q_n$ and the normalized weight function (i.e. $f_{CN}$). iv) See (2.6) of [7]. v) Exercise 12.2(b) and 12.2(c) of [15]. vi) It is the famous Poisson–Mehler formula (see e.g. [15], for the simple proof of it see [21]).

vii) The upper limit follows directly (2.21) and assertion v). To get the lower one let us notice that from (2.12) we have $\frac{f_{CN}(x|y,\rho,q)}{f_N(x|q)} = \prod_{k=0}^{\infty}\frac{1-\rho^2q^k}{w_k(x,y|\rho,q)}$. Now let us notice also that

$$\begin{aligned} w_k(x,y|\rho,q) &= (1-q)\rho^2q^{2k}\bigl(x - \bigl(\rho^{-1}q^{-k} + \rho q^k\bigr)y/2\bigr)^2 \\ &\quad + \bigl(1-\rho^2q^{2k}\bigr)^2\bigl(1-(1-q)y^2/4\bigr) \geqslant 0. \end{aligned}$$

As a nonnegative quadratic form this expression assumes its maximum value for $x \in S(q)$ at the ends of this interval, so

$$\begin{aligned} &\bigl(1-\rho^2q^{2k}\bigr)^2 - (1-q)\rho q^k\bigl(1+\rho^2q^{2k}\bigr)xy + (1-q)\rho^2\bigl(x^2+y^2\bigr)q^{2k} \\ &\quad\leqslant \bigl(1-\rho^2q^{2k}\bigr)^2 + 2(1-q)\bigl(1+\rho^2q^{2k}\bigr)\bigl|y\rho q^k\bigr| + 4\rho^2q^{2k} + (1-q)\rho^2y^2q^{2k} \\ &\quad= \bigl(1+\rho^2q^{2k}\bigr)^2 + 2(1-q)\bigl(1+\rho^2q^{2k}\bigr)\bigl|y\rho q^k\bigr| + (1-q)\rho^2y^2q^{2k}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{f_{CN}(x|y,\rho,q)}{f_N(x|q)} &\geqslant \frac{(\rho^2)_\infty}{\prod_{k=0}^{\infty}(1+\rho^2q^{2k})^2 + 2(1-q)(1+\rho^2q^{2k})|y\rho q^k| + (1-q)\rho^2y^2q^{2k}} \\ &\overset{df}{=} C(y,\rho,q). \qquad \square \end{aligned}$$

**Remark 2.** From the assertion v) of the lemma above it follows that $\phi(x|y,\rho_1,z,\rho_2,q)$ is the conditional density of $X|Y,Z$ if the joint density of $(Y,X,Z)$ is equal to $f_N(y|q)f_{CN}(x|y,\rho_1,q)f_{CN}(z|x,\rho_2,q)$. It is so since then the marginal density of $(Y,Z)$ is equal to $f_N(y|q)f_{CN}(z|y,\rho_1\rho_2,q)$ (which follows directly from assertion iv) of Proposition 1).

Properties of the polynomial sets $\{H_n(x|q)\}_{n\geqslant -1}$ and $\{P_n(x|y,\rho,q)\}_{n\geqslant -1}$ are collected in the second proposition below.

We will use also the following, auxiliary set of polynomials $\{B_n(x|q)\}_{n\geqslant -1}$ defined by

$$B_{n+1}(y|q) = -q^n y B_n(y|q) + q^{n-1}[n]_q B_{n-1}(y|q); \quad n \geqslant 0, \tag{2.22}$$

with $B_{-1}(y|q) = 0$, $B_0(y|q) = 1$. The polynomials $\{B_n\}_{n\geqslant -1}$ with this normalization were introduced and some of their basic properties were exposed in [7]. However they were known earlier with different scaling and normalization (see e.g. [3] or [16] where polynomials $h_n(y|q^{-1})$ are analyzed). In particular it was shown in [7] that $B_n(x|1) = i^n H_n(ix)$. We will need also these polynomials with an another scaling and normalization and also some additional properties of them. Namely we will need 'continuous version' of these polynomials:

$$b_n(y|q) = (1-q)^{n/2} B_n(2y/\sqrt{1-q}|q).$$

It is elementary to notice that the polynomials $b_n$ satisfy 3-term recurrence:

$$b_{n+1}(y|q) = -2q^n y b_n(y|q) + q^{n-1}(1-q^n) b_{n-1}(y|q), \quad n \geqslant 0 \tag{2.23}$$

with $b_{-1}(y|q) = 0$, $b_0(y|q) = 1$. Further let us notice (comparing (2.23) and (2.1)) that

$$(-1)^n q^{-\binom{n}{2}} b_n(y|q) = h_n(y|q^{-1}). \tag{2.24}$$

**Proposition 2.**

i) $\forall n \geqslant 0$: $P_n(x|y,\rho,q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \rho^{n-j} B_{n-j}(y|q) H_j(x|q)$,
ii) $\forall n > 0$: $\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} B_{n-j}(x|q) H_j(x|q) = 0$,
iii) $\forall n \geqslant 0$: $H_n(x|q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \rho^{n-j} H_{n-j}(y|q) P_j(x|y,\rho,q)$.

**Proof.** i) and ii) are proved in [7]. iii) follows after inserting $P_j$ given by i), changing the order of summation and applying ii). However iii) was known earlier, was given by formula (4.7) in [17] for polynomials $h_n$ and $Q_n(x|a,b,q)$. □

We will also need the following additional properties of polynomials $\{H_n(x|q)\}$ and $\{B_n(x|q)\}$.

**Lemma 2.**

i) $n \geqslant 0$:

$$B_n(x|q) = (-1)^n q^{\binom{n}{2}} \sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ k \end{bmatrix}_q [k]_q! q^{-k(n-k)} H_{n-2k}(x|q).$$

*Let us denote* $I_{n,m}(x|q) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q B_{n-i}(x|q) H_{i+m}(x|q)$, *then*

ii) $n, m \geqslant 1$:

$$I_{n,m}(x|q) = -\sum_{k=1}^{n} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q [k]_q! I_{n-k,m-k}(x|q),$$

iii) $n, m \geqslant 0$:

$$I_{n,m}(x|q) = \begin{cases} 0 & \textit{if } n > m, \\ (-1)^n q^{\binom{n}{2}} \frac{[m]_q!}{[m-n]_q!} H_{m-n}(x|q) & \textit{if } n \leqslant m, \end{cases}$$

iv) $\forall n, m \geqslant 1$:

$$H_m(x|q)B_n(x|q) = (-1)^n q^{\binom{n}{2}} \sum_{i=0}^{\lfloor (n+m)/2 \rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} n+m-i \\ i \end{bmatrix}_q [i]_q! q^{-i(n-i)} H_{n+m-2i}(x|q).$$

**Proof.** i) follows basically the formula (13.3.6) in [15] after necessary re-normalization and re-scaling. iv) follows i) and (2.7). Lengthy, detailed proofs of ii) and iii) are shifted to Section 5. □

Since the case $q = 0$ is important to the newly emerging so-called "free probability" (see e.g. nomography [22]) let us see how the considered above sets of polynomials look for $q = 0$. To do this let us introduce the so-called Chebyshev polynomials of the second kind $U_n(x)$ defined e.g. by the following three term recurrence:

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x), \tag{2.25}$$

for $n \geqslant 0$ with $U_{-1}(x) = 0$, $U_0(x) = 1$.

**Remark 3.** Let us set $q = 0$, then $S(0) = [-2, 2]$; $\forall n \geqslant 0$, we have for $n \geqslant 1$:

i) $H_n(x|0) = U_n(x/2)$,
ii) $Q_n(x|a, b, 0) = U_n(x) - (a + b)U_{n-1}(x) + abU_{n-2}(x)$,
iii) $P_n(x|y, \rho, 0) = U_n(x/2) - \rho y U_{n-1}(x/2) + \rho^2 U_{n-2}(x/2)$,
iv) $B_{-1}(y|0) = b_{-1}(y|0) = 0$, $B_0(y|0) = b_0(y|0) = 1$,

$$B_n(y|0) = \begin{cases} -y & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 0 & \text{if } n \geqslant 3 \end{cases} \quad \text{and} \quad b_n(y|0) = \begin{cases} -2y & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 0 & \text{if } n \geqslant 3, \end{cases}$$

v) $f_N(x|0) = \frac{1}{2\pi}\sqrt{4 - x^2} I_{S(0)}$ and

$$f_{CN}(x|y, \rho, 0) = \frac{(1 - \rho^2)\sqrt{4 - x^2}}{2\pi w_0(x, y|\rho, 0)} I_{S(0)},$$

for $|\rho| < 1$, $y \in S(0)$,

vi) $$\phi(x|y, \rho_1, z, \rho_2, 0) = \frac{(1 - \rho_1^2)(1 - \rho_2^2) w_0(y, z|\rho_1\rho_2, 0)\sqrt{4 - x^2}}{(1 - \rho_1^2\rho_2^2) w_0(x, y|\rho_1, 0) w_0(x, z|\rho_2, 0)} \frac{1}{2\pi} I_{S(0)},$$

where $w_0(x, y|\rho_1, 0)$ is given by (2.9).

**Proof.** To get i) compare (2.25) with $x$ replaced by $x/2$ and (2.3) for $q = 0$. To get ii) again compare (2.25) and (2.4) for $q = 0$ and notice that these recursions are the same, however with different initial values. To get iv) we notice that for $q=0$ and $n \geqslant 3$ we get 0. For $n<3$ we get these values directly from (2.22). iii) follows iv) and assertion i) of Proposition 2 or of course from ii) using (2.5). To get v) and vi) we insert $q=0$ in (2.10), (2.12) and (2.15). □

## 3. Main results

We will start this section with the presentation of an alternative form of the AW polynomials. Let $\{D_n(x|a,b,c,d,q)\}_{n\geqslant -1}$ be the sequence the AW polynomials such that $D_n$ has coefficient by $x^n$ equal to $2^n$. Thus the polynomials $\{D_n\}$ are orthogonal with respect to the density $\psi(x|a,b,c,d,q)$ mentioned in Remark 1. Let the polynomials $A_n$ be defined by the change of variables and parameters by the relationship:

$$A_n(x|y,\rho_1,z,\rho_2,q) = D_n(x\sqrt{1-q}/2|a,b,c,d,q),$$

with $a, b, c, d$ related to $y, \rho_1, z, \rho_2$ by (2.17)–(2.20). We have:

**Theorem 1.**

i) $\forall n \geqslant 1$:

$$\begin{aligned} &D_n(x|a,b,c,d,q) \\ &\quad = \frac{(ab,cd)_n}{(abcdq^{n-1})_n}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q b_{n-j}(x|q)\sum_{i=0}^{j}\begin{bmatrix} j \\ i \end{bmatrix}_q \frac{Q_i(x|a,b,q)Q_{j-i}(x|c,d,q)}{(ab)_i(cd)_{j-i}}, \end{aligned}$$

*where the polynomials* $\{Q_n(x|a,b,q)\}$ *and* $\{b_n(x|q)\}$ *are defined by respectively* (2.4) *and* (2.23).

ii) $\forall n \geqslant 1$:

$$\begin{aligned} &A_n(x|y,\rho_1,z,\rho_2,q) \\ &\quad = \frac{(\rho_1^2,\rho_2^2)_n}{(\rho_1^2\rho_2^2q^{n-1})_n}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q B_{n-j}(x|q)\sum_{i=0}^{j}\begin{bmatrix} j \\ i \end{bmatrix}_q \frac{P_i(x|y,\rho_1,q)P_{j-i}(x|z,\rho_2,q)}{(\rho_1^2)_i(\rho_2^2)_{j-i}}, \end{aligned}$$

*where the polynomials* $\{P_n(x|y,\rho,q)\}$ *and* $\{B_n(x|q)\}$ *are defined by respectively* (2.6) *and* (2.22).

iii) $\forall n \geqslant 1$:

$$\begin{aligned} &A_n(x|y,\rho_1,z,\rho_2,q) \\ &\quad = \frac{(\rho_1^2,\rho_2^2)_n}{(\rho_1^2\rho_2^2q^{n-1})_n}\sum_{m=0}^{n}(-1)^m q^{\binom{m}{2}}\begin{bmatrix} n \\ m \end{bmatrix}\rho_1^m \frac{P_{n-m}(x|z,\rho_2,q)P_m(y|x,\rho_1,q)}{(\rho_2^2)_{n-m}(\rho_1^2)_m}. \end{aligned}$$

**Proof.** i) We will use two facts concerning forms of the generating functions of the polynomials $D_n$ and $Q_n$. Namely in [15, formula (15.2.6)] and [18, formula (3.1.13)] we have the following formula adopted for the polynomials $D_n$

$$\sum_{n\geqslant 0}\frac{(abcdq^{n-1})_n D_n(x|a,b,c,d,q)}{(ab,cd,q)_n}t^n = {}_2\phi_1\left(\begin{matrix}ae^{i\theta},be^{i\theta}\\ ab\end{matrix}\Big|q,te^{-i\theta}\right){}_2\phi_1\left(\begin{matrix}c^{-i\theta},de^{-i\theta}\\ cd\end{matrix}\Big|q,te^{i\theta}\right),$$

where $x=\cos\theta$. On the other hand in [18] we have the following formula (3.8.14)

$$\sum_{n\geqslant 0}\frac{Q_n(x|a,b,q)}{(ab,q)_n}t^n = \frac{1}{(te^{i\theta})_\infty}{}_2\phi_1\left(\begin{matrix}ae^{i\theta},be^{i\theta}\\ ab\end{matrix}\Big|q,te^{-i\theta}\right),$$

again with $x=\cos\theta$. Noting that $\cos(-\theta)=\cos(\theta)$ we see that

$$\begin{aligned}&\left(te^{-i\theta},te^{i\theta}\right)_\infty\sum_{i\geqslant 0}\frac{Q_n(x|a,b,q)}{(ab,q)_n}t^n\sum_{i\geqslant 0}\frac{Q_n(x|c,d,q)}{(cd,q)_n}t^n\\ &\quad={}_2\phi_1\left(\begin{matrix}ae^{i\theta},be^{i\theta}\\ ab\end{matrix}\Big|q,te^{-i\theta}\right){}_2\phi_1\left(\begin{matrix}c^{-i\theta},de^{-i\theta}\\ cd\end{matrix}\Big|q,te^{i\theta}\right).\end{aligned}$$

Now it remains to notice that $(te^{-i\theta},te^{i\theta})_\infty=\prod_{k=0}^{\infty}(1-2xtq^k+t^2q^{2k})$, confront it with the formulae (2.22) and (2.23) and given in [7] generating function of the polynomials $B_n(x|q)$ and thus deduce that

$$\left(te^{-i\theta},te^{i\theta}\right)_\infty=\sum_{n\geqslant 0}\frac{b_n(x|q)t^n}{(q)_n}.$$

Next we apply twice the Cauchy formula for the multiplication of power series.

ii) Let us change parameters to ones given by (2.17)–(2.20) and let us also redefine the variable $x$ by introducing instead the variable $\xi=2x/\sqrt{1-q}$ and defining the polynomials $A_n(\xi|y,\rho_1,z,\rho_2,q)=2^{-n}p_n(x,a,b,c,d|q)/(abcdq^{n-1})_n$ where $a$, $b$, $c$, $d$ are given by (2.17)–(2.20).

iii) The proof of this formula is longer and thus is shifted to Section 5. □

As a corollary we get the following property of the ASC polynomials.

**Corollary 1.** $\forall n\geqslant 1$:

$$\sum_{m=0}^{n}(-1)^m q^{\binom{m}{2}}\begin{bmatrix}n\\ m\end{bmatrix}_q\rho_1^m\frac{P_{n-m}(x|z,\rho_2,q)P_m(y|x,\rho_1,q)}{(\rho_2^2)_{n-m}(\rho_1^2)_m},$$

$$\sum_{m=0}^{n}(-1)^m q^{\binom{m}{2}}\begin{bmatrix}n\\ m\end{bmatrix}_q\rho_2^m\frac{P_{n-m}(x|y,\rho_1,q)P_m(z|x,\rho_2,q)}{(\rho_1^2)_{n-m}(\rho_2^2)_m}.$$

**Proof.** Follows symmetry exposed in assertion ii) of the theorem. □

**Corollary 2.** *For $q = 0$ we get*

$$D_1(x|a,b,c,d,0) = 2x - \frac{a+b+c+d-abc-bcd-acd-abd}{1-abcd},$$
$$D_2(x|a,b,c,d,0) = 4x^2 - 2(a+b+c+d)x + ab + ac + ad + bc + bd + cd - 1 - abcd$$

*and generally for $n > 2$*

$$D_n(x|a,b,c,d,0) = \sum_{i=0}^{n} \frac{Q_i(x|a,b,0)Q_{n-i}(x|c,d,0)}{(ab;0)_i(cd;0)_{n-i}} - 2x\sum_{i=0}^{n-1} \frac{Q_i(x|a,b,0)Q_{n-1-i}(x|c,d,0)}{(ab;0)_i(cd;0)_{n-1-i}} + \sum_{i=0}^{n-2} \frac{Q_i(x|a,b,0)Q_{n-2-i}(x|c,d,0)}{(ab;0)_i(cd;0)_{n-2-i}},$$

*where $Q_i(x|a,b,0)$ and $(a;0)_i$ are defined by assertion* ii) *of Remark* 3 *and formulae from the beginning of Section* 2. *Similarly*

$$A_1(x|y,\rho_1,z,\rho_2,0) = x - \frac{y\rho_1(1-\rho_2^2) + z\rho_2(1-\rho_1^2)}{1-\rho_1^2\rho_2^2}$$

*and for $n \geqslant 2$*

$$\frac{A_n(x|y,\rho_1,z,\rho_2,0)}{(1-\rho_1^2)(1-\rho_2^2)} = \sum_{m=0}^{n} \rho_1^m \frac{P_{n-m}(x|z,\rho_2,0)P_m(y|x,\rho_1,0)}{(\rho_2^2;0)_{n-m}(\rho_1^2;0)_m} - \sum_{m=0}^{n-1} \rho_1^m \frac{P_{n-1-m}(x|z,\rho_2,0)P_m(y|x,\rho_1,0)}{(\rho_2^2;0)_{n-1-m}(\rho_1^2;0)_m},$$

*where $P_m(x|y,\rho,0)$ are given by assertion* iii) *of Remark* 3.

The main results of the paper concern calculating values of the functions defined by

$$C_n(y,z|\rho_1,\rho_2,q) = \int_{S(q)} H_n(x|q)\phi(x|y,\rho_1,z,\rho_2,q)\,dx,$$

$n \geqslant 1$. These functions have on one hand nice probabilistic interpretation. Namely assuming that certain 3-dimensional random vector $(Y,X,Z)$ has density equal to $f_{CN}(z|x,\rho_2,q)f_{CN}(x|y,\rho_1,q)f_N(y|q)$, then

$$C_n(y,z|\rho_1,\rho_2,q) = \mathbb{E}\big(H_n(X|q)\big|Y=y,\ Z=z\big), \tag{3.1}$$

for almost all (with respect to measure with density $f_{CN}(y|z,\rho_1\rho_2,q)f_N(z|q)$) $(y,z)\in S(q)\times S(q)$. This fact implies in particular that for almost all $(y,z)\in S(q)\times S(q)$ we have $|C_n(y,z|\rho_1,\rho_2,q)|\leqslant \frac{s_n(q)}{(1-q)^{n/2}}$.

**Remark 4.** In [20] it has been shown that functions $C_n$ are polynomials in $y$ and $z$ of order at most $n$. More precisely it has been shown that

$$C_n(y,z|\rho_1,\rho_2,q)=\sum_{r=0}^{\lfloor n/2\rfloor}\sum_{l=0}^{n-2r}A_{r,-\lfloor n/2\rfloor+r+l}^{(n)}H_l(y|q)H_{n-2r-l}(z|q),$$

where there are $\lfloor\frac{n+2}{2}\rfloor\lfloor\frac{n+3}{2}\rfloor$ constants (depending only on $n$, $q$, $\rho_1$, $\rho_2$) $A_{r,s}^{(n)}$; $r=0,\dots,\lfloor n/2\rfloor$, $s=-\lfloor n/2\rfloor+r,\dots,-\lfloor n/2\rfloor+r+n-2r$. However the exact general form of these constants was not found (except for the cases $n=1,2,3,4$).

As announced in the introduction, in the present paper we will, express the polynomials $C_n$ in terms of the polynomials $H_n$ and (or) $P_n$.

Namely we will prove the following theorem:

**Theorem 2.** $\forall n\geqslant 1$, $|q|<1$, $|\rho_1|,|\rho_2|<1$:

$$C_n(y,z|\rho_1,\rho_2,q)=\frac{1}{(\rho_1^2\rho_2^2)_n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\2k\end{bmatrix}_q\begin{bmatrix}2k\\k\end{bmatrix}_q[k]_q! \tag{3.2}$$

$$\times\rho_2^{2k}\rho_1^{2k}\left(\rho_1^2,\rho_2^2\right)_k\sum_{j=0}^{n-2k}\begin{bmatrix}n-2k\\j\end{bmatrix}_q\left(\rho_1^2q^k\right)_j\left(\rho_2^2q^k\right)_{n-2k-j}$$

$$\times\rho_1^{n-2k-j}\rho_2^jH_j(z|q)H_{n-2k-j}(y|q). \tag{3.3}$$

Before presentation of the proof let us make two immediate remarks.

**Remark 5.** Notice that for, say $\rho_1=0$ we get $C_n(y,z|0,\rho_2,q)=\rho_2^nH_n(z|q)$ which agrees nicely with the probabilistic interpretation of the function $C_n$ given above. Compare also assertion ii) of Proposition 1. It is so since $C_n(y,z|0,\rho,q)=\mathbb{E}(H_n(X|q)|Z=z)=\rho^nH_n(z|q)$ a.s., $(f_N)$ if $(Y,Z)\sim f_{CN}(y|z,\rho,q)f_N(z|q)$ as shown in [6].

**Remark 6.** Keeping in mind the probabilistic interpretation of the functions $C_n$ given in (3.1), notice that the assertion of Theorem 2 enables calculation of all moments of the AW density for complex parameters. Recently S. Corteel at al. in [13] announced that she is going to calculate these moments by some combinatorial methods.

The proof of this theorem is based on the following lemma that in another form and with the different proof (based heavily on the assertion i) of Lemma 2) was presented in [20]. Notice that assertion i) of this lemma is in fact a generalization of an old result of Carlitz [12] (see also [1] or partially [15, Exercise 12.3(d)]). Besides, in this lemma we present an alternative form of the function $C_n$ this time expressed through polynomials $H_n$ and $P_n$.

**Lemma 3.** *Let us denote for $m,k\geqslant 0$ $\gamma_{m,k}(x,y|\rho,q)=\sum_{i=0}^{\infty}\frac{\rho^i}{[i]_q!}H_{i+m}(x|q)H_{i+k}(y|q)$. Then*

i) $$\gamma_{m,k}(x,y|\rho,q)=\gamma_{0,0}(x,y|\rho,q)\sum_{s=0}^{k}(-1)^s q^{\binom{s}{2}}\begin{bmatrix}k\\ s\end{bmatrix}_q \rho^s H_{k-s}(y|q)P_{m+s}(x|y,\rho,q)/\left(\rho^2\right)_{m+s},$$

ii) $$C_n(y,z|\rho_1,\rho_2,q)=\sum_{s=0}^{n}\begin{bmatrix}n\\ s\end{bmatrix}_q \rho_1^{n-s}\rho_2^s\left(\rho_1^2\right)_s H_{n-s}(y|q)P_s(z|y,\rho_1\rho_2,q)/\left(\rho_1^2\rho_2^2\right)_s.$$

**Proof.** The proof is shifted to Section 5. □

As a corollary we get another property of the ASC polynomials:

**Corollary 3.** *For $m\geqslant 0$*

$$P_m(y|x,\rho,q)/\left(\rho^2\right)_m=\sum_{s=0}^{m}(-1)^s\begin{bmatrix}m\\ s\end{bmatrix}_q q^{\binom{s}{2}}\rho^s H_{m-s}(y|q)P_s(x|y,\rho,q)/\left(\rho^2\right)_s.$$

**Proof.** Note that $\gamma_{m,k}(x,y|\rho,q)=\gamma_{k,m}(y,x|\rho,q)$. From assertion ii) of Lemma 3 it follows that on one hand $\gamma_{0,m}(x,y|\rho,q)=\gamma_{0,0}(x,y|\rho,q)P_m(y|x,\rho,q)/(\rho^2)_m$. On the other hand from assertion i) it follows that

$$\gamma_{0,m}(x,y|\rho,q)=\gamma_{0,0}(x,y|\rho,q)\sum_{s=0}^{m}(-1)^s q^{\binom{s}{2}}\begin{bmatrix}m\\ s\end{bmatrix}_q \rho^s H_{m-s}(y|q)P_s(x|y,\rho,q)/\left(\rho^2\right)_s. \qquad \square$$

As another consequence of Theorem 2 and assertions v) and vii) of Proposition 1 we get the following theorem:

**Theorem 3.** $\forall -1<q\leqslant 1$, $x,y,z\in S(q)$, $|\rho_1|,|\rho_2|<1$,

$$\phi(x|y,\rho_1,z,\rho_2,q)=f_N(x|q)\sum_{i=0}^{\infty}\frac{1}{[i]_q!}H_i(x|q)C_i(y,z|\rho_1,\rho_2,q), \tag{3.4}$$

*where convergence is absolute and almost uniform on compact sets.*

**Proof.** Is shifted to Section 5. □

## 4. Open problems

Notice that $\forall n\geqslant 1$:

$$\int_{S(q)}\left(H_n(x)-C_n(y,z|\rho_1,\rho_2,q)\right)\phi(x|y,\rho_1,z,\rho_2,q)\,dx$$
$$=\int_{S(q)}A_n(x|y,\rho_1,z,\rho_2,q)\phi(x|y,\rho_1,z,\rho_2,q)\,dx=0.$$

Hence there must exist polynomials $F_{n,i}(y,z|\rho_1,\rho_2,q)$ such that: $\forall n\geqslant 1$:

$$A_n(x|y,\rho_1,z,\rho_2,q)=\sum_{i=1}^{n}F_{n,i}(y,z|\rho_1,\rho_2,q)\bigl(H_i(x)-C_i(y,z|\rho_1,\rho_2,q)\bigr).$$

(1) One would like to find these polynomials.
(2) We have $F_{n,n}(y,z|\rho_1,\rho_2,q)=1$ (both $\{H_n(x)-C_n(y,z|\rho_1,\rho_2,q)\}$ and $\{A_n(x|y,\rho_1,z,\rho_2,q)\}$ are monic). When say $\rho_2=0$ (the ASC case) we have

$$P_n(x|y,\rho_1,q)=\sum_{i=1}^{n}\begin{bmatrix}n\\ i\end{bmatrix}_q\rho_1^{n-i}B_{n-i}(y|q)\bigl(H_i(x)-\rho_1^iH_i(y|q)\bigr),$$

which is in fact combination of assertions i) and ii) of Proposition 2. Thus one would like to ask if the functions $F_{n,i}(y,z|\rho_1,\rho_2,q)$ also depend on $n-i$?
(3) It was shown in [7] that

$$\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_qB_{n-j}(y|q)H_j(y|q)=0$$

for $y\in S(q)$ and $n\geqslant 1$. Is the same true for the general case. Namely is it true that: $\forall n\geqslant 1$, $y,z\in S(q)$

$$\sum_{j=0}^{n}F_{n,j}(y,z|\rho_1,\rho_2,q)C_i(y,z|\rho_1,\rho_2,q)=0?$$

(4) If $q=1$ we have

$$\frac{1}{\sqrt{2\pi(1-\rho^2)}}\int_{\mathbb{R}}H_n(x)\exp\biggl(-\frac{(x-\rho m)}{2(1-\rho^2)}\biggr)dx=\rho^nH_n(m)$$

hence following observation (2.16) we deduce that the rôle of the parameter $\rho$ is now played by $\sqrt{\frac{\rho_1^2+\rho_2^2-2\rho_1^2\rho_2^2}{1-\rho_1^2\rho_2^2}}$ and of $m$ by $\frac{y\rho_1(1-\rho_2^2)+z\rho_2(1-\rho_1^2)}{\sqrt{1-\rho_1^2\rho_2^2}\sqrt{\rho_1^2+\rho_2^2-2\rho_1^2\rho_2^2}}$. Thus

$$C_n(y,z|\rho_1,\rho_2,1)=\biggl(\sqrt{\frac{\rho_1^2+\rho_2^2-2\rho_1^2\rho_2^2}{1-\rho_1^2\rho_2^2}}\biggr)^nH_n\biggl(\frac{y\rho_1(1-\rho_2^2)+z\rho_2(1-\rho_1^2)}{\sqrt{1-\rho_1^2\rho_2^2}\sqrt{\rho_1^2+\rho_2^2-2\rho_1^2\rho_2^2}}\biggr). \tag{4.1}$$

Is it also true for $|q|<1$ with an obvious modification that $H_n(x)$ is replaced by $H_n(x|q)$. Most certainly not, but may be $C_n(y,z|\rho_1,\rho_2,q)$ can be presented as a linear combination of expression of this type, more compact than (3.2), (3.3). The problem is connected with the problem of expressing $H_n(\alpha x+\beta y|q)$ as a linear combination of $H_i(x|q)H_j(y|q)$, $i+j\leqslant n$. It has known, nice form for $q=1$ and neither nice nor known form for all $n\geqslant 1$ and other values of $q$.

## 5. Proofs

**Proof of Lemma 1.** i) Let us denote $D_n(a) = \sum_{k=0}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q (a)_{n-k} a^k$. Let

$$\phi(t,a) = \sum_{n=0}^{\infty} \frac{t^n}{(q)_n} D_n(a)$$

be a characteristic function of the sequence $\{D_n(a)\}$. We have

$$\phi(t,a) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q (a)_i a_{n-i} = \sum_{i=0}^{\infty} \frac{t^i}{(q)_i}(a)_i \sum_{n=i}^{\infty} \frac{t^{n-i}}{(q)_{n-i}} a^{n-i} = \frac{1}{(at)_\infty} \sum_{i=0}^{\infty} \frac{t^i}{(q)_i}(a)_i$$
$$= \frac{1}{(at)_\infty} \frac{(at)_\infty}{(t)_\infty} = \frac{1}{(t)_\infty} = \sum_{n \geqslant 0} \frac{t^n}{(q)_n},$$

by $q$-binomial theorem. So $D_n(a) = 1$. Convergence was for $|q|, |a|, |t| < 1$. Thus $D_n(a)$ for $|a| < 1$ is constant, but since it is a polynomial we deduce that $D_n(a)$ is constant for all complex $a$.

ii) Using the expansion formula $\sum_{k=0}^{N} (-1)^k \left[\begin{smallmatrix} N \\ k \end{smallmatrix}\right]_q q^{\binom{k}{2}} x^k = (x)_N$,

$$\sum_{i=0}^{n} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q (a)_i b^i (abq^i)_{n-i} = \sum_{i=0}^{n} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q b^i (a)_i \sum_{k=0}^{n-i} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n-i \\ k \end{bmatrix}_q a^k b^k q^{ki}$$
$$= \sum_{s=0}^{n} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} n \\ s \end{bmatrix}_q b^s \sum_{k=0}^{s} \begin{bmatrix} s \\ k \end{bmatrix}_q a^k (a)_{s-k}$$
$$= \sum_{s=0}^{n} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} n \\ s \end{bmatrix}_q b^s = (b)_n$$

by i) and the expansion formula. □

**Proof of Lemma 2.** ii) First let us recall that by assertion ii) of Proposition 2 we have $I_{n,0}(x|q) = 0$ for $n \geqslant 1$. Next we have

$$I_{0,m}(x|q) = H_m(x|q), \qquad I_{1,m}(x|q) = -xH_m(x|q) + H_{m+1}(x|q) = -[m]_q H_{m-1}(x|q).$$

To prove ii) we apply the formula

$$H_n(x|q)H_m(x|q) = H_{n+m}(x|q) + \sum_{k=1}^{\min(n,m)} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q [k]_q! H_{n+m-2k}(x|q)$$

and get

$$\begin{aligned}
I_{n,m}(x|q) &= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q B_{n-i}(x|q) H_{i+m}(x|q) \\
&= H_m(x|q) I_{n,0}(x|q) - \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q B_{n-i}(x|q) \sum_{k=1}^{\min(i,m)} \begin{bmatrix} i \\ k \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q [k]_q! H_{i+m-2k}(x|q) \\
&= -\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q B_{n-i}(x|q) \sum_{k=1}^{\min(i,m)} \begin{bmatrix} i \\ k \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q [k]_q! H_{i+m-2k}(x|q).
\end{aligned}$$

After changing the order of summation we get

$$I_{n,m}(x|q) = -\sum_{k=1}^{n} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q [k]_q! \sum_{s=0}^{n-k} \begin{bmatrix} n-k \\ s \end{bmatrix}_q B_{n-k-s}(x|q) H_{s+m-k}(x|q).$$

iii) will be proved by induction with respect to $n$. Let us assume that the assertion is true for all $n \leqslant k-1$. By ii) we have $I_{k,m}(x|q) = -\sum_{j=1}^{k} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q [j]_q! I_{k-j,m-j}(x|q)$. Now if $m < k$ we see that then $k-j < m-j$ for all $j = 1, \ldots, k$ and thus by induction $I_{k-j,m-j}(x|q) = 0$. If $k \geqslant m$ then by the induction assumption we have $I_{k-j,m-j}(x|q) = (-1)^{k-j} q^{\binom{k-j}{2}} \frac{[m-j]_q!}{[m-k]_q!} H_{m-k}(x|q)$. Hence

$$\begin{aligned}
I_{k,m}(x|q) &= -\sum_{j=1}^{k} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q [j]_q (-1)^{k-j} q^{\binom{k-j}{2}} \frac{[m-j]_q!}{[m-k]_q!} H_{m-k}(x|q) \\
&= -\frac{[m]_q!}{[m-k]_q!} H_{m-k}(x|q) \sum_{j=1}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^{k-j} q^{\binom{k-j}{2}} \\
&= -\frac{[m]_q!}{[m-k]_q!} H_{m-k}(x|q) \sum_{s=0}^{k-1} \begin{bmatrix} k \\ s \end{bmatrix}_q (-1)^{s} q^{\binom{s}{2}} \\
&= -\frac{[m]_q!}{[m-k]_q!} H_{m-k}(x|q) \left( \sum_{s=0}^{k-1} \begin{bmatrix} k \\ s \end{bmatrix}_q (-1)^{s} q^{\binom{s}{2}} + (-1)^k q^{\binom{k}{2}} - (-1)^k q^{\binom{k}{2}} \right) \\
&= (-1)^k q^{\binom{k}{2}} \frac{[m]_q!}{[m-k]_q!} H_{m-k}(x|q),
\end{aligned}$$

since $\sum_{s=0}^{k-1} \begin{bmatrix} k \\ s \end{bmatrix}_q (-1)^s q^{\binom{s}{2}} + (-1)^k q^{\binom{k}{2}} = (1)_k = 0$. □

**Proof of assertion iii) of Theorem 1.** We start with the assertion of Corollary 3 and the assertion iii) of Proposition 1. Using them we get

$$\int_{S(q)} P_m(z|y,t,q) P_k(y|z,t,q) f_{CN}(z|y,t,q)\, dz = \begin{cases} 0 & \text{if } m > k, \\ (-1)^m q^{\binom{m}{2}} \frac{[k]_q!}{[k-m]_q!} t^m H_{k-m}(y|q)(t^2)_k & \text{if } m \leqslant k. \end{cases}$$

Using the assertion ii) of Theorem 1 let us calculate

$$V_{n,m}(x,z,\rho_1,\rho_2|q)=\int_{S(q)}A_n(x|y,\rho_1,z,\rho_2,q)P_m(y|x,\rho_1,q)f_{CN}(y|x,\rho_1,q)\,dy.$$

We have

$$\begin{aligned}
V_{n,m}(x,z,\rho_1,\rho_2|q)&=\frac{(\rho_1^2,\rho_2^2)_n}{(\rho_1^2\rho_2^2q^{n-1})_n}(-1)^m q^{\binom{m}{2}}\rho_1^m\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_q B_{n-j}(x|q)\\
&\quad\times\sum_{i=m}^{j}\begin{bmatrix}j\\ i\end{bmatrix}_q\frac{P_{j-i}(x|z,\rho_2,q)}{(\rho_2^2)_{j-i}}\frac{[i]_q!}{[i-m]_q!}H_{i-m}(x|q)\\
&=\frac{(\rho_1^2,\rho_2^2)_n}{(\rho_1^2\rho_2^2q^{n-1})_n}(-1)^m q^{\binom{m}{2}}\rho_1^m\sum_{j=m}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_q B_{n-j}(x|q)\\
&\quad\times\sum_{k=0}^{j-m}\frac{[j]_q!}{[j-m-k]_q![k]_q!}\frac{P_{j-m-k}(x|z,\rho_2,q)}{(\rho_2^2)_{j-m-k}}H_k(x|q)\\
&=\frac{(\rho_1^2,\rho_2^2)_n}{(\rho_1^2\rho_2^2q^{n-1})_n}(-1)^m q^{\binom{m}{2}}\rho_1^m\frac{[n]_q!}{[n-m]_q!}\\
&\quad\times\sum_{s=0}^{n-m}B_{n-m-s}(x|q)\frac{[n-m]_q!}{[n-m-s]_q![s]_q!}\\
&\quad\times\sum_{k=0}^{s}\frac{[s]_q!}{[s-k]_q![k]_q!}\frac{P_{s-k}(x|z,\rho_2,q)}{(\rho_2^2)_{s-k}}H_k(x|q).
\end{aligned}$$

We change the order of summation and get

$$\begin{aligned}
V_{n,m}(x,z,\rho_1,\rho_2|q)&=\frac{(\rho_1^2,\rho_2^2)_n}{(\rho_1^2\rho_2^2q^{n-1})_n}(-1)^m q^{\binom{m}{2}}\rho_1^m\frac{[n]_q!}{[n-m]_q!}\sum_{k=0}^{n-m}\begin{bmatrix}n-m\\ k\end{bmatrix}_q H_k(x|q)\\
&\quad\times\sum_{s=k}^{n-m}\begin{bmatrix}n-m-k\\ s-k\end{bmatrix}_q\frac{P_{s-k}(x|z,\rho_2,q)}{(\rho_2^2)_{s-k}}B_{n-m-s}(x|q)\\
&=\frac{(\rho_1^2,\rho_2^2)_n}{(\rho_1^2\rho_2^2q^{n-1})_n}(-1)^m q^{\binom{m}{2}}\rho_1^m\frac{[n]_q!}{[n-m]_q!}\sum_{k=0}^{n-m}\begin{bmatrix}n-m\\ k\end{bmatrix}_q H_k(x|q)\\
&\quad\times\sum_{j=0}^{n-m-k}\begin{bmatrix}n-m-k\\ j\end{bmatrix}_q\frac{P_j(x|z,\rho_2,q)}{(\rho_2^2)_j}B_{n-m-k-j}(x|q)
\end{aligned}$$

$$= \frac{(\rho_1^2,\rho_2^2)_n}{(\rho_1^2\rho_2^2q^{n-1})_n}(-1)^m q^{\binom{m}{2}}\rho_1^m \frac{[n]_q!}{[n-m]_q!}\sum_{j=0}^{n-m}\begin{bmatrix} n-m \\ j\end{bmatrix}_q \frac{P_j(x|z,\rho_2,q)}{(\rho_2^2)_j}$$
$$\times \sum_{k=0}^{n-m-j}\begin{bmatrix} n-m-j \\ k\end{bmatrix}_q H_k(x|q)B_{n-m-k-j}(x|q).$$

Now we use the assertion iii) of Lemma 2 and deduce that

$$\sum_{k=0}^{n-m-j}\begin{bmatrix} n-m-j \\ k\end{bmatrix}_q H_k(x|q)B_{n-m-k-j}(x|q) = 0$$

if only $n-m-j>0$, and 1 if $j=n-m$. Hence

$$V_{n,m}(x,z,\rho_1,\rho_2|q) = \frac{(\rho_1^2,\rho_2^2)_n}{(\rho_1^2\rho_2^2q^{n-1})_n}(-1)^m q^{\binom{m}{2}}\rho_1^m \frac{[n]_q!}{[n-m]_q!}\frac{P_{n-m}(x|z,\rho_2,q)}{(\rho_2^2)_{n-m}}.$$

Keeping in mind the assertion iii) of Proposition 1 and the interpretation of $V_{n,m}$ we get

$$A_n(x|y,\rho_1,z,\rho_2,q)$$
$$= \frac{(\rho_1^2,\rho_2^2)_n}{(\rho_1^2\rho_2^2q^{n-1})_n}\sum_{m=0}^{n}\begin{bmatrix} n \\ m\end{bmatrix}_q (-1)^m q^{\binom{m}{2}}\rho_1^m \frac{P_{n-m}(x|z,\rho_2,q)P_m(y|x,\rho_1,q)}{(\rho_2^2)_{n-m}(\rho_1^2)_m}. \qquad \square$$

**Proof of Lemma 3.** i) First notice that $\gamma_{0,0}(x,y|\rho,q)f_N(x|q) = f_{CN}(x|y,\rho,q)$ (compare (2.21)). Besides we will use assertions i) and ii) of Proposition 1. Since for $\forall x,y\in S(q)$, $\gamma_{0,0}(x,y|\rho,q)>0$ we can write

$$\int_{S(q)} P_n(x|y,\rho,q)\gamma_{m,k}(x,y|\rho,q)f_N(x|q)\,dx$$
$$= \int_{S(q)} P_n(x|y,\rho,q)\frac{\gamma_{m,k}(x,y|\rho,q)}{\gamma_{0,0}(x,y|\rho,q)}f_{CN}(x|y,\rho,q)\,dx.$$

Now

$$\int_{S(q)} P_n(x|y,\rho,q)\gamma_{m,k}(x,y|\rho,q)f_N(x|q)\,dx$$
$$= \sum_{i\geqslant 0}\frac{\rho^i}{[i]_q!}H_{i+k}(y|q)\int_{S(q)} P_n(x|y,\rho,q)H_{i+m}(x|q)f_N(x|q)\,dx.$$

Let us recall the assertion i) of Proposition 2. Hence we have

$$\int\limits_{S(q)} P_n(x|y,\rho,q)\gamma_{m,k}(x,y|\rho,q)f_N(x|q)\,dx$$
$$=\sum_{i\geqslant 0}\frac{\rho^i}{[i]_q!}H_{i+k}(y|q)\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q\rho^{n-j}B_{n-j}(y|q)\int\limits_{S(q)} H_j(x|q)H_{i+m}(x|q)f_N(x|q)\,dx.$$

Obviously if $m>n$ we get 0. Otherwise when $n\geqslant m$ we obtain

$$\int\limits_{S(q)} P_n(x|y,\rho,q)\gamma_{m,k}(x,y|\rho,q)f_N(x|q)\,dx$$
$$=\frac{[n]_q!\rho^{n-m}}{[n-m]_q!}\sum_{i=0}^{n-m}\frac{[n-m]_q!}{[i]_q![n-i-m]}H_{i+k}(y|q)B_{n-i-m}(y|q)$$
$$=\frac{[n]_q!\rho^{n-m}}{[n-m]_q!}I_{n-m,k}(y|q)=(-1)^{n-m}q^{\binom{n-m}{2}}\frac{[n]_q!\rho^{n-m}[k]_q!}{[n-m]_q![k+m-n]_q!}H_{k+m-n}(y|q).$$

Hence

$$\frac{\gamma_{m,k}(x,y|\rho,q)}{\gamma_{0,0}(x,y,|\rho,q)}=\sum_{n=m}^{m+k}(-1)^{n-m}q^{\binom{n-m}{2}}\rho^{n-m}\begin{bmatrix}k\\n-m\end{bmatrix}_q H_{k-(n-m)}(y|q)P_n(x|y,\rho,q)/\left(\rho^2\right)_n$$

or equivalently

$$\frac{\gamma_{m,k}(x,y|\rho,q)}{\gamma_{0,0}(x,y|\rho,q)}=\sum_{s=0}^{k}(-1)^s q^{\binom{s}{2}}\begin{bmatrix}k\\s\end{bmatrix}_q\rho^s H_{k-s}(y|q)P_{m+s}(x|y,\rho,q)/\left(\rho^2\right)_{m+s}.$$

ii) We have

$$\begin{aligned}C_n(x,y|\rho_1,\rho_2,q)&=\frac{1}{\gamma_{0,0}(x,y|\rho_1\rho_2,q)}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q\rho_1^{n-i}\rho_2^{i}\gamma_{i,n-i}(x,y,\rho_1\rho_2|q)\\&=\frac{1}{\gamma_{0,0}(x,y|\rho_1\rho_2,q)}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q\rho_1^{n-i}\rho_2^{i}\sum_{j=0}^{n-i}(-1)^j\begin{bmatrix}n-i\\j\end{bmatrix}_q q^{\binom{j}{2}}\rho_1^j\rho_2^j\\&\quad\times H_{n-i-j}(y|q)P_{i+j}(x|y,\rho_1\rho_2,q)/\left(\rho_1^2\rho_2^2\right)_{i+j}\\&=\sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_q H_{n-s}(y|q)P_s(x|y,\rho_1\rho_2,q)/\left(\rho_1^2\rho_2^2\right)_s\\&\quad\times\sum_{j=0}^{s}(-1)^j\begin{bmatrix}s\\j\end{bmatrix}_q q^{\binom{j}{2}}\rho_1^j\rho_2^j\rho_1^{n-s+j}\rho_2^{s-j}.\end{aligned}$$

Now using formula (12.2.27) of [15], that is $(a)_n=\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n\\ k\end{bmatrix}_q q^{\binom{k}{2}}a^k$ we get

$$C_n(x,y|\rho_1,\rho_2,q)=\sum_{s=0}^{n}\begin{bmatrix} n\\ s\end{bmatrix}_q \rho_1^{n-s}\rho_2^{s}\left(\rho_1^2\right)_s H_{n-s}(y|q)P_s(x|y,\rho_1\rho_2,q)/\left(\rho_1^2\rho_2^2\right)_s. \qquad \square$$

**Proof of Theorem 2.**

$$\begin{aligned}
C_n(y,z|\rho_1,\rho_2,q)&=\sum_{s=0}^{n}\begin{bmatrix} n\\ s\end{bmatrix}_q \rho_1^{n-s}\rho_2^{s}\left(\rho_1^2\right)_s H_{n-s}(y|q)P_s(z|y,\rho_1\rho_2,q)/\left(\rho_1^2\rho_2^2\right)_s\\
&=\sum_{s=0}^{n}\begin{bmatrix} n\\ s\end{bmatrix}_q \rho_1^{n-s}\rho_2^{s}\left(\rho_1^2\right)_s H_{n-s}(y|q)\\
&\quad\times\sum_{j=0}^{s}\begin{bmatrix} s\\ j\end{bmatrix}_q \rho_1^{s-j}\rho_2^{s-j}B_{s-j}(y|q)H_j(z|q)/\left(\rho_1^2\rho_2^2\right)_s\\
&=\frac{1}{(\rho_1^2\rho_2^2)_n}\sum_{j=0}^{n}\begin{bmatrix} n\\ j\end{bmatrix}_q \rho_1^{n-j}H_j(z|q)\\
&\quad\times\sum_{s=j}^{n}\begin{bmatrix} n-j\\ s-j\end{bmatrix}_q \left(\rho_1^2\right)_s\rho_2^{2s-j}\left(\rho_1^2\rho_2^2q^s\right)_{n-s}B_{s-j}(y|q)H_{n-s}(y|q)\\
&\quad\times\frac{1}{(\rho_1^2\rho_2^2)_n}\sum_{j=0}^{n}\begin{bmatrix} n\\ j\end{bmatrix}_q \rho_1^{n-j}\rho_2^{j}H_j(z|q)\\
&\quad\times\sum_{m=0}^{n-j}\begin{bmatrix} n-j\\ m\end{bmatrix}_q \left(\rho_1^2\right)_{m+j}\rho_2^{2m}\left(\rho_1^2\rho_2^2q^{m+j}\right)_{n-j-m}B_m(y|q)H_{n-j-m}(y|q).
\end{aligned}$$

Now we apply the formula given in the assertion iv) of Lemma 2 getting

$$\begin{aligned}
C_n(y,z|\rho_1,\rho_2,q)&=\frac{1}{(\rho_1^2\rho_2^2)_n}\begin{bmatrix} n\\ j\end{bmatrix}_q \rho_1^{n-j}\rho_2^{j}\left(\rho_1^2\right)_j H_j(z|q)\\
&\quad\times\sum_{m=0}^{n-j}\begin{bmatrix} n-j\\ m\end{bmatrix}_q \left(\rho_1^2q^j\right)_m\rho_2^{2m}\left(\rho_1^2\rho_2^2q^{m+j}\right)_{n-j-m}\\
&\quad\times(-1)^m q^{\binom{m}{2}}\sum_{k=0}^{\lfloor (n-j)/2\rfloor}\begin{bmatrix} m\\ k\end{bmatrix}_q\begin{bmatrix} n-j-k\\ k\end{bmatrix}_q [k]_q!\,q^{-k(m-k)}H_{n-j-2k}(y|q).
\end{aligned}$$

Now we notice that $\begin{bmatrix} m\\ k\end{bmatrix}_q=0$ if $k>m$. So we split the range of $m$ into two subranges $0,\dots,\lfloor (n-j)/2\rfloor$ and $\lfloor (n-j)/2\rfloor+1,\dots,n-j$. Thus the second sum can be transformed in the following way:

$$\sum_{m=0}^{\lfloor (n-j)/2\rfloor} \begin{bmatrix} n-j \\ m \end{bmatrix}_q \left(\rho_1^2 q^j\right)_m \rho_2^{2m} \left(\rho_1^2\rho_2^2 q^{m+j}\right)_{n-j-m} (-1)^m q^{\binom{m}{2}}$$

$$\times \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n-j-k \\ k \end{bmatrix}_q [k]_q! q^{-k(m-k)} H_{n-j-2k}(y|q)$$

$$+ \sum_{m=\lfloor (n-j)/2\rfloor+1}^{n-j} \begin{bmatrix} n-j \\ m \end{bmatrix}_q \left(\rho_1^2 q^j\right)_m \rho_2^{2m} \left(\rho_1^2\rho_2^2 q^{m+j}\right)_{n-j-m}$$

$$\times (-1)^m q^{\binom{m}{2}} \sum_{k=0}^{\lfloor (n-j)/2\rfloor} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n-j-k \\ k \end{bmatrix}_q [k]_q! q^{-k(m-k)} H_{n-j-2k}(y|q).$$

Now after changing the order of summation we obtain

$$\sum_{k=0}^{\lfloor (n-j)/2\rfloor} \begin{bmatrix} n-j-k \\ k \end{bmatrix}_q [k]_q! H_{n-j-2k}(y|q)$$

$$\times \sum_{m=k}^{\lfloor (n-j)/2\rfloor} (-1)^m q^{\binom{m}{2}} q^{-k(m-k)} \begin{bmatrix} n-j \\ m \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q \left(\rho_1^2 q^j\right)_m \rho_2^{2m} \left(\rho_1^2\rho_2^2 q^{m+j}\right)_{n-j-m}$$

$$+ \sum_{k=0}^{\lfloor (n-j)/2\rfloor} \begin{bmatrix} n-j-k \\ k \end{bmatrix}_q [k]_q! H_{n-j-2k}(y|q)$$

$$\times \sum_{m=\lfloor (n-j)/2\rfloor+1}^{n-j} (-1)^m q^{\binom{m}{2}} q^{-k(m-k)} \begin{bmatrix} n-j \\ m \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q \left(\rho_1^2 q^j\right)_m \rho_2^{2m} \left(\rho_1^2\rho_2^2 q^{m+j}\right)_{n-j-m}$$

$$= \sum_{k=0}^{\lfloor (n-j)/2\rfloor} \begin{bmatrix} n-j-k \\ k \end{bmatrix}_q [k]_q! H_{n-j-2k}(y|q)$$

$$\times \sum_{m=k}^{n-j} (-1)^m q^{\binom{m}{2}} q^{-k(m-k)} \begin{bmatrix} n-j \\ m \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q \left(\rho_1^2 q^j\right)_m \rho_2^{2m} \left(\rho_1^2\rho_2^2 q^{m+j}\right)_{n-j-m}.$$

After changing in the last sum the variable $m$ ranging from $k,\dots,m-j$ to $s$ ranging from 0 to $n-j-k$ and applying firstly formula $\binom{s+k}{2} - sk = \binom{s}{2} + \binom{k}{2}$, then formula $(a)_{n+m} = (a)_n (aq^n)_m$ and finally assertion ii) of Lemma 1 we get

$$C_n(y,z|\rho_1,\rho_2,q) = \frac{1}{(\rho_1^2\rho_2^2)_n} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \rho_1^{n-j} \rho_2^j H_j(z|q)$$

$$\times \sum_{k=0}^{\lfloor (n-j)/2\rfloor} (-1)^k q^{\binom{k}{2}} \rho_2^{2k} \left(\rho_1^2\right)_{k+j} \left(\rho_2^2\right)_{n-j-k} \frac{[n-j]_q!}{[n-j-2k]_q!} H_{n-j-2k}(y|q).$$

Now we change again the order of summing, applying formulae $(a)_{n+m} = (a)_n (aq^n)_m$ applied to $(\rho_1^2)_{k+j}$ and $(\rho_2^2)_{n-j-k}$ we get

$$C_n(y,z|\rho_1,\rho_2,q) = \frac{1}{(\rho_1^2\rho_2^2)_n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k q^{\binom{k}{2}}\begin{bmatrix} n \\ 2k\end{bmatrix}_q \begin{bmatrix} 2k \\ k\end{bmatrix}_q [k]_q!\rho_2^{2k}\rho_1^{2k}\left(\rho_1^2,\rho_2^2\right)_k \\ \times \sum_{j=0}^{n-2k}\begin{bmatrix} n-2k \\ j\end{bmatrix}_q \left(\rho_1^2 q^k\right)_j\left(\rho_2^2 q^k\right)_{n-j-2k}\rho_1^{n-2k-j}\rho_2^j \\ \times H_j(z|q)H_{n-j-2k}(y|q). \qquad \square$$

**Proof of Theorem 3.** For $|q| < 1$ we use the assertion vii) of Proposition 1 and Remark 1 and deduce that $\phi(x|y,\rho_1,z,\rho_2,q)/f_N(x|q)$ is bounded on $S(q)$ hence square integrable with respect to the measure with density $f_N(x|q)$, thus immediately we get $L_2$ convergence in (3.4). To get almost sure convergence let us notice that $\phi(x|y,\rho_1,z,\rho_2,q)/f_N(x|q)$ is also square integrable with respect to the measure that has density equal to $f_N(x|q)f_N(y|q)f_N(z|q)$. Next we notice that polynomials $\{H_i(x|q)H_j(y|q)H_k(z|q)\}_{i,j,k\geqslant 0}$ constitute an orthogonal basis of the space $(S(q)\times S(q)\times S(q), \mathcal{B}, f_N(x|q)f_N(y|q)f_N(z|q))$, where $\mathcal{B}$ denotes $\sigma$-field of Borel subsets of $S(q)\times S(q)\times S(q)$. Moreover we know Fourier coefficients of expansion of $\phi(x|y,\rho_1,z,\rho_2,q)/f_N(x|q)$ in this basis. Namely we can read them from expansion (3.2), (3.3). They are equal to

$$\alpha_{n,j,m} = \int_{S^3(q)} H_n(x|q)H_j(y|q)H_m(z|q)\phi(x|y,\rho_1,z,\rho_2,q)f_N(y|q)f_N(z|q)\,dx\,dy\,dz \\ = \begin{cases} 0 & \text{if } j+m\geqslant n \vee n-j-m \text{ is odd},\\ (-1)^k\frac{q^{\binom{k}{2}}\rho_1^{n-j}(\rho^2)_{j+k}\rho_2^{n-m}(\rho_2^2)_{n-j-k}}{[k]_q!(\rho_1^2\rho_2^2)_n} & \text{if } n-j-m=2k.\end{cases}$$

From the theory of the orthogonal series expansions it follows that $\sum_{n,j,m}\alpha_{n,j,m}^2<\infty$, moreover one can see these coefficients decrease geometrically.

Hence $\sum_{n,j,m}\alpha_{n,j,m}^2(\log n\log j\log m)^2<\infty$ and thus form the Rademacher–Menshov theorem we get almost everywhere convergence of the series:

$$\sum_{n,j,m\geqslant 0}\frac{\alpha_{n,j,m}}{[n]_q![j]_q![m]_q!}H_n(x|q)H_j(y|q)H_m(z|q).$$

On the other hand after regrouping nonzero summands of this series we get (3.4).

For $q=1$ we deal with the normal case. In this case the functions $C_n$ have special form given by (4.1). Thus we deal with summing of special form of a classical Poisson–Mehler kernel

$$\sum_{n\geqslant 0}\frac{t^n}{n!}H_n(x)H_n(u),$$

where $t=\sqrt{\frac{\rho_1^2+\rho_2^2-2\rho_1^2\rho_2^2}{1-\rho_1^2\rho_2^2}}$ and $u=\frac{y\rho_1(1-\rho_2^2)+z\rho_2(1-\rho_1^2)}{\sqrt{1-\rho_1^2\rho_2^2}\sqrt{\rho_1^2+\rho_2^2-2\rho_1^2\rho_2^2}}$. $\square$

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