The Probability that \( k \) Positive Integers are Relatively \( r \)-Prime*

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If \( r, k \) are positive integers, then \( T_k^r(n) \) denotes the number of \( k \)-tuples of positive integers \( (x_1, x_2, \ldots, x_k) \) with \( 1 < x_i < n \) and \( (x_1, x_2, \ldots, x_k) = 1 \). An explicit formula for \( T_k^r(n) \) is derived and it is shown that \( \lim_{n \to \infty} T_k^r(n)/n^k = 1/\zeta(rk) \).

If \( S = \{p_1, p_2, \ldots, p_s\} \) is a finite set of primes, then \( \langle S \rangle = \{p_1^{a_1}p_2^{a_2} \cdots p_s^{a_s} ; p_i \in S \text{ and } a_i \geq 0 \text{ for all } i \} \) and \( T_k^r(S, n) \) denotes the number of \( k \)-tuples \( (x_1, x_2, \ldots, x_k) \) with \( 1 < x_i < n \) and \( (x_1, x_2, \ldots, x_k) \in \langle S \rangle \). Asymptotic formulas for \( T_k^r(S, n) \) are derived and it is shown that \( \lim_{n \to \infty} T_k^r(S, n)/n^k = (p_1 \cdots p_s)^r/\zeta(rk)(p_1^{kr} - 1) \cdots (p_s^{kr} - 1) \).

1. INTRODUCTION

Lehmer [2] and more recently Nymann [3] have both considered the number of \( k \)-tuples of positive integers \( (x_1, x_2, \ldots, x_k) \) with \( 1 < x_i < n \) and \( (x_1, x_2, \ldots, x_k) = 1 \). (We shall use \( (x_1, x_2, \ldots, x_k) \) to denote both the \( k \)-tuple of integers and the greatest common divisor. No confusion will arise from this abuse of notation.) If we denote the number of such \( k \)-tuples by \( T_k(n) \), then both obtain asymptotic formulas for \( T_k(n) \) and show that \( \lim_{n \to \infty} T_k(n)/n^k = 1/\zeta(k) \). This may be interpreted to mean that the probability that \( k \) integers are relatively prime is \( 1/\zeta(k) \).

Theorem 6 extends this result. If \( r \) is a positive integer, then we define the \( r \)th power greatest common divisor of \( a \) and \( b \) by \( (a, b)_r = d \) if \( d \) is the largest \( r \)th power that divides both \( a \) and \( b \). If \( (a, b)_r = 1 \), then we say that \( a \) and \( b \) are relatively \( r \)-prime. Theorem 6 states that the probability that \( k \) integers are relatively \( r \)-prime is \( 1/\zeta(rk) \).

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2. The Totient \( T_r^{*}(n) \)

**Definition.** If \( r \geq 1, s \geq 0 \) are integers, then \( J_s^{*}(n) \) denotes the number of \( s \)-tuples \((x_1, x_2, \ldots, x_s)\) with \( 1 \leq x_i \leq n \) and \((x_1, x_2, \ldots, x_s, n) = 1\).

Note that \( J_1^{*} \) is the ordinary Jordan function and \( J_0^{*} \) is the characteristic function for the set of \( r \)-free integers, i.e.,

\[
J_0^{*}(n) = 1, \quad \text{if } n \text{ is } r\text{-free}
\]
\[
= 0, \quad \text{if } p^r | n \text{ for some prime } p.
\]

A simple inclusion/exclusion argument shows that

\[
J_s^{*}(n) = n^s \prod_{p|n} \left(1 - p^{-rs}\right) = n^s \sum_{d|r} \mu(d) d^{-rs}.
\]

\[
(1)
\]

**Theorem 1.** For \( s \geq 0, r \geq 1 \) we have

\[
\sum_{m=1}^{n} J_s^{*}(m) - n/\zeta(r) + O(n^{1/r}), \quad \text{if } s = 0, r \geq 2,
\]

\[
= n^{s}/2\zeta(2) + O(n \log n), \quad \text{if } s = r = 1,
\]

\[
= n^{s+1}/(s + 1) \zeta(s + 1)) + O(n^{s}), \quad \text{otherwise}.
\]

**Proof.** The proof is similar to the proof for the Jordan function. Details may be found in [1].

**Definition.** If \( r \geq 1, k \geq 1 \) are integers, then \( T_k^{*}(n) \) denotes the number of \( k \)-tuples \((x_1, x_2, \ldots, x_k)\) with \( 1 \leq x_i \leq n \) and \((x_1, x_2, \ldots, x_k, n) = 1\). Also \( T_0^{*}(n) = 0 \) for all \( n, r \).

**Theorem 2.** If \( r \geq 1, k \geq 1 \), then

\[
T_k^{*}(n) = k \sum_{m=1}^{n} J_{k-1}^{*}(m) - \sum_{w=2}^{k} (w - 1) \binom{k}{w} \left( \sum_{m=1}^{n} J_{k-w}^{*}(m) \right) - T_{k-w}^{*}(n) \}
\]

**Proof.** \( J_{k-1}^{*}(m) \) is the number of \( k \)-tuples \((x_1, x_2, \ldots, x_k)\) with \( 1 \leq x_i \leq x_k = m \) and \((x_1, x_2, \ldots, x_k, n) = 1\). Hence \( \sum_{m=1}^{n} J_{k-1}^{*}(m) \) is the number of \( k \)-tuples \((x_1, x_2, \ldots, x_k, n) = 1 \) with \( 1 \leq x_i \leq x_1 \leq n \) and \( k \sum_{m=1}^{n} J_{k-1}^{*}(n) \) is the number of \( k \)-tuples \((x_1, x_2, \ldots, x_k) = 1 \) with \( 1 \leq x_i \leq x_j \leq n \) for some \( 1 \leq j \leq k \). This is not equal to \( T_k^{*}(n) \) since we have counted some of the \( k \)-tuples more than once. For example, if \( k = 4, r = 2 \) and \( n \geq 3 \), then \((2, 2, 2, 1)\) is counted three times by \( k \sum_{m=1}^{n} J_{k-1}^{*}(m) \).
Let \( S^r_{k,w}(n) \) be the number of \( k \)-tuples \((x_1, x_2, \ldots, x_k) = 1\) with \( x_i = x_j = \cdots = x_w \leq n \) for some \( \{i_1, i_2, \ldots, i_w\} \subseteq \{1, 2, \ldots, k\} \) and such that if \( j \neq \{i_1, i_2, \ldots, i_w\} \), then \( x_j < x_{i_1} \). Now each \( k \)-tuple that is counted once by \( S^r_{k,w}(n) \) is counted \( w \) times by \( \sum_{m=1}^{k} J^r_{k-1}(m) \). Hence

\[
T^r_k(n) = k \sum_{m=1}^{n} J^r_{k-1}(m) - \sum_{w=2}^{k} (w - 1) S^r_{k,w}(n). \tag{2}
\]

If \( R^r_{k,w}(n) \) is the number of \( k \)-tuples \((x_1, x_2, \ldots, x_k) = 1\) with \( 1 \leq x_i \leq x_1 \leq n \) and \( x_1 = x_2 = \cdots = x_w \) with \( x_s < x_w \) for \( s > w \), then \( S^r_{k,w}(n) = \binom{k}{w} R^r_{k,w}(n) \). Now if \( x_1 = x_2 = \cdots = x_w \), then \((x_1, x_2, \ldots, x_k) = (x_w, x_{w+1}, \ldots, x_k) \). Hence \( R^r_{k,w}(n) \) is the number of \( k - w + 1 \)-tuples \((x_1, x_2, \ldots, x_{k-w+1}) = 1\) with \( 1 \leq x_i \leq n \) and \( x_s < x_1 \) for \( s > 1 \).

Let \( U^r_s(n) \) be the number of \( s \)-tuples \((x_1, x_2, \ldots, x_s) = 1\) with \( 1 \leq x_i \leq n \) and \( x_i < x_1 \) for \( i \neq 1 \). We see that \( R^r_{k,w}(n) = U^r_{k-w+1}(n) \) and

\[
U^r_s(n) = A^r_s(n) - B^r_s(n),
\]

where \( A^r_s(n) \) is the number of \( s \)-tuples \((x_1, x_2, \ldots, x_s) = 1\) with \( 1 \leq x_i \leq x_1 \leq n \) and \( B^r_s(n) \) is the number of \( s \)-tuples \((x_1, x_2, \ldots, x_s) = 1\) with \( 1 \leq x_i \leq x_1 \leq n \) and \( x_1 = x_j \) for some \( j \neq 1 \). Now \( A^r_s(n) \) is \( \sum_{m=1}^{n} J^r_{s-1}(m) \) and \( B^r_s(n) \) is the number of \( s-1 \)-tuples \((x_1, x_2, \ldots, x_{s-1}) = 1\) with \( 1 \leq x_i \leq n \). Hence, \( B^r_s(n) = T^r_{s-1}(n) \) and

\[
U^r_s(n) = \left( \sum_{m=1}^{n} J^r_{s-1}(m) \right) - T^r_{s-1}(n).
\]

Thus

\[
R^r_{k,w}(n) = U^r_{k-w+1}(n) = \left( \sum_{m=1}^{n} J^r_{k-w}(m) \right) - T^r_{k-w}(n).
\]

Substituting this into (2) completes the proof.

**Theorem 3.** If \( r \geq 2 \), then the number of \( r \)-free integers \( \leq n \) is

\[
\sum_{m=1}^{n} J^r_0(m) = n/\zeta(r) + O(n^{1/r}).
\]

**Proof.** \( T^r_1(n) \) is the number of \( r \)-free integers \( \leq n \). By Theorem 2, \( T^r_1(n) = \sum_{m=1}^{n} J^r_0(m) \) and by Theorem 1

\[
\sum_{m=1}^{n} J^r_0(m) = n/\zeta(r) + O(n^{1/r}).
\]
Theorem 4. The number of ordered pairs \((x_1, x_2) = 1\) with \(1 \leq x_1, x_2 \leq n\) is
\[
2 \sum_{m=1}^{n} J_1^2(m) - 1 = n^2/\zeta(2) + O(n \log n).
\]

Proof. Immediate from Theorems 1 and 2.

Of course, Theorems 3 and 4 are well known, but it is interesting to note that they are special cases of a more general theory.

Theorem 5. If \(k \neq 1\) and \(r_k > 2\), then the number of \(k\)-tuples \((x_1, x_2, \ldots, x_k) = 1\) with \(1 < x_i < n\) is \(n^k/\zeta(r_k) + O(n^{k-1})\).

Proof. The number of such \(k\)-tuples is \(T_k^r(n)\). By Theorem 1, \(\sum_{m=1}^{n} J_{k-1}(m) = n^k/\zeta(r_k) + O(n^{k-1})\). If we note that for \(w > 2\), \(\sum_{m=1}^{n} J_{k-\omega}(m) = O(n^{k-1})\), then the result follows from Theorem 2.

Theorem 6. If \(r_k \geq 2\), then \(\lim_{n \to \infty} T_k^r(n)/n^k = 1/\zeta(r_k)\).

Proof. Immediate from the above theorems.

Theorem 6 may be interpreted to mean that if \(k\) positive integers are chosen at random, then the probability that they are relatively \(r\)-prime is \(1/\zeta(r_k)\).

3. Related Results

We can obtain similar results when we consider a related question.

Definition. If \(S = \{p_1, p_2, \ldots, p_s\}\) is a finite set of primes, then \(\langle S \rangle = \{p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}; p_i \in S \text{ and } a_i \geq 0 \text{ for all } i\}\). Also, \(\overline{S} = \{p; p \text{ is a prime and } p \notin S\}\) and \(\langle \overline{S} \rangle = \{p_1^{b_1} p_2^{b_2} \cdots p_s^{b_s}; p_i \in \overline{S} \text{ and } b_i \geq 0 \text{ for all } i\}\).

The positive integers form a monoid under multiplication and \(\langle S \rangle, \langle \overline{S} \rangle\) are the submonoids generated by \(S, \overline{S}\).

Now we can consider a question that is related to the previous considerations. If \(k\) positive integers \(x_1, x_2, \ldots, x_k\) are chosen at random, what is the probability that \((x_1, x_2, \ldots, x_k)^r \in \langle S \rangle\)? We need a few preliminary results before we can answer this question.

Definition. If \(k > 0, r \geq 1\) and \(S = \{p_1, p_2, \ldots, p_s\}\) is a finite set of primes, then \(J_k^r(S, n)\) denotes the number of \(k\)-tuples \((x_1, x_2, \ldots, x_k)\) with \(1 \leq x_i \leq n\) and \((x_1, x_2, \ldots, x_k, n)^r \in \langle S \rangle\).
As simple application of inclusion/exclusion shows that
\[ J_k^r(S, n) = n^k \prod_{p^r \mid n \atop p \notin S} (1 - p^{-r}). \]

**Theorem 7.** If \( S = \{p_1, p_2, \ldots, p_s\} \) and \( J_k^r(S, n) \) is defined as above, then the following hold:

\[(a) \quad \sum_{m=1}^{n} J_1^r(S, m) = (n p_1 p_2 \cdots p_s r^s/2\zeta(2)(p_1^r - 1) \cdots (p_s^r - 1) + O(n \log n). \]

\[(b) \quad \text{If } r \geq 2, \text{ then } \sum_{m=1}^{n} J_0^r(S, m) = n p_1^r \cdots p_s^r/\zeta(r)(p_1^r - 1) \cdots (p_s^r - 1) + O(n^{1/r}). \]

\[(c) \quad \text{If } r k \geq 2, \text{ then } \sum_{m=1}^{n} J_k^r(S, n) = n^{k+1} (p_1^r \cdots p_s^r)^{r(k+1)/(k + 1)} \zeta(r(k + 1)) \]
\[\times (p_1^{r(k+1)} - 1) \cdots (p_s^{r(k+1)} - 1) + O(n^k). \]

**Proof.** See [1].

**Definition.** If \( S \) is a finite set of primes and \( r \geq 1, k \geq 1 \), then \( T_k^r(S, n) \) denotes the number of \( k \)-tuples \((x_1, x_2, \ldots, x_k) \) with \( 1 < x_i < n \) and \((x_1, x_2, \ldots, x_k) \in \langle S \rangle \). Also, \( T_0^r(S, n) = 0 \) for all \( r \).

Using the same method as in the proof of Theorem 2, we obtain the following theorem.

**Theorem 8.** If \( k \geq 1 \), then
\[ T_k^r(S, n) = k \sum_{m=1}^{n} J_{k-1}^r(S, n) \]
\[- \sum_{w=2}^{k} (w - 1) \binom{k}{w} \left\{ \sum_{m=1}^{n} J_{k-w}^r(S, n) - T_{k-w}^r(S, n) \right\}. \]

Using the asymptotic formulas of Theorem 7 we obtain the following theorem.

**Theorem 9.** If \( r k \geq 2, \) then
\[ \lim_{n \to \infty} T_k^r(S, n)/n^k = (p_1 p_2 \cdots p_s)^{r k}/\zeta(rk)(p_1^{rk} - 1) \cdots (p_s^{rk} - 1). \]
We may interpret this to mean that if $k$ positive integers $x_1, x_2, \ldots, x_k$ are chosen at random, then the probability that $(x_1, x_2, \ldots, x_k) \in \langle S \rangle$ is

\[
(p_1 p_2 \cdots p_s)^{r_k}/\zeta(rk)(p_1^{r_k} - 1) \cdots (p_s^{r_k} - 1).
\]

Similarly, we can show that the probability that $(x_1, x_2, \ldots, x_k) \in \langle S \rangle$ is

\[
(p_1^{r_k} - 1) \cdots (p_s^{r_k} - 1)/p_1^{r_k} \cdots p_s^{r_k}.
\]

REFERENCES