Tolerance stability conjecture revisited

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Abstract

We prove that the strong tolerance stability property is generic in the space of all homeomorphisms of a compact smooth manifold with $C^0$ topology. Actually, it partially resolves Zeeman’s and Taken’s Tolerance Stability Conjecture [F. Takens, in: Lecture Notes in Math., Vol. 197, Springer-Verlag, 1971].

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1. Introduction

We investigate the strong tolerance stability of homeomorphisms (discrete dynamical systems) of a compact smooth manifold. The notion of tolerance stability was introduced by Takens in [13] together with the topological formulation of Zeeman’s Tolerance Stability Conjecture which says that for a set $\mathcal{D} \subset \mathcal{H}(M)$, equipped with the topology not coarser than that of $\mathcal{H}(M)$, the set of all $f \in \mathcal{D}$ having the tolerance stability property (with respect to $\mathcal{D}$) is residual in $\mathcal{D}$, i.e., it includes a countable intersection of open and dense subsets of $\mathcal{D}$. Here $\mathcal{H}(M)$ denotes the space of all homeomorphisms of a compact metric space $M$ with $C^0$ topology.

In [15] White presented the counterexample showing that the set $\mathcal{D}$ cannot be chosen arbitrarily. There were also proved several results in the direction of Zeeman’s Tolerance

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Stability Conjecture (see [3,6,8,11,14]). In this paper we restrict our investigation to the case when the set $D$ is equal to $\mathcal{H}(M)$. To the author’s best knowledge such a problem was studied so far only by Odani [8], who showed that for a compact (smooth) manifold $M$ of the dimension at most 3 the set of all homeomorphisms satisfying the strong tolerance stability condition is residual in $\mathcal{H}(M)$. Our aim is an extension of this theorem to the case of an arbitrary dimension. The proof is based on the technique of a handle decomposition of a manifold, proposed by Pilyugin and Plamenevskaya [12] for proof of $C^0$ genericity of the shadowing property. Additionally, applying this method we prove that for a $C^0$ generic homeomorphism the chain recurrent set is a Cantor set. We recall that the property $P$ of elements of a topological space $X$ is called generic if the set of all $x \in X$ satisfying $P$ is residual in $X$.

The results of this paper are part of author’s Ph.D. Thesis [7] and have already been announced (without proofs) in [9].

2. Basic definitions

Let $M$ be a compact metric space with the metric $d$ and let $\mathcal{H}(M)$ denote the space of all homeomorphisms of $M$ equipped with the metric $\rho_0$, defined by

$$\rho_0(f, g) := \max \left\{ \max_{x \in M} d(f(x), g(x)), \max_{x \in M} d(f^{-1}(x), g^{-1}(x)) \right\},$$

which induces $C^0$ topology and makes $\mathcal{H}(M)$ a complete metric space. We say that a sequence $\{x_i\}_{i \in \mathbb{Z}} \subset M$ is $\varepsilon$-traced ($\varepsilon$-set-traced) by the orbit $O_f(x) := \{f^i(x)\}_{i \in \mathbb{Z}}$ of a homeomorphism $f \in \mathcal{H}(M)$ if $d(f^i(x), x_i) \leqslant \varepsilon$ for every $i \in \mathbb{Z}$ ($\rho(\text{Cl}O_f(x), \text{Cl}\{x_i\}_{i \in \mathbb{Z}}) \leqslant \varepsilon$). Here $\rho$ denotes the Hausdorff metric induced by $d$.

Now, following [8,13], we recall the notions of tolerance stability and strong tolerance stability.

**Definition 1.** A homeomorphism $f \in \mathcal{H}(M)$ is tolerance stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $g \in U_\delta(f)$ each $f$-orbit is $\varepsilon$-set-traced by some $g$-orbit and each $g$-orbit is $\varepsilon$-set-traced by some $f$-orbit. Here $U_\delta(f)$ denotes the $\delta$-neighborhood of $f$ in $\mathcal{H}(M)$.

**Definition 2.** A homeomorphism $f \in \mathcal{H}(M)$ is strongly tolerance stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $g \in U_\delta(f)$ each $f$-orbit is $\varepsilon$-traced by some $g$-orbit and each $g$-orbit is $\varepsilon$-traced by some $f$-orbit.

Obviously, the strong tolerance stability property implies the tolerance stability one. Moreover, it is also stronger than the shadowing property in the case when $M$ is a manifold (see [8]).
3. Handle decomposition

In this section we repeat the relevant material regarding a handle decomposition of a manifold (a more complete theory may be found in [12]). We also make the first step in the direction of the proof of the main result (see Remark 3).

Let $M$ be a compact $n$-dimensional smooth manifold with the metric $d$ induced by the Riemannian structure. We will denote by $D_m^r(a)$ the closed ball in $\mathbb{R}^m$ with the center at $a$ and the radius $r$ (to simplify notation balls centered at the origin will be written as $D_m^r$ and the unit ball as $D_m$). For convenience we consider the maximum norm on $\mathbb{R}^m$, i.e., $\|x\| = \max_{i \in \{1,\ldots,m\}} |x_i|$ for all $x = (x_1,\ldots,x_m) \in \mathbb{R}^m$.

A sequence of sets $\emptyset = M_{-1} \subset M_0 \subset \cdots \subset M_n = M$ is called a handle decomposition of $M$ if for any $m \in \{0,\ldots,n\}$ the following conditions hold:

(1) the set $M_m$ is $n$-dimensional submanifold with boundary;
(2) the set $\text{Cl}(M_m \setminus M_{m-1})$ is a disjoint union of $m$-handles, i.e., sets homeomorphic to $D_m^r \times D_{n-m}^r$;
(3) each $m$-handle is attached to the boundary of $M_{m-1}$ by the image of $S^{m-1} \times D_{n-m}^r$;
(4) for each $m$-handle $H$, the image of $i_H : D_m^r \times D_{n-m}^r \hookrightarrow M$,

there exists an embedding $i_H : D_m^r \times D_{n-m}^r \hookrightarrow M$

such that:
(a) $i_H|_{D_m^r \times D_{n-m}^r} = i_H$,
(b) $i_H(D_m^r \times D_{n-m}^r) \cap M_{m-1} = i_H(S^{m-1} \times D_{n-m}^r)$,
(c) if $G$ is another $m$-handle then the “widened” $m$-handles $\bar{H} := i_H(D_m^r \times D_{n-m}^r)$ and $\bar{G} := i_G(D_m^r \times D_{n-m}^r)$ are disjoint.

We say that a homeomorphism $f \in \mathcal{H}(M)$ preserves a handle decomposition $\mathcal{M}$ if

$f(M_m) \subset \text{Int} M_m$ for all $m \in \{0,\ldots,n\}$.

A subset $V$ of a handle $H = i_H(D_m^r \times D_{n-m}^r)$ of the form

$V = i_H(D_{r_1}(a_1) \times \cdots \times D_{r_m}(a_n))$,

where $r_1,\ldots,r_n \in (0,1)$ and $a_1,\ldots,a_n \in (-1,1)$, will be called a cube in $H$.

Let $\varepsilon > 0$ be fixed. By $\mathcal{B}_\varepsilon$ we denote the set of all homeomorphisms $f \in \mathcal{H}(M)$ for which we can find a handle decomposition $\mathcal{M}_f$ satisfying the following conditions:

(1) $\mathcal{M}_f$ has the diameter less than $\varepsilon$, i.e.,

$|f(M)| := \max\{\text{diam } H \mid H \text{ is a handle of } \mathcal{M}\} < \varepsilon$. 
(2) \( f \) preserves \( \mathcal{M}_f \);
(3) if \( \{H_i\}_{i \in \mathbb{Z}} \) is a sequence of handles with the property that \( f(H_i) \cap H_{i+1} \neq \emptyset \) then there exists a corresponding sequence of cubes \( \{V_i\}_{i \in \mathbb{Z}} \) such that \( V_i \subset H_i \), \( f(V_i) \subset H_{i+1} \) and

\[
\bigcap_{i=-\infty}^{\infty} f^{-i}(V_i) \neq \emptyset.
\]

Now, let \( B_\varepsilon \) be the subset of \( \overline{B}_\varepsilon \) defined as follows: a homeomorphism \( f \in \overline{B}_\varepsilon \) belongs to \( B_\varepsilon \) if there exists \( \delta > 0 \) such that for each \( g \in U_\delta(f) \) the conditions (1)–(3) hold with \( M_g = M_f \) (in particular \( g \in \overline{B}_\varepsilon \)).

**Remark 3.** By the results of [12], especially the definition of the set \( A_\varepsilon \subset \mathcal{H}(M) \) as well as Lemmas 1 and 4 stated there, it is easily seen that the set \( B := \bigcap_{n=1}^{\infty} B_1^n \) is a residual subset of \( \mathcal{H}(M) \) (note that \( B_\varepsilon \) contains the set \( A_\varepsilon \) which was proved to be open and dense in \( \mathcal{H}(M) \)).

4. Main result

Let \( M \) be a compact smooth manifold with the metric \( d \) induced by the Riemannian structure.

**Theorem 4.** A generic \( f \in \mathcal{H}(M) \) has the strong tolerance stability property.

**Proof.** Fix \( \varepsilon > 0 \). Since the set \( B_\varepsilon \), defined in the previous section, is residual in \( \mathcal{H}(M) \) it suffices to prove that for every \( f \in B_\varepsilon \) there exists \( \delta > 0 \) such that for any pair of homeomorphisms \( g_1, g_2 \in U_\delta(f) \) each \( g_1 \)-orbit is \( \varepsilon \)-traced by some \( g_2 \)-orbit.

Choose \( f \in B_\varepsilon \). Let \( \mathcal{M} = \mathcal{M}_f \) be a corresponding handle decomposition of \( M \). Since there is only finite number of handles in \( \mathcal{M} \) we can find \( \delta > 0 \) such that each homeomorphism \( g \in U_\delta(f) \) satisfies the following conditions:

(i) for every pair of handles \( (H, G) \) of \( \mathcal{M} \)
\[
g(H) \cap G = \emptyset \implies f(H) \cap G = \emptyset \implies \text{dist}(g(H), G) > 2\delta;
\]

(ii) \( g \in \overline{B}_\varepsilon \) with \( \mathcal{M}_g = \mathcal{M} \).

Fix \( y \in M \) and \( g_1, g_2 \in U_\delta(f) \). Let \( H_i \) denote a handle of \( \mathcal{M} \) containing \( g_1^i(y) \) \( (i \in \mathbb{Z}) \). Clearly \( \text{dist}(g_2(H_i), H_{i+1}) \leq 2\delta \) and, in consequence,
\[
g_2(H_i) \cap H_{i+1} \neq \emptyset.
\]
From this it follows that there exists a sequence of cubes \( \{V_i\}_{i \in \mathbb{Z}} \) such that \( V_i \subset H_i \) and
\[
\bigcap_{i=-\infty}^{\infty} g_2^{-i}(V_i) \neq \emptyset.
\]
Let \( x \) be an arbitrarily chosen point of the above set. Then \( g_2^i(x) \in V_i \subset H_i \) and so \( d(g_2^i(x), g_1^j(y)) < \varepsilon \) for every \( i \in \mathbb{Z} \) (we recall that \( |\mathcal{M}| < \varepsilon \)).

By the above, we conclude that each \( g_1 \)-orbit is \( \varepsilon \)-traced by some \( g_2 \)-orbit, which completes the proof. \( \square \)

5. Generic asymptotic behavior

Let \( M \) be a compact smooth manifold with the metric \( d \) induced by the Riemannian structure. In this section we apply the technique of a handle decomposition to prove the following theorem, which extends some recent Hurley’s result [5] to the case of an arbitrary dimension. A different and independent proof one can find in [1].

**Theorem 5.** For a generic \( f \in \mathcal{H}(M) \) the chain recurrent set \( CR(f) \) is a Cantor set.

**Proof.** We recall that the chain recurrent set \( CR(f) \) is a collection of all such points \( p \in M \) that for each \( \delta > 0 \) there is a \( \delta \)-chain through \( p \), i.e., a finite sequence \( x_0, x_1, \ldots, x_n \) (\( n \geq 1 \)) with \( x_0 = x_n = p \) and with \( d(f(x_{j-1}), x_j) \leq \delta \) for every \( j \in \{1, \ldots, n\} \). It is a compact, nonempty and invariant set.

By the corollary to Theorem 6.1 in [5], it remains to show that \( CR(f) \) is totally disconnected for a generic \( f \in \mathcal{H}(M) \).

Take \( \varepsilon > 0 \) and \( f \in B_\varepsilon \). Let \( \mathcal{M} = \mathcal{M}_f \) be a corresponding handle decomposition. Since for any point \( p \in M \) lying on the boundary of some handle of \( \mathcal{M} \) no \( \delta \)-chain through \( p \) can be found when \( \delta \) is too small (note that \( f \) preserves \( \mathcal{M} \)), we have

\[
CR(f) \subset \bigcup \{ \text{Int} \ H \mid H \text{ is a handle of } \mathcal{M} \}.
\]

From this it may be concluded that each connected component of \( CR(f) \) does not intersect more than one handle of \( \mathcal{M} \) and therefore its diameter is not greater than \( \varepsilon \). It follows that for each \( f \in B \) the set \( CR(f) \) is completely disjoint (note that its connected components are single points), which makes the proof complete. \( \square \)

**Remark 6.** In [2,4,10] was proved that for a generic \( f \in \mathcal{H}(M) \) the chain recurrent set \( CR(f) \) is the closure of the set of all periodic orbits. So, in the other words, Theorem 5 says that \( C^0 \) generically dynamics of a homeomorphism is, in a specific way, chaotic.

References