Discrete Mathematics 51 (1984) 35–39
North-Holland



# A HELLY THEOREM FOR CONVEXITY IN GRAPHS

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Received 12 April 1983
Revised 9 November 1983

It is shown that for chordless path convexity in any graph, the Helly number equals the size of a maximum clique.

## 1. Introduction

A celebrated theorem of Helly [5] states that if, in a finite family of convex sets in $R^d$, every $d+1$ sets have a point in common, then there is a point common to all the sets. Although numerous extensions and applications of this result are known (cf. [1, 3, 7]), its use in graph theory has thus far been largely confined to (analogues of) the case $d=1$, which is sometimes called the *Helly property* (cf. [4]). The goal here is to establish a version for general graphs in which the dimension $d$ is determined by the maximum clique size.

A family $\mathcal{F}$ of sets has the *k.I.P.* ($k$ a positive integer) if every $k$ or fewer sets in $\mathcal{F}$ have nonvoid intersection. The *Helly number* of a family $\mathcal{F}$ is the smallest $h$ such that any finite subfamily of $\mathcal{F}$ with the $h$.I.P. has nonvoid intersection. A path $x_1$–$x_2$–$\cdots$–$x_n$ in a (simple, loopless, undirected) graph $G$ is *chordless* iff there are no *chords*, i.e., no edges $x_i$–$x_j$ in $G$ with $j>i+1$. A set $K$ of nodes of $G$ is *m-convex* iff for each pair of points $x$ and $y$ in $K$, *all* nodes on *all* chordless paths joining them also lie in $K$. The collection of all $m$-convex sets forms the *monophonic alignment* on $G$ and has been studied extensively for chordal graphs [8, 10]. A *clique* in $G$ is a set of nodes which induce a complete subgraph. The *clique number* of $G$ is the maximum number of nodes in a clique of $G$.

## 2. Results

By applying results of Hoffman [6] and Jamison [7] on the Helly number of antimatroids, and the fact that in the monophonic alignment chordal graphs are

* Research supported in part by NSF Grant 80-02543 and NSF EPSCoR Grant IPS-80-11451.

antimatroids [8], one can deduce that for a chordal graph the Helly number of its $m$-convex sets equals its clique number. Our goal is to show this in fact holds for any connected graph. This will provide the first extension of Hoffman's result beyond the class of antimatroids. A general discussion may be found in [7, 8].

**Theorem 1.** *For any connected graph $G$, the Helly number of the $m$-convex sets equals the clique number of $G$.*

For the proof it is convenient to have two lemmata, the first giving an alternate characterization of $m$-convex sets and the second simplifying the determination of the Helly number. If $S$ is a set of nodes, two nodes $s$ and $t$ in $S$ are *joined externally to $S$* iff there is a path from $s$ to $t$ which contains no nodes of $S$ except $s$ and $t$.

**Lemma 1.** *A set $S$ of nodes in $G$ is $m$-convex iff any pair of nodes in $S$ which are joined externally to $S$ are adjacent.*

**Proof.** Suppose $S$ is $m$-convex and let $s$–$x_1$–$\cdots$–$x_n$–$t$ be a path with $s$ and $t$ in $S$ and $x_i \notin S$ for all $i$. Among all such paths choose a *shortest* one. Such a path need not be shortest in $G$, but it certainly has no chords (or we could shorten it) except possibly $s$–$t$. Hence if $s$ and $t$ are not adjacent, this path is chordless and hence lies in $S$ by $m$-convexity. And since $s$–$t$ is not an edge, there must be *at least* one $x_i$, contradicting $x_i \notin S$. Thus $s$ and $t$ must be adjacent as desired.

To show the converse, suppose $P$ is a chordless path between two nodes of $S$. We must show that $P \subseteq S$. If $P$ contains a node $x$ not in $S$, let $s$ be the last node of $S$ on $P$ before $x$ and $t$ the first node of $S$ on $P$ after $x$. Then the subpath of $P$ from $s$ to $t$ joins $s$ and $t$ externally to $S$. Hence $s$–$t$ must be an edge and thus a chord of $P$, contrary to $P$ being chordless. □

**Lemma 2.** *Suppose $\mathscr{L}$ is a family of sets closed under intersection. If, for some $k$, any family $\mathscr{F}$ of sets in $\mathscr{L}$ with the $k$.I.P. also satisfies the $k+1$.IP., then the Helly number of $\mathscr{L}$ is at most $k$.*

**Proof.** We show that any family $\mathscr{F}$ of sets in $\mathscr{L}$ that has the $k$.I.P. also has the $n$.I.P. for all $n \geq k$. This is true by hypothesis for $n = k+1$. Proceeding by induction on $n$, let $A_0, A_1, \ldots, A_n$ be sets in $\mathscr{F}$ and let $\mathscr{G} = \{A_i \cap A_0 : i \neq 0\}$. Since $\mathscr{F}$ has the $k+1$.I.P., $\mathscr{G}$ has the $k$.I.P. and hence the $n$.I.P. by induction. Thus the intersection of the $n$ sets in $\mathscr{G}$, which is $A_0 \cap A_1 \cap \cdots \cap A_n$, is nonempty. Whence $\mathscr{F}$ has the $n+1$.I.P. □

**Proof of Theorem 1.** Let $k$ be the clique number of $G$. Let $K$ be a clique of $k$ nodes. Clearly $K$ and all its subsets are trivially $m$-convex. Since the $k$ subsets of

$K$ with $k-1$ nodes form a family with the $k-1$.I.P. but empty intersection, the Helly number is at least $k$.

Evidently from the definition of $m$-convexity, the family of $m$-convex sets is closed under intersection. Thus to show the Helly number is at most $k$, it suffices by Lemma 2 to show that the $k$.I.P. implies the $k+1$.I.P. for $m$-convex sets.

Let $A_0, A_1, \ldots, A_k$ be $m$-convex, any $k$ having nonempty intersection. For each $i$, let $D_i = \bigcap_{j \neq i} A_j$, so each $D_i$ is a nonempty $m$-convex set. Suppose $\bigcap_{i \geq 0} A_i = \emptyset$, or equivalently, $A_i \cap D_i = \emptyset$ for each $i$. Select points $x_i$ from $D_i$ in such a way as to maximize the cardinality of the largest clique in $S = \{x_0, x_1, \ldots, x_k\}$. With no loss in generality, we may assume the indices so chosen that $x_0, x_1, \ldots, x_n$ is a clique of largest cardinality in $S$. Since $k$ is the clique number, we should have $n < k$. We aim to contradict this.

*Case* 1. $n = 0$.

Certainly $k \geq 2$ since $G$ is connected, so $x_1, x_2$ exist. Let $P$ and $Q$ be shortest paths from $x_2$ to $x_0$ and from $x_2$ to $x_1$, respectively. Since $x_i \in D_i$ and $A_i \cap D_i = \emptyset$ for each $i$, both $x_0$ and $x_1$ lie in $A_2$ but $x_2$ does not. Let $y$ [resp., $z$] be the last point of $P$ [resp., $Q$] in $A_2$. Since $P$ is shortest, it is certainly chordless and hence lies in any $m$-convex set containing $x_2$ and $x_0$. In particular, $P$ and hence $y$ lies in $A_i$ for $i \neq 0, 2$. But $y \in A_2$ by choice. Hence $y \in D_0$. Likewise $z \in D_1$. Hence if we replace $x_0$ by $y$ and $x_1$ by $z$, we get a new set,

$$S' = (S \setminus \{x_0, x_1\}) \cup \{y, z\},$$

satisfying the choice criterion of one point from each $D_i$. Since $y$ and $z$ are joined externally to $A_2$ along $P$ and $Q$ via $x_2$, and since $A_2$ is $m$-convex, Lemma 1 implies that $y$ and $z$ are adjacent. Thus $S'$ contains a larger clique than $S$, a contradiction.

*Case* 2. $n > 0$.

By choice, all of $x_1, x_2, \ldots, x_{n+1}$ lie in $A_0$ but $x_0$ is not in $A_0$. Let $P$ be a shortest path from $x_0$ to $x_{n+1}$. Let $y$ be the first point of $A_0$ encountered along this path. For each $i$ from 1 to $n$, $x_i$ and $y$ are joined externally to $A_0$ via the edge $x_i$–$x_0$ and the subpath of $P$ from $x_0$ to $y$. By Lemma 1, since $A_0$ is $m$ convex, $y$ is adjacent to all $x_i$, $i = 1, \ldots, n$. As in Case 1, $y$ is in $D_{n+1}$ since it is in $A_0$ by choice and $P$ lies in $A_i$ for $i \neq 0$, $n+1$ by $m$-convexity.

Now $x_1$ is adjacent to $x_0$ and $y$. (It is here that we use $n \geq 1$ to be sure that $y$ is not replacing $x_1$.) Since $x_0$ and $y$ lie in $A_1$ but $x_1$ does not, the path $x_0$–$x_1$–$y$ joins $x_0$ and $y$ externally to $A_1$. Thus $x_0$ and $y$ must be adjacent by Lemma 1. But then $x_0, x_1, \ldots, x_n$, $y$ is a clique. Thus, since

$$S' = (S \setminus \{x_{n+1}\}) \cup \{y\}$$

is a legitimate choice of representatives for the $D_i$, we have increased the largest clique size in such. This contradiction completes the proof. □

**Remark.** The above arguement does not require that the graph $G$ be finite. Clearly, the Helly number will be infinite iff the clique number is infinite.

**Corollary.** *Let $\mathcal{F}$ be a finite family of m-convex sets in a graph G. If $\mathcal{F}$ has the* k.I.P. *and some set A in $\mathcal{F}$ contains no clique on k nodes, then the sets in $\mathcal{F}$ have some node in common.*

**Proof.** The family $\mathcal{G} = \{A \cap B : B \in \mathcal{F}\}$ is a family of $m$-convex subsets of $A$ with the $k-1$.I.P. By hypothesis, the clique number of $A$ is at most $k-1$. Thus the corollary follows by applying Theorem 1 to the subgraph induced by $A$. □

If in our above definition of convexity, we replace 'chordless' by 'shortest' we have what may appear (at least naively) to be a more 'natural' notion of convexity. However, the result on Helly numbers fails drastically. A set $S$ of nodes in $G$ is *geodesically convex* iff $S$ contains all nodes on *all shortest* paths between a pair of nodes in $S$. Geodesic and monophonic convexity coincide on trees, block graphs, and Ptolemaic graphs (cf. [8]), and geodesic convexity is of great importance in the class of median graphs (cf. [9]). However, even bipartite graphs can have arbitrarily large geodesic Helly numbers.

**Example.** Let $K$ be a complete graph on $k$ 'red' nodes. To obtain $G$, subdivide each edge by inserting a new 'blue' node. Any two nodes of $G$ are easily seen to be distance at most 4 apart. For each red node $p$, let $D_p$ be the node set obtaining by deleting $p$ and its blue neighbors from $G$. Any path leaving $D_p$ and re-entering it has length at least 4, with equality only in the case it joins two red nodes of $D_p$. Thus no such path can be shortest, so $D_p$ is geodesically convex. Thus the family of all $D_p$, $p$ a red node, is a family of geodesically convex sets with the $k-1$.I.P. but having empty intersection.

**Remark.** Duchet and Meyniel [2] have recently established a bound on Helly numbers for various convexities in graphs in terms of the Hadwiger number (the largest $n$ such that $G$ is contractictable onto $K_n$). This explains why the Helly number of geodesic convexity in the above example must be so large.

**Note added in proof.** In work continuing [2] Duchet has recently determined the Radon number of $m$-convexity in graphs.

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