On a Class of Matrices with Real Eigenvalues

J. C. Butcher
Department of Computer Science
University of Auckland
Auckland, New Zealand

Submitted by George P. H. Styan

ABSTRACT

It is easy to prove that if A is a real irreducible square matrix and if a real nonsingular diagonal matrix D exists such that AD is symmetric and positive semidefinite, then for any real diagonal matrix Y, AY has only real eigenvalues. This paper proves the converse result that if no such D exists, then for some Y, AY will possess some nonreal eigenvalues.

1. INTRODUCTION

It is easy to see that if A is a real irreducible square matrix for which a nonsingular real diagonal matrix D exists such that AD is symmetric and positive semidefinite, then for any real diagonal matrix Y, AY has only real eigenvalues. If AD is positive definite and has the Cholesky factorization $AD = LL^T$, then eigenvalues of AY are also eigenvalues of the symmetric matrix $L^T(D^{-1}Y)L$, and the result is clear in this case. On the other hand, if AD were positive semidefinite and AY had some nonreal eigenvalues, then a slight perturbation to the diagonals of A would make AD positive definite without shifting the nonreal eigenvalues to the real line. This contradiction shows that the case of singular A is also covered.

This paper considers the converse of this result as stated in the theorem below. It has an immediate application in the analysis of linear and nonlinear stability for a general class of numerical methods for ordinary differential equations [2, 3].

THEOREM. *If A is a real irreducible $n \times n$ matrix for which no nonsingular real $n \times n$ diagonal matrix D exists such that AD is symmetric and*

 0024-3795/88/$3.50

positive semidefinite, then there exists a real $n\times n$ diagonal matrix Y such that AY possesses nonreal eigenvalues.

The justification for this assertion is organized as a sequence of lemmas with a formal proof of the theorem itself at the end. We first consider the case of 2×2 matrices and matrices containing an appropriate 2×2 submatrix.

LEMMA 1. *If $i,j\in\{1,2,\dots,n\}$ exist such that in the $n\times n$ matrix A, $a_{ij}a_{ji}\neq 0$, and such that no nonzero d_i and d_j exist such that the 2×2 matrix*

$$\begin{bmatrix} a_{ii}d_i & a_{ij}d_j \\ a_{ji}d_i & a_{jj}d_j \end{bmatrix}$$

is symmetric and positive semidefinite, then real y_i and y_j exist such that

$$\begin{bmatrix} a_{ii}y_i & a_{ij}y_j \\ a_{ji}y_i & a_{jj}y_j \end{bmatrix}$$

has nonreal eigenvalues.

Proof. We consider three cases:

(i) $a_{ii}a_{jj}a_{ij}a_{ji}<0$,
(ii) $a_{ij}a_{ji}\neq 0$, $a_{jj}=0$,
(iii) $a_{ii}a_{jj}a_{ij}a_{ji}>0$, $|a_{ij}a_{ji}|>|a_{ii}a_{jj}|$,

which we can easily see cover all essentially different possibilities. To obtain nonreal eigenvalues, we may choose y_i and y_j as follows:

(i) $y_i=a_{jj}$, $y_j=a_{ii}$,
(ii) $y_i=4a_{ij}a_{ji}$, $y_j=-(1+a_{ii}^2)$,
(iii) $y_i=a_{jj}$, $y_j=-a_{ii}$. ■

2. THE CASE OF SYMMETRIC MATRICES

We now consider the possibility that although D exists such that AD is symmetric, it is not possible to find such a D for which AD is positive semidefinite. Without loss of generality, we may assume that A itself is symmetric and that it satisfies the requirements of the following lemma.

LEMMA 2. *If A is an irreducible symmetric $n\times n$ matrix with $n\geqslant 2$ such that the principal $(n-1)\times(n-1)$ minor is positive definite, $a_{nn}>0$, and $\det(A)<0$, then a real diagonal Y exists such that AY has nonreal eigenvalues.*

Proof. Let $Y=\mathrm{diag}(1,1,\dots,1,-x)$, where x is positive. Suppose that $\sigma(AY)\subseteq\mathbf{R}$ for all x. If x is sufficiently small, say $x\leqslant x_0$, then $n-1$ of the eigenvalues of AY are positive because their limiting values as $x\to 0$ are the positive eigenvalues of the $(n-1)\times(n-1)$ principal submatrix. Also, because the product of the eigenvalues is $\det(AY)=-x\det(A)>0$, the remaining eigenvalue is also positive. If $x=x_1$, where $x_1>\sum_{i=1}^{n-1}a_{ii}/a_{nn}$, then, because the trace is negative but the determinant is positive, AY has at least two nonpositive eigenvalues. If $\sigma(AY)\subseteq\mathbf{R}$ for all $x\in[x_0,x_1]$, the set of eigenvalues would depend continuously on x. This implies that for some $x\in[x_0,x_1]$, AY has a zero eigenvalue. Since $\det(AY)>0$, this is impossible. ∎

3. THE CASE OF 3×3 MATRICES

In this section we will assume that $n\geqslant 3$ and that the contingencies included within Lemmas 1 and 2 do not arise. We will consider the possibility that distinct $i,\ j,\ k\in\{1,2,3,\dots,n\}$ exist such that $a_{ij}a_{jk}a_{ki}\neq a_{ik}a_{kj}a_{ji}$. By permuting the rows and columns, we may assume that $i=1,\ j=2,\ k=3$, and, because we may choose $y_4=y_5=\cdots=y_n=0$, we can restrict ourselves to $n=3$. It is convenient to divide up the work according as one of the products $a_{12}a_{23}a_{31}$ and $a_{13}a_{32}a_{21}$ is or is not equal to zero.

LEMMA 3.1. *If $n=3$ and $a_{12}a_{23}a_{31}\neq 0$, $a_{13}a_{32}a_{21}=0$, then a real diagonal Y exists such that AY has nonreal eigenvalues.*

Proof. We consider four cases:

(i) $a_{11}=a_{22}=a_{33}=0$,
(ii) for one and only one i, say $i=1$, $a_{ii}\neq 0$,
(iii) for one and only one value of i, say $i=3$, $a_{ii}=0$,
(iv) for $i=1,2,3$, $a_{ii}\neq 0$.

In case (i), because of Lemma 1, we may assume that $a_{13}=a_{32}=a_{21}=0$. It is found that the characteristic polynomial of AY, with $Y=I$, is given by

$$p(\lambda)=\det(A-\lambda I)=a_{12}a_{23}a_{31}-\lambda^3,$$

which has only one real zero. In case (ii), again because of Lemma 1, we assume that $a_{13}=a_{32}=a_{21}=0$. In this case, for $Y=\operatorname{diag}(y_1, y_2, y_3)$,

$$p(\lambda)=\det(AY-\lambda I)=a_{12}a_{23}a_{31}y_1y_2y_3+a_{11}y_1\lambda^2-\lambda^3,$$

which has a nonreal pair of eigenvalues if y_2y_3 is chosen to have the same sign as $a_{11}a_{12}a_{23}a_{31}$. In case (iii), because of Lemma 1, we assume that $a_{13}=a_{32}=0$. The characteristic polynomial of AY is

$$p(\lambda)=a_{12}a_{23}a_{31}y_1y_2y_3-(a_{11}a_{22}-a_{12}a_{21})y_1y_2\lambda+(a_{11}y_1+a_{22}y_2)\lambda^2-\lambda^3.$$

If $y_1\neq 0$ and $y_2\neq 0$ are chosen such that p' has two real zeros, say λ_1 and λ_2, then y_3 can be chosen so that $p(\lambda)<0$ for all $\lambda\in[\lambda_1,\lambda_2]$. It follows that p has only a single real zero. In case (iv), since $a_{21}a_{32}a_{13}=0$, one of these factors is zero. Suppose without loss of generality that $a_{32}=0$, and choose $Y=\operatorname{diag}(-xa_{22}a_{33}/a_{12}a_{23}a_{31}, a_{22}^{-1}, a_{33}^{-1})$ for x positive. We find that

$$p(\lambda)=-x-x\left(\frac{a_{21}a_{33}}{a_{23}a_{31}}+\frac{a_{13}a_{22}}{a_{12}a_{23}}\right)\mu-\left(1+x\frac{a_{11}a_{22}a_{33}}{a_{12}a_{23}a_{31}}\right)\mu^2-\mu^3,$$

where $\mu=\lambda-1$, which has only a single real zero if x is sufficiently small. ■

LEMMA 3.2. *If* $n=3$ *and* $0\neq a_{12}a_{23}a_{31}\neq a_{13}a_{32}a_{21}\neq 0$, *then a real diagonal* Y *exists such that* AY *has nonreal eigenvalues.*

Proof. Let

$$c_1=a_{11}^2a_{23}a_{32}-a_{12}a_{21}a_{13}a_{31},$$

$$c_2=a_{22}^2a_{13}a_{31}-a_{21}a_{12}a_{23}a_{32},$$

$$c_3=a_{33}^2a_{12}a_{21}-a_{31}a_{13}a_{32}a_{23}.$$

We note that if $c_1<0$, then we may assume that $c_2, c_3\geqslant 0$, since otherwise one of the submatrices

$$\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix},\qquad \begin{bmatrix} a_{11} & a_{13}\\ a_{31} & a_{33}\end{bmatrix}$$

would have been covered by Lemma 1 and because we can arbitrarily multiply by a nonsingular diagonal matrix. For the same reason we may assume that $a_{11}, a_{22}, a_{33}>0$ and that $a_{12}=a_{21}, a_{13}=a_{31}, a_{32}/a_{23}>1$. Let

$\theta=(a_{32}/a_{23})^{1/2}$. We also write

$$z_1=a_{23}^2\theta^2a_{11}-a_{12}a_{13}a_{23}\theta,$$

$$z_2=a_{13}^2a_{22}-a_{12}a_{13}a_{23}\theta,$$

$$z_3=a_{12}^2a_{33}-a_{12}a_{13}a_{23}\theta$$

and, in cases (i), (ii) below,

$$Y=\operatorname{diag}\left(\frac{\theta^2a_{23}^2}{z_1},\frac{a_{13}^2}{z_2},\frac{a_{12}^2}{z_3}\right).$$

In cases (ii), (iii), (iv) below, where $a_{12}a_{13}a_{23}>0$, $c_1\leqslant 0$, and $c_2,c_3\geqslant 0$, let $t\in[0,1)$ and $u,v\geqslant 0$ be defined so that

$$a_{11}a_{23}^2\theta^2=a_{12}a_{13}a_{23}\theta(1-t),$$

$$a_{13}^2a_{22}(1-t)=a_{12}a_{13}a_{23}\theta(1+u),$$

$$a_{12}^2a_{33}(1-t)=a_{12}a_{13}a_{23}\theta(1+v),$$

so that

$$A=D_1\tilde{A}D_2,$$

where

$$\tilde{A}=\begin{bmatrix}1 & 1 & 1\\ 1 & 1+u & (1-t)\theta^{-1}\\ 1 & (1-t)\theta & 1+v\end{bmatrix},$$

$$D_1=\operatorname{diag}\left(1,\frac{a_{23}\theta}{a_{13}(1-t)},\frac{a_{23}\theta}{a_{12}(1-t)}\right),$$

$$D_2=\operatorname{diag}\left(\frac{a_{12}a_{13}(1-t)}{a_{23}\theta},a_{12},a_{13}\right).$$

We consider four cases:

(i) $c_1,c_2,c_3>0$ or $a_{12}a_{13}a_{23}<0$,
(ii) $c_1\leqslant 0$, $uv>t^2>0$, $a_{12}a_{13}a_{23}>0$,
(iii) $c_1\leqslant 0$, $t^2\geqslant uv>0$, $a_{12}a_{13}a_{23}>0$,
(iv) $c_1\leqslant 0$, one or more of t, u, v is zero, $a_{12}a_{13}a_{23}>0$.

In case (i), we note that $z_1, z_2, z_3 > 0$. The characteristic polynomial of $AY - I$ is found to be

$$\lambda^3 - (a_{12}a_{13}a_{23}\theta)\left(z_1^{-1} + z_2^{-1} + z_3^{-1}\right)\lambda^2 - \theta^2(\theta - 1)^2$$
$$\times a_{12}^3 a_{13}^3 a_{23}^3 (z_1 z_2 z_3)^{-1} = \lambda^3 - \alpha\lambda^2 - \beta,$$

say, where $\alpha\beta \geqslant 0$, $\beta \neq 0$, and this polynomial has only one distinct real zero.

In cases (ii), (iii), (iv), because $\sigma(AY) = \sigma(\tilde{A}\tilde{Y})$, where $\tilde{Y} = D_2 Y D_1$, we may, without loss of generality, assume that A is equal to $\tilde{A}$. For case (ii), it is found that $z_1 = -t(1-t)$, $z_2 = u + t$, $z_3 = v + t$, so that

$$z_1 z_2 z_3\left(z_1^{-1} + z_2^{-1} + z_3^{-1}\right) = uv - t^2 + t^2(u + v + 2t),$$

and, just as in case (i), $\alpha\beta \geqslant 0$, $\beta \neq 0$.

For case (iii), let the characteristic polynomial of $AY - I$ be $\lambda^3 - \alpha\lambda^2 + \beta\lambda - \gamma$. It is found that

$$\alpha = y_1 + y_2(1+u) + y_3(1+v) - 3,$$
$$\beta = y_1 y_2 u + y_1 y_3 v + y_2 y_3\left[u + v + uv + 2t - t^2\right]$$
$$- 2\left[y_1 + y_2(1+u) + y_3(1+v)\right] + 3,$$
$$\gamma = y_1 y_2 y_3\left[uv - t^2 + T\right] - \beta - \alpha - 1$$
$$+ y_1 y_2 y_3\left[(1-t)(\theta + \theta^{-1} - 2) - T\right],$$

where T is an arbitrary positive number. We will show that it is always possible to select nonzero y_1, y_2, y_3 such that, simultaneously, $\beta = 0$ and $y_1 y_2 y_3(uv - t^2 + T) = \alpha + 1$. Furthermore, in this choice of y_1, y_2, y_3, the coefficient α will not vanish if $T = T_0 = (1-t)(\theta + \theta^{-1} - 2)$, so that, by taking T slightly different from T_0, we can always ensure that $\alpha\gamma > 0$, $\beta = 0$, implying the existence of nonreal eigenvalues.

If $y_1 = (t-1)/t$, it is found that $\beta = 0$ implies

$$\left[t(u + v + uv + 2t - t^2)y_2 - (v + 2t + vt)\right]$$
$$\times\left[t(u + v + uv + 2t - t^2)y_3 - (u + 2t + ut)\right]$$
$$= (t^2 + u)(t^2 + v)$$

with solution

$$y_2 = \left[v + 2t + vt + K(t^2 + v)\right]\left[t(u + v + uv + 2t - t^2)\right]^{-1},$$
$$y_3 = \left[u + 2t + ut + K^{-1}(t^2 + u)\right]\left[t(u + v + uv + 2t - t^2)\right]^{-1},$$

for K a nonzero number. Substitute into $y_1 y_2 y_3(uv - t^2 + T) = \alpha + 1$, and it is found that K must satisfy

$$\begin{aligned} K&\left[(t^2+v)^2(t+u)^2 + T(t^2+v)(1-t)(u+2t+ut)\right] \\ &+ K^{-1}\left[(t^2+u)^2(t+v)^2 + T(t^2+u)(1-t)(v+2t+vt)\right] \\ &+ \left[2(t^2+u)(t^2+v)(t+u)(t+v)\right. \\ &\left.+ T(1-t)\left(t^4 + t^3(4+3u+3v+uv) + t(2u+2v+2uv) + 2uv\right)\right] = 0. \end{aligned} \tag{3.1}$$

Multiply by K and write down the condition for the resulting quadratic equation to have distinct real roots. This condition is found to be $T(PT + Q) > 0$, where

$$P = (1-t)^2 t^2 (t+2)^2 (t^2 - 2t - u - v - uv)^2 \geqslant 0,$$
$$Q = 4(1-t)(t^2+u)(t^2+v)tR(t)$$

with

$$\begin{aligned} R(t) = t^5 - 4t^4 + (4 - 2u - 2v - 2uv)t^3 + (4u + 4v + 3uv)t^2 \\ + \left[(u+v)^2 + uv(u+v+uv)\right]t - u^2v^2. \end{aligned}$$

It is found that

$$R\left((uv)^{1/2}\right) = (uv)^{1/2}(u^{1/2} + v^{1/2})^2\left[u + v + (uv)^{1/2}\left(2 - (uv)^{1/2}\right)\right] > 0$$

and that

$$\begin{aligned} R'(t) = t^2(2-t)(6-5t) + t(u+v)(8-6t) \\ + 6uvt(1-t) + (u+v)^2 + uv(u+v+uv) \\ > 0 \end{aligned}$$

for $t \in [(uv)^{1/2}, 1]$, implying that $T(PT+Q)$ is always positive for $uv \leqslant t^2 < 1$. Hence, (3.1) always has two real distinct roots, and the corresponding values of y_2 and y_3 lead to $\beta = 0$ and $\gamma = y_1 y_2 y_3 (T_0 - T)$. The possibility that $\alpha = 0$ for each of the two values of K can be dismissed, because if this were so, then the left hand side of (3.1), regarded as a function of K, would be a multiple of the equation for $\alpha = 0$, which is

$$K(1+u)(t^2+v) + K^{-1}(1+v)(t^2+u) + \left[uv + t(-2+u+v) - 3t^2 + 2t^3\right] = 0. \quad (3.2)$$

If the coefficients of K and K^{-1} in (3.1) and (3.2) are in proportion, it is found that

$$(u-v)(1-t)^2(uv - t^2 + T) = 0.$$

This implies either $T + uv - t^2 = 0$, giving the value $\alpha = -1$, or else $v = u$. This latter possibility must be rejected because for $u \leqslant t < 1$, the constant term in (3.1) is positive and the constant term in (3.2) is equal to

$$g(t) = u^2 + t(-2+2u) - 3t^2 + 2t^3,$$

which is negative, since $g(u) = -2u(1-u^2) < 0$ and $g'(t) = -2(1-u) - 6t(1-t) < 0$.

For case (iv), in which $tuv = 0$, we assume without loss of generality that $v \geqslant u$. We consider four subcases:

(A) $t = 0,\ u, v > 0$,
(B) $t = u = 0,\ v > 0$,
(C) $t = u = v = 0$,
(D) $u = 0,\ t, v > 0$.

For subcase (A), choose $Y = \operatorname{diag}(1, x/u, x/v)$, where $x > 0$. For λ an eigenvalue of AY, let $\mu = x/\lambda$, so that μ satisfies the equation

$$\left(1 + \frac{\theta + \theta^{-1} - 2}{uv}\right)\mu^3 - \left[2 + x(1 + u^{-1} + v^{-1})\right]\mu^2 + \left[1 + x(2 + u^{-1} + v^{-1})\right]\mu - x = 0.$$

For x sufficiently small, this has only one real root, as can be seen by considering the limiting case.

For subcase (B), choose $Y=\mathrm{diag}(1,1,x)$ and let $\mu=\lambda^{-1}$ for λ an eigenvalue, so that μ satisfies

$$g(\mu)=x(\theta+\theta^{-1}-2)\mu^3-2xv\mu^2+[2+x(1+v)]\mu-1=0.$$

For x sufficiently small, the quadratic

$$g'(\mu)=3x(\theta+\theta^{-1}-2)\mu^2-4xv\mu+[2+x(1+v)]$$

has nonreal zeros, implying that the same holds for g.

For subcase (C), choose $Y=I$ so that the characteristic polynomial is $(\theta+\theta^{-1}-2)+3\lambda^2-\lambda^3$ which has nonreal zeros.

In subcase (D), the argument in case (iii) can be repeated, the only changes being that now $R(0)=0$, $R'(t)>0$, implying that $R(t)>0$ for $t\in(0,1)$, and that now the option $v=u$ does not need to be considered. ∎

4. FURTHER CASES WITH $n \geqslant 4$

In this section we consider the case $n\geqslant 4$ in which no nonsingular diagonal D exists for which AD can be symmetric. We first establish a preliminary result, for which the proof makes use of some standard terminology and some elementary results from the theory of graphs [1].

LEMMA 4.1. *Let A be an irreducible $n\times n$ matrix for which no nonsingular diagonal matrix D exists such that AD is symmetric. Then there exists an integer $m\leqslant n$ and distinct $N(1),N(2),\dots,N(m)\in\{1,2,\dots,n\}$, such that the $m\times m$ matrix B with elements defined by $b_{ij}=a_{N(i),N(j)}$ has the properties*

$$b_{ij}=0 \qquad \textit{unless} \quad |i-j|=0,1,\textit{ or } m-1,$$

$$b_{1m}b_{m,m-1}\cdots b_{32}b_{21}\neq b_{12}b_{23}\cdots b_{m-1,m}b_{m1}\neq 0. \tag{4.1}$$

Proof. Let G denote the digraph with vertex set $V=\{1,2,\dots,n\}$ and edge set $E=\{(i,j): a_{ij}\neq 0, i\neq j\}$. Because A is irreducible, any two members of V are path-connected. We distinguish two cases:

(i) G is not symmetric,
(ii) G is symmetric.

In case (i) there exists a pair of vertices I,J such that $(I,J)\in E$, $(J,I)\notin E$. Choose a path from J to I, and consider the cycle made up from I, J, and the remaining members of the path from J to I. In case (ii), let T denote a tree spanning the vertex set V for which $\{i, j\}$ is an arc of T only if (i, j) is a member of E. For each arc $\{i, j\}$ of T, define the ratio d_i/d_j, so that $a_{ij}d_j = a_{ji}d_i$. Up to an arbitrary nonzero factor, this defines $d_1, d_2, \ldots, d_n$ uniquely. Since AD is not symmetric, there exist I, J such that, with the values of the D elements that have been assigned, $a_{IJ}d_J \neq a_{JI}d_I$. Select a cycle in G made up from (I, J) together with the sequence of edges connecting J to I corresponding to arcs in T.

In both cases (i) and (ii), we have identified a cycle in G, made up from vertices $N(1), N(2), \ldots, N(m)$ such that $b_{1m}b_{m,m-1} \cdots b_{32}b_{21} \neq b_{12}b_{23} \cdots b_{m-1,m}b_{m1} \neq 0$. Without loss of generality, suppose that amongst all cycles with this property, the length m is the minimum possible value. It is easy to verify that this implies the impossibility of an edge $(N(i), N(j))$ in E unless (4.1) is satisfied. ∎

For the type of matrix under consideration in this section, we can, according to Lemma 4.1, restrict ourselves to matrices of the form given in the next result.

LEMMA 4.2. *If A is given by*

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & \cdots & 0 & a_{1n} \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & \cdots & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & 0 & 0 & 0 & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix},$$

where

$$a_{1n}a_{n,n-1}a_{n-1,n-2} \cdots a_{21} \neq a_{12}a_{23} \cdots a_{n1} \neq 0,$$

then Y exists such that AY has nonreal eigenvalues.

Proof. We distinguish two cases:

(i) all diagonal elements of A are zero,
(ii) at least one diagonal element of A, say a_{11}, is nonzero.

In case (i), because of Lemma 1,

$$a_{1n} = a_{n,n-1} = a_{n-1,n-2} = \cdots = a_{21} = 0.$$

Hence, with Y defined so that

$$\det(Y) = (a_{12}a_{23}\cdots a_{n1})^{-1},$$

it is found that AY has eigenvalues equal to $\exp(2\pi ki/n)$, for $k = 0, 1, \ldots, n-1$. In case (ii), we assume that the result of this lemma is already proved for lower integers than n. Since the case $n = 3$ has already been established in Lemmas 3.1 and 3.2, the proof for any n will then follow by an induction argument. Let $\tilde{Y} = \operatorname{diag}(y_2, y_3, \ldots, y_n)$ be chosen such that for the $(n-1)\times(n-1)$ matrix

$$\tilde{A} = \begin{bmatrix} a_{22} - \dfrac{a_{12}a_{21}}{a_{11}} & a_{23} & 0 & \cdots & -\dfrac{a_{21}a_{1n}}{a_{11}} \\ a_{32} & a_{33} & a_{34} & \cdots & 0 \\ 0 & a_{43} & a_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -\dfrac{a_{n1}a_{12}}{a_{11}} & 0 & 0 & \cdots & a_{nn} - \dfrac{a_{n1}a_{1n}}{a_{11}} \end{bmatrix},$$

$\tilde{A}\tilde{Y}$ possesses nonreal eigenvalues. Let $Y = \operatorname{diag}(1, xy_2, xy_3, \ldots, xy_n)$; the characteristic polynomial of AY is found to satisfy the relation

$$\det(AY - x\lambda I) = a_{11}x^{n-1}\det(\tilde{A}\tilde{Y} - \lambda I) - x^n\lambda\det(\hat{A}\tilde{Y} - \lambda I),$$

where $\hat{A}$ is identical with A except for the deletion of the first row and first column. For sufficiently small x it is easy to see (for example by the Rouché theorem) that if $\tilde{\lambda}$ is a complex eigenvalue of $\tilde{A}$, then there is an eigenvalue λ of A such that $x^{-1}\lambda$ is arbitrarily close to $\tilde{\lambda}$. ∎

5. CONCLUDING REMARKS

It remains to present the formal justification of the main result.

Proof of the Theorem. In the case $n = 2$, the existence of Y is shown in Lemma 1.

If a nonsingular diagonal D exists such that AD is symmetric but cannot be made positive semidefinite, then without loss of generality, we may suppose that $D = I$, and that at least one diagonal element of A, say a_{11}, is positive. Since A is irreducible and can be assumed to have no submatrix covered by Lemma 1, it follows that all diagonals are positive. We can now construct a set of indices I such that if $\tilde{A}$ is the submatrix formed by selecting only those rows and columns of A whose index numbers lie in I, then $\tilde{A}$ has the determinants of all its proper principal minors positive, but the determinant of $\tilde{A}$ itself is negative. An algorithm for constructing this index set is to generate in turn index sets $I_1, I_2, \ldots$ such that $I_1 = \{1\}$, and for $k = 2, 3, \ldots,$ $I_k = I_{k-1}$ if $d(I_{k-1} \cup \{k\}) = 0$ and $I_k = I_{k-1} \cup \{k\}$ otherwise, where for an index set J, $d(J)$ is the determinant of the corresponding submatrix. The index set I is then defined as the first I_k for which $d(I_k) < 0$. The matrix $\tilde{A}$ then satisfies the requirements of Lemma 2.

Hence, if nonsingular D exists such that AD is symmetric, but cannot be positive semidefinite, then Y can be found so that AY has nonreal eigenvalues.

We now consider the alternative possibility that no nonsingular D exists for which AD is symmetric. If $n = 3$, or an appropriate submatrix of this size exists, then the existence of Y is proved in Lemmas 3.1 and 3.2. Finally, if $n \geqslant 4$, we have shown in Lemma 4.1 that A must contain a submatrix of a particular form. If the size of the submatrix is less than 4, then the result is covered by Lemmas 1, 3.1, and 3.2. If $n \geqslant 4$, then the existence of Y is proved in Lemma 4.2. ■

Part of this work was carried out during visits to AT&T Bell Laboratories, Murray Hill, New Jersey and to the Centre for Mathematics and Computer Science, Amsterdam, The Netherlands. The author would like to express his appreciation for the assistance he received during those visits.

REFERENCES

1 C. Berge, *The Theory of Graphs and Its Applications*, Methuen, London, 1962.

2 K. Burrage and J. C. Butcher, Non-linear stability of a general class of differential equation methods. *BIT* 20:185–203 (1980).

3 J. C. Butcher, General linear methods: A survey, *Appl. Numer. Math.* 1:273–284 (1985).

Received 1 May 1986; final manuscript accepted 11 June 1987