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Separability in the Strict Topology

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Let X be a completely regular Hausdorff space and E be a locally convex Hausdorff space. Then $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_0)$, $(C_b(X), \beta) \otimes_\epsilon E = (C_b(X) \otimes E, \beta)$ and $(C_b(X), \beta_1) \otimes_\epsilon E = (C_b(X) \otimes E, \beta_1)$. For a separable space E, $(C_b(X, E), \beta_0)$ is separable if and only if X is separably submetrizable. As a corollary, for a locally compact paracompact space X, if $(C_b(X, E), \beta_0)$ is separable, then X is metrizable.

INTRODUCTION

Since the introduction by Buck [1] of the strict topology on the space $C_b(X)$ of bounded continuous functions on a locally compact Hausdorff space X, many studies have been done about the strict topology ([3, 3, 7, 8, 11–13], etc.). In this paper, X denotes a completely regular Hausdorff space, $C_b(X)$ all bounded continuous real-valued functions on X, E a locally convex Hausdorff space over the real numbers, $C_b(X, E)$ all bounded continuous functions from X into E, $C_b(X) \otimes_\epsilon E$ the tensor product, where ϵ is the topology of uniform convergence on sets of the form $S \times T$, S and T being equicontinuous sets of $(C_b(X), \beta_0)'$ and E', respectively.

For a compact subset (zero subset) $Q \subset \beta X \backslash X$, βX is the Stone–Čech compactification of X, let $C_Q(X) = \{f|_X : f \in C(\beta X), f = 0 \text{ on } Q\}$. The topology β_Q on $C_b(X, E)$ is defined by the seminorms $\|\cdot\|_h$, h ranging through the elements of $C_Q(X)$, $\|f\|_h = \sup_{x\in X} \|h(x) f(x)\|$, $f \in C_b(X, E)$. The topology $\beta(\beta_1)$ on $C_b(X, E)$ is defined to be the intersection of the topologies β_Q as Q ranges through compact subsets (zero subsets) of $\beta X \backslash X$. Sentilles [11] showed that $W \in \beta(\beta_1)$ iff $W \in \beta_Q$ for all $Q \subset \beta X \backslash X$, and $W \in \beta(\beta_1)$ iff for each Q there exists $V_Q \in \beta_Q$ such that $W \supset \langle \bigcup_Q V_Q \rangle$, where $\langle \bigcup_Q V_Q \rangle$ denotes the absolutely convex hull of $\bigcup_Q V_Q$.

The strict topology β_0 on $C_b(X, E)$ is defined by the family of seminorms $\|\cdot\|_{h,p}$, as h varies through all real-valued functions on X vanishing at infinity and p ranges over all continuous seminorms on E—$\|f\|_{h,p} = \sup_{x\in X} p(h(x) f(x))$, $f \in C_b(X, E)$. When E is a normed space, it is proved [5] that $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_0)$ and $(C_b(X, E), \beta_0)' = M_t(X, E')$. For a locally convex E, $(C_{rc}(X, E), \beta_0)' = M_t(X, E')$, where $C_{rc}(X, E)$ are those elements of $C_b(X, E)$

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which have relatively compact images in E [7]. $M_t(X, E') = \{\mu: B(X) \to E', \mu$ is a measure and for every $x \in E$, $\mu_x: B(X) \to R$, defined by $\mu_x(B) = \langle \mu(B), x \rangle$, is in $M_t(X)\}$, where $B(X)$ is the family of Borel sets of X and E' is the topological dual of E [2]. Since $C_b(X) \otimes E$ is dense in $C_{rc}(X, E)$ in the topology of uniform convergence on X, a topology finer than β_0, the result $(C_{rc}(X, E), \beta_0)' = M_t(X, E')$ is equivalent to $(C_b(X) \otimes E, \beta_0)' = M_t(X, E')$. Consequently, $(C_b(X, E), \beta_0)' = M_t(X, E')$ (by Theorem 2) which answers the question raised in [5]. Also, it is proved that $(C_b(X) \otimes E, \beta_0) = (C_b(X), \beta_0) \otimes_\epsilon E$ in [2] and $(C_b(X, E), \beta_0)$ is a Mackey space when X is a P-space and E is a normed space [8].

A collection $\{f_\alpha\}_{\alpha \in I}$ of $C_b(X)$ such that $0 \leqslant f_\alpha \leqslant 1$ for each α is called a partition of unity of X if $\sum_{\alpha \in I} f_\alpha = 1$ and the collection $\{\{f_\alpha > 0\}: \alpha \in I\}$ is locally finite [4].

LEMMA 1. *If q is a continuous seminorm on E, $h_0 \in C_b(X, E)$ and $\epsilon > 0$, then there exists a partition of unity $\{f_\alpha\}_{\alpha \in I}$ on X and points $\{x_\alpha\}_{\alpha \in I}$ in X such that $\sup_{x \in X} q(h_1(x) - h_0(x)) < \epsilon$ where $h_1(x) = \sum_{\alpha \in I} h_0(x_\alpha) f_\alpha(x)$, $x \in X$.*

Proof. Define $d: X \times X \to R$ by $d(x, y) = q(h_0(x) - h_0(y))$. Then d is the continuous pseudo-metric on X. The relation $x \sim y$ if and only if $d(x, y) = 0$ is an equivalence relation on X, and the collection of equivalence classes $\bar{x}$ is a metric space X_d by defining $\bar{d}(\bar{x}, \bar{y}) = d(x, y)$. The natural map $\pi_d: X \to X_d$ by $\pi_d(x) = \bar{x}$ is continuous. Let $\{f_\alpha^0\}$ be a continuous partition of unity on X_d subordinate to the covering $\{B(\bar{x}, \epsilon)\}_{\bar{x} \in X_d}$, $B(\bar{x}, \epsilon)$ being the open ball with center at $\bar{x}$ and radius ϵ. For each α, choose $\bar{x}_\alpha \in X_d$ such that $f_\alpha^0(\bar{x}) = 0$ when $\bar{d}(\bar{x}, \bar{x}_\alpha) = d(x, x_\alpha) \geqslant \epsilon$, where $x_\alpha \in X$ with $\pi_d(x_\alpha) = \bar{x}_\alpha$. Now put $f_\alpha = f_\alpha^0 \circ \pi_d$. Then $\{f_\alpha\}$ is a partition of unity on X. Let $h_1(x) = \sum_{\alpha \in I} h_0(x_\alpha) f_\alpha(x)$, $x \in X$. Then a little effort shows that $\sup_{x \in X} q(h_1(x) - h_0(x)) \leqslant \sup_{x \in X} \sum_{\alpha \in I} f_\alpha(x)\, q(h_0(x_\alpha) - h_0(x)) < \epsilon$.

THEOREM 2. *$C_b(X) \otimes E$ is β_0-dense in $C_b(X, E)$.*

Proof. Suppose $A \not\subseteq C_b(X, E)$, A the closure of $C_b(X) \otimes E$ in $(C_b(X, E), \beta_0)$. Then there is $h_0 \in C_b(X, E)$ such that $h_0 \notin A$. Since A is a closed subspace and $\{h_0\}$ is compact, by the Separation Theorem there is a $\mu \in (C_b(X, E), \beta_0)'$ such that $\mu = 0$ on $C_b(X) \otimes E$ and $\mu(h_0) > \epsilon$ for some $\epsilon > 0$. Since the uniform topology is finer than β_0, there exists a continuous seminorm q on E such that $f \in C_b(X, E)$, $\sup_{x \in X} q(f(x)) \leqslant 1$ implies that $|\mu(f)| \leqslant 1$.

If we let B be the closed absolutely convex hull of $h_0(x)$, $E_0 = \bigcup_{n=1}^{\infty} nB$ and $\|\cdot\| = \|\cdot\|_B$ be the Minkowski functional for B, then the topology induced by E on E_0 is weaker than the norm topology on E_0 ($(E_0, \|\cdot\|_B)$ is a normed space [9, p. 26]).

Let d be the continuous pseudo-metric on X, $d(x, y) = q(h_0(x) - h_0(y))$. Then by the lemma, there exists a partition of unity $\{f_\alpha\}_{\alpha \in I}$ and $\{x_\alpha\}_{\alpha \in I} \subset X$ such that

$\sup_{x \in X} q(h_1(x) - h_0(x)) < \epsilon/2$, $h_1(x) = \sum_{\alpha \in I} h_0(x_\alpha) f_\alpha(x)$, $x \in X$. Then $F = C_b(X, E) \cap C_b(X, E_0) \supset (C_b(X) \otimes E_0) \cup \{\sum_{\alpha \in I_0} h_0(x_\alpha) f_\alpha : I_0 \subset I\}$. Considering F as a subspace of $(C_b(X, E_0), \beta_0)$, and using the fact that the $\|\cdot\|$-topology on E_0 is finer than the one induced by E (or that B is bounded) we see that $\mu_0 = \mu|_F \in F'$ and so by the Hahn–Banach theorem, μ_0 can be extended so as to become an element of $(C_b(X, E_0), \beta_0)'$. Since $\mu_0 \equiv 0$ on $C_b(X) \otimes E_0$, it follows from the denseness of $C_b(X) \otimes E_0$ in $(C_b(X, E_0), \beta_0)$ [5, p. 852] that $\mu_0 \equiv 0$ on F, which gives $\mu_0(h_1) = 0$ and so $\mu(h_1) = 0$ (note $h_1 \in F$). Since $\sup_{x \in X} q(h_1(x) - h_0(x)) < \epsilon/2$, we get $|\mu(h_1 - h_0)| < \epsilon/2$ and so $|\mu(h_0)| < \epsilon/2$. This is a contradiction.

THEOREM 3. *$(C_b(X), \beta) \otimes_\epsilon E = (C_b(X) \otimes E, \beta)$, where β is the induced topology.*

Proof. First we want to show that $\epsilon \geqslant \beta$. Let $\{f_\alpha\}_{\alpha \in I}$ be a net in $C_b(X) \otimes E$ such that $f_\alpha \to 0$ in ϵ. $W = \langle \bigcup_{Q \in \beta X \setminus X} V_Q \rangle$, $V_Q = \{f \in C_b(X): \sup_{x \in X} \|f(x) h_Q(x)\| \leqslant 1\}$, $h_Q \in C_Q(X)$. Then W^0, the polar of W, is equicontinuous subset of $(C_b(X), \beta)' = M_\tau(X)$. $W^0 = K = \{\mu \in M_\tau(X): |\mu(f)| \leqslant 1, \forall f \in W\}$.

Let S be an equicontinuous subset of E'. Then $f_\alpha \to 0$ uniformly on $S \times K$, which implies that given $\eta > 0$ there is $\alpha_0 \in I$ such that $\sup_{\mu \in K} \sup_{g \in S} |\mu(g \circ f_\alpha)| \leqslant \eta$, $\forall \alpha \geqslant \alpha_0$. Thus $|((g \circ f_\alpha)/\eta)| \leqslant 1$, $\forall \alpha \geqslant \alpha_0$, $\forall \mu \in K$, $\forall g \in S$. $((g \circ f_\alpha)/\eta) \in K^0 = W^{oo}$, $\forall \alpha \geqslant \alpha_0$, $\forall g \in S$. But W is $(C_b(X), M(X))$-closed [6, p. 34], so by the bipolar theorem $((g \circ f_\alpha)/\eta) \in W = W^{oo}$, $\forall \alpha \geqslant \alpha_0$, $\forall g \in S$. This means that $f_\alpha \to 0$ in β.

To see $\beta \geqslant \epsilon$, let $\{f_\alpha\}_{\alpha \in I}$ be a net in $C_b(X) \otimes E$ such that $f_\alpha \to 0$ in β. Let K be an equicontinuous subset of $(C_b(X), \beta)'$ and let S be an equicontinuous subset of E'. Since K^0 is a zero nhd of $(C_b(X), \beta)$, for every compact set $Q \subset \beta X \setminus X$, there exists an $h_Q \in C_Q(X)$ such that $K^0 \supset \{g \in C_b(X): \sup_{x \in X} \|g(x) h_Q(x)\| \leqslant 1$. Now, let $S^0 = \{x \in E: |f(x)| \leqslant 1, \forall f \in S\}$ and $\| \|_{S^0}$ be the Minkowski functional corresponding to S^0. Since $f_\alpha \to 0$ in β, given $\eta > 0$, there exists $\alpha_0 \in I$ such that $\| h_Q f_\alpha \|_{S^0} \leqslant \eta$, $\forall \alpha \geqslant \alpha_0$ and so $\| h_Q f_\alpha / \eta \|_{S^0} \leqslant 1$, $\forall \alpha \geqslant \alpha_0$, which implies that $\|((h_Q g \circ f_\alpha)/\eta)\| \leqslant 1$, $\forall \alpha \geqslant \alpha_0$, $\forall g \in S$. This proves that $|\mu(g \circ f_\alpha)| \leqslant \eta$, $\forall \alpha \geqslant \alpha_0$, $\forall g \in S$, $\forall \mu \in K$, and hence $f_\alpha \to 0$ in ϵ.

THEOREM 4. $(C_b(X), \beta_1) \otimes_\epsilon E = (C_b(X) \otimes E, \beta_1)$.

Proof. The proof is similar to the above.

A topological space X is called submetrizable if it can be mapped by a one-to-one continuous function onto some metric space Z. If Z is separable, then we say that X is separably submerizable [12].

THEOREM 5. *Let E be a separable space. Then $(C_b(X, E), \beta_0)$ is separable if and only if X is separably submetrizable.*

Proof. If X is separably submetrizable, then $(C_b(X), \beta_0)$ is separable [12, p. 509]. Let $L: (C_b(X), \beta_0) \times E \to (C_b(X), \beta_0) \otimes E$ be the canonical bilinear mapping. Then the image $L(C_b(X) \times E)$ is separable, and so $(C_b(X) \otimes E, \beta_0)$ is separable since $C_b(X) \otimes E$ is the linear hull of $L(C_b(X) \times E)$ and $(C_b(X) \otimes E, \beta_0) = (C_b(X), \beta_0) \otimes_\epsilon E$ [2]. Thus by the density, $(C_b(X, E), \beta_0)$ is separable.

Let $(C_b(X, E), \beta_0)$ be separable. Then fix $f \in E'$, the dual of E and $f \neq 0$ and define $T: (C_b(X, E), \beta_0) \to (C_b(X), \beta_0)$ by $T(g) = f \circ g$ for all g in $C_b(X, E)$. Since $f \in E'$, there is a continuous seminorm q on E such that $|f(y)| \leqslant q(y)$, $\forall y \in E$. Let $\{g_\alpha\} \subset C_b(X, E)$ such that $g_\alpha \to 0$ in β_0. Then $\sup_{x \in X} q(h(x)\, g_\alpha(x)) \to 0$, where h is a real-valued function which vanishes at infinity. For all $x \in X$, $h(x)\, g_\alpha(x) \in E$ and so $|f(h(x)\, g_\alpha(x))| \leqslant q(h(x)\, g_\alpha(x))$ which implies that $\sup_{x \in X} |h(x) f \circ g_\alpha(x)| \leqslant \sup_{x \in X} q(h(x)\, g_\alpha(x))$. This proves that T is continuous, and it is easy to see T is onto. Therefore $(C_b(X), \beta_0)$ is separable, and hence X is separably submetrizable [12, p. 509].

COROLLARY 6. *Let X be locally compact paracompact. If $(C_b(X, E), \beta_0)$ is separable, then X is metrizable.*

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