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A matrix Hilbert transform in Hermitean Clifford analysis

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ABSTRACT

Orthogonal Clifford analysis is a higher dimensional function theory offering both a generalization of complex analysis in the plane and a refinement of classical harmonic analysis. During the last years, Hermitean Clifford analysis has emerged as a new and successful branch of it, offering yet a refinement of the orthogonal case. Recently in [F. Brackx, B. De Knock, H. De Schepper, D. Peña Peña, F. Sommen, submitted for publication], a Hermitean Cauchy integral was constructed in the framework of circulant (2×2) matrix functions. In the present paper, a new Hermitean Hilbert transform is introduced, arising naturally as part of the non-tangential boundary limits of that Hermitean Cauchy integral. The resulting matrix operator is shown to satisfy properly adapted analogues of the characteristic properties of the Hilbert transform in classical analysis and orthogonal Clifford analysis.

1. Introduction

In engineering sciences, and in particular in signal analysis, the one-dimensional Hilbert transform of a real signal $u(t)$, depending on a one-dimensional time variable t, has become a fundamental tool. For a suitable function or distribution $u(t)$ its Hilbert transform is given by the Cauchy Principal Value

$$H[u](t) = -\frac{1}{\pi} \operatorname{Pv} \int_{-\infty}^{+\infty} \frac{u(\tau)}{\tau - t} \, d\tau.$$

Though initiated by Hilbert, the concept of a "conjugated pair" $(u, H[u])$, nowadays called a Hilbert pair, was developed mainly by Titchmarch and Hardy.

The multidimensional approach to the Hilbert transform usually is a tensorial one, considering the so-called Riesz transforms in each of the Cartesian variables separately. As opposed to these tensorial approaches, Clifford analysis (see e.g. [7,11,12]) is particularly suited for a treatment of multidimensional phenomena where all dimensions are encompassed at the same time as an intrinsic feature. Clifford analysis essentially is a higher dimensional function theory offering both a generalization of the theory of holomorphic functions in the complex plane and a refinement of classical harmonic analysis. In the standard case, so-called orthogonal Clifford analysis focuses on monogenic functions, i.e. null solutions of the rotation invariant vector valued Dirac operator

$$\partial_{\underline{X}} = \sum_{j=1}^{m} e_j \partial_{x_j}$$

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where $(e_1, \ldots, e_m)$ forms an orthonormal basis for the quadratic space $\mathbb{R}^{0,m}$ underlying the construction of the real Clifford algebra $\mathbb{R}_{0,m}$. The theory of Hardy spaces and the multidimensional Hilbert transform in the orthogonal Clifford analysis framework is nowadays well established, see [5,10,12]. However we want to draw the reader's attention on the paper [13] of Horváth who, to our knowledge, was the first to define a vector valued Hilbert transform using Clifford algebra.

During the last years another branch of Clifford analysis has emerged, offering yet a refinement of the orthogonal case: Hermitean Clifford analysis, focusing on the simultaneous null solutions of two complex Hermitean Dirac operators. Complex Dirac operators were already studied in [14–16]; however, a systematic development of the associated function theory including the invariance properties with respect to the underlying Lie groups and Lie algebras is still in full progress, see [2,3,9].

Clearly, an essential result in the further development of this function theory is a Cauchy integral formula for Hermitean monogenic functions taking values in the complex Clifford algebra $\mathbb{C}_{2n}$. A first result in this respect has been obtained in [17], however considering null solutions of only one of the Hermitean Dirac operators and moreover presenting a, as termed by the authors themselves, "fake" Cauchy kernel, which fails to be monogenic. In our recent article [6] a Cauchy integral formula for Hermitean monogenic functions, generalizing the traditional Martinelli–Bochner formula for functions of several complex variables, was presented in the framework of circulant (2×2) matrix functions. The aim of this paper is to show that the non-tangential boundary limits of this Cauchy integral reveal a new Hilbert-like matrix operator.

In Section 2 we recall how, in a natural way, Hermitean Clifford analysis can emerge from orthogonal Clifford analysis by means of the introduction of a so-called complex structure on the involved Clifford algebra. Section 3 then recalls the boundary value theory of the orthogonal Cauchy integral, including some properties of the (associated) Hilbert transform. Section 4 is devoted to the Hermitean Cauchy integral, including some aspects of the indispensable framework of circulant (2×2) matrix functions in which it is constructed. In Section 5, a new matrix operator pops up as part of the non-tangential boundary values of the Hermitean Cauchy integral. It is shown that this operator has a close connection to the Hilbert transform in the orthogonal case, showing some quite similar properties as well.

2. Preliminaries

Let $\mathbb{R}^{0,m}$ be endowed with a non-degenerate quadratic form of signature $(0, m)$, let $(e_1, \ldots, e_m)$ be an orthonormal basis for $\mathbb{R}^{0,m}$ and let $\mathbb{R}_{0,m}$ be the real Clifford algebra constructed over $\mathbb{R}^{0,m}$. The non-commutative multiplication in $\mathbb{R}_{0,m}$ is governed by

$$e_j e_k + e_k e_j = -2\delta_{jk}, \quad j, k = 1, \ldots, m. \tag{1}$$

A basis for $\mathbb{R}_{0,m}$ is obtained by considering for a set $A = \{j_1, \ldots, j_h\} \subset \{1, \ldots, m\}$ the element $e_A = e_{j_1} \ldots e_{j_h}$, with $1 \leqslant j_1 < j_2 < \cdots < j_h \leqslant m$. For the empty set $\emptyset$ one puts $e_\emptyset = 1$, the identity element. Any Clifford number a in $\mathbb{R}_{0,m}$ may thus be written as $a = \sum_A e_A a_A$, $a_A \in \mathbb{R}$, or still as $a = \sum_{k=0}^m [a]_k$, where $[a]_k = \sum_{|A|=k} e_A a_A$ is the so-called k-vector part of a $(k = 0, 1, \ldots, m)$. The Euclidean space $\mathbb{R}^{0,m}$ is embedded in $\mathbb{R}_{0,m}$ by identifying $(x_1, \ldots, x_m)$ with the Clifford vector $\underline{X}$ given by

$$\underline{X} = \sum_{j=1}^m e_j x_j.$$

Note that the square of a vector $\underline{X}$ is scalar valued and equals the norm squared up to a minus sign: $\underline{X}^2 = -\langle \underline{X}, \underline{X} \rangle = -|\underline{X}|^2$. The dual of the vector $\underline{X}$ is the vector valued first order differential operator

$$\partial_{\underline{X}} = \sum_{j=1}^m e_j \partial_{x_j}$$

is called Dirac operator. It is precisely this Dirac operator which underlies the notion of monogenicity of a function, a notion which is the higher dimensional counterpart of holomorphy in the complex plane. A function f defined and differentiable in an open region Ω of $\mathbb{R}^{0,m}$ and taking values in $\mathbb{R}_{0,m}$ is called (left) monogenic in Ω if $\partial_{\underline{X}}[f] = 0$. As the Dirac operator factorizes the Laplacian, $\Delta_m = -\partial_{\underline{X}}^2$, monogenicity can be regarded as a refinement of harmonicity. We refer to this setting as the orthogonal case, since the fundamental group leaving the Dirac operator $\partial_{\underline{X}}$ invariant is the special orthogonal group $\mathrm{SO}(m; \mathbb{R})$, which is doubly covered by the $\mathrm{Spin}(m)$ group of the Clifford algebra $\mathbb{R}_{0,m}$. For this reason, the Dirac operator is called a rotation invariant operator.

When allowing for complex constants and moreover taking the dimension to be even, say $m = 2n$, the same set of generators as above, $(e_1, \ldots, e_{2n})$, still satisfying the defining relations (1), may in fact also produce the complex Clifford algebra $\mathbb{C}_{2n}$. As $\mathbb{C}_{2n}$ is the complexification of the real Clifford algebra $\mathbb{R}_{0,2n}$, i.e. $\mathbb{C}_{2n} = \mathbb{R}_{0,2n} \oplus i\mathbb{R}_{0,2n}$, any complex Clifford number $\lambda \in \mathbb{C}_{2n}$ may be written as $\lambda = a + ib$, $a, b \in \mathbb{R}_{0,2n}$, leading to the definition of the Hermitean conjugation $\lambda^\dagger = (a + ib)^\dagger = \bar{a} - i\bar{b}$, where the bar denotes the usual Clifford conjugation in $\mathbb{R}_{0,2n}$, i.e. the main anti-involution for which $\bar{e}_j = -e_j$, $j = 1, \ldots, 2n$. This Hermitean conjugation leads to a Hermitean inner product and its associated norm on $\mathbb{C}_{2n}$ given by $(\lambda, \mu) = [\lambda^\dagger \mu]_0$ and $|\lambda| = \sqrt{[\lambda^\dagger \lambda]_0} = (\sum_A |\lambda_A|^2)^{1/2}$. The above framework will be referred to as the Hermitean

Clifford setting, as opposed to the traditional orthogonal Clifford one. Hermitean Clifford analysis then focuses on the simultaneous null solutions of two Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$, introduced below.

One of the ways for introducing Hermitean Clifford analysis is by considering the complex Clifford algebra $\mathbb{C}_{2n}$ and a so-called complex structure on it, i.e. an $SO(2n;\mathbb{R})$-element J for which $J^2 = -\mathbf{1}$ (see [2]). More specifically, J is chosen to act upon the generators $e_1, \ldots, e_{2n}$ of the Clifford algebra as

$$J[e_j] = -e_{n+j} \quad \text{and} \quad J[e_{n+j}] = e_j, \quad j = 1, \ldots, n.$$

Let us recall that the main objects of the Hermitean setting are then conceptually obtained by considering the projection operators $\frac{1}{2}(\mathbf{1} \pm iJ)$ and letting them act on the corresponding protagonists of the orthogonal framework. First of all, the so-called Witt basis elements $(\mathfrak{f}_j, \mathfrak{f}_j^\dagger)_{j=1}^n$ for the complex Clifford algebra $\mathbb{C}_{2n}$ are obtained through the action of $\frac{1}{2}(\mathbf{1} \pm iJ)$ on the orthogonal basis elements e_j:

$$\mathfrak{f}_j = \frac{1}{2}(\mathbf{1} + iJ)[e_j] = \frac{1}{2}(e_j - ie_{n+j}), \quad j = 1, \ldots, n,$$
$$\mathfrak{f}_j^\dagger = -\frac{1}{2}(\mathbf{1} - iJ)[e_j] = -\frac{1}{2}(e_j + ie_{n+j}), \quad j = 1, \ldots, n.$$

These Witt basis elements satisfy the Grassmann identities

$$\mathfrak{f}_j\mathfrak{f}_k + \mathfrak{f}_k\mathfrak{f}_j = \mathfrak{f}_j^\dagger\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j^\dagger = 0, \quad j,k = 1, \ldots, n,$$

and the duality identities

$$\mathfrak{f}_j\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j = \delta_{jk}, \quad j,k = 1, \ldots, n.$$

Next we identify a vector $\underline{X} = (X_1, \ldots, X_{2n}) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $\mathbb{R}^{0,2n}$ with the Clifford vector $\underline{X} = \sum_{j=1}^n (e_j x_j + e_{n+j} y_j)$ and we denote by $\underline{X}|$ the action of the complex structure J on $\underline{X}$, i.e.

$$\underline{X}| = J[\underline{X}] = \sum_{j=1}^n (e_j y_j - e_{n+j} x_j).$$

Note that the vectors $\underline{X}$ and $\underline{X}|$ are orthogonal w.r.t. the standard Euclidean scalar product, which implies that the Clifford vectors $\underline{X}$ and $\underline{X}|$ anti-commute. The Hermitean Clifford variables $\underline{Z}$ and $\underline{Z}^\dagger$ then arise through the action of the projection operators on the standard Clifford vector $\underline{X}$:

$$\underline{Z} = \frac{1}{2}(\mathbf{1} + iJ)[\underline{X}] = \frac{1}{2}(\underline{X} + i\underline{X}|),$$
$$\underline{Z}^\dagger = -\frac{1}{2}(\mathbf{1} - iJ)[\underline{X}] = -\frac{1}{2}(\underline{X} - i\underline{X}|).$$

They can be rewritten in terms of the Witt basis elements as

$$\underline{Z} = \sum_{j=1}^n \mathfrak{f}_j z_j \quad \text{and} \quad \underline{Z}^\dagger = (\underline{Z})^\dagger = \sum_{j=1}^n \mathfrak{f}_j^\dagger z_j^c$$

where n complex variables $z_j = x_j + iy_j$ have been introduced, with complex conjugates $z_j^c = x_j - iy_j$, $j = 1, \ldots, n$. Finally, the Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ are derived out of the orthogonal Dirac operator $\partial_{\underline{X}}$:

$$\partial_{\underline{Z}^\dagger} = \frac{1}{4}(\mathbf{1} + iJ)[\partial_{\underline{X}}] = \frac{1}{4}(\partial_{\underline{X}} + i\partial_{\underline{X}|}),$$
$$\partial_{\underline{Z}} = -\frac{1}{4}(\mathbf{1} - iJ)[\partial_{\underline{X}}] = -\frac{1}{4}(\partial_{\underline{X}} - i\partial_{\underline{X}|})$$

where we have introduced

$$\partial_{\underline{X}|} = J[\partial_{\underline{X}}] = \sum_{j=1}^n (e_j \partial_{y_j} - e_{n+j} \partial_{x_j}).$$

In terms of the Witt basis elements, the Hermitean Dirac operators are expressed as

$$\partial_{\underline{Z}} = \sum_{j=1}^n \mathfrak{f}_j^\dagger \partial_{z_j} \quad \text{and} \quad \partial_{\underline{Z}^\dagger} = (\partial_{\underline{Z}})^\dagger = \sum_{j=1}^n \mathfrak{f}_j \partial_{z_j^c}$$

involving the classical Cauchy–Riemann operators $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ and their complex conjugates $\partial_{z_j^c} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ in the complex z_j-planes, $j = 1, \ldots, n$.

A continuously differentiable function g on an open subset Ω of $\mathbb{R}^{2n}$ with values in $\mathbb{C}_{2n}$ is called a (left) Hermitean monogenic (or h-monogenic) function in Ω if and only if it satisfies in Ω the system

$$\partial_{\underline{X}} g = 0 = \partial_{\underline{X}|} g$$

or equivalently, the system

$$\partial_{\underline{Z}} g = 0 = \partial_{\underline{Z}^\dagger} g.$$

The Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ are invariant under the action of a realisation, denoted $\widetilde{\mathrm{U}}(n)$, of the unitary group in terms of the Clifford algebra, see [9]. This group $\widetilde{\mathrm{U}}(n) \subset \mathrm{Spin}(2n)$ is given by

$$\widetilde{\mathrm{U}}(n) = \left\{ s \in \mathrm{Spin}(2n) \;\middle|\; \exists \theta \geqslant 0\colon\ \bar{s} I = \exp(-i\theta) I \right\}$$

its definition involving the self-adjoint primitive idempotent $I = I_1 \ldots I_n$, with $I_j = \mathfrak{f}_j \mathfrak{f}_j^\dagger = \frac{1}{2}(1 - i e_j e_{n+j})$, $j = 1, \ldots, n$.

Finally observe for further use that the Hermitean vector variables and Dirac operators are isotropic, i.e.

$$(\underline{Z})^2 = \left(\underline{Z}^\dagger\right)^2 = 0 \quad \text{and} \quad (\partial_{\underline{Z}})^2 = (\partial_{\underline{Z}^\dagger})^2 = 0$$

whence the Laplacian $\Delta_{2n} = -\partial_{\underline{X}}^2 = -\partial_{\underline{X}|}^2$ allows for the decomposition

$$\Delta_{2n} = 4(\partial_{\underline{Z}} \partial_{\underline{Z}^\dagger} + \partial_{\underline{Z}^\dagger} \partial_{\underline{Z}})$$

and one also has that

$$\underline{Z}\,\underline{Z}^\dagger + \underline{Z}^\dagger \underline{Z} = |\underline{Z}|^2 = \left|\underline{Z}^\dagger\right|^2 = |\underline{X}|^2 = |\underline{X}|\,|^2.$$

3. A pair of Cauchy integrals and Hilbert transforms in the orthogonal setting

Both the Cauchy integral and the Hilbert transform are well-known integral operators which have been thoroughly studied in the framework of orthogonal Clifford analysis. In this section we define in the same framework, the so-called associated Cauchy integral and associated Hilbert transform, which are closely related to and satisfy similar properties as their classical counterparts. Both pairs of transforms were recently introduced in [1,8].

As above, we denote by Ω some open subset of $\mathbb{R}^{2n}$. We then consider a $2n$-dimensional compact differentiable and oriented manifold $\Gamma \subset \Omega$ with C^∞ smooth boundary $\partial\Gamma$. Further Γ^+ will stand for $\mathring{\Gamma}$, and Γ^- for $\Omega \setminus \Gamma$.

First of all, the fundamental solutions of the Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$ are respectively given by

$$E(\underline{X}) = \frac{1}{a_{2n}} \frac{\overline{\underline{X}}}{|\underline{X}|^{2n}} \quad \text{and} \quad E|(\underline{X}) = \frac{1}{a_{2n}} \frac{\overline{\underline{X}|}}{|\underline{X}|^{2n}}$$

with a_{2n} the area of the unit sphere S^{2n-1} in $\mathbb{R}^{2n}$ and with

$$\lim_{|\underline{X}| \to \infty} E(\underline{X}) = 0 \quad \text{and} \quad \lim_{|\underline{X}| \to \infty} E|(\underline{X}) = 0. \tag{2}$$

For a function $g \in L_2(\partial\Gamma)$, its Cauchy integral $C[g]$ and associated Cauchy integral $C|[g]$ in $\Gamma^\pm$ are then defined by

$$C[g](\underline{Y}) = \int_{\partial\Gamma} E(\underline{X} - \underline{Y})\, \widetilde{d\sigma}_{\underline{X}}\, g(\underline{X}), \quad \underline{Y} \in \Gamma^\pm,$$

$$C|[g](\underline{Y}) = \int_{\partial\Gamma} E|(\underline{X} - \underline{Y})\, \widetilde{d\sigma}_{\underline{X}|}\, g(\underline{X}), \quad \underline{Y} \in \Gamma^\pm,$$

the boundedness of the integrals being guaranteed by (2). Here, $\widetilde{d\sigma}_{\underline{X}}$ denotes the vector valued oriented surface element on $\partial\Gamma$ and $\widetilde{d\sigma}_{\underline{X}|} = J[\widetilde{d\sigma}_{\underline{X}}]$. They are explicitly given by means of the following differential forms of order $(2n-1)$:

$$\widetilde{d\sigma}_{\underline{X}} = \sum_{j=1}^{n} \left(e_j (-1)^{j-1} \widetilde{\widetilde{dx_j}} + e_{n+j} (-1)^{n+j-1} \widetilde{\widetilde{dy_j}} \right),$$

$$\widetilde{d\sigma}_{\underline{X}|} = \sum_{j=1}^{n} \left(e_j (-1)^{n+j-1} \widetilde{\widetilde{dy_j}} + e_{n+j} (-1)^{j} \widetilde{\widetilde{dx_j}} \right)$$

with

$$\widetilde{\widetilde{dx_j}} = dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n,$$

$$\widetilde{\widetilde{dy_j}} = dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_{j-1} \wedge dy_{j+1} \wedge \cdots \wedge dy_n.$$

If $dS(\underline{X})$ stands for the classical surface element on $\partial\Gamma$ and $\nu(\underline{X})$ for the outward pointing (w.r.t. Γ^+) unit normal vector in $\underline{X}$ on $\partial\Gamma$, then the surface elements $\widetilde{d\sigma}_{\underline{X}}$, respectively $\widetilde{d\sigma}_{\underline{X}|}$, may also be expressed as

$$\widetilde{d\sigma}_{\underline{X}} = \nu(\underline{X})dS(\underline{X})$$

respectively

$$\widetilde{d\sigma}_{\underline{X}|} = \nu|(\underline{X})\,dS(\underline{X}).$$

The Hilbert transform $H[g]$ and its associated Hilbert transform $H|[g]$ then pop up in a natural way when considering the non-tangential boundary values of both Cauchy integrals

$$C^{\pm}[g](\underline{U}) = \lim_{\substack{\underline{Y}\to\underline{U}\\ \underline{Y}\in\Gamma^{\pm}}} C[g](\underline{Y}) = \pm\frac{1}{2}g(\underline{U}) + \frac{1}{2}H[g](\underline{U}), \quad \underline{U}\in\partial\Gamma, \tag{3}$$

$$C|^{\pm}[g](\underline{U}) = \lim_{\substack{\underline{Y}\to\underline{U}\\ \underline{Y}\in\Gamma^{\pm}}} C|[g](\underline{Y}) = \pm\frac{1}{2}g(\underline{U}) + \frac{1}{2}H|[g](\underline{U}), \quad \underline{U}\in\partial\Gamma, \tag{4}$$

where the limits are taken in L_2 sense. Explicitly, both transforms are given by the principal value integrals

$$H[g](\underline{U}) = 2\lim_{\varepsilon\to 0+} \int_{\partial\Gamma\setminus B(\underline{U},\varepsilon)} E(\underline{X}-\underline{U})\,\widetilde{d\sigma}_{\underline{X}}\,g(\underline{X}), \quad \underline{U}\in\partial\Gamma,$$

$$H|[g](\underline{U}) = 2\lim_{\varepsilon\to 0+} \int_{\partial\Gamma\setminus B(\underline{U},\varepsilon)} E|(\underline{X}-\underline{U})\,\widetilde{d\sigma}_{\underline{X}|}\,g(\underline{X}), \quad \underline{U}\in\partial\Gamma,$$

with $B(\underline{U},\varepsilon)$ the open ball in $\mathbb{R}^{2n}$ with centre $\underline{U}$ and radius ε. We recall their main properties, apart from the above defining one (see e.g. [10]).

Proposition 1. *One has*

P(1) *H and $H|$ are bounded linear operators on $L_2(\partial\Gamma)$;*
P(2) *$H^2 = H|^2 = \mathbf{1}$;*
P(3) *$H^* = \nu H\nu$ and $H|^* = \nu|H|\nu|$;*
P(4) *for $g\in L_2(\partial\Gamma)$, we have that $H[g] = g$ (respectively $H|[g] = g$) if and only if $g\in H^2(\partial\Gamma)$ (respectively $g\in H|^2(\partial\Gamma)$).*

The last property P(4) deserves some more explanation. For the open set Γ^+ one can consider the Hardy spaces $H^2(\Gamma^+)$ and $H|^2(\Gamma^+)$ of $\partial_{\underline{X}}$-monogenic, respectively $\partial_{\underline{X}|}$-monogenic Clifford algebra valued functions, viz

$$H^2(\Gamma^+) = \{g:\Gamma^+\to\mathbb{R}_{2n}\colon\ \partial_{\underline{X}}g = 0 \text{ in } \Gamma^+ \text{ and } g_{\partial\Gamma}\in L_2(\partial\Gamma)\},$$

$$H|^2(\Gamma^+) = \{g:\Gamma^+\to\mathbb{R}_{2n}\colon\ \partial_{\underline{X}|}g = 0 \text{ in } \Gamma^+ \text{ and } g_{\partial\Gamma}\in L_2(\partial\Gamma)\}$$

where $g_{\partial\Gamma}$ denotes the non-tangential boundary value of g. It is well known that $H^2(\Gamma^+)$ entails the Hardy space $H^2(\partial\Gamma)$ as the closure in $L_2(\partial\Gamma)$ of the space of all non-tangential boundary values of all functions in $H^2(\Gamma^+)$. Moreover, both spaces $H^2(\Gamma^+)$ and $H^2(\partial\Gamma)$ are isomorphic, the isomorphism being obtained explicitly by means of the Cauchy integral in the following way. For a given $g\in H^2(\partial\Gamma)$ its Cauchy integral $C[g]$ belongs to $H^2(\Gamma^+)$ and

$$\lim_{\substack{\underline{Y}\to\underline{U}\\ \underline{Y}\in\Gamma^{+}}} C[g](\underline{Y}) = g(\underline{U}), \quad \underline{U}\in\partial\Gamma,$$

in the L_2 sense, so that $C[g]$ may be seen as the $\partial_{\underline{X}}$-monogenic extension of g to Γ^+. For the corresponding Hardy space $H|^2(\Gamma^+)$ similar conclusions hold.

4. The Cauchy integral in Hermitean Clifford analysis

A first attempt at constructing Hermitean Hilbert transforms for functions in $L_2(\mathbb{R}^{2n})$ has been undertaken in [4]. However, although the obtained transforms showed some nice and satisfactory properties, one big issue remained unsolved at that moment: it seemed impossible to construct in the Hermitean context an h-monogenic Cauchy integral in $\mathbb{R}^{2n+1}_{\pm}$, such that those Hermitean Hilbert transforms could be retrieved as part of its non-tangential boundary limits.

A partial result to construct such a Hermitean Cauchy integral for functions in $L_2(\partial\Gamma)$, was obtained in [17], however presenting the "fake" – as termed by the authors – Cauchy kernel $\frac{1}{2}(E - iE|)$, which fails to be h-monogenic.

Of course, if only that class of functions $g\in L_2(\partial\Gamma)$ for which $H[g] = H|[g]$ would be considered, then the h-monogenic Cauchy integral is trivially given by $C[g]$ which in this case coincides with $C|[g]$. Indeed, for such functions g we have that

$C[g] - C|[g]$ is a harmonic function in $\Gamma^\pm$ with boundary limit equal to zero, as $C^\pm[g] = C|^\pm[g]$. On account of the maximum and the minimum principle for harmonic functions this yields that $C[g] = C|[g]$ in $\Gamma^\pm$.

In the general case, it appeared that a matrix approach is the key to obtain the desired result, see the recent paper [6]. In what follows we recall the main results.

First of all we introduce, for further use, the Hermitean counterparts of the pair of oriented surface elements $(\widetilde{d\sigma}_{\underline{X}}, \widetilde{d\sigma}_{\underline{X}|})$, applying the same technique as in Section 2, i.e. by means of the action of the projection operators $\frac{1}{2}(\mathbf{1} \pm iJ)$ on $\widetilde{d\sigma}_{\underline{X}}$, up to some deliberately chosen constant factor (see [6] for the specific choice of that constant). The Hermitean oriented surface elements $d\sigma_{\underline{Z}}$ and $d\sigma_{\underline{Z}^\dagger}$ are then defined as

$$d\sigma_{\underline{Z}} = -\frac{1}{4}(-1)^{\frac{n(n+1)}{2}}(2i)^n(\widetilde{d\sigma}_{\underline{X}} - i\widetilde{d\sigma}_{\underline{X}|}),$$
$$d\sigma_{\underline{Z}^\dagger} = -\frac{1}{4}(-1)^{\frac{n(n+1)}{2}}(2i)^n(\widetilde{d\sigma}_{\underline{X}} + i\widetilde{d\sigma}_{\underline{X}|}).$$

Similarly, we introduce $\mathcal{E} = -(E + iE|)$ and $\mathcal{E}^\dagger = (E - iE|)$, starting from the pair of fundamental solutions $(E, E|)$ to the orthogonal Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$. Explicitly this yields

$$\mathcal{E}(\underline{Z}) = \frac{2}{a_{2n}}\frac{\underline{Z}}{|\underline{Z}|^{2n}} \quad \text{and} \quad \mathcal{E}^\dagger(\underline{Z}) = \frac{2}{a_{2n}}\frac{\underline{Z}^\dagger}{|\underline{Z}|^{2n}}$$

with

$$\lim_{|\underline{Z}|\to\infty} \mathcal{E}(\underline{Z}) = 0 \quad \text{and} \quad \lim_{|\underline{Z}|\to\infty} \mathcal{E}^\dagger(\underline{Z}) = 0. \tag{5}$$

Note however that $\mathcal{E}$ and $\mathcal{E}^\dagger$ are not the fundamental solutions to the respective Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$! But surprisingly, introducing the particular circulant (2×2) matrices

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} = \begin{pmatrix} \partial_{\underline{Z}} & \partial_{\underline{Z}^\dagger} \\ \partial_{\underline{Z}^\dagger} & \partial_{\underline{Z}} \end{pmatrix}, \qquad \boldsymbol{\mathcal{E}} = \begin{pmatrix} \mathcal{E} & \mathcal{E}^\dagger \\ \mathcal{E}^\dagger & \mathcal{E} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\delta} = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$$

one obtains that $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\boldsymbol{\mathcal{E}}(\underline{Z}) = \boldsymbol{\delta}(\underline{Z})$, so that $\boldsymbol{\mathcal{E}}$ may be considered as a fundamental solution of $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}$, when this concept is reinterpreted in a matrical context. It was exactly this simple observation which has lead to the idea of following matrix approach in order to establish a Cauchy integral formula in the Hermitean setting, see [6].

Thus, in the same setting of circulant (2×2) matrices we associate, with arbitrary continuously differentiable functions g_1, g_2 and g defined in Ω and taking values in $\mathbb{C}_{2n}$, the respective matrix functions

$$\boldsymbol{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{G}_0 = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}.$$

We then call $\boldsymbol{G}_2^1$ (left) $\boldsymbol{H}$-monogenic if and only if it satisfies the system

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\boldsymbol{G}_2^1 = \boldsymbol{O}$$

where clearly $\boldsymbol{O}$ denotes the matrix with zero entries. This system explicitly reads

$$\begin{cases} \partial_{\underline{Z}}[g_1] + \partial_{\underline{Z}^\dagger}[g_2] = 0, \\ \partial_{\underline{Z}^\dagger}[g_1] + \partial_{\underline{Z}}[g_2] = 0. \end{cases}$$

Choosing in particular $g_1 = g$ and $g_2 = g^\dagger$, it is clear that, in general, the $\boldsymbol{H}$-monogenicity of the corresponding matrix

$$\boldsymbol{G} = \begin{pmatrix} g & g^\dagger \\ g^\dagger & g \end{pmatrix}$$

will not imply the h-monogenicity of the function g and vice versa. An exception to this general remark clearly occurs in the special case of scalar (i.e. complex) valued functions, where h-monogenicity (of g) and $\boldsymbol{H}$-monogenicity (of $\boldsymbol{G}$) are found to be equivalent notions.

Another special, yet very important, case occurs when considering the matrix $\boldsymbol{G}_0$: since its $\boldsymbol{H}$-monogenicity is easily seen to be equivalent with the h-monogenicity of g, this matrix forms the key for the construction of an h-monogenic extension to the function g by means of a matrical Cauchy kernel.

Finally note that we have found above that $\boldsymbol{\mathcal{E}}$ is $\boldsymbol{H}$-monogenic in $\mathbb{R}^{2n} \setminus \{\underline{0}\}$.

From now on we reserve the notations $\underline{Y}$ and $\underline{Y}|$ for Clifford vectors associated to points in $\Gamma^\pm$. Their Hermitean counterparts are denoted

$$\underline{V} = \frac{1}{2}(\mathbf{1} + iJ)[\underline{Y}] = \frac{1}{2}(\underline{Y} + i\underline{Y}|),$$
$$\underline{V}^\dagger = -\frac{1}{2}(\mathbf{1} - iJ)[\underline{Y}] = -\frac{1}{2}(\underline{Y} - i\underline{Y}|).$$

With the Hermitean vector pair $(\underline{Z}, \underline{Z}^\dagger)$, still corresponding as in Section 2 to the orthogonal pair $(\underline{X}, \underline{X}|)$, we associate the volume element $dW(\underline{Z}, \underline{Z}^\dagger)$ defined as

$$dW\big(\underline{Z}, \underline{Z}^\dagger\big) = \big(dz_1 \wedge dz_1^c\big) \wedge \big(dz_2 \wedge dz_2^c\big) \wedge \cdots \wedge \big(dz_n \wedge dz_n^c\big)$$

reflecting integration over each complex z_j-plane, $j = 1, \ldots, n$. The following Hermitean Cauchy–Pompeiu Formula was then established in [6].

Theorem 2 *(Hermitean Cauchy–Pompeiu Formula). If the functions g_1 and g_2 belong to $C^1(\Omega; \mathbb{C}_{2n})$, then*

$$\int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{G}_2^1(\underline{X}) - \int_{\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}) \big[\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{G}_2^1(\underline{X})\big]\, dW\big(\underline{Z}, \underline{Z}^\dagger\big) = \begin{cases} \boldsymbol{O}, & \text{if } \underline{Y} \in \Gamma^-, \\ (-1)^{\frac{n(n+1)}{2}} (2i)^n \boldsymbol{G}_2^1(\underline{Y}), & \text{if } \underline{Y} \in \Gamma^+. \end{cases}$$

This theorem then leads to the following Hermitean Cauchy Integral Formulae for $\boldsymbol{H}$-monogenic matrix functions $\boldsymbol{G}_2^1$ and h-monogenic functions g, respectively. Here we have introduced the additional matrix

$$\boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)} = \begin{pmatrix} d\sigma_{\underline{Z}} & -d\sigma_{\underline{Z}^\dagger} \\ -d\sigma_{\underline{Z}^\dagger} & d\sigma_{\underline{Z}} \end{pmatrix}.$$

Theorem 3 *(Hermitean Cauchy Integral Formula I). If the matrix function $\boldsymbol{G}_2^1$ is $\boldsymbol{H}$-monogenic in Ω then*

$$\int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{G}_2^1(\underline{X}) = \begin{cases} \boldsymbol{O}, & \text{if } \underline{Y} \in \Gamma^-, \\ (-1)^{\frac{n(n+1)}{2}} (2i)^n \boldsymbol{G}_2^1(\underline{Y}), & \text{if } \underline{Y} \in \Gamma^+. \end{cases} \tag{6}$$

Corollary 4 *(Hermitean Cauchy Integral Formula II). If the function g is h-monogenic in Ω then*

$$\int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{G}_0(\underline{X}) = \begin{cases} \boldsymbol{O}, & \text{if } \underline{Y} \in \Gamma^-, \\ (-1)^{\frac{n(n+1)}{2}} (2i)^n \boldsymbol{G}_0(\underline{Y}), & \text{if } \underline{Y} \in \Gamma^+. \end{cases} \tag{7}$$

The matrix function $\boldsymbol{\mathcal{E}}$ appearing in the Hermitean Cauchy Integral Formulae (6) and (7) is then called the Hermitean Cauchy kernel. For functions $g_1, g_2, g \in C^0(\partial\Gamma; \mathbb{C}_{2n})$ the following Hermitean Cauchy integrals $\boldsymbol{\mathcal{C}}[\boldsymbol{G}_0]$ and $\boldsymbol{\mathcal{C}}[\boldsymbol{G}_2^1]$ are then defined

$$\boldsymbol{\mathcal{C}}\big[\boldsymbol{G}_2^1\big](\underline{Y}) = \int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{G}_2^1(\underline{X}), \quad \underline{Y} \in \Gamma^\pm,$$

$$\boldsymbol{\mathcal{C}}[\boldsymbol{G}_0](\underline{Y}) = \int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{G}_0(\underline{X}), \quad \underline{Y} \in \Gamma^\pm,$$

which are both $\boldsymbol{H}$-monogenic in $\Gamma^\pm$, i.e.

$$\boldsymbol{\mathcal{D}}_{(\underline{V},\underline{V}^\dagger)} \boldsymbol{\mathcal{C}}\big[\boldsymbol{G}_2^1\big](\underline{Y}) = \boldsymbol{O} \quad \text{in } \Gamma^\pm$$

and

$$\boldsymbol{\mathcal{D}}_{(\underline{V},\underline{V}^\dagger)} \boldsymbol{\mathcal{C}}[\boldsymbol{G}_0](\underline{Y}) = \boldsymbol{O} \quad \text{in } \Gamma^\pm.$$

Notice that both integrals converge, also in case of Ω being not bounded, on account of (5). It was then shown in [6] that both Hermitean Cauchy integrals can be expressed in terms of the orthogonal Cauchy integrals C and $C|$, viz

$$\boldsymbol{\mathcal{C}}\big[\boldsymbol{G}_2^1\big] = (-1)^{\frac{n(n+1)}{2}} (2i)^n \left[\frac{1}{2} \begin{pmatrix} C[g_1 - g_2] & -C[g_1 - g_2] \\ -C[g_1 - g_2] & C[g_1 - g_2] \end{pmatrix} + \frac{1}{2} \begin{pmatrix} C|[g_1 + g_2] & C|[g_1 + g_2] \\ C|[g_1 + g_2] & C|[g_1 + g_2] \end{pmatrix}\right], \tag{8}$$

$$\boldsymbol{\mathcal{C}}[\boldsymbol{G}_0] = (-1)^{\frac{n(n+1)}{2}} (2i)^n \left[\frac{1}{2} \begin{pmatrix} C[g] & -C[g] \\ -C[g] & C[g] \end{pmatrix} + \frac{1}{2} \begin{pmatrix} C|[g] & C|[g] \\ C|[g] & C|[g] \end{pmatrix}\right]. \tag{9}$$

Taking in particular an $\boldsymbol{H}$-monogenic matrix function $\boldsymbol{G}_2^1$ in Ω, i.e. $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{G}_2^1 = \boldsymbol{O}$, or equivalently

$$\begin{cases} \partial_{\underline{X}}[g_1 - g_2] = 0, \\ \partial_{\underline{X}|}[g_1 + g_2] = 0 \end{cases}$$

then $C[g_1 - g_2] = g_1 - g_2$ and $C|[g_1 + g_2] = g_1 + g_2$ in Γ^+. Taking into account (8), we obtain that

$$\boldsymbol{\mathcal{C}}\big[\boldsymbol{G}_2^1\big] = (-1)^{\frac{n(n+1)}{2}} (2i)^n \boldsymbol{G}_2^1 \quad \text{in } \Gamma^+$$

in accordance with Theorem 3. On the other hand, if we take an h-monogenic function g in Ω, for which

$$C[g] = g = C|[g] \qquad \text{in } \Gamma^+$$

yields

$$\mathcal{C}[\boldsymbol{G}_0] = (-1)^{\frac{n(n+1)}{2}} (2i)^n \boldsymbol{G}_0 \quad \text{in } \Gamma^+$$

on account of (9), thus confirming Corollary 4.

5. A Hilbert transform in Hermitean Clifford analysis

Given functions $g_1, g_2, g \in L_2(\partial\Gamma)$, we will investigate in this section the non-tangential boundary behaviour of the Hermitean Cauchy integrals $\mathcal{C}[\boldsymbol{G}_2^1]$ and $\mathcal{C}[\boldsymbol{G}_0]$. To that end, we introduce the matrix operator

$$\boldsymbol{\mathcal{H}} = \frac{1}{2}\begin{pmatrix} H + H| & -H + H| \\ -H + H| & H + H| \end{pmatrix}$$

its action on the matrix functions $\boldsymbol{G}_2^1$ and $\boldsymbol{G}_0$ being given by matrix multiplication, followed by an operator action on the level of the entries, e.g.

$$\boldsymbol{\mathcal{H}}[\boldsymbol{G}_0] = \frac{1}{2}\begin{pmatrix} (H + H|)[g] & (-H + H|)[g] \\ (-H + H|)[g] & (H + H|)[g] \end{pmatrix}.$$

Now expressing $\mathcal{C}[\boldsymbol{G}_2^1]$ in terms of $C[g_1 - g_2]$ and $C|[g_1 + g_2]$ and $\mathcal{C}[\boldsymbol{G}_0]$ in terms of $C[g]$ and $C|[g]$, as in (8) and (9), respectively, and taking into account the classical Plemelj–Sokhotzki formulae (3) and (4), the following results are obtained.

Proposition 5. *For functions $g_1, g_2 \in L_2(\partial\Gamma)$, the non-tangential boundary values of its Hermitean Cauchy integral $\mathcal{C}[\boldsymbol{G}_2^1]$ are given by*

$$\mathcal{C}^{\pm}\big[\boldsymbol{G}_2^1\big](\underline{U}) = \lim_{\substack{\underline{Y}\to\underline{U} \\ \underline{Y}\in\Gamma^{\pm}}} \mathcal{C}\big[\boldsymbol{G}_2^1\big](\underline{Y}) = (-1)^{\frac{n(n+1)}{2}} (2i)^n \left(\pm\frac{1}{2}\boldsymbol{G}_2^1(\underline{U}) + \frac{1}{2}\boldsymbol{\mathcal{H}}\big[\boldsymbol{G}_2^1\big](\underline{U})\right), \quad \underline{U} \in \partial\Gamma.$$

Corollary 6. *For a function $g \in L_2(\partial\Gamma)$, the non-tangential boundary values of its Hermitean Cauchy integral $\mathcal{C}[\boldsymbol{G}_0]$ are given by*

$$\mathcal{C}^{\pm}[\boldsymbol{G}_0](\underline{U}) = \lim_{\substack{\underline{Y}\to\underline{U} \\ \underline{Y}\in\Gamma^{\pm}}} \mathcal{C}[\boldsymbol{G}_0](\underline{Y}) = (-1)^{\frac{n(n+1)}{2}} (2i)^n \left(\pm\frac{1}{2}\boldsymbol{G}_0(\underline{U}) + \frac{1}{2}\boldsymbol{\mathcal{H}}[\boldsymbol{G}_0](\underline{U})\right), \quad \underline{U} \in \partial\Gamma.$$

We call the matrix operator $\boldsymbol{\mathcal{H}}$ the Hermitean Hilbert transform. The matrix function $\boldsymbol{G}_0$ being a special case of $\boldsymbol{G}_2^1$, we now only focus on the last one. Our aim is to establish for that matrical Hilbert transform the traditional properties, similar to those mentioned in Proposition 1. To this end, we first create the proper framework for dealing with circulant (2×2) matrix functions. First of all, we introduce the vector space

$$\boldsymbol{L_2}(\partial\Gamma) = \left\{\boldsymbol{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix} : \ g_1, g_2 \in L_2(\partial\Gamma)\right\}$$

on which, inspired by the $\mathbb{C}_{2n}$ valued inner product $\langle\cdot,\cdot\rangle$ on $L_2(\partial\Gamma)$ given by

$$\langle f, g\rangle = \int_{\partial\Gamma} f^{\dagger}(\underline{X}) g(\underline{X})\, dS(\underline{X})$$

we introduce the following bilinear form:

$$\langle\cdot,\cdot\rangle_{\boldsymbol{L_2}} : \boldsymbol{L_2}(\partial\Gamma) \times \boldsymbol{L_2}(\partial\Gamma) \longrightarrow (\mathbb{C}_{2n})^{2\times 2};$$
$$\left(\begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix}, \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix}\right) \longmapsto \begin{pmatrix} \langle f_1, g_1\rangle + \langle f_2, g_2\rangle & \langle f_1, g_2\rangle + \langle f_2, g_1\rangle \\ \langle f_1, g_2\rangle + \langle f_2, g_1\rangle & \langle f_1, g_1\rangle + \langle f_2, g_2\rangle \end{pmatrix}.$$

In the lemma below, the proof of which consists of direct calculations, it is stated that $\langle\cdot,\cdot\rangle_{\boldsymbol{L_2}}$ is a $(\mathbb{C}_{2n})^{2\times 2}$ valued inner product. Notice that we introduce there the Hermitean conjugation on circulant elements of $(\mathbb{C}_{2n})^{2\times 2}$, defined as follows:

$$\begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}^{\boldsymbol{\dagger}} = \begin{pmatrix} a_1^{\dagger} & a_2^{\dagger} \\ a_2^{\dagger} & a_1^{\dagger} \end{pmatrix}, \quad a_1, a_2 \in \mathbb{C}_{2n}.$$

Lemma 7. *For $\boldsymbol{F}_2^1, \boldsymbol{G}_2^1, \boldsymbol{K}_2^1 \in \boldsymbol{L_2}(\partial\Gamma)$ and $\lambda \in \mathbb{C}$, we have*

(i) $\langle \boldsymbol{F}_2^1, \lambda \boldsymbol{G}_2^1 + \boldsymbol{K}_2^1\rangle_{\boldsymbol{L_2}} = \lambda\langle \boldsymbol{F}_2^1, \boldsymbol{G}_2^1\rangle_{\boldsymbol{L_2}} + \langle \boldsymbol{F}_2^1, \boldsymbol{K}_2^1\rangle_{\boldsymbol{L_2}}$;
(ii) *if for all $\boldsymbol{F}_2^1 \in \boldsymbol{L_2}(\partial\Gamma)$: $\langle \boldsymbol{F}_2^1, \boldsymbol{G}_2^1\rangle_{\boldsymbol{L_2}} = \boldsymbol{O}$, then $\boldsymbol{G}_2^1 = \boldsymbol{O}$*;
(iii) $(\langle \boldsymbol{F}_2^1, \boldsymbol{G}_2^1\rangle_{\boldsymbol{L_2}})^\dagger = \langle \boldsymbol{G}_2^1, \boldsymbol{F}_2^1\rangle_{\boldsymbol{L_2}}$.

Next, we also consider the Hardy spaces

$$\boldsymbol{H^2}(\Gamma^+) = \left\{\boldsymbol{G}_2^1 : \Gamma^+ \to (\mathbb{C}_{2n})^{2\times 2}\colon\ \boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\boldsymbol{G}_2^1 = \boldsymbol{O} \text{ in } \Gamma^+ \text{ and } g_{1|\partial\Gamma}, g_{2|\partial\Gamma} \in L_2(\partial\Gamma)\right\}$$

and $\boldsymbol{H^2}(\partial\Gamma)$, being the $\boldsymbol{L_2}(\partial\Gamma)$-closure of the set of boundary values of elements of $\boldsymbol{H^2}(\Gamma^+)$.

Finally, we need a notion of a so-called "matrical outward pointing unit vector." An apt choice for our purpose is

$$\boldsymbol{\mathcal{V}} = \frac{1}{2}\begin{pmatrix} \nu + \nu| & -\nu + \nu| \\ -\nu + \nu| & \nu + \nu| \end{pmatrix}$$

observing that indeed $\boldsymbol{\mathcal{V}}^2 = -\boldsymbol{\mathcal{I}}$, $\boldsymbol{\mathcal{I}}$ being the (2×2) identity matrix operator. The Hermitean Hilbert transform $\boldsymbol{\mathcal{H}}$ then satisfies the following properties.

Theorem 8. *One has*

$\boldsymbol{\mathcal{P}}(\mathbf{1})$ *$\boldsymbol{\mathcal{H}}$ is a bounded linear operator on $\boldsymbol{L_2}(\partial\Gamma)$*;
$\boldsymbol{\mathcal{P}}(\mathbf{2})$ $\boldsymbol{\mathcal{H}}^2 = \boldsymbol{\mathcal{I}}$;
$\boldsymbol{\mathcal{P}}(\mathbf{3})$ $\boldsymbol{\mathcal{H}}^* = \boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{V}}$ *(w.r.t. $\langle\cdot,\cdot\rangle_{\boldsymbol{L_2}}$)*;
$\boldsymbol{\mathcal{P}}(\mathbf{4})$ *for $\boldsymbol{G}_2^1 \in \boldsymbol{L_2}(\partial\Gamma)$, we have that $\boldsymbol{\mathcal{H}}[\boldsymbol{G}_2^1] = \boldsymbol{G}_2^1$ if and only if $\boldsymbol{G}_2^1 \in \boldsymbol{H^2}(\partial\Gamma)$.*

Proof. $\boldsymbol{\mathcal{P}}(\mathbf{1})$ Follows from the fact that both H and $H|$ are bounded linear operators on $L_2(\partial\Gamma)$.

$\boldsymbol{\mathcal{P}}(\mathbf{2})$ As both H and $H|$ are involutory operators, we obtain:

$$\begin{aligned}\boldsymbol{\mathcal{H}}^2 &= \frac{1}{2}\begin{pmatrix} H + H| & -H + H| \\ -H + H| & H + H| \end{pmatrix}\frac{1}{2}\begin{pmatrix} H + H| & -H + H| \\ -H + H| & H + H| \end{pmatrix}\\ &= \frac{1}{4}\begin{pmatrix} H^2 + H|^2 + H^2 + H|^2 & -H^2 + H|^2 - H^2 + H|^2 \\ -H^2 + H|^2 - H^2 + H|^2 & H^2 + H|^2 + H^2 + H|^2 \end{pmatrix}\\ &= \boldsymbol{\mathcal{I}}.\end{aligned}$$

$\boldsymbol{\mathcal{P}}(\mathbf{3})$ Let $\boldsymbol{F}_2^1$ and $\boldsymbol{G}_2^1$ belong to $\boldsymbol{L_2}(\partial\Gamma)$ and be given explicitly by

$$\boldsymbol{F}_2^1 = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix}; \qquad \boldsymbol{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix}.$$

We then have

$$\left\langle \boldsymbol{\mathcal{H}}\left[\boldsymbol{F}_2^1\right], \boldsymbol{G}_2^1\right\rangle_{\boldsymbol{L_2}} = \left\langle \frac{1}{2}\begin{pmatrix} H + H| & -H + H| \\ -H + H| & H + H| \end{pmatrix}\begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix}, \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix}\right\rangle_{\boldsymbol{L_2}} = \frac{1}{2}\begin{pmatrix} k_1 & k_2 \\ k_2 & k_1 \end{pmatrix}$$

with

$$\begin{aligned}k_1 &= \left\langle (H + H|)[f_1] + (-H + H|)[f_2], g_1\right\rangle + \left\langle (H + H|)[f_2] + (-H + H|)[f_1], g_2\right\rangle\\ &= \left\langle f_1, \left(H^* + H|^*\right)[g_1] + \left(-H^* + H|^*\right)[g_2]\right\rangle + \left\langle f_2, \left(H^* + H|^*\right)[g_2] + \left(-H^* + H|^*\right)[g_1]\right\rangle\end{aligned}$$

and

$$\begin{aligned}k_2 &= \left\langle (H + H|)[f_1] + (-H + H|)[f_2], g_2\right\rangle + \left\langle (H + H|)[f_2] + (-H + H|)[f_1], g_1\right\rangle\\ &= \left\langle f_1, \left(H^* + H|^*\right)[g_2] + \left(-H^* + H|^*\right)[g_1]\right\rangle + \left\langle f_2, \left(H^* + H|^*\right)[g_1] + \left(-H^* + H|^*\right)[g_2]\right\rangle.\end{aligned}$$

So

$$\begin{aligned}\left\langle \boldsymbol{\mathcal{H}}\left[\boldsymbol{F}_2^1\right], \boldsymbol{G}_2^1\right\rangle_{\boldsymbol{L_2}} &= \left\langle \begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} (H^* + H|^*)[g_1] & (H^* + H|^*)[g_2] \\ (H^* + H|^*)[g_2] & (H^* + H|^*)[g_1] \end{pmatrix}\right\rangle_{\boldsymbol{L_2}}\\ &\quad + \left\langle \begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} (-H^* + H|^*)[g_2] & (-H^* + H|^*)[g_1] \\ (-H^* + H|^*)[g_1] & (-H^* + H|^*)[g_2] \end{pmatrix}\right\rangle_{\boldsymbol{L_2}}\\ &= \left\langle \boldsymbol{F}_2^1, \boldsymbol{\mathcal{H}}^*\left[\boldsymbol{G}_2^1\right]\right\rangle_{\boldsymbol{L_2}}\end{aligned}$$

where we have put

$$\boldsymbol{\mathcal{H}}^* = \frac{1}{2}\begin{pmatrix} H^* + H|^* & -H^* + H|^* \\ -H^* + H|^* & H^* + H|^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \nu H\nu + \nu|H|\nu| & -\nu H\nu + \nu|H|\nu| \\ -\nu H\nu + \nu|H|\nu| & \nu H\nu + \nu|H|\nu| \end{pmatrix} = \boldsymbol{\mathcal{V}\mathcal{H}\mathcal{V}}.$$

$\mathcal{P}(\mathbf{4})$ Let $\boldsymbol{G}_2^1 \in \boldsymbol{L_2}(\partial\Gamma)$. Then $\boldsymbol{G}_2^1 \in \boldsymbol{H^2}(\partial\Gamma)$ if and only if $\boldsymbol{G}_2^1$ is the non-tangential boundary limit of a certain $\boldsymbol{F}_2^1 \in \boldsymbol{H^2}(\Gamma^+)$, i.e. if and only if there exists a matrix function $\boldsymbol{F}_2^1 : \Gamma^+ \to (\mathbb{C}_{2n})^{2\times 2}$ such that

$$\lim_{\Gamma^+ \xrightarrow{NT} \partial\Gamma} \boldsymbol{F}_2^1 = \boldsymbol{G}_2^1 \quad \text{and} \quad \mathcal{D}_{(\underline{Z},\underline{Z}^\dagger)}\boldsymbol{F}_2^1 = \boldsymbol{O}. \tag{10}$$

Putting explicitly

$$\boldsymbol{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix}; \qquad \boldsymbol{F}_2^1 = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix}$$

then the characterization (10) of $\boldsymbol{G}_2^1 \in \boldsymbol{H^2}(\partial\Gamma)$ reads in terms of the matrix entries: there exist functions $f_1, f_2 : \Gamma^+ \to \mathbb{C}_{2n}$ such that

$$\begin{cases} \displaystyle\lim_{\Gamma^+ \xrightarrow{NT} \partial\Gamma} (f_1 - f_2) = g_1 - g_2, \\ \displaystyle\lim_{\Gamma^+ \xrightarrow{NT} \partial\Gamma} (f_1 + f_2) = g_1 + g_2 \end{cases} \quad \text{and} \quad \begin{cases} \partial_{\underline{X}}[f_1 - f_2] = 0, \\ \partial_{\underline{X}|}[f_1 + f_2] = 0 \end{cases}$$

which is equivalent with

$$g_1 - g_2 \in H^2(\partial\Gamma) \quad \text{and} \quad g_1 + g_2 \in H|^2(\partial\Gamma)$$

or with

$$H[g_1 - g_2] = g_1 - g_2 \quad \text{and} \quad H|[g_1 + g_2] = g_1 + g_2 \tag{11}$$

when taking into account P(4) of Proposition 1. This ends the proof since (11) is equivalent to $\boldsymbol{\mathcal{H}}[\boldsymbol{G}_2^1] = \boldsymbol{G}_2^1$. $\square$

These properties then lead to following orthogonal decomposition of $\boldsymbol{L_2}(\partial\Gamma)$ in terms of the Hardy space $\boldsymbol{H^2}(\partial\Gamma)$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\boldsymbol{L_2}}$.

Proposition 9.

$$\boldsymbol{L_2}(\partial\Gamma) = \boldsymbol{H^2}(\partial\Gamma) \oplus \boldsymbol{\mathcal{V}H^2}(\partial\Gamma).$$

Proof. For all $\boldsymbol{F}_2^1 \in \boldsymbol{L_2}(\partial\Gamma)$ one has that $\boldsymbol{F}_2^1 + \boldsymbol{\mathcal{H}}[\boldsymbol{F}_2^1] \in \boldsymbol{H^2}(\partial\Gamma)$ since $\boldsymbol{\mathcal{H}}[\boldsymbol{F}_2^1 + \boldsymbol{\mathcal{H}}[\boldsymbol{F}_2^1]] = \boldsymbol{F}_2^1 + \boldsymbol{\mathcal{H}}[\boldsymbol{F}_2^1]$. Now take an arbitrary matrix function $\boldsymbol{G}_2^1 \in (\boldsymbol{H^2}(\partial\Gamma))^\perp$, then

$$0 = \langle \boldsymbol{F}_2^1 + \boldsymbol{\mathcal{H}}[\boldsymbol{F}_2^1], \boldsymbol{G}_2^1 \rangle_{\boldsymbol{L_2}} = \langle \boldsymbol{F}_2^1, \boldsymbol{G}_2^1 + \boldsymbol{\mathcal{H}}^*[\boldsymbol{G}_2^1] \rangle_{\boldsymbol{L_2}}$$

which leads to

$$\boldsymbol{G}_2^1 \in (\boldsymbol{H^2}(\partial\Gamma))^\perp \iff \boldsymbol{\mathcal{H}}^*[\boldsymbol{G}_2^1] = -\boldsymbol{G}_2^1 \iff \boldsymbol{\mathcal{H}}[\boldsymbol{\mathcal{V}G}_2^1] = \boldsymbol{\mathcal{V}G}_2^1 \iff \boldsymbol{\mathcal{V}G}_2^1 \in \boldsymbol{H^2}(\partial\Gamma)$$
$$\iff \boldsymbol{G}_2^1 \in \boldsymbol{\mathcal{V}H^2}(\partial\Gamma). \qquad \square$$

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